Forum of Mathematics, Sigma (2025), **Vol. 13**:e118 1–18
doi:10.1017/fms.2025.10070

CAMBRIDGE UNIVERSITY PRESS

RESEARCH ARTICLE

Compactness phenomena in HOD

Gabriel Goldberg[1] and Alejandro Poveda[2]

[1]Department of Mathematics, UC Berkeley, Berkeley CA, 94720, USA; E-mail: ggoldberg@berkeley.edu.
[2]Department of Mathematics and CMSA, Harvard University, Cambridge MA, 02138, USA;
E-mail: alejandro@cmsa.fas.harvard.edu (corresponding author).

Received: 23 November 2023; **Revised:** 25 May 2025; **Accepted:** 25 May 2025

2020 Mathematics Subject Classification: *Primary* – 03E35, 03E55

Abstract

We prove two compactness theorems for HOD. First, if κ is a strong limit singular cardinal with uncountable cofinality and for stationarily many $\delta < \kappa$, $(\delta^+)^{\mathrm{HOD}} = \delta^+$, then $(\kappa^+)^{\mathrm{HOD}} = \kappa^+$. Second, if κ is a singular cardinal with uncountable cofinality and stationarily many $\delta < \kappa$ are singular in HOD, then κ is singular in HOD. We also discuss the optimality of these results and show that the first theorem does not extend from HOD to other ω-club amenable inner models.

Contents

1. Introduction

In 1963, Cohen established that Cantor's continuum problem cannot be solved from the accepted ZFC axioms of set theory [Coh63]. This is the problem of determining which among Cantor's transfinite cardinal numbers

$$\aleph_0, \aleph_1, \aleph_2, \ldots \aleph_\omega, \aleph_{\omega+1}, \ldots$$

is the cardinality of the continuum $\mathbb{R}$. More precisely, what Cohen showed is that the axioms cannot rule out that $|\mathbb{R}| = \aleph_2$, while Gödel [Göd39] had already shown that the possibility $|\mathbb{R}| = \aleph_1$ could not be ruled out.

Of course, Cantor himself ruled out that $|\mathbb{R}|$ is $\aleph_0$ by proving that the real numbers form an uncountable set. Later, König [Kön05] showed that $|\mathbb{R}|$ is not equal to $\aleph_\omega$, $\aleph_{\omega+\omega}$, or, more generally, $\aleph_\alpha$ for any limit ordinal α of countable cofinality. Soon after Cohen's theorem, Solovay showed that there are no restrictions on the cardinality of the continuum besides those established by Cantor and König. For example, it is consistent with the ZFC axioms that $|\mathbb{R}| = \aleph_{19}$ or $|\mathbb{R}| = \aleph_{\omega\cdot\omega+1}$ or $|\mathbb{R}| = \aleph_{\omega_5}$.

The cardinality of the continuum is denoted by $2^{\aleph_0}$, recognizing that $\mathbb{R}$ is equinumerous with the set of functions from $\mathbb{N}$ into a set of size 2. For each cardinal number κ, 2^κ denotes the cardinality of the set of functions from a set of size κ to a set of size 2. The function $\kappa \mapsto 2^\kappa$ is known as the *continuum function.*

After Solovay's result classifying all possible values of $2^{\aleph_0}$, set theorists took up the problem of classifying the possibilities for the continuum function itself. Obviously, we have $2^\kappa \leq 2^\lambda$ whenever $\kappa \leq \lambda$. Also, $2^\kappa > \kappa$ by Cantor's theorem, and furthermore $\mathrm{cf}(2^\kappa) > \kappa$ by König's theorem.

Are there any other restrictions on the continuum function, or is the situation analogous to Solovay's theorem for $2^{\aleph_0}$, where no further constraints are possible? In 1966, Easton [Eas70] showed the latter for regular cardinals – that is, those cardinals κ that are not the limit of fewer than κ smaller cardinals. That is, Easton showed that no restrictions on the behavior of the continuum function on regular cardinals can be established in ZFC except the ones mentioned in the previous paragraphs.

Following Easton's theorem, the outstanding open problem in set theory was to generalize the result to all cardinals, showing without restriction that the continuum function obeys no laws other than those discovered by Cantor and König. This paper is inspired by a theorem of Silver [Sil75], which shows such a generalization of Easton's theorem is not possible: in fact, there are intricate and subtle restrictions on the behavior of the continuum function at singular (i.e., non-regular) cardinals. To this day, the problem of completely classifying the possible behavior of the continuum function at singular cardinals remains open, though the theory of singular cardinal arithmetic has since blossomed into one of the deepest subjects in set theory.

Silver's theorem reveals that the value of $2^{\aleph_{\omega_1}}$ is tied to the values of $2^{\aleph_\alpha}$ for ordinals $\alpha < \omega_1$. More precisely, if $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for all limit ordinals $\alpha < \omega_1$, then $2^{\aleph_{\omega_1}} = \aleph_{\omega_1+1}$. He showed moreover that if κ is a singular cardinal of uncountable cofinality and $2^\lambda = \lambda^+$ for a stationary set of $\lambda < \kappa$, then $2^\kappa = \kappa^+$.

Silver's theorem can be construed as a *compactness property* of the continuum function. *Compactness* describes a general pattern in set theory: the properties of a structure are determined by its small substructures. The most familiar compactness phenomena involve infinite structures and their finite substructures: for example, the Compactness Theorem in first-order logic states that the satisfiability of a first-order theory is determined by the satisfiability of its finite fragments. Compactness properties of larger regular cardinals often turn out to be related to large cardinal properties – for instance, the tree property and stationary reflection. However, singular cardinals have been found to have compactness properties that are provable in ZFC – for instance, Shelah's singular compactness theorem in algebra, which led to his solution of Whitehead's problem [She74].

This paper establishes analogs of Silver's theorem in the context of set-theoretic definability. Gödel [Göd46] introduced the concept of *ordinal definability* in an attempt to formalize the intuitive concept of mathematical definability. A set is *ordinal definable* if it is definable over the universe of sets using finitely many ordinal numbers as parameters.

The behavior of ordinal definability is highly sensitive to the structure of the universe of sets, and for this reason, it is subject to the same independence phenomena that hinder our understanding of the continuum function. The main results of this paper show for the first time that ordinal definability at singular cardinals of uncountable cofinality exhibits patterns of compactness parallel to those that Silver identified for the continuum function.

Our theorems concern two invariants of ordinal definability, which play the role of the continuum function in our analogs of Silver's theorem. First, we define the *ordinal definable cofinality* of an ordinal α, denoted by $\mathrm{cf}^{\mathrm{OD}}(\alpha)$, as the least ordinal δ such that there is an ordinal definable cofinal function from δ to α. Second, we define the *ordinal definable successor of* α, denoted by $\alpha^{+\mathrm{OD}}$, as the supremum of all ordinals γ for which there is an ordinal definable surjection from α to γ.[1] With this notation in hand, we can state our compactness theorems for ordinal definability.

[1]Of course, $\mathrm{cf}^{\mathrm{OD}}(\alpha)$ is just the cofinality of α as computed in the inner model HOD, and $\alpha^{+\mathrm{OD}}$ is the least cardinal of HOD that is greater than α.

https://doi.org/10.1017/fms.2025.10070 Published online by Cambridge University Press

Theorem. *Suppose that κ is a singular cardinal with uncountable cofinality and that $\{\delta < \kappa \mid \mathrm{cf}^{\mathrm{OD}}(\delta) < \delta\}$ is stationary. Then $\mathrm{cf}^{\mathrm{OD}}(\kappa) < \kappa$.*

Our theorem on the ordinal definable successor function is significantly harder to prove, and moreover, we do not know how to prove it for arbitrary singular cardinals.

Theorem. *Suppose that κ is a singular strong limit cardinal of uncountable cofinality and $\{\delta < \kappa \mid \delta^{+\mathrm{OD}} = \delta^+\}$ is stationary. Then $\kappa^{+\mathrm{OD}} = \kappa^+$.*

The theorems above are proved by combining the technique of *generic ultrapowers* (see §2.1) with variants of Vopenka's theorem that every set belongs to a forcing extension of HOD. In addition, in §3.1, we employ set-theoretic forcing to show that the hypothesis employed may not be relaxed. Thus, our results are provably optimal.

Finally, we show that the first of our compactness theorems does not extend to arbitrary *ω-club amenable models* (see p.13). This contrasts with the main results of [Gol23] where the first author showed that most known results about HOD – for example, the HOD dichotomy theorem – can actually be proved for an arbitrary inner model that is ω-club amenable.

Theorem. *Assume that every set T belongs to an inner model with a measurable cardinal of Mitchell order 2 above* $\mathrm{rank}(T)$. *Then for every cardinal λ, there is an ω-club amenable inner model M that is correct about cardinals and cofinalities below λ while $(\lambda^+)^M < \lambda^+$.*

The notation of this paper is standard in set theory. In §2, we provide the reader with some preliminaries regarding HOD and the theory of generic ultrapowers. §3 is devoted to prove the above theorems and discuss their optimality. Finally, in §4, we leave some related open questions.

2. Preliminaries and notation

This section collects some set-theoretic tools employed through the paper. The material here is standard and is included just for the benefit of our readers. We also introduce some relevant terminology.

2.1. Generic ultrapowers

Fix a set X. A set $\mathcal{I} \subseteq \mathcal{P}(X)$ is called an *ideal* if $\emptyset \in \mathcal{I}$, $X \notin \mathcal{I}$ and $\mathcal{I}$ is closed under subsets and finite unions. Dually, a set $\mathcal{F} \subseteq \mathcal{P}(X)$ is a *filter* if $\emptyset \notin \mathcal{F}$, $X \in \mathcal{F}$ and $\mathcal{F}$ is closed under supersets and finite intersections. A filter $\mathcal{U}$ is called an *ultrafilter* if it satisfies the following additional property: given $A \in \mathcal{P}(X)$, either $A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$; equivalently, $\mathcal{U}$ is a $\subseteq$-maximal filter.

Given an ideal $\mathcal{I} \subseteq \mathcal{P}(X)$, its *dual filter* $\mathcal{I}^*$ is defined as $\{X \setminus A \mid A \in \mathcal{I}\}$. A set $A \in \mathcal{P}(X)$ has *$\mathcal{I}$-positive measure* if $A \notin I$, and $\mathcal{I}^+$ denotes the collection of all sets with $\mathcal{I}$-positive measure. Note that $\mathcal{I}^*$ is a filter, $\mathcal{I}^* \subseteq \mathcal{I}^+$ and $A \in \mathcal{I}^+$ if and only if $A \cap B \neq \emptyset$ for all $B \in \mathcal{I}^*$. These concepts have natural parallels in the setting of filters $\mathcal{F} \subseteq \mathcal{P}(X)$ as well [Jec03, §7].

Given an ideal $\mathcal{I} \subseteq \mathcal{P}(X)$, define an equivalence relation $\sim_{\mathcal{I}}$ on $P(X)$ as follows:

$$A \sim_{\mathcal{I}} B \text{ if and only if } A \triangle B \in \mathcal{I}.$$

This yields a quotient $\mathcal{P}(X)/\mathcal{I}$, which, endowed with the order

$$[X] \leq [Y] \iff X \setminus Y \in \mathcal{I},$$

gives rise to a Boolean algebra. After removing the zero element from $\mathcal{P}(\kappa)/\mathcal{I}$, the partial ordering $\leq$ becomes a separative order.

There is another presentation of the poset $(\mathcal{P}(X)/\mathcal{I} \setminus \{[\emptyset]\}, \leq)$ as $\mathbb{P} := (\mathcal{I}^+, \subset)$. The two posets are forcing equivalent in the sense that they give rise to the same generic extensions. Indeed, the former poset is the separative quotient of the latter. A V-generic filter $G \subseteq \mathbb{P}$ yields a V-ultrafilter on X extending the dual filter $\mathcal{I}^*$. If $\mathcal{I}$ is κ-complete, then G is also V-κ-complete.[2] For details concerning these facts, see [Jec03, §22].

Suppose that $\mathcal{I} \subseteq \mathcal{P}(\kappa)$ is an ideal containing all singletons and that it is κ-complete (i.e., $\bigcup_{\alpha<\lambda} A_\alpha \in \mathcal{I}$ provided $\langle A_\alpha \mid \alpha < \lambda\rangle \subseteq \mathcal{I}$ with $\lambda < \kappa$).

Let $G \subseteq \mathbb{P}$ be a V-generic filter. Working in $V[G]$, we can define the *generic ultrapower of V by G*. Namely, in $V[G]$, one defines the structure

$$\mathrm{Ult}(V, G) := \langle ({}^X V) \cap V)/{=_G}, \in_G\rangle,$$

where for each two functions $f, g\colon X \to V$ (in V)

$$f =_G g \text{ if and only if } \{x \in X \mid f(x) = g(x)\} \in G$$

and

$$[f]_G \in_G [g]_G \text{ if and only if } \{x \in X \mid f(x) \in g(x)\} \in G.$$

Here and in the future, we will denote by $[f]_G$ the $=_G$ equivalence class of f, omitting the subscript when there is no chance of confusion.

It turns out that $\mathrm{Ult}(V, G)$ is a model of ZFC, yet not necessarily well-founded, even if G is V-κ-complete for an uncountable cardinal κ. As usual, an appropriate version of Łoś's theorem holds. Namely,

$$\mathrm{Ult}(V, G) \models \varphi([f_1], \ldots, [f_n]) \iff \{x \in X \mid V \models \varphi(f_1(x), \ldots, f_n(x))\} \in G,$$

where $\varphi(v_1, \ldots, v_n)$ is a first-order formula in the language $\{=, \in\}$.

This ensures that the map $j_G\colon \langle V, \in\rangle \to \mathrm{Ult}(V, G)$ given by $a \mapsto [c_a]_G$ is an elementary embedding, where $c_a\colon X \to V$ is the constant function with value a.

The combinatorial properties of the ideal $\mathcal{I}$ (in V) are related to the properties of the embedding $j_G : V \to \mathrm{Ult}(V, G)$ in the generic extension $V[G]$. For example, $\mathcal{I}$ is κ-complete if and only if the maximal condition $[X]$ forces that the critical point of j_G is at least κ.

The generic ultrapower construction will play a prominent role in the forthcoming §3. We refer the reader to [Jec03, §22] or Foreman's excellent handbook chapter [For09] for any notion not considered in this account.

Remark 2.1. Let $\mathcal{F} \subseteq \mathcal{P}(X)$ be a filter and $\mathcal{I} := \{X \setminus F \mid F \in \mathcal{F}\}$ be its dual ideal. Since the $\mathcal{F}$-positive sets are exactly the $\mathcal{I}$-positive sets, all the previous comments remain valid starting with a filter $\mathcal{F}$ and taking $\mathbb{P} := (\mathcal{F}^+, \subseteq)$. This will be the approach we take through §3. We decided to phrase the discussion here in the language of ideals just because this is the approach pursued in reference texts, such as [Jec03, For09].

Definition 2.2. Given a filter $\mathcal{F} \subseteq \mathcal{P}(X)$ and functions $f, g\colon X \to V$, denote

$$f <_{\mathcal{F}} g \iff \{x \in X \mid f(x) < g(x)\} \in \mathcal{F}.$$

Similarly, if $\mathcal{I} \subseteq \mathcal{P}(X)$ is an ideal,

$$f <_{\mathcal{I}} g \iff \{x \in X \mid f(x) \geq g(x)\} \in \mathcal{I}.$$

2.2. *Ordinal definability and forcing*

A set X is called *ordinal definable* if it is definable by a formula of the language of set theory using ordinals as parameters. More formally, there is $\varphi(x, \vec{y})$ and $\langle \alpha_*, \alpha_0, \ldots \alpha_n\rangle \in \mathrm{Ord}^{<\omega}$ such that

$$x \in X \Leftrightarrow V_{\alpha^*} \models \varphi(x, \alpha_0, \ldots, \alpha_n).$$

[2] A filter G is V-κ-complete if given $\lambda < \kappa$ and a sequence $\langle A_\alpha \mid \alpha < \lambda\rangle$ in V of sets in G, then $\bigcap_{\alpha<\lambda} A_\alpha \in G$.

https://doi.org/10.1017/fms.2025.10070 Published online by Cambridge University Press

The class of ordinal definable sets is denoted by OD. Since OD need not be transitive, one looks at a special subclass of OD – the *Hereditarily ordinal definable* sets, HOD. A set X is Hereditarily Ordinal Definable (or, simply, in HOD) if $X \in \mathrm{OD}$ and the transitive closure of $\{X\}$ is contained in OD. It turns that HOD is an *inner model*; namely, it is a transitive class containing the ordinals and satisfying all the ZFC axioms.

At many places in this paper, we shall be preoccupied with the following issue. Suppose that $\mathbb{P} \in \mathrm{OD}$ is a forcing poset and $G \subseteq \mathbb{P}$ is V-generic – how does $\mathrm{HOD}^{V[G]}$ compare to HOD^V? Here, HOD^V (resp. $\mathrm{HOD}^{V[G]}$) stands for the class HOD as computed in V (resp. in $V[G]$). In special circumstances, we have $\mathrm{HOD}^{V[G]} \subseteq \mathrm{HOD}$ – for example, if $\mathbb{P}$ is *cone/weakly homogeneous*:

Definition 2.3. A poset $\mathbb{P}$ is *weakly homogeneous* if for for all $p, q \in \mathbb{P}$, there is an automorphism $\varphi\colon \mathbb{P} \longrightarrow \mathbb{P}$ making $\varphi(p)$ and q compatible.

Similarly, $\mathbb{P}$ is *cone-homogeneous* if for all $p, q \in \mathbb{P}$, there are $p^* \le p$ and $q^* \le q$ together with an isomorphism $\varphi\colon \mathbb{P}/p^* \longrightarrow \mathbb{P}/q^*$.

We used $\mathbb{P}/p$ to denote the subposet of $\mathbb{P}$ with universe $\{q \in \mathbb{P} \mid q \le p\}$. It is clear that every weakly homogeneous forcing is cone-homogeneous.

Lemma 2.4 (Folklore). *If $\mathbb{P} \in \mathrm{OD}$ is a cone-homogeneous forcing poset, then $\mathrm{HOD}^{V[G]} \subseteq \mathrm{HOD}^V$ for all V-generic $G \subseteq \mathbb{P}$.*

Sometimes we will need to assume (see, for example, §3.1) that HOD encompasses large cardinals which exist in V. This can be done by forcing "$V = \mathrm{HOD}$" with *McAloon iteration* coding V into the continuum function. The said iteration preserves large cardinals (see [FHR15, BP23]) and produces a model V such that $V \subseteq \mathrm{HOD}^{V^{\mathbb{Q}}}$ for any set-sized forcing $\mathbb{Q}$.

3. Two compactness theorems for HOD

In this section, we prove our compactness theorems for HOD (Theorems 3.4 and 3.5). Our results are very much in the spirit of Silver's classical theorem that the generalized continuum hypothesis cannot first fail at a singular cardinal of uncountable cofinality [Sil75]. The overall idea is to extract some information about ordinal definability from Silver's argument, which heavily uses the technique of generic ultrapowers from §2.1.

Let us begin with the following key result:

Theorem 3.1 (Casey–Goldberg). *For any strong limit cardinal λ,*

$$\mathrm{cf}(\lambda^{+\mathrm{HOD}}) \in \{\omega, \mathrm{cf}(\lambda), \lambda^+\}.$$

Proof. Let $\kappa := \lambda^{+\mathrm{HOD}}$ and let $\delta := \mathrm{cf}(\kappa)$. Let us assume that $\omega < \delta < \lambda^+$.

We must show that $\delta = \mathrm{cf}(\lambda)$. Let $\mathscr{F}$ denote the restriction of the closed unbounded filter on κ to HOD; that is, $\mathscr{F} := \mathrm{Cub}_\kappa \cap \mathrm{HOD}$. Since Cub_κ is ordinal definable it is easy to check that $\mathscr{F} \in \mathrm{HOD}$ and that it is a filter in HOD. We will denote by $\mathscr{F}^+$ the $\mathscr{F}$-positive sets, as computed in HOD – namely, the collection of all $A \in \mathcal{P}(\kappa)^{\mathrm{HOD}}$ intersecting all members of $\mathscr{F}$.

Claim 3.1.1. In HOD, $\mathscr{F}$ is weakly normal in the sense that if $S \in \mathscr{F}^+$ and $f\colon S \longrightarrow \kappa$ is a regressive function in HOD, there is $\beta < \kappa$ such that

$$\{\alpha \in S \mid f(\alpha) \le \beta\} \in \mathscr{F}^+.[3]$$

Moreover, if $\gamma \in \mathrm{Ord}$, $\mathrm{cf}^V(\gamma) \ne \delta$, $S \in \mathscr{F}^+$, and $f\colon S \longrightarrow \gamma$ is any function in HOD, then there is some $\beta < \gamma$ such that

[3] This notion of weak normality is weaker than the one considered in Kan76.

https://doi.org/10.1017/fms.2025.10070 Published online by Cambridge University Press

$$\{\alpha \in S \mid f(\alpha) \leq \beta\} \in \mathscr{F}^+.$$

Finally, $(\mathscr{F}^+, \subset)$ is forcing equivalent in HOD to a poset of size less than $(2^\delta)^{+V}$.

Proof of claim. The bounding properties of $\mathscr{F}$ in HOD follow from the corresponding properties of Cub_κ in V. That is, if $S \in \mathscr{F}^+$ and $f : S \to \kappa$ is a regressive function in V, then f is bounded on a stationary set, and if γ is an ordinal whose cofinality is not δ and $f : S \to \gamma$ is any function in V, then f is bounded on a stationary set. We leave these as exercises with the following hints. First, by restricting to a club in κ of ordertype δ, one can reduce to the more familiar case that $\kappa = \delta$. Second, to prove the statement about functions into γ, one can split into cases based on whether the cofinality of γ is less than δ or greater than δ; in the former case, one appeals to the δ-completeness of the club filter, and in the latter case, one uses that functions from δ to γ are bounded *everywhere*.

Finally, $(\mathscr{F}^+, \subset)$ is equivalent in HOD to a forcing of size less than $(2^\delta)^{+V}$ because its separative quotient $\mathbb{Q}$ has cardinality less than $(2^\delta)^{+V}$: note that the underlying set of $\mathbb{Q}$ is precisely the set of equivalence classes of $\mathscr{F}^+$ modulo the non-stationary ideal on κ (in V). In HOD, choose a set $\mathscr{T} \subseteq \mathscr{F}^+$ such that for each $S \in \mathscr{F}^+$, there is exactly one $S' \in \mathscr{T}$ such that $S \triangle S'$ is non-stationary (in V). Then $|\mathscr{T}|^{\mathrm{HOD}} = |\mathbb{Q}|^{\mathrm{HOD}}$, and $V \models$ "$|\mathscr{T}| \leq 2^\delta$". To see this last inequality, fix a closed unbounded set $C \subseteq \kappa$ of ordertype δ. Then $\langle S \cap C \mid S \in \mathscr{T}\rangle$ is a sequence of distinct subsets of C from which we deduce that $V \models |\mathscr{T}| \leq |\mathcal{P}(C)| = 2^\delta$. □

By forcing over HOD with $(\mathscr{F}^+, \subset)$, we extend $\mathscr{F}$ to a HOD-weakly normal HOD-ultrafilter G on κ with the property that if γ is an ordinal with $\mathrm{cf}^V(\gamma) \neq \delta$ then every $f\colon \kappa \to \gamma$ in HOD is bounded on a set in G. (This is by the moreover part of the claim together with a density argument.) In particular, the generic ultrapower map $i\colon \mathrm{HOD} \to N := \mathrm{Ult}(\mathrm{HOD}, G)$ is continuous at ordinals γ of V-cofinality distinct from δ. Note that N may not be well-founded, so N-ordinals may fail to be ordinals.

Since G is HOD-weakly normal (because so is $\mathscr{F}$), it follows that

$$[\mathrm{id}]_G = \sup i[\kappa] < i(\kappa).$$

(Since N may not be well-founded, these objects are not really ordinals, but elements of Ord^N, and the order relation, which we will simply denote by $<$, is the canonical order of the ordinals as computed in N. If $A \subseteq \mathrm{Ord}^N$, we write $\sup(A)$ for the least upper bound of A in this order, if it exists. Note that this least upper bound exists whenever A belongs to N, and may or may not exist otherwise. In this particular case, $\sup i[\kappa]$ exists and is equal to $[\mathrm{id}]_G$ by weak normality, even though $i[\kappa]$ may not belong to N.)

Assume towards a contradiction that

$$\mathrm{cf}(\lambda) \neq \delta.$$

By this assumption and our previous comments, $i(\lambda) = \sup i[\lambda]$ (i.e., *i is continuous at λ*) as every $f\colon \kappa \to \lambda$ is bounded on a set in G.

Since $\kappa = \lambda^{+\mathrm{HOD}}$, $\{\xi < \kappa \mid \mathrm{cf}^{\mathrm{HOD}}(\xi) \leq \lambda\} \in G$, and so $\mathrm{cf}(\sup i[\kappa]) \leq \sup i[\lambda]$. Let us next argue that this inequality is strict.

Note that $\mathrm{HOD}[G] \models$ "$\mathrm{cf}(\sup i[\kappa]) = \kappa$": Indeed, κ remains regular in $\mathrm{HOD}[G]$ because G comes from a forcing equivalent to another of cardinality less than $(2^\delta)^{+V} < \lambda$.

Because $\mathrm{HOD}[G] \models$ "$\mathrm{cf}(\sup i[\lambda]) = \mathrm{cf}(\lambda) \leq \lambda < \kappa$", it follows that

$$\mathrm{HOD}[G] \models \text{“}\mathrm{cf}(\sup i[\kappa]) \neq \mathrm{cf}(\sup i[\lambda])\text{”}.$$

Therefore, N satisfies the same; namely,

$$N \models \text{“}\mathrm{cf}(\sup i[\kappa]) \neq \mathrm{cf}(\sup i[\lambda])\text{”}.$$

In particular, we must have

$$N \models \text{“}\operatorname{cf}(\sup i[\kappa]) < \sup i[\lambda]\text{''}.$$

Let $C \in N$ be such that

$$N \models \text{“}C \text{ is a closed cofinal subset of } \sup i[\kappa] \text{ of order-type } \operatorname{cf}(\sup i[\kappa])\text{''}.$$

Recall that i is continuous at ordinals whose V-cofinality is not equal to δ. In particular, i is continuous at ordinals whose $V[G]$-cofinality γ lies between $(2^\delta)^+$ and λ: by preservation of regular cardinals, such an ordinal has the same cofinality in V. Thus, for any such γ, a familiar argument shows that $i[\kappa] \cap C$ is γ-closed cofinal in $\sup i[\kappa]$. Hence, $i^{-1}[C]$ is cofinal in κ.

Let $B := i^{-1}[C]$, and note that there is some $A \in \mathrm{HOD}$ unbounded in κ contained in B because G is generic for a partial order which in HOD has size less than $(2^\delta)^{+V} < \lambda < \kappa$. Since $i[A] \subseteq C$, letting $f\colon A \to \kappa$ be the transitive collapse, $i[\kappa] \subseteq \bar{C}$ where $\bar{C} := i(f)[C]$.

Note that

$$N \models \text{“}\operatorname{otp}(\bar{C}) = \operatorname{otp}(C)\text{''}.$$

Fix a $V[G]$-regular cardinal $\gamma \in (\delta, \lambda)$ such that

$$N \models \text{“}\operatorname{otp}(C) < i(\gamma)\text{''}.$$

Then $i[\gamma] \subseteq \bar{C} \cap i(\gamma)$, and so

$$N \models \text{“}i[\gamma] \text{ is bounded above by } \sup(\bar{C} \cap i(\gamma)) < i(\gamma)\text{''}.$$

In particular, i is discontinuous at γ. However, i must be continuous at γ because $\operatorname{cf}(\gamma) \neq \delta$. This yields a contradiction showing that our original assumption that $\operatorname{cf}(\lambda) \neq \delta$ was false. □

The proof of Theorem 3.4 requires another technical result.

Definition 3.2. Let $V \subseteq W$ be two transitive models of ZFC and $\kappa \in V$ be such that $V \models$ ‘κ is a regular cardinal’. We say that the pair (V, W) has the *κ-uniform cover property* if for every function $f \in W$ with $\operatorname{dom}(f) \in V$ and $\operatorname{ran}(f) \subseteq V$, there is yet another function $F \in V$ with $\operatorname{dom}(F) = \operatorname{dom}(f)$, and for all $i \in \operatorname{dom}(f)$, $f(i) \in F(i)$ and $V \models |F(i)| < \kappa$.

If $\mathbb{P} \in V$ is a κ-cc forcing poset and $G \subseteq \mathbb{P}$ is a generic filter, then standard arguments show that $(V, V[G])$ has the κ-uniform cover property. Conversely, a remarkable theorem by Bukovský [Buk73] says that if (V, W) has the κ-uniform cover property, then there is a poset $\mathbb{P} \in V$ that has the κ-cc in V, $W \models$ “$|\mathbb{P}| \leq 2^\kappa$” and W is a generic extension of V by $\mathbb{P}$ (see [Sch20, Theorem 3.11])

Lemma 3.3. *Suppose $\mathbb{P} \in \mathrm{OD}$ is a κ-cc forcing and $G \subseteq \mathbb{P}$ is a V-generic. Let N be the class of sets that are hereditarily definable in the structure $\langle V[G], V, G, \in\rangle$ from ordinal parameters. Then the pair* (HOD, N) *has the κ-uniform cover property.*

In particular, N is a forcing extension of HOD *by a forcing $\mathbb{Q} \in \mathrm{HOD}$ such that* HOD $\models$ *“$\mathbb{Q}$ is κ-cc” and $N \models$ “$|\mathbb{Q}| \leq 2^\kappa$”.*

Proof. Clearly, $\mathrm{HOD} \subseteq N$. We verify that (HOD, N) has the κ-uniform cover property. Fix an ordinal λ and a function $f : \lambda \to \lambda$ that is definable in the structure $\langle V[G], V, G, \in\rangle$ from ordinal parameters. Let $\varphi(x_0, x_1, x_2)$ be a formula in the language of $\langle V[G], V, G, \in\rangle$ such that for some ordinal β, $f(\xi) = \zeta$ if and only if $\langle V[G], V, G, \in\rangle$ satisfies $\varphi(\xi, \zeta, \beta)$. Then let

$$F(\xi) = \{\zeta < \lambda \mid \exists p \in \mathbb{P}\, (p \Vdash_{\mathbb{P}} \varphi(\xi, \zeta, \beta))\}.$$

https://doi.org/10.1017/fms.2025.10070 Published online by Cambridge University Press

Note that F is ordinal definable (because so is $\mathbb{P}$) and that $f(\xi) \in F(\xi)$. Since $\mathbb{P}$ is κ-cc, it also follows that $\text{HOD} \models |F(\xi)| < \kappa$. $\square$

We are now in a position to prove our first main result:

Theorem 3.4. *If κ is a strong limit singular cardinal of uncountable cofinality and $\{\delta < \kappa \mid (\delta^+)^{\text{HOD}} = \delta^+\}$ is stationary, then $(\kappa^+)^{\text{HOD}} = \kappa^+$.*

Proof. The first attempt at a proof, on which the correct proof will elaborate, proceeds as follows. Let $\iota = \text{cf}(\kappa)$ and fix $f : \iota \to \kappa$ an increasing continuous cofinal function. Let $\mathcal{F}$ be the club filter on ι. Then, by assumption,

$$S := \{\xi < \iota \mid f(\xi)^{+\text{HOD}} = f(\xi)^+\} \in \mathcal{F}^+.$$

By forcing with $\mathcal{F}^+$ below S, one produces a generic filter $G \subseteq \mathcal{F}^+$ extending the filter $\mathcal{F}$, which is V-ι-complete and V-normal. In particular,

$$(\dagger) \;\; \{\xi < \iota \mid f(\xi)^{+\text{HOD}} = f(\xi)^+\} \in G.$$

Then we take the generic ultrapower $j_G : V \to M_G$, using only functions $f : \iota \to V$ in the ground model V (see §2.1). By V-normality of G,

$$(\dagger\dagger) \;\; X \in G \iff \iota \in j_G(X).$$

The ultrapower M_G has its own version of κ, the unique ordinal κ_* of M_G that is 'κ-like' in the sense that each of its predecessors has cardinality less than κ, whereas the set of predecessors of κ_* has cardinality exactly κ. Indeed, $\kappa_* = j_G(f)(\iota)$, where as above $f : \iota \to \kappa$ is any cofinal continuous function in V. Note that if M_G is well-founded, then $\kappa_* = \kappa$, but we must deal with the possibility that M_G is ill-founded.

Let us begin with an easy (yet useful) observation.

Claim 3.4.1. In $V[G]$, $|(\kappa_*^+)^{M_G}| \geq \kappa^+$.

Proof of claim. In V, fix a sequence of functions $\langle h_\alpha \rangle_{\alpha<\kappa^+} \subseteq \prod_{\xi<\iota} f(\xi)^+$ that is increasing in the order of domination modulo the bounded ideal on ι; namely, for each $\alpha < \beta < \kappa^+$, $\{\xi < \iota \mid h_\alpha(\xi) \geq h_\beta(\xi)\}$ is bounded in ι. Such a sequence exists because this reduced product is κ^+-directed. Note that in $V[G]$, $\langle j_G(h_\alpha)(\iota) \rangle_{\alpha<\kappa^+}$ is an increasing sequence of length κ^+ consisting of predecessors of $(\kappa_*^+)^{M_G}$. Thus, $|(\kappa_*^+)^{M_G}| \geq \kappa^{+V}$. But $\kappa^{+V} = \kappa^{+V[G]}$ since G is added by $(\mathcal{F}^+, \subseteq)$, which is a forcing of size $2^\iota < \kappa$. $\square$

Let $H := \text{HOD}^{M_G}$. By $(\dagger)$ and $(\dagger\dagger)$ above, $(\kappa_*^+)^H = (\kappa_*^+)^{M_G}$.

Let N denote the inner model of $V[G]$ consisting of all sets hereditarily ordinal definable in the structure $\langle V[G], V, G, \in \rangle$. The model N is a κ-cc forcing extension of HOD by Lemma 3.3, and so $(\kappa^+)^{\text{HOD}} = (\kappa^+)^N$.[4] If the structure H were a subclass of N, then we could finally conclude that

$$(\kappa^+)^{\text{HOD}} = (\kappa^+)^N \geq |(\kappa_*^+)^H| = |(\kappa_*^+)^{M_G}| \geq \kappa^+.$$

The intuition that H should be a subclass of N comes from our experience with well-founded ultrapowers. The structure M_G is definable over the structure $\langle V[G], V, G, \in \rangle$, and so if M_G were well-founded, then any element of H, being ordinal definable in M_G, would be ordinal definable in $\langle V[G], V, G, \in \rangle$; this would yield $H \subseteq N$. If H is ill-founded, however, then ordinals of M_G are not really ordinals, so it is not clear that ordinal definable elements of M_G are ordinal definable in $\langle V[G], V, G, \in \rangle$. To handle the possibility that H is not well-founded, we take a different approach.

Instead, we consider the V-ultrafilter $\mathcal{U}$ on κ given by $\mathcal{U} := f_*(G)$, where

$$f_*(G) := \{A \in \mathcal{P}(\kappa)^V \mid f^{-1}[A] \in G\},$$

[4]Here, and hereafter, HOD is in the sense of V.

https://doi.org/10.1017/fms.2025.10070 Published online by Cambridge University Press

and the ultrapower

$$H_0 := \mathrm{Ult}(\mathrm{HOD}, \mathcal{U} \cap \mathrm{HOD})$$

of HOD by $\mathcal{U} \cap \mathrm{HOD}$, using only functions in HOD.

An important observation is that $\mathcal{U}$ is ordinal definable in the structure $\langle V[G], V, G, \in\rangle$. This is because $f'_*(G) = \mathcal{U}$ for any increasing continuous cofinal map $f' : \iota \to \kappa$. Therefore, $H_0 \subseteq N$: The point is that the structure H_0 has for its universe the class of ordinal definable functions from κ into HOD, which is a subclass of HOD and hence of N; the (possibly non-standard) membership and equality predicates of H_0 are ordinal definable over $\langle V[G], V, G, \in\rangle$ as they are definable from $\mathcal{U} \cap \mathrm{HOD}$, which belongs to N.

Let $\kappa_0 := [\mathrm{id}]_{\mathcal{U}}$. Then κ_0 is the unique κ-like ordinal of H_0, in the same sense that κ_* is the unique κ-like ordinal of M_G. In fact, there is an embedding $k : H_0 \to H$ defined by

$$k([g]_{\mathcal{U}}) := j_G(g)(\kappa_*)$$

such that $k \circ j_{\mathcal{U}} = j_G \restriction \mathrm{HOD}$, which one can easily check is well-defined and elementary. This embedding restricts to an injective map from κ_0 to κ_*, so that the κ-likeness of κ_0 follows from that of κ_*.

The argument from above shows that $|(\kappa_0^+)^{H_0}| = |(\kappa^+)^{\mathrm{HOD}}|$ in N. We also obtain the following:

Claim 3.4.2. $N \models \kappa^+ = |(\kappa_0^+)^{H_0}|$.

Proof of claim. By Lemma 3.3, $(\kappa^+)^N = (\kappa^+)^{\mathrm{HOD}}$. Thus, as $H_0, \mathrm{HOD} \subseteq N$ and $|(\kappa_0^+)^{H_0}| = |(\kappa^+)^{\mathrm{HOD}}|$ in N, $N \models |(\kappa_0^+)^{H_0}| = |(\kappa^+)^{\mathrm{HOD}}| = \kappa^+$. $\square$

Next, we work towards showing that the previous claim is incompatible with "$(\kappa^+)^{\mathrm{HOD}} < \kappa^+$". This will yield the desired contradiction and as a result will lead to the proof of the theorem.

Let us begin with an auxiliary claim:

Claim 3.4.3. There is a $<_{\mathcal{U}}$-increasing sequence $\langle g_\alpha \mid \alpha < (\kappa^+)^{\mathrm{HOD}}\rangle \subseteq \prod_{\delta<\kappa}(\delta^+)^V$ in HOD, such that letting $\gamma_\alpha := [g_\alpha]_{\mathcal{U}}$, $\langle \gamma_\alpha \mid \alpha < (\kappa^+)^{\mathrm{HOD}}\rangle$ is an increasing cofinal sequence in $(\kappa_0^+)^{H_0}$.

Proof of claim. We note first that there is such a sequence in N. This is simply because N satisfies that $|(\kappa_0^+)^{H_0}| = (\kappa^+)^{\mathrm{HOD}} = \kappa^+$, and moreover by the proof of this fact, N satisfies that $\mathrm{cf}((\kappa_0^+)^{H_0}) = \kappa^+$, so in N, one can choose representatives for an increasing cofinal sequence in $(\kappa_0^+)^{H_0}$, which is simply a $<_{\mathcal{U}}$-increasing sequence $\langle g_\alpha \mid \alpha < (\kappa^+)^{\mathrm{HOD}}\rangle \subseteq \prod_{\delta<\kappa}(\delta^+)^V$ such that letting $\gamma_\alpha := [g_\alpha]_{\mathcal{U}}$, $\langle \gamma_\alpha \mid \alpha < (\kappa^+)^{\mathrm{HOD}}\rangle$ is cofinal in $(\kappa_0^+)^{H_0}$.

Now we pull the sequence down to HOD. Let $\mathbb{P} := (\mathcal{F}^+, \subseteq)$ denote our poset. Since κ is a strong limit cardinal (in V) and $|\mathbb{P}|^V < \kappa$, there is some V-regular $\gamma < \kappa$ such that $\mathbb{P}$ is γ-cc. By Lemma 3.3, the pair (HOD, N) has the γ-uniform cover property, so there is $\mathbb{Q} \in \mathrm{HOD}$ with $N \models$ "$|\mathbb{Q}| = 2^\gamma$" and $N = \mathrm{HOD}[F]$ where F is a HOD-generic filter for $\mathbb{Q}$. Note that

$$(2^\gamma)^N \le (2^\gamma)^{V[G]} < \kappa,$$

because κ remains a strong limit cardinal in $V[G]$.

Let $\langle \dot{g}_\alpha \mid \alpha < (\kappa^+)^{\mathrm{HOD}}\rangle \in \mathrm{HOD}$ be a sequence of $\mathbb{Q}$-names for functions in HOD such that $(\dot{g}_\alpha)_F = g_\alpha$. Since $|\mathbb{Q}| < \kappa$, there is a condition $p \in F$ deciding the value of $\dot{g}_\alpha$ for unboundedly many $\alpha < \kappa^{+\mathrm{HOD}}$; that is, for an unbounded set $S \subseteq (\kappa^+)^{\mathrm{HOD}}$ in HOD, for each $\alpha \in S$, $p \Vdash_{\mathbb{P}} \dot{g}_\alpha = \check{g}_\alpha$. Now $\langle g_\alpha \mid \alpha \in S\rangle \in \mathrm{HOD}$ is as desired.[5] $\square$

Assume towards a contradiction that $(\kappa^+)^{\mathrm{HOD}} < \kappa^+$. We would like to derive a uniform H_0-ultrafilter $\mathcal{D}$ on $\kappa_0^{+H_0}$ from the factor embedding $k : H_0 \to H$. The next claim allows us to ensure that $\mathcal{D}$ will be ordinal definable in the structure $\langle V[G], V, G, \in\rangle$:

[5]We thank the second referee for pointing that the proof of this claim that appeared in the first draft of this paper was garbled to the point of incorrectness.

https://doi.org/10.1017/fms.2025.10070 Published online by Cambridge University Press

Claim 3.4.4. $k[\kappa_0^{+H_0}]$ has a least upper bound $\nu < (\kappa_*^+)^H$ in H.

Proof of claim. By Theorem 3.1, $\mathrm{cf}(\kappa^{+\mathrm{HOD}}) \leq \iota$. Let $\rho := \mathrm{cf}(\kappa^{+\mathrm{HOD}})$ and $A \subseteq \kappa^{+\mathrm{HOD}}$ be a cofinal set of ordertype ρ. Then $\langle \gamma_\alpha \rangle_{\alpha \in A}$ is cofinal in $(\kappa_0^+)^{H_0}$, and hence, $\langle k(\gamma_\alpha) \rangle_{\alpha \in A}$ is cofinal in $k[\kappa_0^{+H_0}]$. But $\langle k(\gamma_\alpha) \rangle_{\alpha \in A} \in M_G$: Letting $\langle g_\alpha^* \rangle_{\alpha < j_G(\kappa^{+\mathrm{HOD}})} = j_G(\langle g_\alpha \rangle_{\alpha < \kappa^{+\mathrm{HOD}}})$,

$$\langle k(\gamma_\alpha) \rangle_{\alpha \in A} = \langle g_\alpha^*(\kappa_*) \rangle_{\alpha \in j_G[A]}$$

with $j_G[A] \in M_G$. (As $\mathrm{crit}(j_G) = \iota$, $A \in V$, and $|A| \leq \iota$.) Since $\langle k(\gamma_\alpha) \rangle_{\alpha \in A}$ is a set of ordinals in M_G, it has a least upper bound ν, and since $\langle k(\gamma_\alpha) \rangle_{\alpha \in A}$ is cofinal in $k[\kappa_0^{+H_0}]$, ν is the least upper bound of $k[\kappa_0^{+H_0}]$.

Note that $\nu < (\kappa_*^+)^H$: First, by our comments after Claim 3.4.1, $(\kappa_*^+)^H = (\kappa_*^+)^{M_G}$, so $(\kappa_*^+)^H$ is regular in M_G. Second, the previous argument shows that $\mathrm{cf}^{M_G}(\nu) \leq \iota$, which is less than κ. Therefore, $\nu < (\kappa_*^+)^H$. □

Let $\mathcal{D}$ be the H_0-ultrafilter on $(\kappa_0^+)^{H_0}$ derived from k using ν; namely,

$$\mathcal{D} = \{S \in \mathcal{P}^{H_0}((\kappa_0^+)^{H_0}) \mid \nu \in k(S)\}.$$

Let $H_1 := \mathrm{Ult}(H_0, \mathcal{D})$, again using only functions in H_0. Let $i : H_0 \to H_1$ be the ultrapower embedding, and let $\bar{\nu} := [\mathrm{id}]_{\mathcal{D}}$. Then,

$$\bar{\nu} = \sup i[\kappa_0^{+H_0}] < i(\kappa_0^{+H_0}).$$

Note that $\mathcal{D}$ is ordinal definable in the structure $\langle V[G], V, G, \in \rangle$, and hence, $H_1 \subseteq N$, by the same argument as for H_0. Since the ultrapower embedding $i : H_0 \to H_1$ is definable over $\langle V[G], V, G, \in \rangle$ from ordinal parameters,

$$i \restriction \kappa_0^{+H_0} \in N.$$

The next claim yields the desired contradiction with Claim 3.4.2:

Claim 3.4.5. $N \models |(\kappa_0^+)^{H_0}| \leq \kappa$.

Proof of claim. Since $i[\kappa_0^{+H_0}] \subseteq \bar{\nu}$, it follows that $|\kappa_0^{+H_0}|^N \leq |\bar{\nu}|^N$. Also, $\bar{\nu} < i(\kappa_0^{+H_0}) = i(\kappa_0)^{+H_1}$. Since $H_1 \subseteq N$, we have the following inequalities:

$$|\kappa_0^{+H_0}|^N \leq |\bar{\nu}|^N \leq |i(\kappa_0)|^N = \kappa.$$

The latter equality being true in that $i(\kappa_0)$ is κ-like, as it embeds into κ_*. □

Since we get a contradiction, our initial assumption that "$(\kappa^+)^{\mathrm{HOD}} < \kappa^+$" was false, and this proves the theorem. □

Let us now prove our second compactness theorem. This uses a slightly different technique (due to Casey–Goldberg) to prove the theorem for an arbitrary singular cardinal of uncountable cofinality; a direct adaptation of the argument from Theorem 3.4 would only prove the result for strong limit cardinals.

Theorem 3.5. *If κ is a singular cardinal of uncountable cofinality and $\{\delta < \kappa \mid \mathrm{cf}^{\mathrm{HOD}}(\delta) < \delta\}$ is stationary in κ, then $\mathrm{cf}^{\mathrm{HOD}}(\kappa) < \kappa$.*

Proof. Assume towards a contradiction that κ is a regular cardinal in HOD. For the rest of the proof, $\mathcal{F}$ denotes the closed unbounded filter on κ.

We claim that in HOD, the filter $\bar{\mathcal{F}} = \mathcal{F} \cap \mathrm{HOD}$ is weakly normal in the sense that every regressive function $f : A \to \kappa$ in HOD defined on a set $A \in \bar{\mathcal{F}}$ admits some $\gamma < \kappa$ such that $\{\alpha \in A \mid f(\alpha) < \gamma\} \in \bar{\mathcal{F}}$.[6] Fix $f \in \mathrm{HOD}$, and assume towards a contradiction that no such γ exists.

By Fodor's Lemma, it is not hard to see that any regressive function defined on a stationary subset of κ is bounded on a stationary set. (This is the argument used in Claim 3.1.1.) Therefore, let γ_0 be least such that the function $f : A \to \kappa$ is bounded by γ_0 on a stationary subset of A. By our assumption, the set A_1 of ordinals $\alpha \in A$ such that $f(\alpha) \geq \gamma_0$ is stationary as well. Let γ_1 be least such that $f \restriction A_1$ is bounded below γ_1 on a stationary set. Continuing this way, we can produce a continuous sequence $\langle \gamma_i \mid i < \kappa \rangle$ such that for all $i < \delta$,

$$\{\alpha \in A \mid f(\alpha) \in [\gamma_i, \gamma_{i+1})\}$$

is stationary. We use our assumption that κ is regular in HOD to ensure that the process can be continued at limit ordinals $i < \kappa$. (Note that the entire construction is internal to HOD.) But since $\mathrm{cf}(\kappa) < \kappa$, there cannot be κ-many disjoint stationary subsets of κ.

A similar argument shows that if $\gamma < \kappa$ is regular in HOD, greater than $\mathrm{cf}(\kappa)$, and of a different V-cofinality from κ, then $\bar{\mathcal{F}}$ is γ-indecomposable in HOD in the following sense: Working in HOD, any function $f : B \to \gamma$ with $B \in \bar{\mathcal{F}}$ is bounded below γ on a set in $\bar{\mathcal{F}}$.

Until further notice, let us work in HOD and denote

$$S := \{\delta < \kappa \mid \mathrm{cf}(\delta) < \delta\}.$$

Since $S \in \bar{\mathcal{F}}^+$, there is an ultrafilter U extending

$$\bar{\mathcal{F}} \cup \{S\}.$$

Since $\bar{\mathcal{F}}$ is weakly normal, U is weakly normal, and since $S \in U$, U concentrates on singular cardinals. Therefore, by [Ket72, Theorem 1.3], U is (ν, κ)-regular for some $\nu < \kappa$.[7]

Claim 3.5.1. U is γ-decomposable for every regular cardinal in (ν, κ).

Proof of claim. Let $\langle A_\alpha \rangle_{\alpha<\kappa}$ be a witness for "U is (ν, κ)-regular". Namely, this is a collection of U-measure one sets such that $\bigcap_{\alpha \in I} A_\alpha = \emptyset$ for all $I \subseteq \kappa$ with $|I| = \nu$. Let $\gamma \in (\nu, \kappa)$ be regular, and define a function $f : \kappa \to \gamma$ as $f(\alpha) := \sup\{\beta < \gamma \mid \alpha \in A_\beta\}$. The fact that $\langle A_\alpha \rangle_{\alpha<\kappa}$ witnesses (ν, κ)-regularity ensures that f is well-defined. Note that f cannot be bounded below γ on a set in U: Otherwise, $A := \{\alpha < \kappa \mid f(\alpha) < \beta\} \in U$, for some $\beta < \gamma$, but by definition, $A \cap A_\beta = \emptyset$. Therefore, U is γ-decomposable. $\square$

Now we return to V. Since $\bar{\mathcal{F}}$ is γ-indecomposable in HOD for all ordinals γ that are regular in HOD, greater than $\mathrm{cf}(\kappa)$, and of different V-cofinality from κ, U is γ-indecomposable for such ordinals.

It follows that for all ordinals $\gamma \in (\max\{\nu, \mathrm{cf}(\kappa)\}, \kappa)$, if γ is regular in HOD, then $\mathrm{cf}(\gamma) = \mathrm{cf}(\kappa)$; otherwise, the previous paragraph implies U is γ-indecomposable while the paragraph preceding it implies U is γ-decomposable. But κ is a limit of V-regular cardinals, and these are certainly regular in HOD and do not have the same cofinality as κ. This is a contradiction. $\square$

3.1. Optimality

In this section, we discuss the optimality of Theorems 3.4 and 3.5. Some of our arguments require rather technical Prikry-type forcings. Instead of elaborating on their precise definitions (which are fairly

[6] Note that this is a stronger form of normality than the one proved in Claim 3.1.1.

[7] Here, one cannot directly use Ketonen's result since U is not countably complete. Ketonen's result refers to the 'first function' of an ultrafilter, which in the case of a weakly normal ultrafilter is the identity. As Ketonen remarks after the proof of Theorem 1.3, the result only requires the existence of a first function, not the countable completeness of U. Since our ultrafilter does have a first function – namely, the identity – the required result is true.

https://doi.org/10.1017/fms.2025.10070 Published online by Cambridge University Press

long), we give appropriate references. Let us begin with Theorem 3.4. The next shows that the cofinality assumption is necessary in Theorem 3.4:

Proposition 3.6. *Assume that κ is a κ^+-supercompact cardinal. Then, there is a generic extension where*

1. *κ is a strong limit cardinal with $\mathrm{cf}(\kappa) = \omega$,*
2. *$\delta^{+\mathrm{HOD}} = \delta^+$ for all $\delta < \kappa$,*
3. *and $(\kappa^{+\mathrm{HOD}}) < \kappa^+$.*

Proof. By forcing with McAloon iteration, we may assume that "$V = \mathrm{HOD}$" holds (see p.5). Let $\mathcal{U}$ be a κ-complete, normal and fine ultrafilter over $\mathcal{P}_\kappa(\kappa^+)$. Let us force with the *Supercompact Prikry forcing* with respect to $\mathcal{U}$ ([Git10, §1]). This forcing is easily shown to be cone-homogeneous so that $\mathrm{HOD}^{V[G]} = V$ holds for all V-generic $G \subseteq \mathbb{P}$. This forcing does not introduce bounded subsets of κ so, in $V[G]$, $V[G]_\kappa = V_\kappa$. Also, κ becomes a strong limit cardinal with $\mathrm{cf}(\kappa) = \omega$. This gives (1) and (2) above. Finally, in $V[G]$, $(\kappa^+)^V$ is collapsed to κ, and hence, $(\kappa^+)^{\mathrm{HOD}^{V[G]}} = (\kappa^+)^V < \kappa^+$. □

The hypothesis '$\{\delta < \kappa \mid \delta^{+\mathrm{HOD}} = \delta^+\}$ is stationary' is also necessary:

Theorem 3.7. *Suppose that κ is a κ^{+2}-supercompact cardinal such that $2^{\kappa^{+n}} = \kappa^{n+1}$ for $n < \omega$. Then, for each regular uncountable cardinal $\mu < \kappa$, there is a generic extension where*

1. *κ is a strong limit cardinal with cofinality μ.*
2. *$\kappa^{+\mathrm{HOD}} < \kappa^+$.*
3. *There is a club $C \subseteq \kappa$ with $\mathrm{otp}(C) = \mu$ such that*

$$\delta^{+\mathrm{HOD}} = \delta^+ \text{ for all cardinals } \delta < \kappa \text{ not in } \mathrm{acc}(C).$$

Proof. By preliminarily forcing with McAloon's iteration, we may assume that $V \subseteq \mathrm{HOD}^{V^{\mathbb{Q}}}$ for any set-sized forcing $\mathbb{Q}$. If this iteration is started at a sufficiently large regular cardinal, our hypothesis on κ are maintained.

Suppose that $j\colon V \to M$ is a κ^{+2}-supercompact embedding. Let $\mu < \kappa$ be a regular cardinal and let u be the (κ, κ^+)-measure sequence of length μ derived from j. Namely, $u = \langle u_\alpha \mid \alpha < \mu\rangle$, where $u_0 := j``\kappa^+$ and $u_\alpha := \{X \subseteq V_\kappa \mid u \restriction \alpha \in j(X)\}$ for $\alpha > 0$. Notice that M contains every (κ, κ^+)-measure sequence of length less than μ so u above indeed exists. In addition, by the argument in [CFG15, Lemma 3.2], u belongs to $\mathcal{U}^{\sup}_\infty$.[8]

Let $\mathbb{R}^{\sup}_u$ be the supercompact Radin forcing defined from u [CFG15]. Let $G \subseteq \mathbb{R}^{\sup}_u$ a V-generic filter. Combining [CFG15, Corollary 4.2] with our forcing preparation,

$$V \subseteq \mathrm{HOD}^{V[G]} \subseteq V[G^\phi],$$

where G^ϕ is a V-generic for a plain Radin forcing $\mathbb{R}_u$ – hence, for a cardinal-preserving poset. In particular, the following inequalities hold:

$$(\kappa^+)^V \leq (\kappa^+)^{\mathrm{HOD}^{V[G]}} \leq (\kappa^+)^{V[G^\phi]} = (\kappa^+)^V < (\kappa^+)^{V[G]}.$$

The above yields item (2) of the theorem.

Let $\langle w_\alpha \mid \alpha < \mu\rangle$ be an injective enumeration of $\{w \mid w \text{appears in} p \in G\}$. Denote $\kappa_{w_\alpha} := \min(\mathrm{Ord} \setminus w_\alpha(0))$ and $\lambda_{w_\alpha} := \mathrm{otp}(w_\alpha(0))$. The increasing enumeration of $\{\kappa_{w_\alpha} \mid \alpha < \mu\}$ yields a club $C \subseteq \kappa$ of order-type μ and by forcing below an appropriate condition μ remains regular in $V[G]$. In addition, standard arguments show that κ remains a strong limit in $V[G]$ (see [CFG15, Lemma

[8]Our assumption that '$2^{\kappa^{+n}} = \kappa^{+n+1}$ holds for all $n < \omega$' is used precisely at this stage. A close inspection of the proof of [CFG15, Lemma 3.2] indicates that less instances of the GCH suffice to run the argument. However, we opted for this slightly stronger assumption for the sake of a more neat presentation.

https://doi.org/10.1017/fms.2025.10070 Published online by Cambridge University Press

3.10(6)]). These two observations combined yield item (1) of the theorem. Finally, [CFG15, Lemma 3.10(8)] shows that the only V-cardinals $\leq\kappa$ that survive after passing to $V[G]$ are those outside

$$\bigcup_{\alpha\in\mathrm{Lim}\cap\mu}(\kappa_{w_\alpha},\lambda_{w_\alpha}].$$

Thus, for every $V[G]$-cardinal $\delta\notin\mathrm{acc}(C)$, we have that $(\delta^+)^V$ does not belong to the above union and thus $(\delta^+)^V=(\delta^+)^{V[G]}$. By our previous observations, this yields $(\delta^+)^{\mathrm{HOD}^{V[G]}}=(\delta^+)^V=(\delta^+)^{V[G]}$, as claimed. □

Remark 3.8. The exact consistency strength of the configuration described above is unclear to us. Since the configuration violates the weak covering theorem for K [JS13, MSS97], one obtains the lower bound of a Woodin cardinal, but presumably one can obtain a stronger lower bound.

The following theorem shows that the assumptions of Theorem 3.5 are provably optimal. Starting with appropriate large cardinal assumptions, it is consistent for $\{\delta<\kappa\mid \mathrm{cf}^{\mathrm{HOD}}(\delta)<\delta\}$ to be nonstationary and κ to be inaccessible in HOD. This is a special case of a recent theorem of the authors [GOP24, Theorem 4.6] which utilizes the Supercompact Radin forcing of Theorem 3.7. We state the theorem without proof:

Theorem 3.9. *Suppose that δ is a supercompact cardinal and the* GCH *holds. Then, there is a generic extension where δ remains supercompact and there is a club $D\subseteq\delta$ consisting of cardinals κ for which*

$$\mathrm{HOD}\models\text{“}\kappa\textit{ is regular”}.$$

In particular, every $\lambda\in E^\kappa_{\omega_1}\cap\mathrm{acc}(D)$ is a singular of uncountable cofinality for which $\{\theta<\lambda\mid \mathrm{cf}^{\mathrm{HOD}}(\theta)=\theta\}$ contains a club and $\mathrm{cf}^{\mathrm{HOD}}(\lambda)=\lambda$.

To obtain the above configuration, it would be natural to utilize Ben-Neria-Unger's method from [BNU17, Theorem 1.3]. However, an apparent drawback of this alternative approach is that it does not seem amenable to preserving large cardinals at the level of strong compactness. This limitation is primarily caused by the club-shooting poset used after the nonstationary support iteration of Prikry-type forcings. It turns out that the preservation of supercompacts becomes interesting in light of the HOD dichotomy theorem proved in [Gol23].

3.2. On ω-club amenability

The first author showed that many of the known results on HOD – for example, the HOD dichotomy theorem – can actually proved for an arbitrary inner model that is ω-club amenable [Gol23].

A set $C\subseteq\delta$ is an *ω-club in δ* (for $\mathrm{cf}(\delta)\geq\omega_1$) if it is unbounded (in δ) and whenever S is a countable subset of C, $\sup(S)\in C$. The ω-club filter on δ, denoted by $\mathscr{C}_\delta$, is the collection of all subsets of δ that contain an ω-club.

Definition 3.10. An inner model M is *ω-club amenable* if $\mathscr{C}_\delta\cap M\in M$ for all ordinals δ with uncountable cofinality.

Not much is known about the size of HOD that does not already hold of any ω-club amenable model, so it is natural to seek properties that are more specific to HOD. In this section, we show that Theorem 3.4 does not generalize to an arbitrary ω-club amenable model.

Let $\vec{\mathscr{C}}$ denote the proper class $\{(\delta,S)\mid \mathrm{cf}(\delta)\geq\omega_1, S\in\mathscr{C}_\delta\}$. If one builds the constructible universe relative to the sequence $\vec{\mathscr{C}}$, then one obtains an ω-club amenable model. More generally,

Lemma 3.11. *For any class A, $M=L[A,\vec{\mathscr{C}}]$ is ω-club amenable.*

To construct ω-club amenable models that do not satisfy the conclusion of Theorem 3.4, we need a mild large cardinal hypothesis. Namely, we will assume that for all sets X, *X-sword exists*. Let us now define this hypothesis precisely.

https://doi.org/10.1017/fms.2025.10070 Published online by Cambridge University Press

If X is a set of rank λ, we say that *X-sword exists* if there is a *coarse X-sword mouse*, which is an iterable transitive structure $(M, \vec{U}, U, W)$ with the following properties:

- M is an transitive model of ZFC$^-$ with largest cardinal κ.
- $X \cap M \in M$ and $\vec{U} \in M$.
- $M \vDash \vec{U}$ is a coherent sequence of normal ultrafilters of length κ.
- $o^{\vec{U}}(\delta) = 0$ whenever $\delta \leq \lambda$
- $o^{\vec{U}}(\delta) \leq 1$ whenever $\lambda < \delta < \kappa$.
- W is a weakly amenable M-normal M-ultrafilter on κ
- $j^M_W(\vec{U}) \restriction \kappa + 1 = \vec{U}^\frown U$.

The notation for coherent sequences comes from Mitchell's handbook article [Mit09]. Things are a bit simpler here since all measures on $\vec{U}$ have order 0. One may therefore think of $\vec{U}$ as a partial function with $o^{\vec{U}}(\alpha) = 0$ if $\vec{U}$ is not defined on α, and $o^{\vec{U}}(\alpha) = 1$ if it is.

We emphasize that the notion of a coarse X-sword mouse is defined for an arbitrary set X, not necessarily a set of ordinals, and this will be relevant below.

The hypothesis that X-sword exists for every set X follows from the existence of a proper class of measurable cardinals of Mitchell order 2. To see this, assume W is a measure on κ of order 1 and $\vec{U}$ is a sequence of measures of order 0 defined on all measurable cardinals between γ and κ. Let $U = [\alpha \mapsto U_\alpha]_W$. Then $(H_{\kappa^+}, \vec{U}, U, W)$ is a coarse X-sword mouse for every $X \in V_\gamma$.

In terms of consistency strength, the hypothesis that X-sword exists for every set X is a bit weaker than the existence of a single measurable cardinal of Mitchell order 2: it is not hard to show that it holds in V_κ if κ is a measurable cardinal of Mitchell order 2.

Suppose $\mathcal{M} = (M, \vec{U}, U, W)$ is a coarse X-sword mouse and $\mathcal{H}$ is transitive with a Σ_1-elementary $\pi : \mathcal{H} \to \mathcal{M}$ such that $X \cap M \in \operatorname{ran}(\pi)$. Then $\mathcal{H}$ is a coarse $\pi^{-1}(X \cap M)$-sword mouse. Similarly, if $\mathcal{M}$ is a coarse X-sword mouse and there is a cofinal Σ_0-elementary $\pi : \mathcal{M} \to \mathcal{N}$, then $\mathcal{N}$ is a coarse $\pi(X \cap M)$-sword mouse if it is iterable. These facts are straightforward, except that the statement that a structure M satisfies ZFC$^-$ may seem too complicated to be preserved under such weak forms of elementarity. This is not really an issue, however, since the fact that M satisfies ZFC$^-$ follows from the fact that M satisfies Σ_0-Separation and the Well-Ordering Theorem combined with the weak amenability of W, which yields that $M = H(\kappa^+)^{\mathrm{Ult}_0(M,W)}$, which is a model of ZFC$^-$.

The following lemmas, suggested by one of the anonymous referees, vastly simplify the proof of Lemma 3.14 below.

Lemma 3.12. *Suppose X is a set and $\mathcal{M} = (M, \vec{U}, U, W)$ is a coarse X-sword mouse such that $o(M)$ is as small as possible. Then* $\operatorname{cf}(o(M)) = \omega$.

Proof. Let $\theta > \lambda$ be a sufficiently large regular cardinal, and let H be a countable transitive set admitting an elementary embedding $\pi : H \to H(\theta)$ with $\{X, \mathcal{M}\} \subseteq \operatorname{ran}(\pi)$. Let $\bar{X} = \pi^{-1}(X)$ and $\bar{\mathcal{M}} = \pi^{-1}(\mathcal{M})$, and note that $\bar{\mathcal{M}}$ is countable and π restricts to an elementary embedding $i : \bar{\mathcal{M}} \to \mathcal{M}$. We claim $i[o(\bar{M})]$ is cofinal in $o(M)$, which will establish the lemma. Suppose not, and let $N \subsetneq M$ be the transitive closure of $i[\bar{M}]$. Then $o(N) < o(M)$. To get a contradiction, it suffices to show that $\mathcal{N} = (N, \vec{U}, U \cap N, W \cap N)$ is a coarse X-sword mouse. Everything except iterability follows from the fact that $i : \bar{\mathcal{M}} \to \mathcal{N}$ is a cofinal Σ_0-elementary embedding. The iterability of $\mathcal{N}$ follows from that of $\mathcal{M}$ since any iterate of $\mathcal{N}$ admits a Σ_0-elementary embedding into an iterate of $\mathcal{M}$. □

Lemma 3.13. *Suppose X is a set and $\mathcal{M} = (M, \vec{U}, U, W)$ is a coarse X-sword mouse with largest cardinal κ. Assume the following hold:*

- $M = \mathrm{Hull}^{\mathcal{M}}_{\Sigma_1}(\alpha \cup \{p\})$ *for some $p \in M$ and $\alpha \geq \operatorname{rank}(X)$.*
- $M = \bigcup_{\xi < o(M)} M_\xi$, *where $\langle M_\xi \rangle_{\xi < o(M)}$ is an increasing sequence of transitive sets in M that contain V^M_κ.*

Then every ordinal $\delta > \alpha$ that is regular in M satisfies $\operatorname{cf}(\delta) = \operatorname{cf}(o(M))$.

https://doi.org/10.1017/fms.2025.10070 Published online by Cambridge University Press

Proof. For $\xi < o(M)$, let $\mathcal{M}_\xi = (M_\xi, \vec{U}, U \cap M_\xi, W \cap M_\xi)$ and let $H_\xi = \mathrm{Hull}_{\Sigma_1}^{\mathcal{M}_\xi}(\alpha \cup \{p\})$. Note that for all $\xi < o(M)$, $H_\xi \in M$ and $M \vDash |H_\xi| \leq \alpha$; also for $\xi \leq \xi' < o(M)$, $H_\xi \subseteq H_{\xi'}$. Moreover, for any Σ_1-formula $\varphi(x)$ and any $a \in M$, $\mathcal{M} \vDash \varphi(a)$ if and only if $\mathcal{M}_\xi \vDash \varphi(a)$ for all sufficiently large ξ if and only if $\mathcal{M}_\xi \vDash \varphi(a)$ for some ξ. It follows that

$$M = \mathrm{Hull}_{\Sigma_1}^{\mathcal{M}}(\alpha \cup \{p\}) = \bigcup_{\xi < o(M)} H_\xi.$$

Now suppose $\delta > \alpha$ is regular in M. For $\xi < o(M)$, let $\beta_\xi = \sup(\delta \cap H_\xi)$. Then $\beta_\xi < \delta$ since δ is regular in M and $M \vDash |H_\xi| \leq \alpha < \delta$. Since $\langle \beta_\xi \rangle_{\xi < o(M)}$ is weakly increasing and cofinal in δ, it follows that $\mathrm{cf}(\delta) = \mathrm{cf}(o(M))$. □

Lemma 3.14. *Assume that for all sets X, X-sword exists. Then for any cardinal λ and any set $A \subseteq \lambda$, $L[A, \vec{\mathscr{C}}]$ does not correctly compute λ^+.*

Proof. Fix a set $A \subseteq \lambda$. Using a pairing function on λ, it is not hard to construct a family X of subsets of λ such that for any class E, $L[X, E] = L[A, \mathscr{C} \restriction (\lambda + 1), E]$. In particular,

$$L[X, \vec{\mathscr{C}} \restriction (\lambda, \infty)] = L[A, \vec{\mathscr{C}}],$$

where $(\lambda, \infty) = \{\xi \in \mathrm{Ord} : \xi > \lambda\}$.

Let $\mathcal{M} = (M, \vec{U}, U, W)$ be a coarse X-sword mouse such that $o(M)$ is as small as possible. Note that letting $\bar{M} = L_{o(M)}[X, \vec{U}, U, W]$, the structure $\bar{\mathcal{M}} = (\bar{M}, \vec{U} \cap \bar{M}, U \cap \bar{M}, W \cap \bar{M})$ is also a coarse X-sword mouse, and so we may assume $\mathcal{M} = \bar{\mathcal{M}}$. This guarantees that $M = \bigcup_{\xi < o(M)} M_\xi$, where $\langle M_\xi \rangle_{\xi < o(M)}$ is an increasing sequence of transitive sets in M that contain V_κ^M: take $M_\xi = L_{\gamma + \xi}[X, \vec{U}, U, W]$ where $\gamma < o(M)$ is least such that $V_\kappa^M \subseteq L_\gamma[X, \vec{U}, U, W]$. Similarly, we may assume that

$$\mathcal{M} = \mathrm{Hull}_{\Sigma_1}^{\mathcal{M}}(\lambda \cup \{X \cap M\})$$

since the transitive collapse of $\mathrm{Hull}_{\Sigma_1}^{\mathcal{M}}(\lambda \cup \{X \cap M\})$ is again a coarse X-sword mouse. Thus, we may ensure that $\mathcal{M}$ satisfies the hypotheses of Lemmas 3.12 and 3.13. In particular, every M-regular cardinal $\delta > \lambda$ has countable cofinality in V.

We claim that $L[A, \vec{\mathscr{C}}]$ is contained in a proper initial segment N of a proper class iterate of $\mathcal{M}$. Granting this, we have $\lambda^{+L[A, \vec{\mathscr{C}}]} \leq \lambda^{+N} = \lambda^{+M} < \lambda^+$. (The final inequality comes from the fact that λ^{+M} has countable cofinality in V.) Thus, our claim suffices to finish the proof.

The idea is to iterate $\mathcal{M}$ to a model $\mathcal{N}$ with the following property. For each ordinal $\delta > \lambda$, $o^{\mathcal{N}}(\delta) > 0$ if and only if δ is regular in $\mathcal{N}$ and has uncountable cofinality in V; moreover, in this case, the unique measure on δ on the sequence of $\mathcal{N}$ is equal to $\mathscr{C}_\delta \cap \mathcal{N}$.

The iteration is defined by selecting at each stage the first total measure on the sequence of the current iterate that lies on an ordinal of countable cofinality. More formally, we define an iterated ultrapower

$$\langle (\mathcal{M}_\alpha, U_\alpha) \mid \alpha \in \mathrm{Ord} \rangle$$

of $\mathcal{M}$ by setting U_α equal to the first measure on the sequence $\vec{U}^{\mathcal{M}_\alpha} {}^\frown U^{\mathcal{M}_\alpha}$ that lies on an ordinal κ_α of countable cofinality in V; if there is no such measure, set U_α equal to the top measure $W^{\mathcal{M}_\alpha}$. For $\alpha \leq \beta \in \mathrm{Ord}$, let

$$j_{\alpha\beta} : \mathcal{M}_\alpha \to \mathcal{M}_\beta$$

denote the iterated ultrapower embedding.

For any ordinal ξ, the structure $M_\alpha \cap V_\xi$ is eventually constant, and therefore, we can define an inner model N of ZFC such that for all ordinals ξ, $N \cap V_\xi$ is equal to the eventual value of $M_\alpha \cap V_\xi$. Similarly,

https://doi.org/10.1017/fms.2025.10070 Published online by Cambridge University Press

we can define a sequence of N-ultrafilters $\vec{U}^{\mathcal{N}} : \mathrm{Ord} \to N$ by setting $\vec{U}^{\mathcal{N}}(\delta)$ equal to the eventual value of $U^{\mathcal{M}_\alpha}(\delta)$. We let $\mathcal{N} = (N, \vec{U}^{\mathcal{N}})$.

By the definition of the iteration, it is clear that if for some ordinal δ, $o^{\mathcal{N}}(\delta) > 0$, then δ has uncountable cofinality.

We claim that conversely, if $\delta > \lambda$ is a regular cardinal of N that has uncountable cofinality in V, then $o^{\mathcal{N}}(\delta) = 1$ and $\vec{U}^{\mathcal{N}}(\delta) = \mathscr{C}_\delta \cap N$. Since the models $\mathcal{M}_\alpha$ converge to $\mathcal{N}$, to prove the claim, it suffices to show, by induction on $\alpha \in \mathrm{Ord}$, that if $\delta > \lambda$ is a regular cardinal of M_α that has uncountable cofinality in V, then either $o^{\vec{U}^{\mathcal{M}_\alpha}}(\delta) = 1$ and $\vec{U}^{\mathcal{M}_\alpha}(\delta) = \mathscr{C}_\delta \cap M_\alpha$ or δ is the largest cardinal of $\mathcal{M}_\alpha$ and $U^{\mathcal{M}_\alpha} = \mathscr{C}_\delta \cap M_\alpha$.

For the case that $\alpha = 0$, note that our choice of $\mathcal{M}_0 = \mathcal{M}$, we have $\mathcal{M} = \mathrm{Hull}_{\Sigma_1}^{\mathcal{M}}(\lambda \cup \{X \cap M\})$, and so by the referee's Lemmas 3.12 and 3.13, if $\delta > \lambda$ is a regular cardinal of M, then δ has countable cofinality in V. Thus, the base case holds vacuously.

Now assume the induction hypothesis holds for $\mathcal{M}_\alpha$, and we claim it is true for $\mathcal{M}_{\alpha+1}$. Suppose therefore that $\delta > \lambda$ is a regular cardinal of $M_{\alpha+1}$. Note that $\mathcal{M}_\alpha | \kappa_\alpha = \mathcal{M}_{\alpha+1} | \kappa_\alpha$, so for ordinals δ in the open interval (λ, κ_α), the induction hypothesis for $\mathcal{M}_\alpha$ easily implies the induction hypothesis for $\mathcal{M}_{\alpha+1}$. A slightly more complicated variation of this argument establishes the induction hypothesis for $\mathcal{M}_{\alpha+1}$ in the case that $\delta = \kappa_\alpha$: by the definition of the iteration, either κ_α has countable cofinality, in which case we have nothing to show, or else $U_\alpha = W^{\mathcal{M}_\alpha}$, in which case the fact that $U^{\mathcal{M}_\alpha} = \vec{U}^{\mathcal{M}_{\alpha+1}}(\kappa_\alpha)$ implies the induction hypothesis holds for $\mathcal{M}_{\alpha+1}$ with respect to $\delta = \kappa_\alpha$.

To finish the successor case, we show that if $\delta > \kappa_\alpha$ is regular in $M_{\alpha+1}$, then $\mathrm{cf}(\delta) = \omega$, so the induction hypothesis holds vacuously in the interval $(\kappa_\alpha, o(M_{\alpha+1}))$. Since $M_{\alpha+1}$ is generated by the critical points of the iteration along with the range of $j_{0\alpha+1}$, $\mathcal{M}_{\alpha+1} = \mathrm{Hull}_{\Sigma_1}^{\mathcal{M}_{\alpha+1}}((\kappa_\alpha + 1) \cup \{X \cap M\})$. Since there is a cofinal embedding from M to $M_{\alpha+1}$, Lemma 3.12 implies that $\mathrm{cf}(o(M_{\alpha+1})) = \omega$. Therefore, we can apply Lemma 3.13 to obtain that $\mathrm{cf}(\delta) = \omega$, as desired.

Finally, we consider the limit case. Suppose α is a limit ordinal and $\delta > \lambda$ is a regular cardinal of M_α. Let $\gamma = \sup_{\beta<\alpha} \kappa_\beta$. As in the previous case, if $\delta < \gamma$, the desired conclusion is immediate from the induction hypothesis. Moreover, since $\mathcal{M}_\alpha = \mathrm{Hull}_{\Sigma_1}^{\mathcal{M}_\alpha}(\gamma \cup \{X \cap M\})$, if $\delta > \gamma$, then by Lemma 3.13, $\mathrm{cf}(\delta) = \omega$, and there is nothing to prove.

It therefore suffices to consider the case that $\delta = \gamma$.

For each $\beta < \alpha$, let $\delta_\beta = j_{\beta\alpha}^{-1}[\delta]$. Note that $\delta_\beta \geq \kappa_\beta$ for all $\beta < \alpha$. Moreover, for sufficiently large $\beta < \alpha$, $j_{\beta\alpha}(\delta_\beta) = \delta$, and therefore for any $\beta' \geq \beta$ with $\beta' < \alpha$, $j_{\beta\beta'}(\delta_\beta) = \delta_{\beta'}$.

Assume first that for sufficiently large $\beta < \alpha$, $\delta_\beta \neq \kappa_\beta$. Then in fact $\delta_\beta > \kappa_\beta$, and so by Lemma 3.13, $\mathrm{cf}(\delta_\beta) = \omega$. Moreover, it follows that for sufficiently large $\beta < \alpha$, $j_{\beta\alpha}$ is continuous at δ_β and $j_{\beta\alpha}(\delta_\beta) = \delta$, so δ has countable cofinality as well, and we are done.

To finish the induction, assume instead that $\kappa_\beta = \delta_\beta$ for cofinally many $\beta < \alpha$. In this case, the set $C = \{\kappa_\beta : \beta < \alpha \text{ and } j_{\beta\alpha}(\kappa_\beta) = \delta\}$ is unbounded in δ. Moreover, it is ω-closed in δ, since if $\langle \beta_n \rangle_{n<\omega}$ is an increasing sequence of ordinals with $\kappa_{\beta_n} \in C$ and β is their supremum, then we claim $\kappa_\beta = \sup_{n<\omega} \kappa_{\beta_n}$. Clearly, $\kappa_\beta \geq \sup_{n<\omega} \kappa_{\beta_n}$, and the reverse inequality follows from the choice of κ_β in our construction of the iterated ultrapower, noting that $\sup_{n<\omega} \kappa_{\beta_n}$ has countable cofinality and carries a measure on the $\mathcal{M}_\beta$-sequence. The latter fact uses that $\sup_{n<\omega} \kappa_{\beta_n} = j_{\beta_0\beta}(\kappa_{\beta_0})$.

Similarly, $o^{\mathcal{M}_\alpha}(\delta) \geq 1$. Let $Z = \vec{U}^{\mathcal{M}_\alpha}(\delta)$ if δ is less than the largest cardinal of $\mathcal{M}_\alpha$, and let $Z = U^{\mathcal{M}_\alpha}$ otherwise. The standard argument (going back to Kunen) shows that for any $A \in P(\delta) \cap M_\alpha$, $A \in Z$ if and only if $C \setminus \eta \subseteq A$ for some $\eta < \delta$. Therefore, if δ has uncountable cofinality, then $Z = \mathscr{C}_\delta \cap \mathcal{M}_\alpha$.

This completes our transfinite induction and establishes that if $\delta > \lambda$ is a regular cardinal of N that has uncountable cofinality in V, then $o^{\mathcal{N}}(\delta) = 1$ and $\vec{U}^{\mathcal{N}}(\delta) = \mathscr{C}_\delta \cap N$.

From this, it follows that $L[A, \vec{\mathscr{C}}]$ is a definable inner model of $\mathcal{N}$, since in fact $L[A, \vec{\mathscr{C}}] = L[X, \vec{\mathscr{C}} \restriction (\lambda, \infty)]$ is definable over $\mathcal{N}$ using the parameter $X \cap N$ and the sequence $\vec{U}^{\mathcal{N}}$. As explained above, it follows that $\lambda^{+L[A,\vec{\mathscr{C}}]} < \lambda^+$, which completes the proof. □

Putting everything together, we arrive at the following corollary:

https://doi.org/10.1017/fms.2025.10070 Published online by Cambridge University Press

Corollary 3.15. *Suppose that for every set X, X-sword exists. Then for every cardinal λ, there is an ω-club amenable inner model M that is correct about cardinals and cofinalities $\leq \lambda$ while $(\lambda^+)^M < \lambda^+$.*

Proof. Fix a sequence $\langle a_\alpha \rangle_{\alpha \leq \lambda}$ such that for every limit ordinal $\alpha \leq \lambda$, a_α is a cofinal subset of α ordertype $\mathrm{cf}(\alpha)$. This sequence can be coded by a set $A \subseteq \lambda$, and by Lemma 3.14, the inner model $M = L[A, \vec{\mathscr{C}}]$ is an ω-club amenable model such that $\lambda^{+M} < \lambda^+$. □

4. Open questions and remarks

The following is a configuration not handled by our arguments:

Question 1. Suppose that κ is a strong limit singular cardinal of uncountable cofinality and that $\{\delta < \kappa \mid (\delta^{++})^{\mathrm{HOD}} \geq \delta^+\}$ is stationary. Is it true that $(\kappa^{++})^{\mathrm{HOD}} \geq \kappa^+$?

We do not know either if other HOD-related properties behave in a compact-like way. For instance, the following is open.

Question 2. Suppose that κ is a singular strong limit cardinal with uncountable cofinality and that $\{\delta < \kappa \mid \delta^+$ is not ω-strongly measurable in HOD$\}$ is stationary. Is it true that κ^+ is not ω-strongly measurable in HOD?

There is another intriguing question connecting Woodin's HOD Conjecture with Theorem 3.4. Assuming the existence of strong enough large cardinals, in [Pov23, Theorem 3.1] it was proved that a cardinal κ can be $<\lambda$-extendible for a singular a strong limit cardinal λ with $\mathrm{cf}(\lambda) = \omega$ and $(\lambda^+)^{\mathrm{HOD}_x} < \lambda^+$ for all subsets $x \subseteq \lambda$. In simple terms, the HOD Conjecture can *fail locally*.[9]

A natural speculation is whether this failure can take place at a strong limit singular of uncountable cofinality.

Question 3. Is the following configuration consistent with ZFC?

1. κ is $<\lambda$-extendible.
2. λ strong limit with $\mathrm{cf}(\lambda) \geq \omega_1$.
3. $(\lambda^+)^{\mathrm{HOD}} < \lambda^+$

Granting the HOD Conjecture, Theorem 3.4 suggests that the answer to Question 3 is negative. For suppose Clause (3) above holds. Then, by Theorem 3.4, the set $\{\delta < \lambda \mid (\delta^+)^{\mathrm{HOD}} < \delta^+\}$ contains a club. In particular, the degree of extendibility of κ overlaps a singular cardinal $\delta < \lambda$ witnessing $(\delta^+)^{\mathrm{HOD}} < \delta^+$. This is on the verge of refuting the HOD Conjecture. Note, however, that it does not outright preclude it, the reason being that V_λ may not satisfy ZF. A negative answer would point out yet another difference between singular cardinals of countable and uncountable cofinality.

Acknowledgements. We thank the referees for their detailed comments and corrections to previous drafts of this paper.

Competing interest. The authors have no competing interests to declare.

Funding statement. Goldberg was supported in part by NSF Grant No. 2401789. Poveda was supported by the Department of Mathematics and the Center of Mathematical Sciences and Applications at Harvard University.

References

[BNU17] O. Ben-Neria and S. Unger, 'Homogeneous changes in cofinalities with applications to HOD', *J. Math. Log.* **17**(02) (2017), 1750007.

[BP23] J. Bagaria and A. Poveda, 'More on the preservation of large cardinals under class forcing', *J. Symb. Log.* **88**(1) (2023), 290–323.

[Buk73] L. Bukovskỳ, 'Characterization of generic extensions of models of set theory', *Fund. Math.* **83**(1) (1973), 35–46.

[CFG15] J. Cummings, S. D. Friedman and M. Golshani, 'Collapsing the cardinals of hod', *J. Math. Log.* **15**(02) (2015), 1550007.

[9] A related result, yet this time requiring the failure of AC, has been proved in [Sch22, Theorem 3.7].

[Coh63] P. J. Cohen, 'The independence of the continuum hypothesis', *Proc. Nat. Acad. Sci. U.S.A.* **50** (1963), 1143–1148.
[Eas70] W. B. Easton, 'Powers of regular cardinals', *Ann. Math. Logic* **1** (1970), 139–178.
[FHR15] G. Fuchs, J. D. Hamkins and J. Reitz, 'Set-theoretic geology', *Ann. Pure Appl. Logic* **166**(4) (2015), 464–501.
[For09] M. Foreman, 'Ideals and generic elementary embeddings', in *Handbook of Set Theory* (Springer, 2009), 885–1147.
[Git10] M. Gitik, 'Prikry-type forcings', in *Handbook of Set Theory. Vols. 1, 2, 3* (Dordrecht, Springer, 2010), 1351–1447.
[Göd39] K. Gödel, 'Consistency-proof for the generalized continuum-hypothesis', *Proc. Natl. Acad. Sci.* **25**(4) (1939), 220–224.
[Göd46] K. Gödel, 'Remarks before the Princeton bicentennial conference on problems in mathematics', 1946.
[Gol23] G. Goldberg, 'A note on Woodin's HOD dichotomy', to appear in *Journal of Mathematical Logic*, 2023.
[GOP24] G. Goldberg, J. Osinski and A. Poveda, 'On the optimality of the HOD hypothesis', Submitted, 2024.
[Jec03] T. Jech, *Set Theory: The Third Millennium Edition, Revised and Expanded* (Springer, 2003).
[JS13] R. Jensen and J. Steel, 'K without the measurable', *J. Symb. Log.* **78**(3) (2013), 708–734.
[Kan76] A. Kanamori, 'Weakly normal filters and irregular ultrafilters', *Trans. Amer. Math. Soc.* **220** (1976), 393–399.
[Ket72] J. Ketonen, 'Strong compactness and other cardinal sins', *Ann. Math. Log.* **5**(1) (1972), 47–76.
[Kön05] J. König, 'Zum Kontinuum-Problem', *Math. Ann.* **60**(2) (1905), 177–180.
[Mit09] W. J. Mitchell, 'Beginning inner model theory', In *Handbook of Set Theory* (Springer, 2009), 1449–1495.
[MSS97] W. J. Mitchell, E. Schimmerling and J. R. Steel, 'The covering lemma up to a Woodin cardinal', *Ann. Pure Appl. Logic* **84**(2) (1997), 219–255.
[Pov23] A. Poveda, 'Two results on extendible cardinals', accepted in *Proc. Amer. Math. Soc.*, 2023.
[Sch20] R. Schindler, 'From set theoretic to inner model theoretic geology', Preprint, 2020.
[Sch22] F. Schlutzenberg, 'Reinhardt cardinals and iterates of V', *Ann. Pure Appl. Logic* **173**(2) (2022), 103056.
[She74] S. Shelah, 'Infinite abelian groups, Whitehead problem and some constructions', *Israel J. Math.* **18** (1974), 243–256.
[Sil75] J. Silver, 'On the singular cardinals problem', (1975), 435–438.