A THEORY OF NORMAL CHAINS

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Introduction. In this paper we deal with a group-theoretic configuration of the following type: G is an additive group (not necessarily abelian) for which an operator system M and a complete lattice ϕ of M admissible subgroups are defined; we call G and M-ϕ group. In §1 we make various definitions and note that analogues of some of the classical theorems of group theory hold.

Our main interest is in the normal chains for an M-ϕ group. We first discuss normal chains in general, and obtain results which hold if the factors of the chain fulfil suitable conditions (§3). In the remainder of the paper these results are applied to three particular types of normal chain and the relation between these chains is discussed.

The first type of chain discussed is the so-called Loewy chain. This type is of especial importance because it is intimately related to the other two types considered. It is shown how the existence of a Loewy chain connecting the group to 0 may be used in place of chain conditions. Furthermore, if such a chain exists for a nilpotent group, then it is actually a central chain.

We have adopted Hirsch's definition of solubility (or rather its analogue for M-ϕ groups) rather than the customary definition. For the chains usually employed do not meet the general requirements needed to apply our theory. On the other hand, the chains introduced by Hirsch do satisfy these requirements, provided that the group possesses a Loewy chain connecting it to 0.

1. Definitions and basic theorems. Let G be an additive group which is not necessarily abelian. If A_α, for each α in a set $\mathfrak{A}$, is a subgroup of G, then we denote the intersection of the A_α by $\bigcap A_\alpha$ $(\alpha \in \mathfrak{A})$. The subgroup of G generated by the A_α we call the compositum of the A_α and denote this subgroup by $\complement A_\alpha (\alpha \in \mathfrak{A})$. In the case of a finite number of subgroups, $A_1, \ldots, A_n$, we denote the intersection and compositum by

$$A_1 \cap \ldots \cap A_n \text{ (or } \bigcap_{i=1}^{n} A_i) \quad \text{and} \quad \{A_1, \ldots, A_n\} \text{ (or } \complement_{i=1}^{n} A_i)$$

respectively.

Let M be a system of operators for G, so that each element of M induces an endomorphism in G, i.e., we have:

(i) αg is in G, whenever α is in M and g is in G.

(ii) $\alpha(g_1 \pm g_2) = \alpha g_1 \pm \alpha g_2$, for α in M and g_1, g_2 in G.

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We let $(\alpha\beta)g = \alpha(\beta g)$, for α and β in M and g in G.

A subgroup S of G is called M admissible if $MS \subseteq S$. We shall restrict our attention to a family of M admissible subgroups ϕ which form a complete lattice relative to intersection and compositum, i.e., we assume about the subgroups of ϕ that:

(i) If A is in ϕ, A is M admissible.

(ii) 0 and G are in ϕ.

(iii) If A_α is in ϕ, for each α in the set $\mathfrak{A}$, then $\bigcap A_\alpha(\alpha \in \mathfrak{A})$ is in ϕ.

(iv) If A_α is in ϕ, for each α in the set $\mathfrak{A}$, then $\mathsf{C}\, A_\alpha(\alpha \in \mathfrak{A})$ is in ϕ.

We note that if the subgroups of ϕ satisfy the descending chain condition, (iii) may be replaced by (iii′), and if they satisfy the ascending chain condition, (iv) may be replaced by (iv′), where:

(iii′) If A and B are in ϕ, then $A \cap B$ is in ϕ.

(iv′) If A and B are in ϕ, then $\{A, B\}$ is in ϕ.

We call G an M-ϕ group if a particular system of operators M, and a particular complete lattice ϕ of M subgroups are to be distinguished; if ϕ consists of all M admissible subgroups we call G an M group. If a subgroup S belongs to ϕ, we say that S is a ϕ subgroup of G; we note that S is also an M-ϕ group. If G is an M-ϕ group, we denote by ψ the set of all normal ϕ subgroups of G; since the ϕ subgroups of G form a complete lattice, the normal ϕ subgroups of G also form a complete lattice. Hence we may also consider G as an M-ψ group. We make the following definitions:

Definition. If the ϕ subgroup S of G has no normal ϕ subgroups, S is ϕ ***simple.***

Definition. Let G and G' be M-ϕ groups. σ is an M-ϕ *isomorphism* (*homomorphism*) of G onto G' if

(i) σ is an isomorphism (homomorphism) of G onto G'. (Hence $G' = G\sigma$).

(ii) $(\alpha g)\sigma = \alpha(g\sigma)$, for all α in M and for all g in G.

(iii) If S is a ϕ subgroup of G, $S\sigma$ is a ϕ subgroup of G'; if S' is a ϕ subgroup of G', the inverse image of S', $S'\sigma^{-1}$ is a ϕ subgroup of G. We say that G is M-ϕ *isomorphic* to G' if there exists an M-ϕ isomorphism of G onto G', and we write $G \cong G'(M\text{-}\phi)$.

Definition. Let G and G' be M-ϕ groups. σ is an M-ϕ *isomorphism* (*homomorphism*) of G into G' if $G\sigma \subseteq G'$ and σ is an M-ϕ isomorphism (homomorphism) of G onto $G\sigma$.

In the last two definitions, the systems of distinguished M admissible subgroups for the groups G and G' are both denoted by ϕ, although in general they are different systems. At first sight this would seem to lead to confusion, but it is always clear from the context what is meant and the notation proves to be a convenient one.

Let G be an M-ϕ group, and N a normal ϕ subgroup. Then G/N is an M group, and a system of M admissible subgroups ϕ in G/N may be defined in this way: if U/N is an M admissible subgroup of G/N and U is in ϕ, then U/N is in ϕ. It is clear that this system of subgroups of G/N forms a complete lattice and hence G/N is an M-ϕ group.

The following analogues to the classical theorems hold:

THEOREM 1.1 (The Homomorphism Theorem). *If σ is an M-ϕ homomorphism of the M-ϕ group G onto the M-ϕ group G', then the kernel N is a normal ϕ subgroup of G and G/N is M-ϕ isomorphic to G'. Conversely, if N is a normal ϕ subgroup of the M-ϕ group G, then there exists an M-ϕ homomorphism τ of G onto G/N; τ maps g onto the coset $N + g$, for all g in G, and is called the natural mapping of G onto G/N.*

THEOREM 1.2 (The First Isomorphism Theorem). *If S and T are ϕ subgroups of the M-ϕ group G, and if S is normal in $\{S, T\}$, then $S \cap T$ is a normal ϕ subgroup of T and*

$$\{S, T\}/S \cong T/S \cap T \qquad (M\text{-}\phi).$$

THEOREM 1.3 (The Second Isomorphism Theorem). *Let G be an M-ϕ group and N, H normal ϕ subgroups of G with $N \subseteq H$, then we have:*

$$\frac{G/N}{H/N} \cong \frac{G}{H} \qquad (M\text{-}\phi).$$

Definition. Let A and B be ϕ subgroups of the M-ϕ group G with $A \subseteq B$. If there exists a chain

$$(0) \qquad A = A_0 \subseteq \ldots \subseteq A_i \subseteq A_{i+1} \subseteq \ldots \subseteq A_n = B,$$

where A_i is a normal ϕ subgroup of A_{i+1} $(i = 1, \ldots, n-1)$, (0) is called a *normal* ϕ chain from A to B, or a normal ϕ chain *connecting* A and B. If $A_i \neq A_{i+1}$ for each i, (0) has *length n;* the M-ϕ groups A_{i+1}/A_i are called the *factors* of (0). If all the factors of (0) are ϕ simple, (0) is called a ϕ *composition* chain.

Definition. Let G be an M-ϕ group. A normal ϕ chain (ϕ composition chain) connecting 0 and G is called a normal ϕ chain *for G* (ϕ composition *series*).

Definition. Let G be an M-ϕ group. The ϕ subgroup S of G is M-ϕ *characteristic* if every M-ϕ automorphism (M-ϕ isomorphism of G onto itself) leaves S invariant, i.e., if $S\sigma = S$ for every M-ϕ automorphism σ of G. S is M-ϕ *fully invariant* if every M-ϕ endomorphism of G (M-ϕ homomorphism of G into itself) leaves S invariant, i.e., if $S\tau \subseteq S$ for every M-ϕ endomorphism τ of G.

It is important to notice that the inner automorphisms of a group are not necessarily M-ϕ automorphisms and hence a ϕ subgroup may be M-ϕ characteristic without being normal. In some of our arguments we consider the map of a ϕ subgroup under an inner automorphism. Thus in some cases we make the assumption that ϕ contains conjugates, i.e., if S is in ϕ and g is any element of G, then $-g + S + g$ is in ϕ. If ϕ contains conjugates, we say that ϕ is normal.

2. K-chains. Let (K) be a property which has meaning for each ϕ subgroup of an M-ϕ group, i.e., if S is a ϕ subgroup of the M-ϕ group G, then one of the following statements must be true; S satisfies (K) in G; S does not satisfy (K)

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in G. We shall consider properties (K) which satisfy some or all of the following conditions:

(k_1) *If G is an M-ϕ group, then the ϕ subgroup* 0 *satisfies* (K) *in G.*

(k_2) *If for each α in a set $\mathfrak{A}$, A_α is a normal ϕ subgroup of the M-ϕ group G which satisfies* (K) *in G, then $\bigcup A_\alpha (\alpha \in \mathfrak{A})$ satisfies* (K) *in G.*

(k_3) *If A and A_α, for each α in the set $\mathfrak{A}$, are normal ϕ subgroups of the M-ϕ group G with $A \supset A_\alpha$, and if A/A_α satisfies* (K) *in G/A_α, for each α in $\mathfrak{A}$, then $A/\bigcap A_\alpha(\alpha \in \mathfrak{A})$ satisfies* (K) *in $G/\bigcap A_\alpha(\alpha \in \mathfrak{A})$.*

(k_4) *If A, B, C are normal ϕ subgroups of the M-ϕ group G with $A \supset B$, and if A/B satisfies* (K) *in G/B, then $A \cap C/B \cap C$ satisfies* (K) *in $G/B \cap C$.*

(k_5) *If A, B, C are normal ϕ subgroups of the M-ϕ group G with $A \supset B$, and if A/B satisfies* (K) *in G/B, then $\{A, C\}/\{B, C\}$ satisfies* (K) *in $G/\{B, C\}$.*

Let G be an M-ϕ group and ψ the lattice of normal ϕ subgroups of G. If N is in ψ, G/N is an M-ϕ group and hence (K) is defined not only for the ϕ subgroups of G (in G) but for the ϕ subgroups of G/N (in G/N).

We now consider two chains. We construct first the ascending chain:

$$(1) \qquad 0 = T_0 \subseteq T_1 \subseteq \ldots \subseteq T_i \subseteq T_{i+1} \subseteq \ldots,$$

where, for $i = 0, 1, 2, \ldots$, T_{i+1} is the compositum of all N in ψ such that $N \supseteq T_i$ and N/T_i satisfies (K) in G/T_i. Then T_{i+1}/T_i satisfies (K) in G/T_i by (k_2). T_{i+1} is well defined, since by (k_1), T_i/T_i satisfies (K) in G/T_i. We note that in order to construct the chain (1), we need only use the properties (k_1) and (k_2) of (K). Similarly, we construct the descending chain:

$$(2) \qquad G = S_0 \supseteq S_1 \supseteq \ldots \supseteq S_j \supseteq S_{j+1} \supseteq \ldots,$$

where, for $j = 0, 1, 2, \ldots$, S_{j+1} is the intersection of all N in ψ such that $N \subseteq S_j$ and S_j/N satisfies (K) in G/N. Then S_j/S_{j+1} satisfies (K) in G/S_{j+1} by (k_3). S_{j+1} is well defined, since by (k_1), S_j/S_j satisfies (K) in G/S_j. For the construction of the chain (2) only (k_1) and (k_3) are used.

THEOREM 2.1. *Let G be an M-ϕ group.*

(i) *Assume that the property* (K) *satisfies* (k_1) *and* (k_2). *If the ψ subgroups of G satisfy the ascending chain condition, and if for A in ψ, $A \neq G$, there exists a subgroup B in ψ such that $B \supset A$ and B/A satisfies* (K) *in G/A, the chain* (1) *is finite and $T_t = G$ for some integer t.*

(ii) *Assume that the property* (K) *satisfies* (k_1) *and* (k_3). *If the ψ subgroups of G satisfy the descending chain condition, and if for B in ψ, $B \neq 0$, there exists a subgroup A in ψ such that $A \subset B$ and B/A satisfies* (K) *in G/A, the chain* (2) *is finite and $S_s = 0$ for some integer s.*

Proof. (i) The groups T_i of (1) are ψ subgroups of G by definition. Hence by the ascending chain condition, there exists an integer t such that $T_t = T_{t+1}$. If T_t is different from G, there exists a ψ subgroup N of G such that N/T_t satisfies (K) in G/T_t and $N \supset T_t$; but this is impossible, since then T_{t+1} would be different from T_t. Hence $T_t = G$. (ii) is established in a similar fashion.

Definition. Let G be an M-ϕ group. A chain

$$N_0 \subseteq N_1 \subseteq \ldots \subseteq N_i \subseteq N_{i+1} \subseteq \ldots, \tag{3}$$

where, for $i = 0, 1, \ldots, N_i$ is in ψ, and N_{i+1}/N_i satisfies (K) in G/N_i, is called a K-*chain for* G (an *ascending* K-*chain*). A chain

$$M_0 \supseteq M_1 \supseteq \ldots \supseteq M_j \supseteq M_{j+1} \supseteq \ldots, \tag{4}$$

where, for $j = 0, 1, \ldots, M_j$ is in ψ, and M_j/M_{j+1} satisfies (K) in G/M_{j+1}, will also be called a K-*chain for* G (a *descending* K-*chain*). The K-chain

$$N_0 \subset N_1 \subset \ldots \subset N_i \subset \ldots \subset N_n \tag{5}$$

has length n, if for $i = 0, \ldots, n-1$, $N_i \neq N_{i+1}$. The chains (1) and (2) are called the *upper* and *lower* K-chains for G.

THEOREM 2.2. *Let G be an M-ϕ group.*

(i) *Assume that* (K) *satisfies* (k_1), (k_2), *and* (k_5). *If*

$$0 = N_0 \subseteq \ldots \subseteq N_i \subseteq N_{i+1} \subseteq \ldots$$

is an ascending K-*chain for G, then, for $i = 0, 1, \ldots, N_i \subseteq T_i$, where the T_i are the terms of the upper* K-*chain* (1).

(ii) *Assume that* (K) *satisfies* (k_1), (k_3), *and* (k_4). *If*

$$G = M_0 \supseteq \ldots \supseteq M_j \supseteq M_{j+1} \supseteq \ldots$$

is a descending K-*chain for G, then, for $j = 0, 1, \ldots, M_j \supseteq S_j$, where the S_j are the terms of the lower* K-*chain* (2).

Proof. (i). We prove by induction on i that $N_i \subseteq T_i$ for $i = 0, 1, \ldots$. Since $0 = N_0 = T_0$, it is obvious that $N_0 \subseteq T_0$. Assume that $N_i \subseteq T_i$. N_{i+1}/N_i satisfies (K) in G/N_i; therefore, by (k_5),

$$\{N_{i+1}, T_i\}/\{N_i, T_i\} = \{N_{i+1}, T_i\}/T_i$$

satisfies (K) in G/T_i. Hence by the definition of T_{i+1}, $\{N_{i+1}, T_i\} \subseteq T_{i+1}$, or $N_{i+1} \subseteq T_{i+1}$. Thus $N_i \subseteq T_i$ for $i = 0, 1, \ldots$.

(ii). We prove by induction on j that $M_j \supseteq S_j$ for $j = 0, 1, \ldots$. $G = M_0 = S_0$; hence $M_0 \supseteq S_0$. Assume that $M_j \supseteq S_j$. M_j/M_{j+1} satisfies (K) in G/M_{j+1}; therefore, by (k_4), $M_j \cap S_j/M_{j+1} \cap S_j$ satisfies (K) in $G/M_{j+1} \cap S_j$. But $M_j \cap S_j = S_j$ by the induction assumption. Hence by the definition of S_{j+1},

$$S_{j+1} \subseteq M_{j+1} \cap S_j \subseteq M_{j+1}.$$

Thus $S_j \subseteq M_j$ for $j = 0, 1, \ldots$.

COROLLARY 2.1. *Let G be an M-ϕ group and assume that* (k_1)-(k_5) *hold for the property* (K). *Then if there exists a (finite)* K-*chain which connects* 0 *and G, the upper and lower* K-*chains are* K-*chains of shortest length connecting* 0 *and G. If*

$$0 = U_0 \subset \ldots \subset U_i \subset \ldots \subset U_n = G \qquad (\textit{of length } n)$$

is any K-*chain of shortest length, $S_{n-i} \subseteq U_i \subseteq T_i$, for $i = 1, \ldots, n$.*

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THEOREM 2.3. *Assume that* (k_1)-(k_5) *hold for the property* (K). *Let G be an M-ϕ group and assume that G has upper and lower* K-*chains of length n connecting* 0 *and G. Then the chains*

(6) $0 = T_0 \cap S_{n-1} \subset T_1 \cap S_{n-2} \subset \ldots \subset T_i \cap S_{n-i-1} \subset \ldots \subset T_{n-1} \cap S_0$
$= T_{n-1} \subset G$

and

(7) $0 \subset \{T_0, S_{n-1}\} = S_{n-1} \subset \ldots \subset \{T_i, S_{n-i-1}\} \subset \ldots \subset \{T_{n-1}, S_0\} = G$

are K-*chains for G.*

Proof. To show that (6) is a K-chain, we have to verify that $T_{i+1} \cap S_{n-i-2} / T_i \cap S_{n-i-1}$ satisfies (K) in $G/T_i \cap S_{n-i-1}$ for $i = 0, \ldots, n-2$. By the definition of T_i, T_{i+1}/T_i satisfies (K) in G/T_i for $i = 0, \ldots, n-1$. Therefore, by (k_4),

$$T_{i+1} \cap S_{n-i-2}/T_i \cap S_{n-i-2}$$

satisfies (K) in $G/T_i \cap S_{n-i-2}$. By the definition of S_j, S_{n-i-2}/S_{n-i-1} satisfies (K) in G/S_{n-i-1} for $i = 0, \ldots, n-2$. Therefore, by (k_4),

$$T_{i+1} \cap S_{n-1-2}/T_{i+1} \cap S_{n-i-1}$$

satisfies (K) in $G/T_{i+1} \cap S_{n-i-1}$. Hence, by (k_3),

$$T_{i+1} \cap S_{n-i-2}/T_i \cap S_{n-i-2} \cap T_{i+1} \cap S_{n-i-1} = T_{i+1} \cap S_{n-i-2}/T_i \cap S_{n-i-1}$$

satisfies (K) in $G/T_i \cap S_{n-i-1}$, and (6) is a K-chain for G.

To show that (7) is a K-chain, we have to verify that $\{T_{i+1}, S_{n-i-2}\}/T_i, S_{n-i-1}\}$ satisfies (K) in $G/\{T_i, S_{n-i-1}\}$ for $i = 0, \ldots, n-2$. Since T_{i+1}/T_i satisfies (K) in G/T_i, we deduce from (k_5) that

$$\{T_{i+1}, S_{n-i-1}\}/\{T_i, S_{n-i-1}\}$$

satisfies (K) in $G/\{T_i, S_{n-i-1}\}$. Also since S_{n-i-2}/S_{n-i-1} satisfies (K) in G/S_{n-i-1}, $\{T_i, S_{n-i-2}\}/\{T_i, S_{n-i-1}\}$ satisfies (K) in $G/\{T_i, S_{n-i-1}\}$. Hence by (k_2),

$$\{\{T_{i+1}, S_{n-i-1}\}, \{T_i, S_{n-i-2}\}\}/\{T_i, S_{n-i-1}\} = \{T_{i+1}, S_{n-i-2}\}\{/T_i, S_{n-i-1}\}$$

satisfies (K) in $G/\{T_i, S_{n-i-1}\}$, and (7) is a K-chain for G.

The K-chains (1), (2), (6) and (7) are shown in the accompanying Hasse diagram.

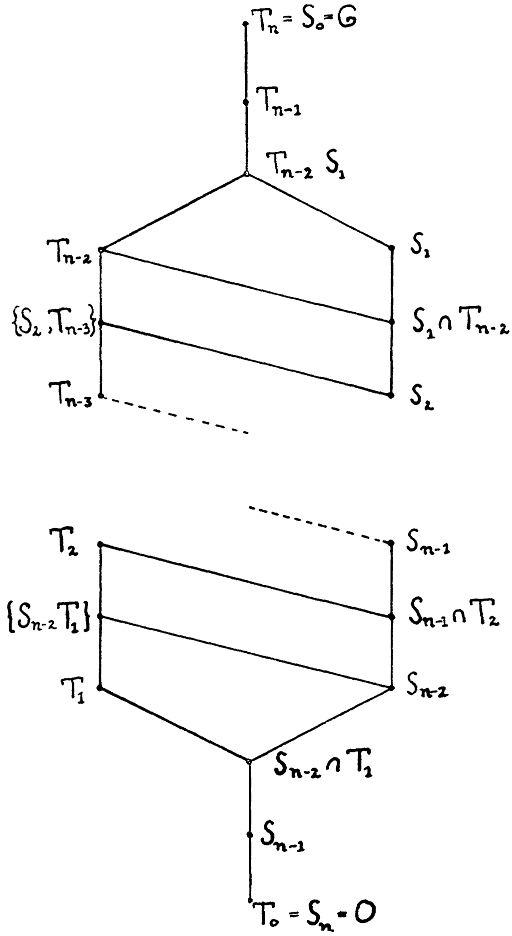
THEOREM 2.4. *Let* (K) *be a property for which* (k_4) *holds. If the M-ϕ group G has a* K-*chain connecting* 0 *and G, let*

(8) $$0 = U_0 \subset \ldots \subset U_i \subset U_{i+1} \subset \ldots \subset U_n = G,$$

and

(9) $$0 = V_0 \subset \ldots \subset V_i \subset V_{i+1} \subset \ldots \subset V_n = G$$

be K-*chains of shortest length. Then* U_{i+1} not $\subseteq V_i$, *for* $i = 0, \ldots, n-1$.



Proof. Suppose that $U_{i+1} \subseteq V_i$; then

$$\begin{aligned} 0 = V_0 \cap U_{i+1} &\subseteq \ldots \subseteq V_j \cap U_{i+1} \subseteq \ldots \subseteq V_i \cap U_{i+1} \\ &= U_{i+1} \subset U_{i+2} \subset \ldots \subset U_n = G \end{aligned}$$

is a K-chain, and its length is less than n. But this is impossible and hence U_{i+1} not $\subseteq V_i$.

Consequently if G has a K-chain connecting 0 and G we have for the upper and lower K-chains, provided that (K) satisfies (k_1)-(k_5):

(i) S_{n-i-1} not $\subseteq T_i$ $\quad (i = 0, \ldots, n-1)$,

(ii) T_{i+1} not $\subseteq S_{n-i}$ $\quad (i = 0, \ldots, n-1)$.

The properties (K) which we shall discuss are also invariant under M-ϕ isomorphisms of the group. That is:

(k_6) *Let G be an M-ϕ group and σ an M-ϕ isomorphism of G. Then if the normal ϕ subgroup A satisfies* (K) *in G, $A\sigma$ satisfies* (K) *in $G\sigma$.*

The conditions (k_5) and (k_6) are equivalent to the following condition:

(k'_5) *Let G be an M-ϕ group and η an M-ϕ homomorphism of G. If A and B are normal ϕ subgroups with $A \supset B$, and if A/B satisfies* (K) *in G/B, then $A\eta/B\eta$ satisfies* (K) *in $G\eta/B\eta$.*

Proof. Clearly (k'_5) implies (k_6). We show next that (k'_5) also implies (k_5). We assume that A and B are normal ϕ subgroups of the M-ϕ group G with $A \supset B$ and that A/B satisfies (K) in G/B. If C is a normal ϕ subgroup of G, let η be the natural mapping of G onto G/C. Then $A\eta = \{A, C\}/C$ and $B\eta = \{B,C\}/C$. Thus by (k'_5),

$$\frac{\{A, C\}/C}{\{B, C\}/C} \text{ satisfies (K) in } \frac{G/C}{\{B, C\}/C}.$$

But (K) is invariant under M-ϕ isomorphism and hence $\{A, C\}/\{B, C\}$ satisfies (K) in $G/\{B,C\}$.

Conversely, we show that (k_5) and (k_6) imply (k'_5). Assume that A and B are normal ϕ subgroups of the M-ϕ group G with $A \supset B$ and that A/B satisfies (K) in G/B. Let η be an M-ϕ homomorphism of G, and let C be the kernel of η. Then C is a normal ϕ subgroup of G and the natural homomorphism of G onto G/C takes A onto $\{A, C\}/C$ and B onto $\{B, C\}/C$. Hence by the Homomorphism Theorem there exists an M-ϕ isomorphism of $G\eta$ onto G/C which takes $A\eta$ onto $\{A, C\}/C$ and B onto $\{B, C\}/C$.

By the Second Isomorphism Theorem there exists an M-ϕ isomorphism of

$$\frac{G/C}{\{B, C\}/C} \text{ onto } G/\{B, C\}$$

which takes

$$\frac{\{A, C\}/C}{\{B, C\}/C} \text{ onto } \{A, C\}/\{B, C\}.$$

Therefore, there exists an M-ϕ isomorphism σ of $G/\{B, C\}$ onto $G\eta/B\eta$ with

$$A\eta/B\eta = (\{A, C\}/\{B, C\})\sigma.$$

By (k_5), $\{A, C\}/\{B, C\}$ satisfies (K) in $G/\{B, C\}$, and from (k_6) it follows that $A\eta/B\eta$ satisfies (K) in $G\eta/B\eta$.

THEOREM 2.5. *Let G be an M-ϕ group and* (K) *a property for which* (k_1)-(k_6) *hold. The terms of the upper and lower* K-*chains are M-ϕ characteristic.*

Proof. We prove by induction that the terms of the upper K-chain are M-ϕ characteristic. $T_0 = 0$ and hence is M-ϕ characteristic. Assume that T_i is M-ϕ characteristic; and let η be an M-ϕ automorphism of G. Then η induces an M-ϕ automorphism $\bar{\eta}$ of G/T_i, since $T_i\eta = T_i$. We deduce from (k_6) that

$$T_{i+1}\eta/T_i = (T_{i+1}/T_i)\bar{\eta}$$

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satisfies (K) in $G/T_i = (G/T_i)\bar{\eta}$. Hence by the definition of T_{i+1}, $T_{i+1}\eta \subseteq T_{i+1}$. Similarly, $T_{i+1}\eta^{-1} \subseteq T_{i+1}$ so that $T_{i+1}\eta = T_{i+1}$.

We prove by induction that the terms of the lower K-chain are M-ϕ characteristic. $S_0 = G$ and hence is M-ϕ characteristic. Assume that S_j is M-ϕ characteristic and let η be an M-ϕ automorphism of G. By (k'_5), $S_j\eta/S_{j+1}\eta$ satisfies (K) in $G\eta/S_{j+1}\eta$ which implies that $S_{j+1} \subseteq S_{j+1}\eta$. Since a similar argument shows that $S_{j+1} \subseteq S_{j+1}\eta^{-1}$, $S_{j+1}\eta = S_{j+1}$.

In this section we have often made the hypothesis that the property (K) satisfies certain ones of the conditions (k_1)-(k_6). It may happen, of course, that (K) satisfies these (k_i) for some M-ϕ groups but not for others. In the following sections we shall often restrict the class of M-ϕ groups considered, and discuss particular properties (K) for this class. It is clear that the results of this section may be applied to this class of groups, provided that (K) satisfies suitable conditions (k_i) for groups in this class, and provided that the ψ subgroups and quotient groups of a group in the class also belong to the class.

3. Loewy chains.

The first property (K) which we shall consider gives rise to the so-called Loewy chains [**2**, pp. 506-509]. Following Remak, we make the following definitions:

Definition. Let G be an M-ϕ group. If F is a minimal normal ϕ subgroup ($\neq 0$) of G, we call F a *foot* of G.

Definition. The compositum of all feet of the M-ϕ group G is called the *socle* and is denoted by $S = S(G)$. (If G has no feet, the socle is defined to be 0.)

Before defining Loewy chains we state the following results [**7**]:

LEMMA 3.1. *If T is a normal ϕ subgroup of the M-ϕ group G and if $T = \complement F(\alpha \in \mathfrak{A})$ where F_α is a foot of G for each α of the set $\mathfrak{A}$, then there exists a subset $\mathfrak{B}$ of $\mathfrak{A}$ such that $T = \sum^\circ F_\beta(\beta \in \mathfrak{B})$. (The notation $\sum^\circ$ is used for direct sum.)*

Remak proves this in the case where $\mathfrak{A}$ is finite. The same method of proof is valid in the infinite case, using transfinite induction.

COROLLARY 3.1. *Let G be an M-ϕ group with socle $S \neq 0$. S is the direct sum of feet of G.*

LEMMA 3.2. *If N is a normal ϕ subgroup of the M-ϕ group G contained in the socle S of G, N is the direct sum of feet of G. Furthermore, there exists a normal ϕ subgroup N' of G such that $S = N \oplus N'$.*

Proof. Let K be the compositum of all feet F of G with $F \subseteq N$. By Lemma 3.1, there exist sets $\mathfrak{A}$ and $\mathfrak{B}$ such that

$$K = \sum{}^\circ F_\alpha\ (\alpha \in \mathfrak{U}),\quad S = K \oplus \sum{}^\circ F_\beta\ (\beta \in \mathfrak{B})$$

where F_α and F_β are feet of G for α in $\mathfrak{A}$ and β in $\mathfrak{B}$ respectively. $N \supseteq K$ and hence $N = K \oplus (N \cap \sum^\circ F_\beta(\beta \in \mathfrak{B}))$.

Assume that $N \cap \sum^\circ F_\beta \neq 0$ $(\beta \in \mathfrak{B})$. Let x be a non-zero element of

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$N \cap \sum F_\beta (\beta \in \mathfrak{B})$; then $x = f_1 + \ldots + f_n$, where, for $i = 1, \ldots, n$, f_i is in F_{β_i} and β_i is in $\mathfrak{B}$. Hence

$$L = N \cap \sum_{i=1}^{n}{}^{\circ} F_{\beta_i} \neq 0.$$

$\sum_{i=1}^{n}{}^{\circ} F_{\beta_i}$ is a ψ subgroup of G and its ψ subgroups satisfy the minimum condition. Hence there exists a minimal ψ subgroup $F \neq 0$ contained in L. Thus F is a foot of G and is contained in N. But this is impossible because then $F \subseteq K$ and $K \cap L = 0$. Therefore

$$N \cap \sum{}^{\circ} F_\beta = 0 \ (\beta \in \mathfrak{B}), N = K = \sum{}^{\circ} F_\alpha \ (\alpha \in \mathfrak{A}),$$

so that N is the direct sum of feet of G.

Let $N' = \sum^{\circ} F_\beta (\beta \in \mathfrak{B})$; then N' is a normal ϕ subgroup of G and $S = N \oplus N'$.

COROLLARY 3.2. *If N is normal ϕ subgroup of the M-ϕ group G contained in the socle S of G, then S/N is the direct sum of feet of G/N.*

Proof. By Lemma 3.2, $S = N \oplus N'$ and $N' = \sum^{\circ} F_\beta (\beta \in \mathfrak{B})$, where F_β is a foot of G, for β in $\mathfrak{B}$. S/N is therefore M-ψ isomorphic to N' and hence is the direct sum of feet of G/N.

Consider now the following property of ϕ subgroups of an M-ϕ group:

(R). Let A be a ϕ subgroup of the M-ϕ group G. A satisfies (R) in G if A is contained in the socle of G.

Definition. A normal ϕ subgroup N of the M-ϕ group G is *fully reducible* with respect to G if it is the compositum of feet of G. (We assume that 0, which is the sum of no feet, is fully reducible.)

From Lemma 3.2 we see that a normal ϕ subgroup satisfies (R) in G if and only if it is fully reducible with respect to G.

We call an R-chain a Loewy chain. The property (R) obviously satisfies (k_1) and (k_2) so that the upper Loewy chain may be constructed. We denote the upper Loewy chain by:

$$(10) \qquad 0 = S_0 \subseteq \ldots \subseteq S_i \subseteq S_{i+1} \subseteq \ldots.$$

We verify that (R) satisfies (k_5):

THEOREM 3.1. *Let A, B, and C be normal ϕ subgroups of the M-ϕ group G with $A \supset B$. If A/B is fully reducible with respect to G/B, then $\{A, C\}/\{B, C\}$ is fully reducible with respect to $G/\{B,C\}$.*

Proof. Since A/B is fully reducible with respect to G/B, $A/B = \mathsf{C}\ (A_\alpha/B)$ $(\alpha \in \mathfrak{A})$, where A_α/B is a foot of G/B, for each α in the set $\mathfrak{A}$.

$$\frac{\{A, C\}}{\{B, C\}} = \frac{\{\mathsf{C} A_\alpha, C\}}{\{B, C\}} = \frac{\mathsf{C}\{A_\alpha, C\}}{\{B, C\}} \qquad (\alpha \in \mathfrak{A}).$$

Now

$$\frac{\{A_\alpha, C\}}{\{B, C\}} = \frac{\{A_\alpha, \{B, C\}\}}{\{B, C\}} \cong \frac{A_\alpha}{A_\alpha \cap \{B, C\}} \qquad (M\text{-}\psi);$$

A_α is a minimal ψ subgroup of G which contains B. Hence since $A_\alpha \cap \{B, C\}$ is a ψ subgroup of G and $B \subseteq A_\alpha \cap \{B, C\} \subseteq A_\alpha$, either

$$B = A_\alpha \cap \{B, C\} \quad \text{or} \quad A_\alpha = A_\alpha \cap \{B, C\} \quad \text{so that } A_\alpha \subseteq \{B, C\}.$$

In the first case,

$$\frac{\{A_\alpha, C\}}{\{B, C\}} \cong \frac{A_\alpha}{B} \qquad (M\text{-}\psi)$$

and hence $\{A_\alpha, C\}/\{B, C\}$ is a foot of $G/\{B, C\}$. In the second case, $\{A_\alpha, C\} = \{B, C\}$. Therefore, $\{A, C\}/\{B, C\}$ is fully reducible with respect to $G/\{B, C\}$.

Thus the condition (R) satisfies (k_1), (k_2), and (k_5) and hence as a consequence of Theorem 2.2 we have:

THEOREM 3.2. *Let G be an M-ϕ group which possesses a Loewy chain*

$$0 = N_0 \subseteq \ldots \subseteq N_i \subseteq N_{i+1} \subseteq \ldots$$

(that is, N_i is normal in G, and N_{i+1}/N_i is fully reducible with respect to G/N_i, for $i = 0, 1, \ldots$), then $N_i \subseteq S_i$, where the S_i are the terms of the upper Loewy chain. Hence if $N_n = G$ for some integer n, $S_n = G$ so that the upper Loewy chain connects 0 and G and has length $\leqslant n$.

If J is a maximal ψ subgroup of the M-ϕ group G, then G/J is ϕ simple and hence is fully reducible with respect to G/J. Hence if (k_3) were satisfied by (R) we should have G/N fully reducible (with respect to G/N) for N the intersection of maximal ψ subgroups of G. That this is not in general the case is shown by a simple example:

EXAMPLE 3.21. Let G be the additive group of integers, M void and let ϕ consist of all subgroups of G. Then if p is any prime, (p), the group generated by p, is a maximal normal subgroup of G. Furthermore,

$$\bigcap_{i=1}^{\infty} (p_i) = 0$$

if $p_1, p_2, \ldots$ is an infinite sequence of different primes. But G contains no minimal subgroups, and hence is certainly not fully reducible; in fact, the upper Loewy chain has as its only term 0.

We shall need the following theorem to show that (k_3) holds for the property (R) in an M-ϕ group G, provided that the ψ subgroups of G satisfy the minimum condition:

THEOREM 3.3. *Let A and B be normal ϕ subgroups of the M-ϕ group G with $A \supset B$. If A/B is fully reducible with respect to G/B, then B is the intersection of maximal ψ subgroups of A.*

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Proof. (i) Assume that $B = 0$. By Lemma 3.1, there exists a set $\mathfrak{B}$ such that $A = \Sigma^{\circ} A_{\beta}(\beta \in \mathfrak{B})$ where, for β in $\mathfrak{B}$, A_{β} is a foot of G. For δ in $\mathfrak{B}$ define

$$J_{\delta} = \sum_{\beta \neq \delta} A_{\beta} \ (\beta \in \mathfrak{B}).$$

Then $A = J_{\delta} \oplus A_{\delta}$, and J_{δ} is a maximal ψ subgroup of A.

Let $K = \bigcap J_{\delta}$; K is a ψ subgroup of A and hence if $K \neq 0, K = \Sigma A_{\gamma}(\gamma \in \mathfrak{C})$, where $\mathfrak{C}$ is non-void, and $\mathfrak{C} \subset \mathfrak{B}$. If γ is in $\mathfrak{C}$, $A_{\gamma} \subseteq \bigcap J_{\delta} = K$ $(\delta \in \mathfrak{B})$ so that in particular $A_{\gamma} \subseteq J_{\gamma}$. But this is impossible. Therefore, $K = \bigcap J_{\delta} = 0$, and $B = 0$ is the intersection of maximal ψ subgroups of A.

(ii) In the general case if we apply the result of (i) to the quotient group, G/B, we have: There exists a set $\mathfrak{B}$ such that $B/B = \bigcap (J_{\gamma}/B)$ $(\gamma \in \mathfrak{B})$, where J_{γ}/B is a maximal ψ subgroup of A/B, for γ in $\mathfrak{B}$. But then

$$B = \bigcap_{\gamma} J_{\gamma},$$

and J_{γ} is a maximal ψ subgroup of A, for γ in $\mathfrak{B}$.

THEOREM 3.4. *Assume that the ψ subgroups of the M-ϕ group G satisfy the minimum condition. Let A, A_{α} be ψ subgroups of G for each α in the set $\mathfrak{A}$, with $A_{\alpha} \subset A$. Then if A/A_{α} is fully reducible with respect to G/A_{α}, for α in $\mathfrak{A}$, $A/\bigcap A_{\alpha}$ is fully reducible with respect to $G/\bigcap A_{\alpha}$.*

Proof. By Theorem 3.3, A_{α} is the intersection of maximal ψ subgroups of A. Hence $\bigcap A_{\alpha}$ is the intersection of maximal ψ subgroups of A, and is therefore the intersection of a finite number of maximal ψ subgroups of A, since the ψ subgroups of G satisfy the minimum condition.

Let

$$C = \bigcap A_{\alpha} = \bigcap_{i=1}^{n} M_i,$$

where M_i $(i = 1, \ldots, n)$ is a maximal ψ subgroup of A, and assume that $n > 1$ and that

$$K_j = \bigcap_{i(\neq j)=1}^{n} M_i \neq C \qquad (j = 1, \ldots, n).$$

Then since M_1 is a maximal ψ subgroup of A and K_1 is not contained in M_1, $A = \{K_1, M_1\}$. M_1 has the maximal ψ subgroups

$$M_1 \cap M_2, \ \ldots, \ M_1 \cap M_{ni}; \text{ and } K_2 = \bigcap_{i=3}^{n} (M_1 \cap M_i)$$

so that the same argument applied to M_1 shows that $M_1 = \{K_2, M_1 \cap M_2\}$. Continuing in this manner we obtain

$$A = \bigcup_{j=1}^{n} K_j.$$

Hence

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$$A/C = \bigcup_{j=1}^{n} (K_j/C),$$
$$\frac{K_j}{C} = \frac{K_j}{K_j \cap M_j} \cong \frac{\{M_j, K_j\}}{M_j} \qquad (M\text{-}\psi)$$
$$= A/M_j,$$

which is ψ simple. Thus K_j/C is a foot of G/C, so that $A/\bigcap A_a = A/C$ is fully reducible with respect to G/C.

Hence (R) satisfies (k_3) for M-ϕ groups whose ψ subgroups satisfy the minimum condition, and the lower Loewy chain may be constructed for these groups. We denote the lower Loewy chain by:

$$G = R_0 \supseteq \ldots \supseteq R_j \supseteq R_{j+1} \supseteq \ldots . \tag{11}$$

THEOREM 3.5. *Assume that the ψ subgroups of the M-ϕ group G satisfy the minimum condition. Let A and B be ψ subgroups of G with $B \subset A$. If A/B is fully reducible with respect to G/B, then $A \cap C/B \cap C$ is fully reducible with respect to $G/B \cap C$.*

Proof. By Theorem 3.3,

$$B = \bigcap_{i=1}^{n} M_i,$$

where M_i ($i = 1, \ldots, n$) is a maximal ψ subgroup of A. Hence

$$B \cap C = \bigcap_{i=1}^{n} (M_i \cap C).$$

If $A \cap C \neq M_i \cap C$,

$$\frac{A \cap C}{M_i \cap C} = \frac{A \cap C}{M_i \cap (A \cap C)} \cong \frac{\{M_i, A \cap C\}}{M_j} \qquad (M\text{-}\psi),$$

which is ψ simple. Hence $M_i \cap C$ is a maximal ψ subgroup of $A \cap C$; and Theorem 3.4 shows that $A \cap C/B \cap C$ is fully reducible with respect to $G/B \cap C$.

COROLLARY 3.3. *If the ψ subgroups of the M-ϕ group G satisfy the minimum condition, then for G the property* (R) *satisfies the conditions* (k_1)-(k_5).

Hence we have:

THEOREM 3.6. *Let G be an M-ϕ group whose ψ subgroups satisfy the minimum condition, and assume that G possesses a Loewy chain:*

$$G = K_0 \supseteq \ldots \supseteq K_j \supseteq K_{j+1} \supseteq \ldots$$

(that is, K_j is in ψ, and K_j/K_{j+1} is fully reducible with respect to G/K_{j+1}, for $j = 0, 1, \ldots$) Then $K_j \supseteq R_j$ for $j = 0, 1, \ldots$, where the R_j are the terms of the lower Loewy chain (11). *Hence if $K_n = 0$ for some integer n, $R_n = 0$ so that the lower Loewy chain connects G and 0 and has length $\leqslant n$.*

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COROLLARY 3.4. *Under the hypotheses of the preceding theorem (that is, the ψ subgroups of G satisfy the minimum condition and $K_n = 0$ for some integer n), the upper and lower Loewy chains connect* 0 *and G and have equal lengths.*

THEOREM 3.7. *Let G be an M-ϕ group and assume that the upper Loewy chain connects* 0 *and G so that*

$$0 = S_0 \subset \ldots \subset S_i \subset \ldots \subset S_n = G.$$

Then if we define the chain

$$G = M_0 \supseteq \ldots \supseteq M_j \supseteq M_{j+1} \supseteq \ldots,$$

where M_{j+1} is the intersection of M_j with all maximal ψ subgroups of M_j, there exists an integer $m \leqslant n$ such that $M_m = 0$.

Proof. We use induction to prove that $M_i \subseteq S_{n-i}$. Since S_{n-1} is the intersection of maximal ψ subgroups of G, $M_1 \subsetneqq S_{n-1}$. Assume that $M_j \subset S_{n-j}$; by Theorem 3.3, S_{n-j-1} is the intersection of maximal ψ subgroups of S_{n-j} so that there exists a set $\mathfrak{A}$ such that N_α is a maximal ψ subgroup of S_{n-j} for α in $\mathfrak{A}$, and

$$S_{n-j-1} = \bigcap N_\alpha \qquad (\alpha \in \mathfrak{A}).$$

Either $\{M_j, N_\alpha\} = N_\alpha$ or S_{n-j}. In the first case, $M_j \cap N_\alpha = M_j$; in the second, $M_j \cap N_\alpha$ is a maximal ψ subgroup of M_j, since

$$\frac{M_j}{M_j \cap N_\alpha} \cong \frac{\{M_j, N_\alpha\}}{M_j} = \frac{S_{n-j}}{M_j} \qquad (M\text{-}\psi)$$

which is ψ simple. Thus

$$\bigcap_\alpha (M_j \cap N_\alpha)$$

is the intersection with M_j of maximal ψ subgroups of M_j so that

$$M_{j+1} \subseteq \bigcap_\alpha (M_j \cap N_\alpha) \subseteq \bigcap_\alpha N_\alpha = S_{n-j-1}.$$

Hence $M_i \subseteq S_{n-i}$ $(i = 0, 1, \ldots, n)$. In particular, $M_n \subseteq S_0 = 0$ and $M_n = 0$.

As we have seen the converse of Theorem 3.7 is not true, for Example 3.21 shows that even if $M_1 = 0$, there may be no Loewy chain connecting 0 and G. Under the hypothesis of Theorem 3.7, it is not possible to prove that if $M_m = 0$, $S_m = G$, as the following example shows:

EXAMPLE 3.71. Let W be the direct sum of the cyclic groups generated by b, $b_1, \ldots, b_i, \ldots$, elements of prime order p; thus

$$W = (b) \oplus (b_1) \oplus \ldots \oplus (b_i) \oplus \ldots.$$

Let M consist of the endomorphisms ρ_i, where $\rho_i(b) = b_i$, $\rho_i(b_i) = b_i$ and $\rho_i(b_j) = 0$ $(j \neq i)$. Let ϕ consist of all M admissible subgroups of W.

Then $V = (b_1) \oplus \ldots \oplus (b_i) \oplus \ldots$ is the socle of W and $0 \subset V \subset W$ is the upper Loewy chain for W. If

$$V_i = (b_1) \oplus \ldots \oplus (b_{i-1}) \oplus (b - b_i) \oplus (b_{i+1}) \oplus \ldots,$$

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V_i is a maximal ϕ subgroup of W, and

$$\bigcap_{i=1}^{n} V_i = (b - b_1 - \ldots - b_n) \oplus (b_{n+1}) + \ldots,$$

so that

$$\bigcap_{i=1}^{\infty} V_i = 0.$$

Hence although the length of the shortest Loewy chain for W is 2, the intersection of all maximal (normal) ϕ subgroups is 0.

4. Central chains. The centre of an M-ϕ group G is not necessarily a ϕ subgroup of G. However, if for α in the set $\mathfrak{A}$, S_α is a ϕ subgroup contained in the centre of G, the compositum of the S_α is a ϕ subgroup which is contained in the centre of G.

Definition. Let G be an M-ϕ group. The ϕ *centre* of G is the compositum of all the ϕ subgroups which are contained in the centre of G, and is denoted by $Z_\phi(G)$.

The ϕ centre is the uniquely determined greatest ϕ subgroup of G all of whose elements are centre elements, and is obviously normal in G.

In this section we shall consider Z-chains, or central chains, where the property (Z) is defined by:

(Z) The ϕ subgroup A of the M-ϕ group G satisfies (Z) in G if $A \subseteq Z_\phi(G)$.

Clearly (k_1) and (k_2) hold for (Z).

THEOREM 4.1. *If A and B are normal ϕ subgroups of the M-ϕ group G with $A \supset B$, and if A/B is contained in $Z_\phi(G/B)$, then for any M-ϕ homomorphism η of G, $A\eta/B\eta$ is contained in $Z_\phi(G\eta/B\eta)$. Hence* (k'_5) *holds for* (Z).

Proof. Let a be an element of A, g an element of G; then

$$-a\eta - g\eta + a\eta + g\eta = (-a - g + a + g)\eta,$$

which is in $B\eta$, since $-a - g + a + g$ is in B. Hence $A\eta/B\eta \subseteq Z_\phi(G\eta/B\eta)$.

Definition. Let G be an M-ϕ group. We make the inductive definition:

$$Z_0 = Z_0(G) = 0, \quad Z_{\nu+1}/Z_\nu = Z_{\nu+1}(G)/Z_\nu(G) = Z_\phi(G/Z_\nu)$$

for all ordinals $\nu \geqslant 0$, and

$$Z_\lambda = Z_\lambda(G) = \bigcup_{\nu<\lambda} Z_\nu(G),$$

for limit ordinals λ.

The groups Z_i, for positive integral i, are the terms of the upper central chain and hence are M-ϕ characteristic by Theorem 2.5; it is easily verified (by transfinite induction) that Z_ν is normal and M-ϕ characteristic, for each ordinal ν.

THEOREM 4.2. *Assume that the M-ϕ group G possesses a central chain,*

$$0 = N_0 \subseteq \ldots \subseteq N_i \subseteq N_{i+1} \subseteq \ldots,$$

then $N_i \subseteq Z_i$, *for* $i = 0, 1, \ldots$. *If* $N_n = G$ *for some integer* n, *the upper central chain is finite of length* $c \leqslant n$, *and connects* 0 *and* G.

Proof. It has been shown that (k'_5) implies (k_5) so that (Z) satisfies (k_1), (k_2), and (k_5). Hence the theorem follows from Theorem 2.2 (i).

Definition. Let A and B be normal ϕ subgroups of the M-ϕ group G. Then (A, B) is the intersection of all normal ϕ subgroups of $\{A, B\}$ which contain $-a - b + a + b$, for all a in A and all b in B.

Thus (A, B) is the smallest normal ϕ subgroup of $\{A, B\}$ which contains all the commutators $-a - b + a + b$.

LEMMA 4.1. *Let* A *and* B *be normal* ϕ *subgroups of the* M-ϕ *group* G *with* $B \subset A$. A/B *is contained in* $Z_\phi(G/B)$ *if and only if* (A, G) *is contained in* B.

Proof. Assume that $A/B \subseteq Z_\phi(G/B)$. If a and g are elements of A and G respectively, $-a - g + a + g$ is an element of B. Thus B is a normal ϕ subgroup of $G = \{A, G\}$ which contains $-a - g + a + g$, for all a in A and all g in G; hence $(A, G) \subseteq B$. Conversely, if $B \supseteq (A, G)$, the element $-a - g + a + g$ is in B, for all a in A and all g in G; hence

$$a + g \equiv g + a \pmod{B},$$

or $A/B \subseteq Z_\phi(G/B)$.

THEOREM 4.3. *Let* G *be an* M-ϕ *group.*

(i) *If* A *and* A_α, *for each* α *in the set* $\mathfrak{A}$, *are normal* ϕ *subgroups with* $A \supset A_\alpha$, *and if* $A/A_\alpha \subseteq Z_\phi(G/A_\alpha)$, *for* α *in* $\mathfrak{A}$, *then*

$$A / \bigcap A_\alpha \subseteq Z_\phi(G/ \bigcap A_\alpha);$$

hence (k_3) *holds for* (Z).

(ii) *If* A,B *and* C *are normal* ϕ *subgroups with* $A \supset B$, *and if* $A/B \subseteq Z_\phi(G/B)$, *then*

$$A \cap C/B \cap C \subseteq Z_\phi(G/B \cap C);$$

hence (k_4) *holds for* (Z).

Proof. (i) By Lemma 4.1, $A_\alpha \supseteq (A, G)$, for α in $\mathfrak{A}$; therefore, $\bigcap A_\alpha \supseteq (A, G)$ so that

$$A/ \bigcap A_\alpha \subseteq Z_\phi(G/ \bigcap A_\alpha).$$

(ii) Since, by Lemma 4.1, $(A, G) \subseteq B$, $(A \cap C, G) \subseteq (A, G) \subseteq B$. Since C is normal in G, the element $-c - g + c + g$ is in C, for all c in C and g in G. Thus $(A \cap C, G) \subseteq C$. Therefore, $(A \cap C, G) \subseteq B \cap C$, and by Lemma 4.1,

$$A \cap C/B \cap C \subseteq Z_\phi(G/B \cap C).$$

Definition. Let G be an M-ϕ group. We define by transfinite induction:

$$C^0(G) = G, \quad C^{\nu+1}(G) = (C^\nu(G), G)$$

for all ordinals $\nu \geqslant 0$, and

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$$C^{\lambda}(G) = \bigcap_{\nu<\lambda} C^{\nu}(G)$$

for limit ordinals λ.

The groups $C^i(G)$, for positive integral i, are the terms of the lower central chain. For by Lemma 4.1, $C^{i+1}(G)$ is the smallest normal ϕ subgroups of G in $C^i(G)$ such that $C^i(G)/C^{i+1}(G)$ is contained in $Z_{\phi}(G/C^{i+1}(G))$. It is easily verified (by transfinite induction) that $C^{\nu}(G)$ is M-ϕ fully invariant for each ordinal ν; $C^{\nu+1}(G)$ is normal in G (by definition) and, for λ a limit ordinal, $C^{\lambda}(G)$ is obviously normal in G.

LEMMA 4.2. *If N is a normal ϕ subgroup of the M-ϕ group G, then*

$$C^i(G/N) = \{C^i(G), N\}/N \qquad (i = 0, 1, \ldots).$$

Proof. We use induction on i. The lemma is true for $i = 0$, since

$$C^0(G/N) = G/N = \{G, N\}/N = \{C^0(G), N\}/N.$$

Assume that the lemma is true for $i = j$, that is, assume that

$$C^j(G/N) = \{C^j(G), N\}/N,$$

and let $C^{j+1}(G/N) = K/N$. Then

$$\frac{C^j(G/N)}{C^{j+1}(G/N)} \subseteq Z_{\phi}\left(\frac{G/N}{C^{j+1}(G/N)}\right)$$

or

$$\frac{\{C^j(G), N\}/N}{K/N} \subseteq Z_{\phi}\left(\frac{G/N}{K/N}\right);$$

hence $\{C^j(G), N\}/K \subseteq Z_{\phi}(G/K)$. Thus

$$K \supseteq (\{C^j(G), N\}, G) \supseteq (C^j(G), G) = C^{j+1}(G)$$

so that

(12) $$K \supseteq \{C^{j+1}(G), N\}.$$

On the other hand, since $C^j(G)/C^{j+1}(G)$ is contained in $Z_{\phi}(G/C^{j+1}(G))$, we deduce from property (k_5) that

$$\frac{\{C^j(G), N\}}{\{C^{j+1}(G), N\}} \subseteq Z_{\phi}\left(\frac{G}{\{C^{j+1}(G), N\}}\right);$$

hence

$$\frac{\{C^j(G), N\}/N}{\{C^{j+1}(G), N\}/N} \subseteq Z_{\phi}\left(\frac{G/N}{\{C^{j+1}(G), N\}/N}\right).$$

Thus

(13) $$C^{j+1}(G/N) \subseteq \{C^{j+1}(G), N\}/N,$$

and combining (12) and (13) we obtain $K/N = C^{j+1}(G/N) = \{C^{j+1}(G), N\}/N$. The induction is thus complete, and $C^i(G/N) = \{C^i(G), N\}/N$ $(i = 0, 1, \ldots)$.

THEOREM 4.4. *Let G be an M-ϕ group which possesses a central chain*

$$G = M_0 \supseteq \ldots \supseteq M_j \supseteq M_{j+1} \ldots,$$

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then $M_j \supseteq C^j(G)$, *for* $j = 0, 1, \ldots$. *If* $M_n = 0$ *for some integer* n, *then the lower central chain connects* G *and* 0 *and has length* $\leqslant n$.

Proof. Since (Z) satisfies (k_1), (k_3), and (k_4), this follows from Theorem 2.2 (ii).

COROLLARY 4.1. *If the M-ϕ group G possesses a central chain of length n connecting* 0 *and G, the upper and lower central chains are of equal length $c \leqslant n$ and both connect* 0 *and G.*

Definition. The M-ϕ group G is ϕ *nilpotent of finite class* c, if the upper central chain connects 0 and G and has length c.

THEOREM 4.5. *If the M-ϕ group G is ϕ nilpotent of finite class c, then*

(i) *Any ϕ subgroup S is ϕ nilpotent of finite class $\leqslant c$.*

(ii) *If N is a normal ϕ subgroup of G, G/N is ϕ nilpotent of finite class $\leqslant c$.*

Proof. (i) We prove by induction that $Z_j(G) \cap S \subseteq Z_j(S)$ $(j = 0, 1, \ldots, c)$. Since $Z_\phi(G) \cap S \subseteq Z_\phi(S)$, the assertion is true for $j = 0$. We assume that $Z_i(G) \cap S \subseteq Z_i(S)$ and show that

$$Z_{i+1}(G) \cap S \subseteq Z_{i+1}(S).$$

Let z and s be elements of $Z_{i+1}(G) \cap S$ and S respectively; then $-s-z+s+z$ is in S, and is in $Z_i(G)$, since

$$(G, Z_{i+1}(G)) \subseteq Z_i(G).$$

Hence $-s-z+s+z$ is an element of $Z_i(G) \cap S \subseteq Z_i(S)$ so that z is in $Z_{i+1}(S)$ and $Z_{i+1}(G) \cap S \subseteq Z_{i+1}(S)$.

(ii) Since G is ϕ nilpotent of finite class c, $C^c(G) = 0$. By Lemma 4.2,

$$C^c(G/N) = \{C^c(G), N\}/N = N/N.$$

Hence G/N is ϕ nilpotent of finite class $\leqslant c$.

THEOREM 4.6. *Let G be an M-ϕ group. G is ϕ nilpotent of finite class if and only if a central chain connecting* 0 *and G may be obtained from any normal ψ chain for G by a suitable refinement.*

Proof. Assume that $Z_c(G) = G$ and let

$$0 = N_0 \subseteq \ldots \subseteq N_i \subseteq N_{i+1} \subseteq \ldots \subseteq N_n = G \tag{14}$$

be any ψ chain for G. Consider the chain

$$\begin{aligned} 0 \subseteq \ldots \subseteq N_i = \{N_i, Z_0 \cap N_{i+1}\} &\subseteq \ldots \subseteq \{N_i, Z_j \cap N_{i+1}\} \\ &\subseteq \{N_i, Z_{j+1} \cap N_{i+1}\} \subseteq \ldots \subseteq N_{i+1} \\ &= \{N_i, Z_c \cap N_{i+1}\} = \{N_{i+1}, Z_0 \cap N_{i+2}\} \subseteq \ldots \subseteq G. \end{aligned} \tag{15}$$

Clearly $\{N_i, Z_j \cap N_{i+1}\}$ is normal in G. Furthermore,

$$\{N_i, Z_{j+1} \cap N_{i+1}\}/\{N_i, Z_j \cap N_{i+1}\} \subseteq Z_\phi(G/\{N_i, Z_j \cap N_{i+1}\}),$$

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as can be seen by using properties (k_4) and (k_5). Hence (15) is a central chain. This proves that the condition is necessary. The sufficiency is obvious.

COROLLARY 4.2. *Assume that the M-ϕ group G is ϕ nilpotent of finite class. If the ψ subgroups of G satisfy the double chain condition, then a ψ composition series is necessarily a central chain.*

5. M-ϕ groups with a finite Loewy chain. We now consider an M-ϕ group G which has a finite Loewy chain connecting 0 and G and show that in this case the upper and lower central chains are finite. Furthermore, if G is ϕ nilpotent of finite class, the upper Loewy chain, if it exists, is a central chain.

Definition. Let G be an M-ϕ group. If τ is the first ordinal such that $Z_\tau(G) = Z_{\tau+1}(G)$, then $Z_\tau(G)$ is the *hypercentre* of G and is denoted by $H(G)$. If σ is the first ordinal such that $C^\sigma(G) = C^{\sigma+1}(G)$, then $C^\sigma(G)$ is the *hypercommutator* of G and is denoted by $H^*(G)$. G is *ϕ nilpotent* if $H(G) = G$ and $H^*(G) = 0$.

Let us suppose for the moment that G is an M-ϕ group whose ψ subgroups satisfy the double chain condition. Then the hypercentre $H(G) = Z_n(G)$ for some integer n, and the hypercommutator $H^*(G) = C^m(G)$ for some integer m. Hence G is ϕ nilpotent if and only if G is ϕ nilpotent of finite class so that either of the following conditions is necessary and sufficient for G to be ϕ nilpotent:

$$\text{(i)}\ \ H(G) = G \quad \text{or} \quad \text{(ii)}\ \ H^*(G) = 0.$$

In this section we shall show that these results hold for an M-ϕ group which possesses a Loewy chain connecting 0 and G. Furthermore, if G is ϕ nilpotent then any Loewy chain connecting 0 and G (if one exists) is a central chain. This is an analogue to Corollary 4.2, which asserts that a ψ composition series (if one exists) is a central chain.

THEOREM 5.1. *Let J be a minimal normal ϕ subgroup of the M-ϕ group G which is not contained in the hypercommutator of G, then J is contained in the ϕ centre of G.*

Proof. J is contained in $G = C^0(G)$ but is not contained in $H^*(G) = C^\sigma(G)$. Hence there exists a first ordinal ν such that J is not contained in $C^\nu(G)$. Since $J \subseteq C^\mu(G)$ for all $\mu < \nu$ implies

$$J \subseteq \bigcap_{\mu<\nu} C^\mu(G),$$

ν is not a limit ordinal. Let $\nu = \lambda + 1$. Thus J is contained in $C^\lambda(G)$ but not in $C^{\lambda+1}(G)$.

Since $C^{\lambda+1}(G)$ is normal in G, $J \cap C^{\lambda+1}(G)$ is a normal ϕ subgroup of G. $J \cap C^{\lambda+1}(G)$ is contained in the minimal normal ϕ subgroup J and is not equal to J, since J is not contained in $C^{\lambda+1}(G)$. Hence

$$J \cap C^{\lambda+1}(G) = 0.$$

Let g be an element of G, and a an element of J; then

$$-g - a + g + a = (-g - a + g) + a$$

is in J, and is also in $C^{\lambda+1}(G)$, since $J \subseteq C^{\lambda}(G)$. Therefore $-g-a+g+a=0$, or a commutes with g. Thus $J \subseteq Z_\phi(G)$.

COROLLARY 5.1. *If $S(G)$ is the socle of the M-ϕ group G,*

$$S(G) \subseteq Z_\phi(G) + H^*(G).$$

In particular, if G is ϕ nilpotent, $S(G) \subseteq Z_\phi(G)$.

COROLLARY 5.2. *If, for every ψ subgroup N of the M-ϕ group G, G/N is ϕ nilpotent, any Loewy chain is a central chain. In particular, if G is ϕ nilpotent of finite class, every Loewy chain is a central chain.*

Proof. Let

$$0 = N_0 \subseteq \ldots \subseteq N_i \subseteq N_{i+1} \subseteq \ldots \tag{16}$$

be a Loewy chain. Then $N_{i+1}/N_i \subseteq S(G/N_i) \subseteq Z_\phi(G/N_i)$, since G/N_i is ϕ nilpotent. Hence (16) is a central chain.

THEOREM 5.2. *If the M-ϕ group G has a Loewy chain of length n which connects 0 and G, then $H^*(G) = C^n(G)$.*

Proof. From the theory of Loewy chains we know that the upper Loewy chain connects 0 and G and has length $\leqslant n$. Let

$$0 = S_0 \subset \ldots \subset S_j \subset \ldots \subset G$$

be the upper Loewy chain. For each positive integer i, $G/C^i(G)$ is ϕ nilpotent of finite class since, by Lemma 4.2,

$$C^i(G/C^i(G)) = \{C^i(G), C^i(G)\}/C^i(G) = C^i(G)/C^i(G).$$

The chain

$$\{S_0, C^i(G)\}/C^i(G) \subseteq \ldots \subseteq \{S_j, C^i(G)\}/C^i(G) \subseteq \{S_{j+1}, C^i(G)\}/C^i(G) \subseteq \ldots$$
$$\subseteq \{S_n, C^i(G)\}/C^i(G) = G/C^i(G)$$

is a Loewy chain (of length $\leqslant n$), since

$$\frac{\{S_{j+1}, C^i(G)\}/C^i(G)}{\{S_j, C^i(G)\}/C^i(G)} \cong \frac{\{S_{j+1}, C^i(G)\}}{\{S_j, C^i(G)\}} \qquad (M\text{-}\phi),$$

which is contained in the socle of $G/\{S_j, C^i(G)\}$. By Corollary 5.2, this is a central chain for $G/C^i(G)$. Hence $G/C^i(G)$ is ϕ nilpotent of finite class $\leqslant n$. Therefore

$$C^n(G/C^i(G)) = C^i(G)/C^i(G).$$

But on the other hand, by Lemma 4.2, $C^n(G/C^i(G)) = \{C^n(G), C^i(G)\}/C^i(G)$. Thus $C^n(G) = C^i(G)$, for $i \geqslant n$, and $H^*(G) = C^n(G)$.

COROLLARY 5.3. *If the M-ϕ group G has a Loewy chain of length n which connects 0 and G, and if $H^*(G) = 0$, then G is ϕ nilpotent of finite class $\leqslant n$.*

A theorem about maximal normal ϕ subgroups analogous to Theorem 5.1 about minimal normal ϕ subgroups is:

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THEOREM 5.3. *If J is a maximal normal ϕ subgroup of the M-ϕ group G which does not contain the hypercentre of G, then J contains $C^1(G)$.*

Proof. $H(G) = Z_\tau(G)$ is not contained in J. Hence there exists a first ordinal ν such that $Z_\nu(G)$ not $\subseteq J$. Since $Z_\mu(G) \subseteq J$ for all $\mu < \nu$,

$$\bigcup_{\mu<\nu} Z_\mu(G) \subseteq J$$

and therefore ν is not a limit ordinal. Let $\nu = \kappa + 1$. $Z_\kappa(G)$ is contained in J but $Z_{\kappa+1}(G)$ is not contained in J. $Z_{\kappa+1}(G)$ is normal in G, and J is a maximal normal ϕ subgroup of G; therefore

$$G = J + Z_{\kappa+1}(G).$$

Let z and z' be elements of $Z_{\kappa+1}(G)$; the element $-z - z' + z + z'$ is in $Z_\kappa(G) \subseteq J$. Hence G/J is abelian, or $C^1(G) \subseteq J$.

In Theorem 5.2 we have given a sufficient condition that the hypercommutator, $H^*(G)$ equal $C^n(G)$ for some integer n. We now find that under a somewhat weaker condition the hypercentre, $H(G)$ equals $Z_m(G)$ for some integer m. We need first a lemma.

LEMMA 5.1. *If the normal ϕ subgroup N of the M-ϕ group G is contained in $Z_r(G)$ for some integer r, and if G possesses a chain*

$$G = D_0 \supset \ldots \supset D_i \supset D_{i+1} \supset \ldots \supset D_m = 0,$$

where D_{i+1} is the intersection of maximal ψ subgroups of D_i, then

$$N \cap D_i / N \cap D_{i+1} \subseteq Z_\phi(G/N \cap D_{i+1}).$$

Proof. For fixed i $(0 \leqslant i < m)$ consider the chain

$$0 = Z_0(G) \cap N \cap D_i \subseteq \ldots \subseteq Z_j(G) \cap N \cap D_i \subseteq \ldots$$
$$\subseteq Z_r(G) \cap N \cap D_i = N \cap D_i.$$

If J is a maximal ψ subgroup of $N \cap D_i$, J contains the first subgroup of the chain but does not contain the last. Hence there exists an integer j such that

$$Z_j(G) \cap N \cap D_i \subseteq J;\ Z_{j+1}(G) \cap N \cap D_i \text{ not } \subseteq J.$$

Thus $J \subset (Z_{j+1}(G) \cap N \cap D_i) + J \subseteq N \cap D_i$, and, since J is maximal,

$$N \cap D_i = (Z_{j+1}(G) \cap N \cap D_i).$$

Let g and z be elements of G and $Z_{j+1}(G) \cap N \cap D_i$ respectively. The element $-g - z + g + z$ is in $N \cap D_i$ (since N and D_i are normal subgroups of G), and is also in $Z_j(G)$, since by definition

$$Z_{j+1}(G)/Z_j(G) = Z_\phi(G/Z_j(G)).$$

Thus $-g - z + g + z$ is in $Z_j \cap N \cap D_i \subseteq J$. Hence $J \supseteq (G, N \cap D_i)$. But $N \cap D_{i+1}$ is the intersection of maximal ψ subgroups of $N \cap D_i$. Therefore

$$N \cap D_{i+1} \supseteq (G, N \cap D_i)$$

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and thus by Lemma 4.1, $N \cap D_i/N \cap D_{i+1} \subseteq Z_\phi(G/N \cap D_{i+1})$.

THEOREM 5.4. *If the M-ϕ group G possesses a chain*

$$G = D_0 \supset \ldots \supset D_i \supset D_{i+1} \supset \ldots \supset D_m = 0,$$

where D_{i+1} is the intersection of maximal ψ subgroups of D_i, then $H(G) = Z_m(G)$.

Proof. Let r be any positive integer and let $Z_r = Z_r(G)$. Then by Lemma 5.1,

$$Z_r \cap D_i/Z_r \cap D_{i+1} \subseteq Z_\phi(G/Z_r \cap D_{i+1}).$$

Hence

$$0 = N_0 = Z_r \cap D_m \subseteq \ldots \subseteq N_j = Z_r \cap D_{m-j} \subseteq \ldots \subseteq N_m = Z_r \cap D_0 = Z_r$$

is a central chain for G; and by Theorem 4.2, $Z_r = N_m \subseteq Z_m$. But r was arbitrary so that the relation holds for each r. Hence $Z_m = Z_r$ for $r \geqslant m$, and $H(G) = Z_m(G)$.

COROLLARY 5.4. *If the M-ϕ group G possesses a Loewy chain of length n which connects* 0 *and* G, $H(G) = Z_n(G)$. *Hence if* $H(G) = G$, *G is ϕ nilpotent of finite class $\leqslant n$, and the Loewy chain is a central chain.*

Proof. By Theorem 3.7, if G has a Loewy chain of length n connecting 0 and G, and if we define the chain

$$G = M_0 \supseteq \ldots \supseteq M_j \supseteq M_{j+1} \supseteq \ldots,$$

where M_{j+1} is the intersection of M_j with all maximal ψ subgroups of M_j, there exists an integer $m \leqslant n$ such that $M_m = 0$. Thus by Theorem 5.4, $H(G) = Z_m(G)$. But $n \geqslant m$, so that $H(G) = Z_n(G)$.

6. ϕ-solubility. In this section we study another property of the type discussed in §3. However, before defining the property, we prove some further results about ϕ nilpotency which we shall need.

LEMMA 6.1. *Let G be an M-ϕ group and assume that ϕ is normal. If N is a normal ϕ subgroup of G which is ϕ nilpotent of finite class, N is ψ nilpotent of finite class.*

Proof. It is sufficient to show that the ϕ subgroups $Z_i(N)$ are normal in G. To show that $Z_\phi(N)$ is normal in G, we note that $Z_\phi(N)$ is a subgroup of the centre $Z(N)$ of N, and that $Z(N)$ as a characteristic subgroup of N is normal in G. Hence if g is any element of G,

$$-g + Z_\phi(N) + g \subseteq -g + Z(N) + g = Z(N).$$

Since ϕ is normal, $-g + Z_\phi(N) + g$ is a ϕ subgroup of G; hence

$$-g + Z_\phi(N) + g = Z_\phi(N).$$

It may be shown by induction that $Z_i(N)$ is normal in G.

THEOREM 6.1. *Let G be an M-ϕ group and assume that ϕ is normal. If M and*

N are normal ϕ subgroups of G which are ϕ nilpotent of finite class, then $M + N$ is ϕ nilpotent of finite class.

Proof. (i) Assume that $M \cap N = 0$ so that $M + N = M \oplus N$. Since (by Lemma 6.1) M and N are ψ nilpotent, there exist chains:

$$0 = N_0 \subseteq \ldots \subseteq N_i \subseteq N_{i+1} \subseteq \ldots \subseteq N_n = N, \tag{17}$$

$$0 = M_0 \subseteq \ldots \subseteq M_i \subseteq M_{i+1} \subseteq \ldots \subseteq M_n = M, \tag{18}$$

with N_i and M_i ψ subgroups of G $(i = 1, \ldots, n)$, and

$$N_{i+1}/N_i \subseteq Z_\phi(N/N_i);\ M_{i+1}/M_i \subseteq Z_\phi(M/M_i).$$

Let m_{i+1}, n_{i+1}, m, and n be elements of M_{i+1}, N_{i+1}, M, and N respectively; the element

$$-(m + n) - (m_{i+1} + n_{i+1}) + (m + n) + (m_{i+1} + n_{i+1})$$
$$= -m - m_{i+1} + m + m_{i+1} - n - n_{i+1} + n + n_{i+1}$$

is in $M_i + N_i$, since $(M, M_{i+1}) \subseteq M_i$ and $(N, N_{i+1}) \subseteq N_i$. Hence

$$M_{i+1} + N_{i+1}/M_i + N_i \subseteq Z_\phi(M + N/M_i + N_i),$$

and the chain $0 = M_0 + N_0 \subseteq \ldots \subseteq M_i + N_i \subseteq M_{i+1} + N_{i+1} \subseteq \ldots \subseteq M + N$ is a central chain for $M + N$; thus $M + N$ is ϕ nilpotent of finite class.

(ii) We consider the general case (i.e., no longer assume that $M \cap N = 0$). Since $M/M \cap N$ and $N/M \cap N$ are ϕ nilpotent of finite class, it follows from (i) that $M + N/M \cap N$ is ϕ nilpotent of finite class and hence there exists a chain

$$M \cap N = Q_0 \subseteq \ldots \subseteq Q_i \subseteq Q_{i+1} \subseteq \ldots \subseteq Q_q = M + N, \tag{19}$$

where Q_i is in ψ, and $Q_{i+1}/Q_i \subseteq Z_\phi(M + N/Q_i)$. By Theorem 4.6, there exists a chain

$$0 = K_0 \subseteq \ldots \subseteq K_j \subseteq K_{j+1} \subseteq \ldots \subseteq K_k = M \cap N, \tag{20}$$

where K_j is in ψ and $K_{j+1}/K_j \subseteq Z_\phi(M/K_j)$, and there exists a refinement of (20):

$$0 = K_0 = K_{0,0} \subseteq \ldots \subseteq K_j = K_{j,0} \subseteq \ldots \subseteq K_{j,p} \subseteq \ldots \subseteq K_{j,n_j} \tag{21}$$
$$= K_{j+1} \subseteq \ldots \subseteq M \cap N,$$

where $K_{j,p}$ is in ψ, and $K_{j,p+1}/K_{j,p} \subseteq Z_\phi(N/K_{j,p})$. Clearly,

$$K_{j,p+1}/K_{j,p} \subseteq Z_\phi(M + N/K_{j,p}).$$

Combining (19) and (21) we obtain the chain

$$0 = K_0 \subseteq \ldots \subseteq K_{j,p} \subseteq \ldots \subseteq M \cap N = Q_0 \subseteq \ldots \subseteq Q_i \subseteq \ldots \subseteq Q_q$$
$$= M + N.$$

This is a central chain for $M + N$. Thus $M + N$ is ϕ nilpotent of finite class.

COROLLARY 6.1. *Let G be an M-ϕ group and assume that ϕ is normal. If the*

ψ subgroups of G satisfy the ascending chain condition, the compositum of normal ϕ nilpotent ϕ subgroups of finite class is ϕ nilpotent of finite class.

This result can also be obtained under the hypothesis that there exists a Loewy chain connecting 0 and G.

THEOREM 6.2. *Assume that the M-ϕ group G possesses a Loewy chain connecting 0 and G, and assume that ϕ is normal. If A_α, for each α in a set $\mathfrak{A}$, is a normal ϕ subgroup of G which is ϕ nilpotent of finite class, then $\complement A_\alpha (\alpha \in \mathfrak{A})$ is ϕ nilpotent of finite class.*

Proof. Let $0 = S_0 \subseteq \ldots \subseteq S_i \subseteq \ldots \subseteq S_n = G$ be a Loewy chain for G. If

$$A = \complement A_\alpha \qquad (\alpha \in \mathfrak{A})$$

the chain

$$(22)\quad 0 = A \cap S_0 = T_0 \subseteq \ldots \subseteq A \cap S_i = T_i \subseteq \ldots \subseteq A \cap S_n = T_n = A$$

is a Loewy chain for the M-ψ group A, that is, each T_{i+1}/T_i is the sum of minimal ψ subgroups of A/T_i. For

$$\frac{T_{i+1}}{T_i} = \frac{A \cap S_{i+1}}{A \cap S_i} \simeq \frac{\{A \cap S_{i+1}, S_i\}}{S_i} \qquad (M\text{-}\psi),$$

which is the sum of minimal ψ subgroups since it is contained in S_{i+1}/S_i. We now show that the chain (22) is a central chain.

By Lemma 3.1, T_{i+1}/T_i can be written as the direct sum of minimal ψ subgroups; let

$$T_{i+1}/T_i = \sum{}^{\circ} F_\lambda/T_i \qquad (\gamma \in \mathfrak{C}),$$

where F_γ/T_i, for each γ in the set $\mathfrak{C}$, is a minimal ψ subgroup. For fixed γ in $\mathfrak{C}$ and for fixed α in $\mathfrak{A}$, we show that $T_i \supseteq (F_\gamma, A_\alpha)$. Since F_γ/T_i is a minimal ψ subgroup, either

$$F_\gamma \cap (A + T_i) = T_i \quad \text{or} \quad F_\gamma \cap (A + T_i) = F_\gamma.$$

In the first case, $F_\gamma \cap A_\alpha \subseteq T_i$; and $(F_\gamma, A_\alpha) \subseteq F_\gamma \cap A_\alpha$ so that $(F_\gamma, A_\alpha) \subseteq T_i$. In the second case,

$$F_\gamma \subseteq A_\alpha + T_i \quad \text{or} \quad F_\gamma/T_i \subseteq A_\alpha + T_i/T_i.$$

Since F_γ/T_i is a minimal ψ subgroup of the ψ nilpotent group $A_\alpha + T_i/T_i$, it is contained in $Z_\phi(A_\alpha + T_i/T_i)$. Therefore, $(A_\alpha + T_i, F_\gamma) \subseteq T_i$ so that $(A_\alpha, F_\gamma) \subseteq T_i$. It follows that, for each γ in $\mathfrak{C}$ and for each α in $\mathfrak{A}$, $(A_\alpha, F_\gamma) \subseteq T_i$. It follows that, for each γ in $\mathfrak{C}$, $(\complement A_\alpha, F_\gamma) \subseteq T_i$ or equivalently, $F_\gamma/T_i \subseteq Z_\phi(A/T_i)$. This in turn implies that

$$T_{i+1}/T_i = \sum{}^{\circ} F_\gamma/T_i \subseteq Z_\phi(A/T_i),$$

which shows that (22) is a central chain. Hence $A = \complement A_\alpha$ is ϕ nilpotent of finite class.

THEOREM 6.3. *Assume that the hypercommutator $H^*(G)$ of the M-ϕ group G, is equal to $C^n(G)$ for some integer n. If N_α is a normal ϕ subgroup of G, for each α in the set $\mathfrak{A}$, and if G/N_α is ϕ nilpotent of finite class, then $G/\bigcap N_\alpha$ $(\alpha \in \mathfrak{A})$ is ϕ nilpotent of finite class.*

Proof. There exists a central chain for G of finite length n_α connecting N_α to G, for each α. Hence

$$C^{n_\alpha}(G) \subseteq N_\alpha.$$

But

$$H^*(G) = C^n(G) \subseteq C^{n_\alpha}(G)$$

for each α. Hence $H^*(G) \subseteq N_\alpha$, for each α, and

$$H^*(G) \subseteq \bigcap_\alpha N_\alpha = N.$$

$G/H^*(G)$ is ϕ nilpotent of finite class and hence G/N is ϕ nilpotent of finite class.

COROLLARY 6.2. *Under the hypotheses of the previous theorem, $H^*(G)$ is the intersection of all normal ϕ subgroups N such that G/N is ϕ nilpotent of finite class.*

LEMMA 6.2. *Let A and B be normal ϕ subgroups of the M-ϕ group G with $A \supset B$. If A/B is ϕ nilpotent of finite class, $A\eta/B\eta$ is ϕ nilpotent of finite class for any M-ϕ homomorphism η of G.*

Proof. There exists a chain $B = B_0 \subseteq \ldots \subseteq B_i \subseteq B_{i+1} \subseteq \ldots \subseteq B_n = A$, where B_i is a normal ϕ subgroup of A and $B_{i+1}/B_i \subseteq Z_\phi(A/B_i)$. By Theorem 4.1,

$$B_{i+1}\eta/B_i\eta \subseteq Z_\phi(A\eta/B_i\eta),$$

and thus $A\eta/B\eta$ is ϕ nilpotent of finite class.

It may be shown in a similar fashion that the following is a consequence of Theorem 4.3 (ii):

LEMMA 6.3. *Let A, B, and C be normal ϕ subgroups of the M-ϕ group G with $A \supset B$. If A/B is ϕ nilpotent of finite class, then $A \cap C/B \cap C$ is ϕ nilpotent of finite class.*

Consider now the property (S) of M-ϕ groups:

(S) The ϕ subgroup A of the M-ϕ group G satisfies (S) (in G), if it is ϕ nilpotent of finite class.

In order to apply our theory of normal chains we must verify that (S) satisfies the conditions (k_1)-(k_6). (k_1) obviously holds. The validity of (k_4) follows from Lemma 6.3. Lemma 6.2 shows that (k'_5) holds; and (k'_5) is equivalent to (k_5) and (k_6). In order to ensure that (k_2) and (k_3) hold we make further hypotheses about the groups under consideration.

Assume that ϕ is normal. It follows from Corollary 6.1 that (k_2) is satisfied if the ascending chain condition holds for the ψ subgroups. On the other hand, in virtue of Theorem 6.3, (k_3) is satisfied if the descending chain condition holds for the ψ subgroups. Hence (k_2) and (k_3) hold if we assume the double chain condition for ψ subgroups. However, this condition may be replaced by the weaker condition that G possesses a Loewy chain connecting 0 and G. This follows from Theorem 6.2 (for (k_2)); and from Theorems 5.2 and 6.3 (for (k_3)). So we have:

THEOREM 6.4. *Let G be an M-ϕ group. Assume that ϕ is normal and that G possesses a Loewy chain connecting* 0 *and G. Then* (S) *satisfies* (k_1)-(k_6).

Therefore, the upper and lower S-chains may be constructed, and the results of §2 hold for S-chains. The terms of the lower S-chain are:

$$G \supseteq H^*(G) \supseteq H_2^*(G) = H^*[H^*(G)] \supseteq \ldots \supseteq H_{n+1}^*(G) = H^*[H_n^*(G)] \supseteq \ldots .$$

This follows from Corollary 6.2. However, the terms of the upper S-chain are not necessarily the successive hypercentres, for the hypercentre $H(G)$ is not necessarily the maximal ϕ nilpotent normal ϕ subgroup of G.

Definition. If the *M-ϕ group G* possesses an S-chain that connects 0 and G, G is *ϕ soluble.*

THEOREM 6.5. *Let G be an M-ϕ group. Assume that ϕ is normal and that G possesses a Loewy chain connecting* 0 *and G. If G is ϕ soluble, any Loewy chain connecting* 0 *and G has abelian factors and consequently is an* S-*chain.*

Proof. Let $0 = U_0 \subseteq \ldots \subseteq U_i \subseteq U_{i+1} \subseteq \ldots \subseteq U_m = G$ be a Loewy chain for G; then U_{i+1}/U_i is the direct sum of feet of G/U_i. Hence in order to show that U_{i+1}/U_i is abelian, it is sufficient to show that any foot of G/U_i is abelian.
Let F/U_i be a foot of G/U_i. Since G is ϕ soluble, there exists a chain

$$U_i = T_0 \subseteq \ldots \subseteq T_j \subseteq T_{j+1} \subseteq \ldots \subseteq T_m = G,$$

where T_j is in ψ and T_{j+1}/T_j is ψ nilpotent of finite class. Choose j so that F is not contained in T_j but is contained in T_{j+1}. Then

$$U_i \subseteq F \cap T_j \subset F$$

and hence, since F/U_i is a minimal ψ subgroup, $U_i = F \cap T_j$. Now $F + T_j/T_j$ is a minimal ψ subgroup of the ψ nilpotent group T_{j+1}/T_j; by Corollary 5.1, $F + T_j/T_j$ is in the centre of T_{j+1}/T_j. This implies that F/U_i is abelian, since

$$F/U_i \cong F + T_j/T_j.$$

The definition of solubility that we have used was discussed by Hirch [6]. It is customary to proceed somewhat differently.

Definition. For the M-ϕ group G we define

$$G^{(0)} = G, G^{(n+1)} = (G^{(n)}, G^{(n)}),$$

for $n \geqslant 0$.

$$G = G^{(0)} \supseteq \ldots \supseteq G^{(i)} \supseteq G^{(i+1)} \supseteq \ldots \tag{23}$$

is a descending normal ϕ chain, and the factors $G^{(i+1)}/G^{(i)}$ are abelian; in fact,

$G^{(i+1)}$ is the smallest normal ϕ subgroup of $G^{(i)}$ such that the quotient group is abelian. However the dual construction does not yield an ascending normal ϕ chain with abelian factors; for the compositum of abelian normal ϕ subgroups is not necessarily abelian.

The following theorem shows that the definition given for ϕ solubility coincides with the customary one:

THEOREM 6.6. *The M-ϕ group G is ϕ soluble, if and only if $G^{(s)} = 0$ for some integer s.*

Proof. The chain $G = G^{(0)} \supset \ldots \supset G^{(i)} \supset \ldots \supset G^{(s)} = 0$ has abelian factors and hence is an S-chain.

Conversely, assume that $G = R_0 \supseteq \ldots \supseteq R_i \supseteq R_{i+1} \supseteq \ldots \supseteq R_n = 0$ is an S-chain so that R_i/R_{i+1} is ϕ nilpotent of finite class. Then the chain

$$R_i/R_{i+1} = C^0(R_i/R_{i+1}) \supseteq \ldots \supseteq C^j(R_i/R_{i+1}) \supseteq \ldots \supseteq C^{n_i}(R_i/R_{i+1}) = R_{i+1}/R_{i+1}$$

joins R_i/R_{i+1} to R_{i+1}/R_{i+1} and has abelian factors. Hence if

$$C^j(R_i/R_{i+1}) = R_{i,j}/R_{i+1}, \quad R_{i,n_i} = R_{i+1},$$

the chain

$$G = R_0 \supseteq \ldots \supseteq R_i \supseteq \ldots \supseteq R_{i,j} \supseteq \ldots \supseteq R_{i,n_i} = R_{i+1} \supseteq \ldots \supseteq R_n = 0$$

is a normal ϕ chain for G with abelian factors. It is easy to verify that if there exist a normal ϕ chain with abelian factors connecting G and 0, then $G^{(s)} = 0$ for some integer s.

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