E. Carpenter's Proof of Taylor's Theorem.

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The idea of the following proof was communicated to me some years ago by Mr Edward Carpenter of Millthorpe, Derbyshire, formerly Fellow of Trinity Hall, Cambridge; who remarked that it seemed to afford a demonstration of Taylor's Theorem which came very naturally and directly from the definition of a differential coefficient. The chief difficulty seemed to arise in dealing with the negligible small quantities which are produced in great numbers. However, I found it not difficult to complete the proof for the case when *all* the successive differential coefficients of $f(x)$ are finite and continuous.

It occurred to me lately that this proof might interest the Society: and it is here given with the addition of a modified proof leading to an expansion in m terms with a remainder.

1. If $f(x)$ possesses a differential coefficient $f'(x)$, then

$$\operatorname*{Lim}_{h=0} \frac{f(x+h)-f(x)}{h} = f'(x)$$

that is

$$f(x+h) = f(x) + hf'(x) + h\alpha_1 \qquad \ldots \qquad \ldots \qquad (1)$$

where α_1 is a quantity which vanishes with h.
Similarly

$$f'(x+h) = f'(x) + hf''(x) + h\alpha_2 \qquad \ldots \qquad \ldots \qquad (2)$$

$$f''(x+h) = f''(x) + hf'''(x) + h\alpha_3 \qquad \ldots \qquad \ldots \qquad (3)$$

and so on, $\alpha_2\ \alpha_3$ etc. being quantities which vanish with h.

Writing $x+h$ for x in (1) we have

$$f(x+2h) = f(x+h) + hf'(x+h) + h\alpha_1'$$

where α_1' also vanishes with h; and hence, by (2)

$$f(x+2h) = f(x) + 2hf'(x) + h^2f''(x) + h(\alpha_1+\alpha_1') + h^2\alpha_2$$

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Repeating this process n times, we get

$$f(x+nh)=f(x)+{}^nC_1hf'(x)+{}^nC_2h^2f''(x)+\ldots+{}^nC_nh^nf^n(x)$$
$$+h(a_1+a_1'+a_1''+\ldots)+h^2(a_2+a_2'+\ldots)+\ldots+h^na_n \qquad (4)$$

where ${}^nC_r$ is the number of r combinations of n things, and the number of the quantities $a_r, a_r', a_r''\ldots$ is ${}^nC_r$.

Now if η be the numerical value of the numerically greatest of the quantities a, the sum of the terms in the 2nd line of (4) is

$$<\eta({}^nC_1h+{}^nC_2h^2+\ldots+{}^nC_nh^n)$$
$$<\eta(\overline{1+h}^n-1) \qquad <\eta(e^{nh}-1).$$

Now write y for nh and (4) becomes

$$f(x+y)=f(x)+\frac{y}{1}f'(x)+\frac{y.(y-h)}{1\cdot 2}f''(x)+\ldots$$
$$+\frac{y(y-h)\ldots(y-\overline{n-1}h)}{n!}f^n(x)+\epsilon \qquad (5)$$

where $\epsilon<\eta(e^y-1)$.

It is clear that if we take any finite number of the terms of the series (5), say $m+1$, these terms will differ from the first $m+1$ terms of Taylor's Expansion of $f(x+y)$ by a finite number of quantities which vanish with h; the sum of which we may denote by ϵ', a quantity which likewise vanishes with h.

If then Taylor's Expansion is absolutely convergent, it is clear that the above investigation affords a complete proof of his Theorem. In fact if R be the remainder after $m+1$ terms of Taylor's Expansion, and L the corresponding remainder (excluding ϵ) of the expansion in (5), then each term of L is numerically less than the corresponding term of R and L can be made less than $\frac{\delta}{3}$ when δ is an arbitrary small finite quantity, by taking m great enough, but still finite.

Thus from (5) we have

$$f(x+y)=f(x)+y.f'(x)+\frac{y^2}{2!}f''(x)+\ldots+\frac{y^m}{m!}f^m(x)+L+\epsilon+\epsilon'.$$

Now take m so that $L < \frac{\delta}{3}$.

Then take n so that $\epsilon < \frac{\delta}{3}$ and $\epsilon' < \frac{\delta}{3}$.

Thus $$f(x+y) = f(x) + yf'(x) + \ldots + \frac{y^m}{m!}f^m(x)$$
$$+ \text{ a quantity less than } \delta.$$

This investigation will apply only when *all* the differential coefficients of $f(x)$ are finite and continuous, and Taylor's Expansion an absolutely convergent series for all values of the independent variable from x up to $x+y$.

2. In order to get a formula with a remainder after m terms, m being *any* integer, without assuming anything about the differential coefficients after the m^{th} we may modify our procedure thus:—

Proceed as before until we get

$$f(x+mh) = f(x) + {}^mC_1hf'(x) + \ldots + {}^mC_mh^mf^m(x)$$
$$+ h(a_1 + a_1' + \ldots) + h^2(a_2 + a_2' + \ldots) + \ldots + h^m a_m.$$

In taking the succeeding steps up to the n^{th}, proceed as before except that the m^{th} differential coefficient is left unaltered each time, when we are substituting for $f(x+h), f'(x+h)\ldots$ the values given by (1), (2) ...

After the n^{th} step we have

$$f(x+nh) = f(x) + {}^nC_1hf'(x) + \ldots + {}^nC_{m-1}h^{m-1}f^{m-1}(x)$$
$$+ {}^{n-1}C_{m-1}h^mf^m(x) + {}^{n-2}C_{m-1}h^mf^m(x+h) + \ldots$$
$$+ {}^mC_{m-1}h^mf^m(x+\overline{n-m-1}h) + {}^mC_mh^mf^m(x+\overline{n-m}h)$$
$$+ h(a_1 + a_1' + \ldots) + h^2(a_2 + a_2' + \ldots) + \ldots + h^m(a_m + \ldots) \qquad (6)$$

Here the number of the quantities a_1 is ${}^nC_1$ and that of the quantities a_r is ${}^nC_r$. Hence if η is the numerical value of the numerically greatest a, the last line of (6) is

$$< \eta\{{}^nC_1h + {}^nC_2h^2 + \ldots + {}^nC_mh^m\} < \eta\{\overline{1+h}^n - 1\} < \eta(e^y - 1)$$

where $y = nh$. Denoting the sum of these small terms by ϵ, we see that ϵ is less than a quantity which vanishes with h.

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Again the sum of the coefficients of the terms which contain differential coefficients of the m^{th} order is

$$^{n-1}C_{m-1} + {}^{n-2}C_{m-1} + \dots + {}^{m}C_{m-1} + {}^{m-1}C_{m-1}, \quad \text{since } {}^{m}C_{m} = {}^{m-1}C_{m-1}$$

and this is $= {}^{n}C_{m}$. Multiplying it by h^m we get

$$y(y-h)(y-2h)\dots(y-\overline{m-1}h)/m\,!$$

which differs from $y^m/m\,!$ by a quantity ζ which vanishes with h. Hence the terms in question are

$$= (y^m/m\,! - \zeta) \times \text{a mean of the values } f^m(x), f^m(x+h)\dots f^m(x+y-mh).$$

This mean may obviously be written $f^m(x+\theta y)$, where θ is a proper fraction, so that we have $y^m f^m(x+\theta y)+\epsilon'$ where $\epsilon' = -\zeta f^m(x+\theta y)$, a quantity which vanishes with h.

Again the first line of (6) differs from

$$f(x) + yf'(x) + \frac{y^2}{2\,!}f''(x) + \dots + \frac{y^{m-1}}{(m-1)\,!}f^{m-1}(x)$$

by a quantity ϵ'' which also vanishes with h.

Now take n so large that $\epsilon < \frac{\delta}{3}$, $\epsilon' < \frac{\delta}{3}$, and $\epsilon'' < \frac{\delta}{3}$ where δ is a finite quantity but may be as small as we please.

Thus $$f(x+y) = f(x) + yf'(x) + \dots + \frac{y^{m-1}}{(m-1)\,!}f^{m-1}(x)$$
$$+ \frac{y^m}{m\,!}f^m(x+\theta h) + \text{a quantity less than } \delta.$$

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