Canad. Math. Bull. Vol. **60** (2), 2017 pp. 372–380
http://dx.doi.org/10.4153/CMB-2016-083-x

Coaxer Lattices

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Abstract. The notion of coaxers is introduced in a pseudo-complemented distributive lattice. Boolean algebras are characterized in terms of coaxer ideals and congruences. The concept of coaxer lattices is introduced in pseudo-complemented distributive lattices and characterized in terms of coaxer ideals and maximal ideals. Finally, the coaxer lattices are also characterized in topological terms.

Introduction

The theory of pseudo-complements in lattices, and particularly in distributive lattices, was developed by M. H. Stone [8], O. Frink [4], and G. Gratzer [5]. Later, many authors extended the study of pseudo-complements to characterize Stone lattices; see, for example R. Balbes [1] and O. Frink [4]. Properties of various types of congruences were studied in distributive lattices by W. H. Cornish [3] and T. P. Speed [6].

In this paper, the concept of coaxer ideals is introduced in pseudo-complemented distributive lattices. Some properties of coaxer ideals are observed and then a set of equivalent conditions is established for every coaxer ideal to be a principal ideal, which leads to a characterization of Boolean algebras. The concept of coaxer lattices is introduced in pseudo-complemented distributive lattices. Coaxer lattices are then characterized in terms of maximal ideals and coaxer ideals. A necessary and sufficient condition is derived for every sublattice of a coaxer lattice to be a coaxer lattice, which is not true in general. Coaxer ideals are also characterized in terms of kernels and antikernels of congruences. Finally, coaxer lattices are also characterized in topological terms.

The reader is refered to [5] for definitions and notation. However, some preliminary definitions and results are presented for the ready reference of the reader. Throughout the rest of this note all lattices are bounded and pseudo-complemented distributive lattices.

The pseudo-complement b^* of an element b is the greatest element disjoint from b, if such an element exists. The defining property of b^* is

$$a \wedge b = 0 \Longleftrightarrow a \wedge b^* = a \Longleftrightarrow a \leq b^*,$$

where $\leq$ is a partial ordering relation on the lattice L.

A distributive lattice L in which every element has a pseudo-complement is called a *pseudo-complemented distributive lattice.* For any two elements a, b of a pseudo-complemented lattice, we have

Received by the editors February 21, 2014; revised October 27, 2015.
Published electronically February 28, 2017.
AMS subject classification: **06D99**.
Keywords: pseudo-complemented distributive lattice, coaxer ideal, coaxer lattice, maximal ideal, congruence, kernel, antikernel.

(1) $a \le b$ implies $b^* \le a^*$,
(2) $a \le a^{**}$,
(3) $a^{***} = a^*$,
(4) $(a \vee b)^* = a^* \wedge b^*$,
(5) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$.

An element a of L is called *dense* if $a^* = 0$. The set D of all dense elements of L forms a filter in L.

A prime ideal P of a lattice L is called a *minimal prime ideal* if there is no prime ideal Q such that $Q \subset P$. A prime ideal P of L is minimal [2] if and only if for each $x \in P$ there exists $y \notin P$ such that $x \wedge y = 0$. Let P be a prime ideal of L. Then the following conditions are equivalent:

(a) P is a minimal prime ideal;
(b) $x \in P$ implies $x^* \notin P$;
(c) $P \cap D = \emptyset$.

1 Coaxer Ideals and Coaxer Lattices

In this section, the concepts of coaxer ideals and coaxer lattices are introduced in a pseudo-complemented distributive lattice. Coaxer lattices are characterized in terms of maximal ideals and coaxer ideals. Finally, coaxer lattices are characterized in topological terms.

Definition 1.1 Let L be a pseudo-complemented distributive lattice. For any $a \in L$, the coaxer of a is defined as the set $(a)^\circ = \{x \in L \mid x^* \vee a = 1\}$.

Evidently, $(0)^\circ = \{0\}$ and $(1)^\circ = L$.

Lemma 1.2 *For any $a \in L$, $(a)^\circ$ is an ideal of L.*

Proof Clearly, $0 \in (a)^\circ$. Let $x, y \in (a)^\circ$. Then $x^* \vee a = 1$ and $y^* \vee a = 1$. Therefore, $(x \vee y)^* \vee a = (x^* \wedge y^*) \vee a = (x^* \vee a) \wedge (y^* \vee a) = 1 \wedge 1 = 1$. Hence, $x \vee y \in (a)^\circ$. Let $x \in (a)^\circ$ and $r \in L$. Then $x^* \vee a = 1$, so $1 = x^* \vee a \le (x \wedge r)^* \vee a$. Thus, $x \wedge r \in (a)^\circ$. Therefore, $(a)^\circ$ is an ideal of L. ∎

In the following lemma, some more properties of coaxers can be observed.

Lemma 1.3 *For any $a, b \in L$, we have the following:*

(i) $a \le b \Rightarrow (a)^\circ \subseteq (b)^\circ$;
(ii) $a \vee b = 1$ *implies* $a^* \in (b)^\circ$;
(iii) $(a)^\circ \subseteq (a]$;
(iv) $(a)^\circ \cap (b)^\circ = (a \wedge b)^\circ$;
(v) $(a)^\circ = L$ *if and only if* $a = 1$.

Proof (i) Suppose $a \le b$. Let $x \in (a)^\circ$. Hence, $1 = x^* \vee a \le x^* \vee b$.

(ii) This is clear.

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(iii) Let $x \in (a)^\circ$. Then $x^* \vee a = 1$. Hence, we get $a \wedge x = 0 \vee (a \wedge x) = (x^* \wedge x) \vee (a \wedge x) = (x^* \vee a) \wedge x = 1 \wedge x = x$. Therefore, $x \in (a]$.

(iv) Clearly, $(a \wedge b)^\circ \subseteq (a)^\circ \cap (b)^\circ$. Conversely, let $x \in (a)^\circ \cap (b)^\circ$. Hence, $x^* \vee (a \wedge b) = (x^* \vee a) \wedge (x^* \vee b) = 1 \wedge 1 = 1$. Hence, $x \in (a \wedge b)^\circ$.

(v) Suppose $(a)^\circ = L$. Then we get $1 \in (a)^\circ$. Hence, we obtain $a = 0 \vee a = 1^* \vee a = 1$. The converse is clear. ■

The ideals $(a)^\circ$, $a \in L$ in the above result are called coaxer ideals. Then we have the following lemma.

Lemma 1.4 *Every proper coaxer ideal of L is contained in a minimal prime ideal.*

Proof Let $(a)^\circ$ be a proper coaxer ideal of L. Suppose $(a)^\circ \cap D \neq \emptyset$ and $d \in (a)^\circ \cap D$. Then $0 \vee a = d^* \vee a = 1$. Hence, $a = 1$. Therefore, $(a)^\circ = L$, which is a contradiction. Hence, $(a)^\circ \cap D = \emptyset$. Then there exists a prime ideal P of L such that $(a)^\circ \subseteq P$ and $P \cap D = \emptyset$. Let $x \in P$. Then $x \vee x^* \in D$ and hence $x \vee x^* \notin P$. Thus, $x^* \notin P$. Therefore, P is a minimal prime ideal of L such that $(a)^\circ \subseteq P$. ■

Definition 1.5 For any maximal ideal M of L, define

$$\pi(M) = \{x \in L \mid x^* \notin M\}.$$

Proposition 1.6 *For any maximal ideal M of L, $\pi(M)$ is an ideal of L such that $\pi(M) \subseteq M$.*

Proof Assume that M is a maximal ideal of L. Then M is a prime ideal. Clearly $0 \in \pi(M)$. Let $x, y \in \pi(M)$. Then $x^* \notin M$ and $y^* \notin M$. Since M is prime, we get $(x \vee y)^* = x^* \wedge y^* \notin M$. Hence, $x \vee y \in \pi(M)$. Let $x \in \pi(M)$ and $r \in L$. Then $x^* \notin M$. Hence, $(x \wedge r)^* \notin M$, otherwise $x^* \in M$. Thus, $x \wedge r \in \pi(M)$. Therefore, $\pi(M)$ is an ideal of L. Let $x \in \pi(M)$. Then $x^* \notin M$. Since M is prime and $x \wedge x^* = 0 \in M$, we get $x \in M$. Therefore, $\pi(M) \subseteq M$. ■

Let us denote the class of all maximal ideals of L by μ and let $\mu_a = \{M \in \mu \mid a \in M\}$. Then we have the following theorem.

Theorem 1.7 *For any $a \in L$, $(a)^\circ = \bigcap_{M \in \mu_a} \pi(M)$.*

Proof Let $I_0 = \bigcap_{M \in \mu_a} \pi(M)$. Let $x \in (a)^\circ$ and $M \in \mu_a$. Then $x^* \vee a = 1$. If $x^* \in M$, then $1 = x^* \vee a \in M$, which is a contradiction. Hence, $x \in \pi(M)$. This is true for all $M \in \mu_a$. Hence $(a)^\circ \subseteq I_0$. Conversely, let $x \in I_0$. Then $x \in \pi(M)$ for all $M \in \mu_a$. Suppose $x^* \vee a \neq 1$. Then there exists a maximal ideal M_0 of L such that $x^* \vee a \in M_0$. Hence, $x^* \in M_0$ and $a \in M_0$. Since $a \in M_0$, by our assumption $x \in \pi(M_0)$. Hence, it yields that $x^* \notin M_0$, which is a contradiction. Hence, $x^* \vee a = 1$. Thus, $x \in (a)^\circ$ and hence $I_0 \subseteq (a)^\circ$. This completes the proof of the theorem. ■

Corollary 1.8 *Let $M \in \mu$. If $a \in M$, then $(a)^\circ \subseteq \pi(M)$.*

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In general, the set $\mathcal{C}^{\circ}(L)$ of all coaxer ideals of L is not a sublattice of the lattice $\mathcal{I}(L)$ of all ideals of L. However, we will establish a set of equivalent conditions for $\mathcal{C}^{\circ}(L)$ to be a sublattice of $\mathcal{I}(L)$. For this we need the following lemma.

Lemma 1.9 *A proper ideal M of L is maximal if and only if for each $x \notin M$, there exists $y \in M$ such that $x \vee y = 1$.*

Proof Assume that M is maximal and let $x \notin M$. Then $M \vee (x] = L$. Hence, $1 = a \vee x$ for some $a \in M$. Conversely assume the condition and suppose M is not maximal. Then there exists a proper ideal Q such that $M \subset Q$. Choose $x \in Q - M$. Then there exists $y \in M$ such that $x \vee y = 1$. Since $x \in Q$ and $y \in M \subset Q$, we get $1 = x \vee y \in Q$, which is a contradiction. ∎

We now introduce the concept of coaxer lattices.

Definition 1.10 A pseudo-complemented distributive lattice L is called a *coaxer lattice* if $\pi(M) = M$ for every $M \in \mu$.

Example 1.11 Every Boolean algebra is a coaxer lattice. Let L be a Boolean algebra and M a maximal ideal of L. Clearly $\pi(M) \subseteq M$. Conversely, let $x \in M$. Then there exists some $y \in L$ such that $x \wedge y = 0$ and $x \vee y = 1$. Hence $y \leq x^*$ and $x \vee y = 1 \notin M$, which implies that $y \notin M$. Thus, $x^* \notin M$, and hence $x \in \pi(M)$. Therefore, L is coaxer.

We first characterize the class of coaxer lattices in the following theorem.

Theorem 1.12 *The following conditions are equivalent:*

(i) *L is coaxer;*
(ii) *for any $a, b \in L$, $a \vee b = 1$ implies $(a)^{\circ} \vee (b)^{\circ} = L$;*
(iii) *for any $a, b \in L$, $(a)^{\circ} \vee (b)^{\circ} = (a \vee b)^{\circ}$;*
(iv) *for any two distinct maximal ideals M and N, $\pi(M) \vee \pi(N) = L$;*
(v) *for any $M \in \mu$, M is the unique member of μ such that $\pi(M) \subseteq M$;*
(vi) *for any $M \in \mu$, $\pi(M)$ is maximal.*

Proof (i)⇒(ii) Assume that L is coaxer. Let $a, b \in L$ be such that $a \vee b = 1$. Suppose $(a)^{\circ} \vee (b)^{\circ} \neq L$. Then there exists a maximal ideal M such that $(a)^{\circ} \vee (b)^{\circ} \subseteq M$. Hence, $(a)^{\circ} \subseteq M$ and $(b)^{\circ} \subseteq M$. Now

$$\begin{aligned} (a)^{\circ} \subseteq M &\Rightarrow \bigcap_{M \in \mu_a} \pi(M) \subseteq M \\ &\Rightarrow \pi(M_i) \subseteq M && \text{for some } M_i \in \mu_a \text{ (since } M \text{ is prime)} \\ &\Rightarrow M_i \subseteq M && \text{since } L \text{ is coaxer} \\ &\Rightarrow a \in M \\ &\Rightarrow a \notin L - M \end{aligned}$$

Similarly, we can get $b \notin L - M$. Since $L - M$ is a prime filter, we get $1 = a \vee b \notin L - M$, which is a contradiction. Therefore, $(a)^{\circ} \vee (b)^{\circ} = L$.

(ii)⇒(iii) Assume condition (ii). Let $a, b \in L$. Clearly, $(a)^{\circ} \vee (b)^{\circ} \subseteq (a \vee b)^{\circ}$. Let $x \in (a \vee b)^{\circ}$. Then $(x^* \vee a) \vee (x^* \vee b) = x^* \vee a \vee b = 1$. Hence, by condition (ii), we

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get $(x^* \vee a)^\circ \vee (x^* \vee b)^\circ = L$. Thus, $x \in L = (x^* \vee a)^\circ \vee (x^* \vee b)^\circ$. Hence, $x = r \vee s$ for some $r \in (x^* \vee a)^\circ$ and $s \in (x^* \vee b)^\circ$. Now

$$\begin{aligned} r \in (x^* \vee a)^\circ &\Rightarrow r^* \vee x^* \vee a = 1 \\ &\Rightarrow 1 = (r^* \vee x^*) \vee a \leq (r \wedge x)^* \vee a \\ &\Rightarrow (r \wedge x)^* \vee a = 1 \\ &\Rightarrow r \wedge x \in (a)^\circ. \end{aligned}$$

Similarly, we can get $s \wedge x \in (b)^\circ$. Hence,

$$x = x \wedge x = x \wedge (r \vee s) = (x \wedge r) \vee (x \wedge s) \in (a)^\circ \vee (b)^\circ.$$

Hence, $(a \vee b)^\circ \subseteq (a)^\circ \vee (b)^\circ$. Therefore, $(a)^\circ \vee (b)^\circ = (a \vee b)^\circ$.

(iii)⇒(iv) Assume condition (iii). Let M, N be two distinct maximal ideals of L. Choose $x \in M - N$ and $y \in N - M$. Now

$$\begin{aligned} x \notin N &\Rightarrow \text{ there exists } x_1 \in N \text{ such that } x \vee x_1 = 1 \\ y \notin M &\Rightarrow \text{ there exists } y_1 \in M \text{ such that } y \vee y_1 = 1. \end{aligned}$$

Hence, $(x \vee y_1) \vee (y \vee x_1) = (x \vee x_1) \vee (y \vee y_1) = 1$. Now

$$\begin{aligned} L = (1)^\circ &= \{(x \vee y_1) \vee (y \vee x_1)\}^\circ \\ &= (x \vee y_1)^\circ \vee (y \vee x_1))^\circ \\ &\subseteq \pi(M) \vee \pi(N) \qquad \text{since } x \vee y_1 \in M, y \vee x_1 \in N \end{aligned}$$

Therefore, $\pi(M) \vee \pi(N) = L$.

(iv)⇒(v) Assume condition (iv). Let $M \in \mu$. Suppose $N \in \mu$ such that $N \neq M$ and $\pi(N) \subseteq M$. Since $\pi(M) \subseteq M$, by hypothesis, we get $L = \pi(M) \vee \pi(N) = M$, which is a contradiction. Therefore, M is the unique maximal ideal such that $\pi(M)$ is contained in M.

(v)⇒(vi) Assume condition (v). Let $M \in \mu$. Suppose $\pi(M)$ is not maximal. Let M_0 be a maximal ideal of L such that $\pi(M) \subseteq M_0$. We have always $\pi(M_0) \subseteq M_0$, which is a contradiction to the hypothesis.

(vi)⇒(i) This is clear. ∎

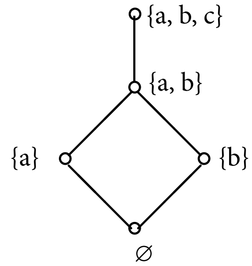
Corollary 1.13 *If every chain of a lattice L has at most three elements, then L is a coaxer lattice.*

Proof Assume that every chain of L contains at most three elements. Let $x, y \in L$ be such that $x \vee y = 1$. If $x = 1$ or $y = 1$, then clearly $(x)^\circ \vee (y)^\circ = L$. Suppose $x \neq 1$ and $y \neq 1$. Then $x \wedge y \leq x < 1$. If $x \wedge y = x$, then $y = (x \wedge y) \vee y = x \vee y = 1$, which is a contradiction. Hence, $x \wedge y < x < 1$. Therefore, by hypothesis, $x \wedge y = 0$. Thus, $x \leq y^*$ and $y \leq x^*$. Therefore $1 = x \vee y \leq x \vee x^*$. Thus, $x \in (x)^\circ$. Similarly, we get $y \in (y)^\circ$. Therefore, $1 = x \vee y \in (x)^\circ \vee (y)^\circ$. Thus, $(x)^\circ \vee (y)^\circ = L$. Therefore, L is coaxer. ∎

In the following, we observe that every sublattice of a coaxer lattice need not be a coaxer lattice. Consider $X = \{a, b, c, d\}$ and let

$$L = \Big\{\varnothing, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\Big\}$$

be a subset of the power set $\wp(X)$ of X. Then clearly L is a sublattice of the Boolean algebra $\wp(X)$ (whose Hasse diagram is given in the following figure).



Since $\wp(X)$ is a Boolean algebra, it is coaxer. But L is not coaxer, because $M = \{\varnothing, \{a\}, \{b\}, \{a,b\}\}$ is a maximal ideal of L and $\pi(M) = \{\varnothing\}$.

However, we have the following theorem.

Theorem 1.14 *Let L be a pseudo-complemented distributive lattice. Then every sublattice of L is coaxer if and only if for all $x, y \in L - \{1\}$, $x \vee y = 1$ implies that $x \wedge y = 0$.*

Proof Assume that every sublattice of L is a coaxer lattice. Let $x, y \in L - \{1\}$ such that $x \vee y = 1$. Suppose there exists $z \in L$ such that $0 < z < x \wedge y$. Consider $L_1 = \{0, z, x \wedge y, x, y, 1\}$. Then clearly L_1 is a sublattice of L. But L_1 is not a coaxer, because $M = \{0, z, x \wedge y, x, y\}$ is a maximal ideal of L_1 and $\pi(M) = \{0\} \neq M$, which is a contradiction to the hypothesis. Therefore, we must have $x \wedge y = 0$.

Conversely, assume the condition. Let L_1 be a sublattice of L. Let $x, y \in L_1$ be such that $x \vee y = 1$. Define $(a)^\circ_{L_1} = (a)^\circ \cap L_1$ for any $a \in L_1$. If $x = 1$ or $y = 1$, then clearly $(x)^\circ_{L_1} \vee (y)^\circ_{L_1} = L_1$. Suppose $x \neq 1$ and $y \neq 1$. Then by the hypothesis, we get $x \wedge y = 0$. Hence, $x \leq y^*$ and $y \leq x^*$. Now $1 = x \vee y \leq x \vee x^*$. Hence, $x \in (x)^\circ_{L_1}$. Similarly, we get $y \in (y)^\circ_{L_1}$. Thus, $1 = x \vee y \in (x)^\circ_{L_1} \vee (y)^\circ_{L_1}$. Therefore, $(x)^\circ_{L_1} \vee (y)^\circ_{L_1} = L_1$. Therefore, L_1 is a coaxer lattice. ■

In the following proposition, we derive some equivalent conditions for a pseudo-complemented distributive lattice to be a Boolean algebra. For that we first observe the following proposition whose proof is routine.

Proposition 1.15 *For any $a \in L$, define a relation ψ^a on L as follows:*

$$(x, y) \in \psi^a \text{ if and only if } x \wedge a^* = y \wedge a^*.$$

Then ψ^a is a congruence on L.

We now establish a set of equivalent conditions for every coaxer ideal of a pseudo-complemented distributive lattice to be a principal ideal, which will lead to a characterization of Boolean algebras.

Theorem 1.16 *The following conditions are equivalent:*

(i) *for each $a \in L$, $(a)^\circ = (a]$;*
(ii) *for each $a \in L$, $(a)^\circ = \operatorname{Ker} \psi^a$;*

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(iii) *L is a Boolean algebra;*
(iv) *L is a Stone lattice in which every element is closed;*
(v) *every prime ideal is maximal;*
(vi) *every prime ideal is minimal.*

Proof (i)⇒(ii): Assume that $(a)^\circ = (a]$ for all $a \in L$. By the above result, ψ^a is a congruence on L. Let $x \in (a)^\circ$. Then $x^* \vee a = 1$. Hence, $x \wedge a^* \leq x^{**} \wedge a^* = (x^* \vee a)^* = 1^* = 0$. Hence, $x \in \operatorname{Ker}\psi^a$. Conversely, let $x \in \operatorname{Ker}\psi^a$. Then $x \wedge a^* = 0$. Hence, $a^* \leq x^*$. By hypothesis, $a \in (a] = (a)^\circ$ which implies that $a \vee a^* = 1$. Hence, $1 = a \vee a^* \leq a \vee x^*$. Therefore $x \in (a)^\circ$. Thus, $(a)^\circ = \operatorname{Ker}\psi^a$.

(ii)⇒(iii): Assume condition (ii). Let $a \in L$. Since $a \wedge a^* = 0$, we get $a \in \operatorname{Ker}\psi^a = (a)^\circ$. Hence, $a \vee a^* = 1$. Thus, a^* is the complement of a in L. Therefore, L is a Boolean algebra.

(iii)⇒(iv): Assume that L is a Boolean algebra. Let $x \in L$. Then there exists $y \in L$ such that $x \wedge y = 0$ and $x \vee y = 1$. Since $x \wedge y = 0$, we get $y \leq x^*$. Now $1 = x \vee y \leq x^{**} \vee x^*$. Thus, L is a Stone lattice. Since $x \wedge y = x^{**} \wedge y = 0$ and $x \vee y = x^{**} \vee y = 1$, we get $x = x^{**}$. Therefore, every element of L is closed.

(iv)⇒(v): Assume that L is a Stone lattice in which every element is closed. Let P be a prime ideal of L, and suppose P is not maximal. Then there exists a proper ideal Q of L such that $P \subset Q$. Choose $x \in Q - P$. Then $1 = x^* \vee x^{**} = x^* \vee x \in P \vee Q = Q$, which is contradiction.

(v)⇒(vi): This is clear.

(vi)⇒(i): Assume that every prime ideal is minimal and let $a \in L$. We always have $(a)^\circ \subseteq (a]$. Let $x \in (a]$. Suppose $x^* \vee a \neq 1$. Then there exists a maximal ideal M such that $x^* \vee a \in M$. Since M is prime, by hypothesis M is a minimal prime ideal. Now $x^* \vee a \in M$ implies $x^* \in M$ and $a \in M$. Hence, $x \notin M$ and $x \in (a] \in M$, which is a contradiction. Hence, $x^* \vee a = 1$. Thus, $x \in (a)^\circ$. Therefore, $(a)^\circ = (a]$. ∎

We now characterize the coaxer ideals of L in terms of congruences. For this we first need the following congruences.

Theorem 1.17 *For any $a \in L$, define a relation ψ_a on L as follows:*

$$(x, y) \in \psi_a \text{ if and only if } x \vee a = y \vee a.$$

Then ψ_a is a congruence on L.

Proof Clearly, ψ_a is an equivalence relation on L. Let $(x, y) \in \psi_a$. For any $c \in L$, we get $(x \wedge c) \vee a = (x \vee a) \wedge (c \vee a) = (y \vee a) \wedge (c \vee a) = (y \wedge c) \vee a$. Hence, $(x \wedge c, y \wedge c) \in \psi_a$. Also, $(x \vee c) \vee a = (x \vee a) \vee c = (y \vee a) \vee c = (y \vee c) \vee a$. Hence, $(x \vee c, y \vee c) \in \psi_a$. Therefore, ψ_a is a congruence on L. ∎

Theorem 1.18 *For any $\wedge$-closed subset F of L, define a relation ψ^F on L as follows:*

$$(x, y) \in \psi^F \text{ if and only if } x \wedge a = y \wedge a \text{ for some } a \in F.$$

Then ψ^F is a congruence on L.

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It is a known fact that for any congruence θ of a distributive lattice, the kernel $\theta^* = \{x \in L \mid (x,0) \in \theta\}$ is an ideal of L and the antikernel $\theta^+ = \{x \in L \mid (x,1) \in \theta\}$ is a filter of L. For the above two congruences, we clearly have $(\psi_a)^+ = \{x \in L \mid x \vee a = 1\}$ and $(\psi^F)^* = \{x \in L \mid x \wedge f = 0 \text{ for some } f \in F\}$. Now we have the following theorem.

Theorem 1.19 *For any $a \in L$, $(a)^\circ = (\psi_a)^{+*}$.*

Proof Let $x \in (a)^\circ$. Then $x^* \vee a = 1$. Hence, $x^* \in (\psi_a)^+$. Now $x \wedge x^* = 0$ and $x^* \in (\psi_a)^+$ imply that $x \in (\psi_a)^{+*}$. Therefore $(a)^\circ \subseteq (\psi_a)^{+*}$. Conversely, let $x \in (\psi_a)^{+*}$. Then $x \wedge f = 0$ for some $f \in (\psi_a)^+$. Now $f \in (\psi_a)^+$ implies that $a \vee f = 1$. Then $1 = a \vee f \leq a \vee x^*$(since $x \wedge f = 0$). Thus, $x \in (a)^\circ$. Therefore, $(a)^\circ = (\psi_a)^{+*}$. ∎

We now characterize a coaxer lattice in topological terms. For that we make use of the following hypothesis from [7].

Let L be a pseudo-complemented bounded distributive lattice. Let us denote the set of all prime ideals of L by $\operatorname{Spec} L$. For any $A \subseteq L$, let $K(A) = \{P \in \operatorname{Spec} L \mid A \nsubseteq P\}$, and for any $a \in L, K(a) = K(\{a\})$. Then from [7], we have the following lemma.

Lemma 1.20 *For any $x, y \in L$, the following hold:*

(i) $\bigcup_{x \in L} K(x) = L$,
(ii) $K(x) \cap K(y) = K(x \wedge y)$,
(iii) $K(x) \cup K(y) = K(x \vee y)$.

From this lemma we can immediately say that the collection $\{K(x) \mid x \in L\}$ forms a base for a topology on $\operatorname{Spec} L$. The topology generated by this base is precisely $\{K(A) \mid A \subseteq L\}$ and is called the hull-kernel topology on $\operatorname{Spec} L$. We denote the set of all maximal ideals of L by $\operatorname{Max} L$. Then $\operatorname{Max} L \subseteq \operatorname{Spec} L$, and for any $x \in L$, we write $K_m(x) = K(x) \cap \operatorname{Max} L$.

Theorem 1.21 *If the intersection of all maximal ideals of L is $\{0\}$ and for any $a, b \in L$ with $a \vee b = 1$ there exists a minimal prime ideal P such that $(a)^\circ \vee (b)^\circ \subseteq P$, then the following conditions are equivalent:*

(i) *L is coaxer;*
(ii) *$\operatorname{Max} L$ is a Hausdorff space;*
(iii) *every prime ideal is contained in a unique maximal ideal;*
(iv) *every minimal prime ideal is contained in a unique maximal ideal.*

Proof (i)⇒(ii): Assume that L is coaxer. Let M and N be two distinct elements of $\operatorname{Max} L$. Choose $x \in M - N$. Then $N \vee (x] = L$. Hence, $a \vee x = 1$ for some $a \in N$. Since L is coaxer, we get $(a)^\circ \vee (x)^\circ = L$. Thus, there exists two elements $s, t \in L$ such that $s^* \vee a = 1, t^* \vee x = 1$ and $s \vee t = 1$. If $s^* \in N$, then $1 = s^* \vee a \in N$, which is a contradiction. If $t^* \in M$, then $1 = t^* \vee x \in M$ (since $x \in M$), which is also a contradiction. Hence, $N \in K_m(s^*)$ and $M \in K_m(t^*)$. Now

$$K_m(s^*) \cap K_m(t^*) = K_m(s^* \wedge t^*) = K_m((s \vee t)^*) = K_m(0) = \varnothing$$

Hence, $\operatorname{Max} L$ is a Hausdorff space.

https://doi.org/10.4153/CMB-2016-083-x Published online by Cambridge University Press

(ii)⇒(iii): Assume that Max L is a Hausdorff space. Let P be a prime ideal of L. Let M_1 and M_2 be two maximal ideals of L such that $P \subseteq M_1$ and $P \subseteq M_2$. Suppose $M_1 \neq M_2$. Since Max L is Hausdorff, there exists two elements $x, y \in L$ such that $M_1 \in K_m(x), M_2 \in K_m(y)$ and $K_m(x) \cap K_m(y) = \emptyset$. Hence, $K_m(x \wedge y) = \emptyset$. Thus, $x \wedge y = 0$. Otherwise, there exists a maximal ideal M such that $M \in H'_m(x \wedge y)$. Since $x \notin M_1$ and $y \notin M_2$, we get that $x \notin P$ and $y \notin P$. Therefore, we get $0 = x \wedge y \notin P$, which is a contradiction. Hence, P is contained in a unique maximal ideal.

(iii)⇒(iv): This is obvious.

(iv)⇒(i): Assume that every minimal prime ideal is contained in a unique maximal ideal. Let $a, b \in L$ be such that $a \vee b = 1$. Suppose $(a)^\circ \vee (b)^\circ \neq L$. Then by hypothesis, there exists a minimal prime ideal P such that $(a)^\circ \vee (b)^\circ \subseteq P$. Since $1 = a \vee b \notin P$, we get that $a \notin P$ and $b \notin P$. Suppose $1 \in P \vee (a]$. Then $1 = x \vee a \leq x^{**} \vee a$ for some $x \in P$. Hence, $x^* \in (a)^\circ \subseteq (a)^\circ \vee (b)^\circ \subseteq P$, which is a contradiction to that P is minimal. Hence, $P \vee (a]$ is a proper ideal of L. Similarly, we get that $P \vee (b]$ is a proper ideal. Then there exist maximal ideals M, N such that $P \vee (a] \subseteq M$ and $P \vee (b] \subseteq N$. Since $a \vee b = 1$, we get M, N must be distinct, which is a contradiction. Hence, $(a)^\circ \vee (y)^\circ = L$. Therefore, L is coaxer. ∎

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