QUASI-ERGODICITY OF COMPACT STRONG FELLER SEMIGROUPS ON L^2

KAMIL KALETA [1] AND RENÉ L. SCHILLING [2]

[1] *Wrocław University of Science and Technology, Faculty of Pure and Applied Mathematics, Wyb. Wyspiańskiego 27, 50-370 Wrocław, Poland*
(kamil.kaleta@pwr.edu.pl)

[2] *TU Dresden, Fakultät Mathematik, Institut für Mathematische Stochastik, 01062 Dresden, Germany*
(rene.schilling@tu-dresden.de)

Abstract We study the quasi-ergodicity of compact strong Feller semigroups U_t, $t > 0$, on $L^2(M,\mu)$; we assume that M is a locally compact Polish space equipped with a locally finite Borel measue μ. The operators U_t are ultracontractive and positivity preserving, but not necessarily self-adjoint or normal. We are mainly interested in those cases where the measure μ is infinite and the semigroup is not intrinsically ultracontractive. We relate quasi-ergodicity on $L^p(M,\mu)$ and uniqueness of the quasi-stationary measure with the finiteness of the heat content of the semigroup (for large values of t) and with the progressive uniform ground state domination property. The latter property is equivalent to a variant of quasi-ergodicity which progressively propagates in space as $t \uparrow \infty$; the propagation rate is determined by the decay of $U_t \mathbb{1}_M(x)$. We discuss several applications and illustrate our results with examples. This includes a complete description of quasi-ergodicity for a large class of semigroups corresponding to non-local Schrödinger operators with confining potentials.

1. Introduction

Let (M,d,μ) be a metric measure space. The central theme of our contribution is the question whether a semigroup of compact operators $\{U_t : t \geqslant 0\}$ acting on the scale of Lebesgue spaces $L^p(\mu) = L^p(M,\mu)$ admits some 'ergodic' measure. More precisely, we

look at expressions of the from

$$\lim_{t \to \infty} \left| \frac{\sigma(U_t f)}{\sigma(U_t \mathbb{1}_M)} - m(f) \right| = 0 \quad \text{where} \quad m(f) := \int f \, dm, \quad \sigma(g) := \int g \, d\sigma, \qquad (1.1)$$

and we establish conditions on $\{U_t : t \geqslant 0\}$ such that the limit in (1.1) i) exists, ii) exists uniformly for certain families of measures σ and functions f, iii) converges with a certain rate (in time and space), and iv) that m is the unique quasi-stationary measure. In many situations, the rate can be explicitly given, and it is even exponential in time t. The measure m, which is sometimes called a *quasi-ergodic*, has a very concrete form

$$m(E) = \frac{\int_E \psi_0 \, d\mu}{\int_M \psi_0 \, d\mu},$$

where ψ_0 is the eigenfunction of the $L^2(\mu)$-adjoint semigroup $\{U_t^* : t \geqslant 0\}$ for the eigenvalue $e^{-\lambda_0 t}$. In the context of Schrödinger operators and their semigroups, λ_0 is the ground state eigenvalue, and ψ_0 is the corresponding eigenfunction (for the adjoint). If $\psi_0 \in L^1(\mu)$, the measure m is a finite measure, and it is a quasi-stationary (or quasi-invariant) measure of the semigroup $\{U_t : t \geqslant 0\}$.

Since we do not assume that $U_t \mathbb{1}_M \equiv \mathbb{1}_M$ – that is, we do not assume conservativeness – we can expect only some kind of quasi-ergodic behavior. This is reflected in the structure of (1.1) where we normalize $\sigma(U_t f)$ by the total mass $\sigma(U_t \mathbb{1}_M)$; if U_t is the transition semigroup of some non-conservative stochastic process with initial distribution σ, this can be seen as conditioning on survival, – that is, $\sigma(U_t f)/\sigma(U_t \mathbb{1}_M) = \int \mathbb{E}^x [f(X_t) \mid \zeta > t)] \, \sigma(dx)$, $\zeta = \inf\{s > 0 : X_s \notin M\}$ being the life-time.

Large time evolution phenomena occur in various areas of mathematics and science (e.g., in PDEs, dynamical systems, stochastic processes, statistical and quantum physics), and they are often modeled by semigroups of operators. These applications require quite different state spaces, ranging from discrete spaces (lattices, graphs, ...) to manifolds, fractals and, of course, classical Euclidean spaces. In order to cover these situations, we work in a rather general setting, on a locally compact Polish space N and μ is a locally finite Borel measure on the Borel sets $\mathcal{B}(M)$, and we consider sufficiently regular (but not necessarily self-adjoint or normal) operator semigroups acting (initially) on $L^2(M, \mu)$ This includes the most interesting case when M is unbounded and/or μ is an infinite measure. We consider semigroups of compact integral operators which are positivity improving and ultracontractive, and have the strong Feller property – see (**A0**)–(**A3**) and (**A4**) in Section 2 for a precise statement. This framework accomodates many situations which are typically discussed in the literature.

We focus on Schrödinger semigroups – with confining potentials, based both on local (second order differential operators) [16, 15, 58, 57, 56, 42, 43] and non-local kinetic term operators (Lévy operators, fractional and relativistic Laplacians, etc.) [6, 22, 36, 39, 8, 25, 26] – and on Feynman–Kac semigroups of Markov processes whose state spaces are general locally compact Polish spaces; this includes processes on fractals [19, 37] and Markov chains on graphs and discrete spaces [4, 14, 46, 47]. Some specific cases and various examples are discussed in Section 7. There are many other examples that fit into our framework, but which we do not discuss here in detail (e.g., evolution semigroups

of non-homogeneous second order differential operators (generating diffusion processes) and non-local Lévy-type operators (generating jump Lévy-type processes) with confining coefficients [1, 45, 44, 64, 55] or Markov processes killed upon exiting an unbounded domain [3, 38, 8, 9]).

We are aware of only a few papers that work in a similar direction: Takeda [62], Knobloch and Partzsch [31] and Zhang, Li and Song [65]. Takeda uses a Dirichlet form approach, his processes and semigroups are symmetric and his main objective is the existence and uniqueness of the quasi-invariant measure. The key technical assumption is a tightness condition on the resolvent (kernel) of the semigroup; this is, essentially, a compactness condition which is weaker than IUC (intrinsic ultracontractivity). The contributions by Knobloch and Partzsch and Zhang et al. are, in some sense, extremes: Knobloch and Partzsch strive for a high level of generality in the basic setup, starting from a general Markov kernel assuming what they call 'compact domination'. In the end, they prove that intrinsic ultracontractivity (IUC) implies uniform quasi-ergodicity at an exponential rate. One should mention that the only known examples satisfying the compact dominaton property are IUC kernels and semigroups. However, Zhang, Li and Song consider Markov processes; they are topologically quite general, but assume that the underlying measure is finite.

From the point of view of applications, both IUC and the finiteness of the underlying measure are rather restrictive conditions; for Schrödinger semigroups with confining potentials, IUC requires very fast growth of the potential at infinity; see [16, 2, 25, 10]. For non-self-adjoint operators, IUC was first studied by Kim and Song [29, 30].

Recently, we introduced in [26] the notion of *progressive intrinsic ultracontractivity* in order to obtain sharp two-sided heat kernel estimates for non-IUC or not necessarily IUC semigroups; see also Definition 4.2 and Remark 4.3 further down. As it turns out, it is this notion that helps us to avoid IUC as well as the finiteness of the underlying base measure. Consequently, we are able to characterize the quasi-ergodic regularity for a fairly general class of compact semigroups and underlying state spaces, including necessary and sufficient conditions for the existence and uniqueness of a quasi-stationary measure.

Let us briefly summarize the main contributions of this paper. Recall that the heat content is defined as $Z(t) = \|U_t \mathbb{1}_M\|_{L^1(\mu)} = \|U_t^* \mathbb{1}_M\|_{L^1(\mu)}$.

Finite heat content and exponential quasi-ergodicity

We begin with a fairly general, yet easily verified, sufficient condition on the semigroup $\{U_t : t \geqslant 0\}$ such that the convergence in (1.1) is exponential in time, uniform for $f \in L^p$, $\|f\|_p \leqslant 1$, for any fixed $p \in [1, \infty]$, and it holds for any finite measure σ. In fact (cf. Theorem 3.4), if the heat content of the semigroup is finite for some $t_1 > 0$, then ψ_0 is integrable, the measure m is the unique quasi-stationary measure of the semigroup and the first term appearing in (1.1) is dominated by $c_1 e^{-c_2 t} \sigma(U_{t_1} \mathbb{1}_M)/\sigma(\phi_0) \|f\|_p$ – here, $t \gg 1$ is large, the constants $c_1, c_2 > 0$ do not depend on σ or $f \in L^p$, and ϕ_0 is the ground state of $\{U_t : t \geqslant 0\}$. A similar reasoning applies to the adjoint semigroup $\{U_t^* : t \geqslant 0\}$. Taking $\sigma(dy) = \delta_x(dy)$ as the Dirac-delta measure concentrated in $x \in M$, we obtain pointwise convergence with an explicit space-rate which is uniform on compact sets;

see Corollary 3.5. We will see in Section 7.1 that for a very general class of Feynman–Kac semigroups with potential $V(x)$, the heat content $Z(t_1)$ is finite if the function $\exp(-t_1 V(x))$ is integrable 'at infinity'; this condition is often also a necessary condition. This means that the finiteness of the heat content is independent of the underlying Markov process. Since our proof relies only on the Feynman–Kac formula (see Demuth and van Casteren [17]), it also applies to evolution semigroups of very general Schrödinger operators $H = -L + V$ – in this case, the finiteness of the heat content is determined solely by the behavior of the potential V at infinity, and it is independent of the kinetic term (i.e., the free Feller operator L).

By definition, the heat content is finite if $U_t \mathbb{1}_M$ is integrable. The latter is guaranteed by the asymptotic ground state domination (aGSD) of the semigroup (that is, $U_t \mathbb{1}_M \leqslant ce^{-t\lambda_0}\phi_0$ for some $t > 0$) and the integrability of the ground state ϕ_0. In [26], we defined the notion of progressive ground state domination (pGSD), which requires only that $\mathbb{1}_{K_t} U_t \mathbb{1}_M \leqslant ce^{-t\lambda_0}\phi_0$ for some increasing family of compact sets $K_t \uparrow M$. In [26], pGSD was obtained as a necessary condition of sharp two-sided heat kernel estimates for Schrödinger semigroups; we will now see that pGSD is closely related to quasi-ergodicity.

Equivalence of aGSD/pGSD and quasi-ergodicity

If pGSD holds for some family of compact sets $K_t \uparrow M$, one can show that (1.1) is dominated by some rate $\kappa(t)$ which is essentially given by decay of $U_t \mathbb{1}_M(x)$ and $U_t^* \mathbb{1}_M(x)$ for $x \notin K_{bt}$ (for some fixed $b \in (0, 1/2)$). In fact, the bound $\kappa(t)$ is uniform for all $f \in L^p(\mu)$, $\|f\|_p \leqslant 1$, and all measures σ with support in K_{at} (where $a \in (0,1)$ such that $a + 2b = 1$); see Theorem 5.2. Specializing again to $\sigma = \delta_x$ with $x \in K_{at}$, we see in Theorem 5.4 that pGSD is equivalent to the convergence of (1.1) to zero (locally uniformly for $x \in K_t$) as $t \to \infty$. The latter property is best described as 'progressive uniform quasi-ergodicity'. As a consequence (see Theorem 5.5), we also see that the quasi-ergodic measure is necessarily given by m, resp., m^*, hence it is unique.

In Section 4, we see that the finiteness of the heat content always implies pGSD, but without good control on the growth of the family of compact sets $K_t \uparrow M$. In this connection, it is interesting to compare Theorem 3.4 and Theorems 5.2 and 5.4. Theorem 3.4 gives an easy-to-check sufficient condition (to wit: finite heat content) for the exponential (in time) quasi-ergodicity, but we may not be able to control the space behavior. Theorem 5.2 and 5.4 contain a necessary and sufficient condition (to wit: pGSD) for a somewhat stronger property, which is progressively uniform in space, and with a time-rate depending on the semigroup. The examples in Section 7.3 illustrate that the finiteness of the heat content requires stronger assumptions on the generator – for instance, for Schrödinger or Feynman–Kac semigroups, one needs that the potential V grows at infinity at least logarithmically.

Assuming that the heat content is finite, our methods also yield results on the asymptotics of compact semigroups and the heat content as $t \to \infty$; cf. Section 6. For example, one can see that $Z(t)$ is exponentially close to $e^{-\lambda_0 t}\|\phi_0\|_1\|\psi_0\|_1/\mu(\psi_0\phi_0)$ as $t \to \infty$; see Corollary 6.2.

The final section (Section 7) contains a number of examples which illustrate the applicability and the limitations of our results from the previous sections. The examples in 7.1 are related to Theorem 3.4 and Corollary 3.5, and the examples in Sections 7.2 and 7.3 mainly illustrate the pGSD-property from Theorems 5.2 and 5.4. In fact, Section 7.3 contains a complete characterization, giving necessary and sufficient conditions, of quasi-ergodic regularity of the Schrödinger semigroups corresponding to a large class of non-local Schrödinger operators with confining potentials; this can be seen as a direct continuation of our recent work [26].

Notation. Most of our notation is standard or self-explanatory. We use 'positive' in the non-strict sense (e.g., a function $f \in L^2(M,\mu)$ is called positive if $f \geqslant 0$ μ-a.e.), and we say that f is 'strictly positive' if $f > 0$ μ-a.e. We will frequently use the shorthand $m(f)$ to denote $\int f \, dm$. We use $a \wedge b$ and $a \vee b$ to indicate $\min(a,b)$ and $\max(a,b)$, and we write $f \asymp g$ if $c^{-1}f(x) \leqslant g(x) \leqslant cg(x)$ for all x (in a given range) and with some fixed constant $c \in (0,\infty)$. Finally, we call a strongly continuous contraction semigroup $\{U_t : t \geqslant 0\}$ a *Feller semigroup* if it maps the continuous functions vanishing at infinity, $C_\infty(M)$ into itself, and a *strong Feller semigroup* if it maps the bounded Borel measurable functions $B_b(M)$ (or, sometimes, the set $L^\infty(M)$) into the bounded continuous functions $C_b(M)$.

2. Setup and basic assumptions

Topological preliminaries.

For the convenience of our readers, we summarize a few topological preliminaries with references. We will work in a *Polish space M* (i.e., a completely metrizable topological space such that the metric defines the topology, and the topology has a countable base (second countable)). Being a metric space, M is also first countable (each point has a countable neighborhood basis), and it is separable (i.e., it contains a countable dense subset); cf. [18, Corollary 1.3.8, Theorem 4.1.15]. We call *M locally compact* if each $x \in M$ has a relatively compact open neighborhood $U(x)$. If a Polish space is locally compact, it is σ-compact (or countable at infinity) (i.e., there is an increasing sequence of compact sets K_n such that $\bigcup_{n \in \mathbb{N}} K_n = M$); cf. [18, Theorem 4.1.15.iii)]. In a metric space, compact sets are closed, and σ-compactness allows us to write any closed set as a countable union of compact sets. Therefore, the Borel σ-algebra $\mathcal{B}(M)$ of a locally compact Polish space is generated by the compact sets. A Borel measure μ is a positive measure defined on the Borel sets $\mathcal{B}(M)$. It is locally finite if each point $x \in M$ has an open neighborhood with finite measure. This implies that μ is finite on compact sets. Thus, a locally finite measure on a locally compact Polish space is σ-finite.

Setting and assumptions.

Throughout, we assume that M is a locally compact Polish space and μ a positive, locally finite Borel measure with full topological support. We do not assume that the space M is compact or that the measure μ is finite. On $L^2(M,\mu)$, we consider a strongly continuous

semigroup of bounded linear operators, $\{U_t : t \geqslant 0\}$ and its adjoint semigroup $\{U_t^* : t \geqslant 0\}$, which is given by

$$\int U_t f(x) \cdot \overline{g(x)} \, \mu(dx) = \int f(x) \cdot \overline{U_t^* g(x)} \, \mu(dx), \quad f, g \in L^2(M, \mu), \, t > 0.$$

Moreover, we assume that U_t and U_t^* are integral operators, which are given by a real-valued measurable kernel $u_t(x, y)$:

$$U_t f(x) = \int u_t(x, y) f(y) \, \mu(dy), \qquad U_t^* g(y) = \int u_t(x, y) g(x) \, \mu(dx), \quad t > 0. \qquad (2.1)$$

Moreover, we use the following assumptions:

Assumptions.

> The linear operators $U_t : L^2(M, \mu) \to L^2(M, \mu), t > 0$, are compact;   **(A0)**

> For some $t_0 > 0$, the operator U_{t_0} is positivity improving: for every positive $f \in L^2(M, \mu), f \not\equiv 0$, we have $U_{t_0} f(x) > 0$ for μ-almost all $x \in M$.   **(A1)**

> The semigroups $\{U_t : t \geqslant 0\}$ and $\{U_t^* : t \geqslant 0\}$ have the strong Feller property: for every $t > 0$ and $f \in L^\infty(M, \mu)$, we have $U_t f, U_t^* f \in C_b(M)$;   **(A2)**

> For some $t_0 > 0$, the operators $U_{t_0}, U_{t_0}^* : L^2(M, \mu) \to L^\infty(M, \mu)$ are bounded (i.e., ultracontractive):   **(A3)**
> $$\int u_{t_0}^2(\cdot, y) \, \mu(dy) \in L^\infty(M, \mu) \quad \text{and} \quad \int u_{t_0}^2(x, \cdot) \, \mu(dx) \in L^\infty(M, \mu).$$

Our results in Sections 4 and 5 need, when considering non-compact spaces M, a further *strong decay* condition.

> There is some $t_0 > 0$ such that for every $t \geqslant t_0$ and $\epsilon > 0$, there exists a compact set $K \subset M$ satisfying   **(A4)**
> $$U_t \mathbb{1}_M(x) \leqslant \epsilon \quad \text{or} \quad U_t^* \mathbb{1}_M(x) \leqslant \epsilon, \quad \text{for every } x \in M \setminus K.$$

Let us comment on the assumptions (**A0**)–(**A4**) and how they are related.

Remark 2.1. (**A0**) is equivalent to assuming that the adjoint operators U_t^* are compact on $(L^2(M, \mu))^* = L^2(M, \mu)$.

(**A1**) is equivalent to assuming that $U_{t_0}^*$ is positivity improving for some $t_0 > 0$. Moreover, U_{t_0} and $U_{t_0}^*$ are positivity improving if, and only if, the integral kernel $u_{t_0}(x, y)$ is for $\mu \otimes \mu$ almost all (x, y) strictly positive.

(**A2**) is equivalent to the condition that $U_t \mathbb{1}_M(x)$ and $U_t^* \mathbb{1}_M(x)$ are bounded continuous functions of x. These functions will play a central role in our investigations.

(**A3**) implies, because of the reflexivity of the space $L^2(M,\mu)$, that (the restrictions of the second duals to $L^1(M,\mu) \subset (L^\infty(M,\mu))^*$ of) the operators U_{t_0} and $U_{t_0}^*$ are bounded operators from $L^1(M,\mu)$ to $L^2(M,\mu)$. Due to the semigroup property, the operators U_{2t_0} and $U_{2t_0}^*$ are bounded operators from $L^1(M,\mu)$ to $L^\infty(M,\mu)$; this is equivalent to saying

$$u_{2t_0}(\cdot,\cdot) \in L^\infty(M \times M, \mu \otimes \mu). \tag{2.2}$$

Using the semigroup property, these boundedness properties extend to all $t \geqslant t_0$, resp. $t \geqslant 2t_0$. Conversely, if there is some $t_0 > 0$, for which (2.2) holds, then the condition (**A3**) is also satisfied.

Let $t > 2t_0$. From (**A2**), we know that U_t and U_t^* are continuous operators from $L^\infty(M,\mu) \to L^\infty(M,\mu)$, and (**A3**) means that U_t and U_t^* are continuous operators from $L^2(M,\mu) \to L^\infty(M,\mu)$; by duality, we see that U_t and U_t^* map $L^1(M,\mu)$ continuously into $L^2(M,\mu)$, and using again (**A3**) and the semigroup property, we conclude that U_t and U_t^* are continuous from $L^1(M,\mu)$ into $L^\infty(M,\mu)$. Therefore, a standard interpolation argument shows that U_t and U_t^* are continuous $L^p(M,\mu) \to L^\infty(M,\mu)$ for any $p \in (1,\infty)$ and, in view of (**A2**), we also have

$$U_t(L^p(M,\mu)), U_t^*(L^p(M,\mu)) \subseteq C_b(M), \quad t > 2t_0, \, p \in [1,\infty]. \tag{2.3}$$

Let A, and A^*, denote the generators of semigroups $\{U_t : t \in T\}$, and $\{U_t^* : t \geqslant 0\}$, respectively. A version of the Jentzsch theorem (see [50, Theorem V.6.6]) shows that $\lambda_0 := \inf \mathrm{Re}\left(\mathrm{Spec}(-A)\right) = \inf \mathrm{Re}\left(\mathrm{Spec}(-A^*)\right)$ is common isolated and simple eigenvalue of $-A$ and $-A^*$. Moreover, the corresponding normalized eigenfunctions $\phi_0, \psi_0 \in L^2(M,\mu)$ can be assumed to be strictly positive. In particular,

$$U_t\phi_0 = e^{-\lambda_0 t}\phi_0 \quad \text{and} \quad U_t^*\psi_0 = e^{-\lambda_0 t}\psi_0, \quad t > 0. \tag{2.4}$$

The number λ_0 is called the *ground state eigenvalue*, and the functions ϕ_0, ψ_0, are the *ground states* of $-A$, and $-A^*$, respectively.

By (2.3), we have $\phi_0, \psi_0 \in C_b(M)$, and both eigenequations in (2.4) are also true in a pointwise sense. We will frequently use the following constant

$$\Lambda := \int \phi_0(x)\psi_0(x)\,\mu(dx)$$

and set

$$\mathcal{N} := \{x \in M : \phi_0(x) = 0\} \cup \{x \in M : \psi_0(x) = 0\}. \tag{2.5}$$

As mentioned above, $\mu(\mathcal{N}) = 0$. Due to the continuity of ϕ_0 and ψ_0, the set $\mathcal{N}$ is a closed set. If for some $t > 0$, the kernel $u_t(x,y)$ is defined pointwise (e.g., if the map $(x,y) \mapsto u_t(x,y)$ is continuous) and strictly positive for all $x,y \in M$, then $\mathcal{N} = \emptyset$. We are mainly interested in examples which come from the theory of stochastic processes, and for these examples, $\mathcal{N} = \emptyset$ is always satisfied. Since the original question comes from the L^2-theory, we do not want to assume *a priori* $\mathcal{N} = \emptyset$.

The following result is a straightforward adaptation of a Lemma by Zhang, Li and Song [65, Lemma 2.1]; it is a variant of the estimate which was originally obtained for intrinsic semigroups by Kim and Song [28, Theorem 2.7]. The very first result in this

direction is due to Pinsky [48, Theorem 3] for diffusion semigroups on bounded domains in $\mathbb{R}^d$. The result of Zhang et al. is valid for rather general nonsymmetric semigroups. These authors use sub-Markovian semigroups of standard Markov processes with strictly positive kernels of Hilbert-Schmidt type satisfying (**A3**). A close inspection of their proof reveals that only the compactness of the operators U_t and U_t^* is needed and that the result is valid in the more general setting, which is described by our present conditions (**A0**)–(**A3**). Note that Zhang et al. use $t_0 = 1$ and that their proof requires $t/2 > 1 = t_0$.

Lemma 2.2 [65, Lemma 2.1]. *If (**A0**)–(**A3**) hold for some $t_0 > 0$, then there exist $C, \gamma > 0$ such that*

$$\left| e^{\lambda_0 t} u_t(x,y) - (1/\Lambda)\phi_0(x)\psi_0(y) \right| \leqslant C e^{-\gamma t}, \quad t > 2t_0, \quad \text{for } \mu \otimes \mu \text{ a.a. } x, y \in M. \quad (2.6)$$

This estimate leads to a rate of convergence (as $t \to \infty$) for compact semigroups, which can be seen as a variant of the classical result of Chavel and Karp [7]; see also [59, 27].

Unless otherwise stated, we take the same $t_0 > 0$ in (A1), (A3) and (A4).

We will now briefly discuss the relation between (**A4**) and (**A0**).

Lemma 2.3. *Assume (**A2**) and (**A3**), (**A4**) for some $t_0 > 0$. Then the operators $U_t, U_t^* : L^2(M,\mu) \to L^2(M,\mu)$ are compact for all $t \geqslant t_0$. In particular, if (**A3**), (**A4**) are true for every $t_0 > 0$, then (**A0**) holds.*

Proof. We consider only the operators U_t. The proof for U_t^* is similar. For a compact set $K \subset M$, we define

$$U_t^K f(x) := \mathbb{1}_K(x) U_t f(x) = \int_M \mathbb{1}_K(x) u_t(x,y) f(y) \, \mu(dy), \quad f \in L^2(M,\mu), \ t > 0$$

(i.e., U_t^K is an integral operator with the kernel $u_t^K(x,y) = \mathbb{1}_K(x) u_t(x,y)$). Observe that by (**A3**),

$$\int_M \int_M u_t^K(x,y)^2 \, \mu(dy)\, \mu(dx) = \int_K \int_M u_t(x,y)^2 \, \mu(dy)\, \mu(dx)$$
$$\leqslant \mu(K) \cdot \operatorname{ess\,sup}_{x \in M} \int_M u_t(x,y)^2 \, \mu(dy) < \infty, \quad t \geqslant t_0,$$

which means that U_t^K, $t \geqslant t_0$, are Hilbert–Schmidt operators. Moreover, for $f \in L^2(M,\mu)$, by the Cauchy–Schwarz inequality and the Tonelli theorem,

$$\int_M \left| U_t f(x) - U_t^K f(x) \right|^2 \mu(dx) = \int_{M \setminus K} |U_t f(x)|^2 \, \mu(dx)$$
$$\leqslant \int_{M \setminus K} U_t \mathbb{1}_M(x) \int_M u_t(x,y) |f(y)|^2 \, \mu(dy)\, \mu(dx)$$
$$\leqslant \sup_{x \in M \setminus K} U_t \mathbb{1}_M(x) \, \|U_t^* \mathbb{1}_M\|_\infty \, \|f\|_2^2.$$

The assumptions (**A2**) and (**A4**) yield that for every $t > t_0$ and $\epsilon > 0$, we can find a compact set K such that

$$\left\| U_t f - U_t^K f \right\|_2 \leqslant \epsilon \| f \|_2.$$

Letting $\epsilon \to 0$, we see that every operator U_t, $t \geqslant t_0$, is a strong limit of compact operators, and hence compact. This completes the proof. $\qquad\square$

If the operators U_t, $t > 0$, are self-adjoint, then compactness for some $t > 0$ implies compactness for every $t > 0$ – this is due to spectral theorem. In that case, (**A4**) implies (**A0**), even if (**A3**), (**A4**) are true for some $t_0 > 0$ only. For non-self-adjoint operators, this is not clear. Therefore, in Sections 4 and 5, we have to assume both (**A0**) and (**A4**).

In unbounded spaces, it is often true that (**A0**) and (**A4**) (holding for all $t_0 > 0$) are equivalent see, for example, [38, Lemma 1] and [23, Lemma 9]. Some general sufficient conditions for compactness in various settings, including Markov, Schrödinger and Feynman–Kac semigroups, and more general perturbations can be found in [63, 61, 40].

3. Heat content and exponential quasi-ergodicity of compact semigroups

If $\phi_0, \psi_0 \in L^1(M, \mu)$,[1] we can define probability measures on $(M, \mathcal{B}(M))$ by

$$m(E) = \frac{\int_E \psi_0(x) \, \mu(dx)}{\int_M \psi_0(x) \, \mu(dx)} \quad \text{and} \quad m^*(E) = \frac{\int_E \phi_0(x) \, \mu(dx)}{\int_M \phi_0(x) \, \mu(dx)}. \tag{3.1}$$

It is rather easy to check that the measures m and m^* are *quasi-stationary* (*probability*) *measures* of the semigroups $\{U_t : t \geqslant 0\}$ and $\{U_t^* : t \geqslant 0\}$.

Definition 3.1 (*Quasi-stationary measure on L^∞*). Let (**A2**) hold. A probability measure $\bar{m}$ on $(M, \mathcal{B}(M))$ is said to be a *quasi-stationary measure* of the semigroup $\{U_t : t \geqslant 0\}$ (on $L^\infty(M, \mu)$), if for all $t > 0$ and $f \in L^\infty(M, \mu)$, we have $\bar{m}(U_t \mathbb{1}_M) > 0$ and

$$\frac{\bar{m}(U_t f)}{\bar{m}(U_t \mathbb{1}_M)} = \bar{m}(f). \tag{3.2}$$

Because of (2.3) – this follows from (**A2**), (**A3**) – we can extend the definition of quasi-stationarity to $L^p(M, \mu)$ for any $p \in [1, \infty)$, at least for large t.

Lemma 3.2. *Assume (A1)–(A3) for $t_0 > 0$ and let $p \in [1, \infty)$. Let $\bar{m}$ be a probability measure on $(M, \mathcal{B}(M))$ and let $t > 2t_0$ be such that $\bar{m}(U_t \mathbb{1}_M) > 0$. Then*

$$\forall_{g \in L^p(M,\mu)} \quad \frac{\bar{m}(U_t g)}{\bar{m}(U_t \mathbb{1}_M)} = \bar{m}(g) \quad \Longleftrightarrow \quad \forall_{f \in L^\infty(M,\mu)} \quad \frac{\bar{m}(U_t f)}{\bar{m}(U_t \mathbb{1}_M)} = \bar{m}(f).$$

Proof. Fix $p \in [1, \infty)$ and $t > 2t_0$. It is enough to show both implications for non-negative functions.

'$\Leftarrow$' Let $0 \leqslant g \in L^p(M, \mu)$ and $f_n := g \wedge n \in L^\infty(M, \mu)$, $n \in \mathbb{N}$. We have $f_n \uparrow g$ as $n \to \infty$, $\frac{\bar{m}(U_t f_n)}{\bar{m}(U_t \mathbb{1}_M)} = \bar{m}(f_n)$, $n \in \mathbb{N}$, and the implication follows with the monotone convergence theorem.

---

[1] In general, $\phi_0, \psi_0 \in L^1(M, \mu)$ need not be satisfied since μ can be an infinite measure.

'⇒' Let $0 \leqslant f \in L^\infty(M,\mu)$ and set $g_n := \mathbb{1}_{K_n} f \in L^p(M,\mu)$, $n \in \mathbb{N}$, where (K_n) is an increasing sequence of compact sets such that $\bigcup_{n \in \mathbb{N}} K_n = M$; see the discussion of topological preliminaries at the beginning of Section 2. We have $g_n \uparrow f$ as $n \to \infty$, $\frac{\bar{m}(U_t g_n)}{\bar{m}(U_t \mathbb{1}_M)} = \bar{m}(g_n)$, $n \in \mathbb{N}$, and we use monotone convergence once again. □

If (**A3**) holds for any $t_0 > 0$, then it also makes sense to speak about quasi-stationarity on L^p, $p < \infty$, for all $t > 0$. In particular, the implications in Lemma 3.2 hold true for every $t > 0$.

We come back to the measures m, m^* defined in (3.1). Here, the condition (3.2) reads

$$m(U_t f) = e^{-\lambda_0 t} m(f) \quad \text{and} \quad m^*(U_t^* f) = e^{-\lambda_0 t} m^*(f), \quad t > 0, \, f \in L^\infty(M,\mu). \quad (3.3)$$

Since we assume that $\phi_0, \psi_0 \in L^\infty(M,\mu)$, (3.3) remains valid for $f \in L^p(M,\mu)$, $p \in [1,\infty]$. This can be easily seen without Lemma 3.2. Indeed, it follows from the Hölder inequality and the fact that $\phi_0, \psi_0 \in L^1(M,\mu) \cap L^\infty(M,\mu) \subset L^q(M,\mu)$ for any $q \in [1,\infty]$.

Our arguments in Sections 3 and 5 are based on the following observation, which is a direct consequence of the estimate in Lemma 2.2.

Lemma 3.3. *Let (A0)–(A3) hold for some $t_0 > 0$ and denote by C, γ the constants appearing in Lemma 2.2. For every $0 \leqslant s, r < t$ with $t - s - r > 2t_0$*

$$\left| e^{\lambda_0 t} u_t(x,y) - (1/\Lambda) \phi_0(x) \psi_0(y) \right| \leqslant C \kappa(t,s,r,x,y), \quad \mu\text{-a.e. } x,y \in M, \quad (3.4)$$

holds true with the function

$$\kappa(t,s,r,x,y) = \begin{cases} e^{-\gamma t} & \text{if } s = r = 0; \\ e^{-\gamma(t-s)} e^{\lambda_0 s} U_s \mathbb{1}_M(x) & \text{if } s > 0, r = 0; \\ e^{-\gamma(t-r)} e^{\lambda_0 r} U_r^* \mathbb{1}_M(y) & \text{if } s = 0, r > 0; \\ e^{-\gamma(t-s-r)} e^{\lambda_0 s} U_s \mathbb{1}_M(x) e^{\lambda_0 r} U_r^* \mathbb{1}_M(y) & \text{if } s > 0, r > 0. \end{cases}$$

Proof. If $s = r = 0$, this is just the estimate from Lemma 2.2. We will only work out the case $s, r > 0$. The remaining two cases follow with very similar arguments. Using the eigenequations

$$\phi_0(x) = e^{\lambda_0 s} \int_M u_s(x,z) \phi_0(z) \, \mu(dz) \quad \text{and} \quad \psi_0(y) = e^{\lambda_0 r} \int_M u_r(w,y) \psi_0(w) \, \mu(dw),$$

and the Chapman-Kolmogorov identities, we see that

$$\begin{aligned} & e^{\lambda_0 t} u_t(x,y) - (1/\Lambda) \phi_0(x) \psi_0(y) \\ &= e^{\lambda_0 s} \int_M u_s(x,z) \left[e^{\lambda_0(t-s)} u_{t-s}(z,y) - (1/\Lambda) \phi_0(z) \psi_0(y) \right] \mu(dz) \\ &= e^{\lambda_0 s} \int_M u_s(x,z) \left[e^{\lambda_0 r} \int_M \left[e^{\lambda_0(t-s-r)} u_{t-s-r}(z,w) \right. \right. \\ & \qquad \left. \left. - (1/\Lambda) \phi_0(z) \psi_0(w) \right] u_r(w,y) \, \mu(dw) \right] \mu(dz). \end{aligned}$$

By Lemma 2.2, we have

$$\left| e^{\lambda_0(t-s-r)} u_{t-s-r}(z,w) - (1/\Lambda)\phi_0(z)\psi_0(w) \right| \leqslant C e^{-\gamma(t-s-r)},$$

which gives

$$\left| e^{\lambda_0 t} u_t(x,y) - (1/\Lambda)\phi_0(x)\psi_0(y) \right|$$
$$\leqslant C e^{-\gamma(t-s-r)} \left(e^{\lambda_0 s} \int_M u_s(x,z)\,\mu(dz) \right) \left(e^{\lambda_0 r} \int_M u_r(w,y)\,\mu(dw) \right).$$

This is the claimed bound. $\qquad\qquad\square$

Our next theorem is the first main result of this section. It relates the exponential quasi-ergodicity of the semigroups $\{U_t : t \geqslant 0\}$ and $\{U_t^* : t \geqslant 0\}$ with the finiteness of their heat content. It also shows that the corresponding quasi-stationary probability distributions are unique. Recall that

$$Z(t) := \int U_t \mathbb{1}_M(x)\,\mu(dx) = \int U_t^* \mathbb{1}_M(y)\,\mu(dy) = \iint u_t(x,y)\,\mu(dx)\,\mu(dy), \quad t > 0$$

is the *heat content* of $\{U_t : t \geqslant 0\}$ or $\{U_t^* : t \geqslant 0\}$. In general, $Z(t) \in (0,\infty]$. For each $t > 0$, the condition $Z(t) < \infty$ is equivalent to the boundedness of the operators $U_t, U_t^* : L^\infty(M,\mu) \to L^1(M,\mu)$.

Unless otherwise stated, we use the value $t_0 > 0$ from Lemma 3.3 in all further statements as well as for the assumptions (A1), (A3) and (A4), and $\mathcal{N}$ denotes the exceptional set defined in (2.5).

Theorem 3.4. *Let M be a locally compact Polish space and assume that (A0)–(A3) hold. If there exists some $t_1 \geqslant t_0$ such that the heat content is finite, that is,*

$$Z(t_1) < \infty,$$

then we have the following:

a) *$\phi_0, \psi_0 \in L^1(M,\mu)$ and both measures m and m^* in (3.1) are well defined.*

b) *For every $p \in [1,\infty]$, there exists $C > 0$ such that for every finite measure σ on M with $\operatorname{supp}\sigma \cap (M \setminus \mathcal{N}) \neq \emptyset$, we have for all $t > 6t_1$ and $t_1 \leqslant s \leqslant t/2$,*

$$\sup_{\substack{f \in L^p(M,\mu) \\ \|f\|_p \leqslant 1}} \left| \frac{\sigma(U_t f)}{\sigma(U_t \mathbb{1}_M)} - m(f) \right| \leqslant C e^{-\gamma(t-s)} \frac{e^{\lambda_0 s} \sigma(U_s \mathbb{1}_M)}{\sigma(\phi_0)},$$

and

$$\sup_{\substack{f \in L^p(M,\mu) \\ \|f\|_p \leqslant 1}} \left| \frac{\sigma(U_t^* f)}{\sigma(U_t^* \mathbb{1}_M)} - m^*(f) \right| \leqslant C e^{-\gamma(t-s)} \frac{e^{\lambda_0 s} \sigma(U_s^* \mathbb{1}_M)}{\sigma(\psi_0)}.$$

c) *The measures m and m^* are the unique quasi-stationary probability measures of $\{U_t : t \geqslant 0\}$ and $\{U_t^* : t \geqslant 0\}$, respectively, such that $m(M \setminus \mathcal{N}) \cdot m^*(M \setminus \mathcal{N}) > 0$.*

Proof. Recall that $U_t \mathbb{1}_M, U_t^* \mathbb{1}_M \in L^1(M, \mu)$, $t \geqslant t_1$. Thus, a) follows from the inequalities $e^{-\lambda_0 t} \phi_0(x) \leqslant \|\phi_0\|_\infty U_t \mathbb{1}_M(x)$ and $e^{-\lambda_0 t} \psi_0(x) \leqslant \|\psi_0\|_\infty U_t^* \mathbb{1}_M(x)$, $x \in M$, $t > 0$.

We prove b) only for the semigroup $\{U_t : t \geqslant 0\}$. The proof for $\{U_t^* : t \geqslant 0\}$ is literally the same. Let $p \in [1, \infty]$ and let σ be a finite measure on M such that $\mathrm{supp}\, \sigma \cap (M \setminus \mathcal{N}) \neq \emptyset$. Note that $0 < \sigma(\phi_0) \leqslant e^{\lambda_0 t} \|\phi_0\|_\infty \sigma(U_t \mathbb{1}_M)$ and $0 < \sigma(\psi_0) \leqslant e^{\lambda_0 t} \|\psi_0\|_\infty \sigma(U_t^* \mathbb{1}_M)$, $t > 0$. For every $f \in L^p(M, \mu)$ and $t > 6t_1$ (in particular, $t - t_1 - s > 2t_0$), we may write

$$
\begin{aligned}
\frac{\sigma(U_t f)}{\sigma(U_t \mathbb{1}_M)} - m(f) &= \frac{e^{\lambda_0 t} \sigma(U_t f) - m(f) e^{\lambda_0 t} \sigma(U_t \mathbb{1}_M)}{e^{\lambda_0 t} \sigma(U_t \mathbb{1}_M)} \\
&= \frac{e^{\lambda_0 t} \sigma(U_t f) - (1/\Lambda) \sigma(\phi_0) \int_M f(y) \psi_0(y) \, \mu(dy)}{e^{\lambda_0 t} \sigma(U_t \mathbb{1}_M)} \\
&\quad + m(f) \frac{(1/\Lambda) \sigma(\phi_0) \int_M \psi_0(y) \, \mu(dy) - e^{\lambda_0 t} \sigma(U_t \mathbb{1}_M)}{e^{\lambda_0 t} \sigma(U_t \mathbb{1}_M)}.
\end{aligned}
$$

Since for all $x \in M$,

$$
\begin{aligned}
e^{\lambda_0 t} U_t f(x) &- (1/\Lambda) \phi_0(x) \int_M f(y) \psi_0(y) \, \mu(dy) \\
&= \int_M f(y) \left(e^{\lambda_0 t} u_t(x, y) - (1/\Lambda) \phi_0(x) \psi_0(y) \right) \mu(dy),
\end{aligned}
$$

and

$$
\begin{aligned}
e^{\lambda_0 t} U_t \mathbb{1}_M(x) &- (1/\Lambda) \phi_0(x) \int_M \psi_0(y) \, \mu(dy) \\
&= \int_M \left(e^{\lambda_0 t} u_t(x, y) - (1/\Lambda) \phi_0(x) \psi_0(y) \right) \mu(dy),
\end{aligned}
$$

we can apply Lemma 3.3 (with $t_1 \leqslant s \leqslant t/2$ and $r = t_1$) and integrate with respect to σ to get

$$
\begin{aligned}
\left| e^{\lambda_0 t} \sigma(U_t f) - (1/\Lambda) \sigma(\phi_0) \int_M f(y) \psi_0(y) \, \mu(dy) \right| \\
\leqslant C e^{-\gamma(t - t_1 - s)} e^{\lambda_0 (t_1 + s)} \sigma(U_s \mathbb{1}_M) \int_M |f(y)| U_{t_1}^* \mathbb{1}_M(y) \, \mu(dy),
\end{aligned}
$$

and

$$
\begin{aligned}
\left| e^{\lambda_0 t} \sigma(U_t \mathbb{1}_M) - (1/\Lambda) \sigma(\phi_0) \int_M \psi_0(y) \, \mu(dy) \right| \\
\leqslant C e^{-\gamma(t - t_1 - s)} e^{\lambda_0 (t_1 + s)} \sigma(U_s \mathbb{1}_M) Z(t_1).
\end{aligned}
$$

Note that

$$
e^{-\lambda_0 t} \sigma(\phi_0) = \sigma(U_t \phi_0) \leqslant \|\phi_0\|_\infty \sigma(U_t \mathbb{1}_M) \implies e^{\lambda_0 t} \sigma(U_t \mathbb{1}_M) \geqslant \frac{\sigma(\phi_0)}{\|\phi_0\|_\infty}. \tag{3.5}
$$

This gives

$$
\left| \frac{\sigma(U_t f)}{\sigma(U_t \mathbb{1}_M)} - m(f) \right| \leqslant \widetilde{C} e^{-\gamma(t - s)} \left(\int_M |f(y)| U_{t_1}^* \mathbb{1}_M(y) \, \mu(dy) + m(|f|) Z(t_1) \right) \frac{e^{\lambda_0 s} \sigma(U_s \mathbb{1}_M)}{\sigma(\phi_0)},
$$

with $\widetilde{C} := Ce^{(\lambda_0+\eta)t_1} \|\phi_0\|_\infty$, where C,γ are the constants from Lemma 3.3. Finally, by the Hölder inequality,

$$\left| \frac{\sigma(U_t f)}{\sigma(U_t \mathbb{1}_M)} - m(f) \right| \leqslant \widetilde{C} \left(\left\| U_{t_1}^* \mathbb{1}_M \right\|_q + Z(t_1) \frac{\|\psi_0\|_q}{\|\psi_0\|_1} \right) e^{-\gamma(t-s)} \frac{\sigma(U_s \mathbb{1}_M)}{\sigma(\phi_0)} \|f\|_p,$$

where q and p are conjugate exponents. Notice that $\left\| U_{t_1}^* \mathbb{1}_M \right\|_q < \infty$ and $\|\psi_0\|_q < \infty$. Indeed, if $p=1$, then $q=\infty$, and the claim follows from (**A2**) and (**A3**) combined with (2.4). If $p=\infty$ and $q=1$, then the claim follows from the fact that $\left\| U_{t_1}^* \mathbb{1}_M \right\|_1 = Z(t_1) < \infty$ and from a). If $1 < p < \infty$, then also $1 < q < \infty$, and we have

$$\left\| U_{t_1}^* \mathbb{1}_M \right\|_q^q = \int_M U_{t_1}^* \mathbb{1}_M(x) (U_{t_1}^* \mathbb{1}_M(x))^{q-1} \mu(dx) \leqslant \left\| U_{t_1}^* \mathbb{1}_M \right\|_\infty^{q-1} Z(t_1) < \infty$$

and

$$\|\psi_0\|_q^q = \int_M \psi_0(x) \psi_0^{q-1}(x) \mu(dx) \leqslant \|\psi_0\|_\infty^{q-1} \|\psi_0\|_1 < \infty.$$

This completes the proof of b).

In order to show c), assume that both m and $\bar{m}$ are quasi-stationary probability measures and $\operatorname{supp} \bar{m} \cap (M \setminus \mathcal{N}) \neq \emptyset$. Since $\bar{m}$ is finite, we can use $\bar{m} = \sigma$ in Part b) and we get, because of the quasi-stationary property (3.2),

$$0 = \lim_{t\to\infty} \left| \frac{\bar{m}(U_t f)}{\bar{m}(U_t \mathbb{1}_M)} - m(f) \right| = |\bar{m}(f) - m(f)|$$

(uniformly) for all $f \in L^\infty(M,\mu)$ with $\|f\|_\infty \leqslant 1$. By our assumptions, μ has full topological support, it is locally finite – hence finite on all compact sets – and the compact sets generate the Borel σ-algebra on M. Moreover, M is σ-compact (i.e., there is an increasing sequence of compact sets K_n such that $M = \bigcup_{n\geqslant 1} K_n$). Taking $f = \mathbb{1}_K$ for any compact set K, we conclude that $m = \bar{m}$ first on the compact sets and, using the standard uniqueness theorem for measures, on all Borel sets. The proof of the uniqueness of m^* is similar. $\square$

Corollary 3.5. *Let M be a locally compact Polish space and assume that (**A0**)–(**A3**) hold, and assume that there exists some $t_1 \geqslant t_0$ such that $Z(t_1) < \infty$.*

Both semigroups $\{U_t : t \geqslant 0\}$ and $\{U_t^* : t \geqslant 0\}$ are quasi-ergodic with exponential time rate $e^{-\gamma t}$ and space rates $U_{t_1} \mathbb{1}_M(x)/\phi_0(x)$ and $U_{t_1}^* \mathbb{1}_M(x)/\psi_0(x)$, respectively. More precisely, for every $p \in [1,\infty]$, there is a constant $C > 0$ such that for every $f \in L^p(M,\mu)$ with $\|f\|_p \leqslant 1$,

$$\left| \frac{U_t f(x)}{U_t \mathbb{1}_M(x)} - m(f) \right| \leqslant C \, e^{-\gamma t} \frac{U_{t_1} \mathbb{1}_M(x)}{\phi_0(x)}, \quad t > 6t_1, \ x \in M \setminus \mathcal{N},$$

and

$$\left| \frac{U_t^* f(x)}{U_t^* \mathbb{1}_M(x)} - m^*(f) \right| \leqslant C \, e^{-\gamma t} \frac{U_{t_1}^* \mathbb{1}_M(x)}{\psi_0(x)}, \quad t > 6t_1, \ x \in M \setminus \mathcal{N}.$$

Moreover, the measures m, and m^*, are the only quasi-stationary measures of $\{U_t : t \geqslant 0\}$, and $\{U_t^* : t \geqslant 0\}$, respectively, such that $m(M \setminus \mathcal{N}) \cdot m^*(M \setminus \mathcal{N}) > 0$.

Note that, since $U_t \mathbb{1}_M, U_t^* \mathbb{1}_M \in C_b(M)$ and ϕ_0, ψ_0 are continuous and strictly positive on M, the exponential quasi-ergodicity in Corollary 3.5 is uniform for all x in compact subsets of $M \setminus \mathcal{N}$.

If $\mu(M) < \infty$, then the heat content is automatically finite for large times, and we recover some of the results from [65].

4. Progressive ground state domination and heat content

In this section, we will need the notions of *hemi-compact space* and *exhausting* family of sets.

Definition 4.1. Let M be a topological space. An *exhausting family of compact sets* is a family of compact sets $\{K_t \subset M : t \geqslant t_1\}$ (defined for some $t_1 \geqslant 0$) such that $K_t \subset K_s$, for every $t_1 \leqslant t < s$ and $\bigcup_{t \geqslant t_1} K_t = M$.

The space M is called *hemi-compact* if there is an exhausting family of compact sets $\{K_t : t \geqslant t_1\}$, and for every compact set $K \subset M$, there exists some $t \geqslant t_1$ such that $K \subset K_t$.

Since M is a locally compact Polish space, M is hemi-compact; moreover, the concepts of σ-compactness and hemi-compactness coincide in such spaces; see, for example, [18, p. 195]. If M is compact, it is also hemi-compact with a trivial exhausting family $\{K_t : t \geqslant t_1\}$ such that $K_t = M$ for all $t \geqslant t_1$. Assume now that **(A0)**–**(A3)** hold with some $t_0 > 0$. The following properties will be central to our investigations in this section.

Definition 4.2 (aGSD and pGSD)**.** Let $\{U_t : t \geqslant 0\}$ be a semigroup of operators on $L^2(M, \mu)$ satisfying the conditions **(A0)**–**(A3)** and let λ_0 and ϕ_0 be the corresponding ground state eigenvalue and normalized eigenfunction.

a) The semigroup $\{U_t : t \geqslant 0\}$ is said to be *asymptotically ground state dominated* (aGSD, for short) if there exist constants $C, t_1 > 0$ such that

$$U_t \mathbb{1}_M(x) \leqslant C e^{-\lambda_0 t} \phi_0(x), \quad t \geqslant t_1, \, x \in M. \tag{4.1}$$

b) The semigroup $\{U_t : t \geqslant 0\}$ is said to be *progressively ground state dominated* (pGSD, for short) if there exist constants $C, t_1 > 0$ and an exhausting family of compact sets $\{K_t : t \geqslant t_1\}$ such that

$$\mathbb{1}_{K_t}(x) \cdot U_t \mathbb{1}_M(x) \leqslant C e^{-\lambda_0 t} \phi_0(x), \quad t \geqslant t_1, \, x \in M. \tag{4.2}$$

Remark 4.3.

a) Because of the eigenequation $e^{-\lambda_0 t} \phi_0 = U_t \phi_0$, there is some kind of reverse inequality to (4.1): $e^{-\lambda_0 t} \phi_0(x) \leqslant \|\phi_0\|_\infty U_t \mathbb{1}_M(x)$, $x \in M$, $t > 0$.

b) Notice that both $U_t \mathbb{1}_M(x)$ and $\phi_0(x)$ are continuous functions. Since $\mu(\mathcal{N}) = 0$ and since μ charges every open set, the interior of $\mathcal{N}$ is empty and $\overline{M \setminus \mathcal{N}} = M$. This means that we can replace in (4.1) and (4.2) the condition '$x \in M$' by '$x \in M \setminus \mathcal{N}$'.

c) aGSD was studied in the papers [25, 24] in relation to intrinsic ultracontractivity and hypercontractivity. It is shown in [26] that aGSD is a much more restrictive condition than the finiteness of heat content for large times. It follows directly from Lemma 3.3 that aGSD of $\{U_t : t \geqslant 0\}$ implies $\phi_0 \in L^1(M, \mu)$ as well as the finiteness of the heat content for large times; this is also true for $\{U_t^* : t \geqslant 0\}$ and ψ_0.

d) pGSD was recently introduced in [26] as a direct consequence of the concept of progressive intrinsic ultracontractivity (pIUC) for Schrödinger semigroups. In this and the next section, we will explain the impact of pGSD on the quasi-ergodic behavior of compact semigroups and discuss the relation between pGSD and the finiteness of the heat content.

e) From the definition, we see that aGSD always implies pGSD. Recall that $U_t \mathbb{1}_M$, $U_t^* \mathbb{1}_M$, ϕ_0, $\psi_0 \in C_b(M)$. If the space M is compact and ϕ_0, ψ_0 are strictly positive everywhere (i.e., $\mathcal{N} = \emptyset$), then $\inf_M \phi_0 > 0$ and $\inf_M \psi_0 > 0$; therefore, aGSD always holds. In particular, pGSD holds with an exhausting family $\{K_t : t \geqslant t_1\}$ such that $K_t = M$ for all $t \geqslant t_1$.

We first show that aGSD of $\{U_t : t \geqslant 0\}$ implies the exponential uniform (for the whole space M) quasi-ergodicity of this semigroup on every $L^p(M, \mu)$, $p \geqslant 1$; as before, this also holds for $\{U_t^* : t \geqslant 0\}$. In particular, this gives a relatively short and elementary proof of the main results of [31, Theorem 1, Corollary 2] in our framework. We also have some converse statement.

Corollary 4.4. *Let M be a general locally compact (but not necessarily compact) Polish space and assume that $(A0)$–$(A3)$ hold. Then we have the following assertions.*

a) *If the semigroup $\{U_t : t \geqslant 0\}$ is aGSD for some $t_1 \geqslant t_0$, then there is for every $p \in [1, \infty]$ a constant $C > 0$, such that for every $f \in L^p(M, \mu)$ with $\|f\|_p \leqslant 1$,*

$$\sup_{x \in M \setminus \mathcal{N}} \left| \frac{U_t f(x)}{U_t \mathbb{1}_M(x)} - m(f) \right| \leqslant C e^{-\gamma t}, \quad t > 6t_1. \tag{4.3}$$

b) *If there is a constant $C > 0$ such that*

$$\sup_{x \in M \setminus \mathcal{N}} \left| \frac{U_t \phi_0(x)}{U_t \mathbb{1}_M(x)} - C \right| \to 0 \quad as \quad t \to \infty,$$

then the semigroup $\{U_t : t \geqslant 0\}$ is aGSD.

Both assertions a) and b) are also true for the semigroup $\{U_t^* : t \geqslant 0\}$ and the quasi-stationary measure m^*. Under aGSD, the measures m, and m^*, are the only quasi-stationary measures of $\{U_t : t \geqslant 0\}$, and $\{U_t^* : t \geqslant 0\}$, respectively, such that $m(M \setminus \mathcal{N}) \cdot m^*(M \setminus \mathcal{N}) > 0$.

Proof. Assertion a) follows directly from a combination of Remark 4.3.c) and Corollary 3.5. Let us show b). Because of the eigenequation $U_t \phi_0 = e^{-\lambda_0 t} \phi_0$, we see that there is some $t_1 > 0$ such that

$$\left| \frac{e^{-\lambda_0 t} \phi_0(x)}{U_t \mathbb{1}_M(x)} - C \right| \leqslant \frac{C}{2}, \quad t \geqslant t_1, \ x \in M \setminus \mathcal{N},$$

which implies

$$\frac{e^{-\lambda_0 t}\phi_0(x)}{U_t \mathbb{1}_M(x)} \geqslant \frac{C}{2}, \quad t \geqslant t_1, \ x \in M \setminus \mathcal{N}.$$

Since $C > 0$ and both functions ϕ_0 and $U_t \mathbb{1}_M$ are bounded and continuous, this completes the proof. $\qquad \square$

As explained in Remark 4.3.e), if the space M is compact (hence, $\mu(M) < \infty$, due to the local finiteness of μ) and the ground states are everywhere strictly positive, the semigroups $\{U_t : t \geqslant 0\}$, $\{U_t^* : t \geqslant 0\}$ are automatically aGSD, and the heat content is finite for large times. In particular, Corollary 4.4 implies that both semigroups are exponentially uniformly (on the whole space M) quasi-ergodic.

If the space M is non-compact, then aGSD is a rather restrictive condition. However, as illustrated in [26], one should expect that in this case, a wide range of non-aGSD compact semigroups are still pGSD. From now on, we focus on the quasi-ergodicity of pGSD semigroups in the setting of non-compact spaces.

We first show that the finiteness of the heat content implies pGSD for both $\{U_t : t \geqslant 0\}$ and $\{U_t^* : t \geqslant 0\}$; consequently, we get progressive uniform quasi-ergodicity with an exponential time-rate. To do so, we need a lemma. To keep the proofs simple, we assume in Lemma 4.5 and Theorem 4.6 that $\mathcal{N} = \emptyset$ (i.e., that the ground states are everywhere strictly positive).

Lemma 4.5. *Let M be a locally compact, but not compact, Polish space. Assume that (**A0**)–(**A4**) hold, $\mathcal{N} = \emptyset$, and let $\{K_t : t \geqslant t_0\}$ be an exhausting family of compact sets in M. Set*

$$h(t) := \min \left\{ \inf_{x \in K_t} \phi_0(x), \ \inf_{x \in K_t} \psi_0(x) \right\}, \quad t \geqslant t_0,$$

and consider the generalized inverse of h,

$$h^{-1}(s) := \inf \{ t \geqslant t_0 : h(t) < s \}, \quad s \in (0, h(t_0)].$$

Define the function $\eta : [-\gamma^{-1} \log h(t_0), \infty) \to [t_0, \infty)$ by

$$\eta(t) := h^{-1}(s)\big|_{s = e^{-\gamma t}} = h^{-1}(e^{-\gamma t}). \tag{4.4}$$

Then η is increasing on $[-\gamma^{-1} \log h(t_0), \infty)$, $\eta(t) \to \infty$ as $t \to \infty$ and

$$e^{\gamma t} \min \left\{ \inf_{x \in K_{\eta(t)}} \phi_0(x), \ \inf_{x \in K_{\eta(t)}} \psi_0(x) \right\} = e^{\gamma t} h(\eta(t)) = 1, \qquad t \geqslant -\gamma^{-1} \log h(t_0).$$

Proof. Because of the eigenequations, our assumptions (**A1**)–(**A3**) and $\mathcal{N} = \emptyset$, the ground state eigenfunctions ϕ_0 and ψ_0 are continuous and strictly positive on M. Together with the inequality in Remark 4.3.a) and the assumption (**A4**), this implies that the function h is continuous, strictly positive and decreasing on $[t_0, \infty)$ such that $h(t) \to 0$ as $t \uparrow \infty$. In particular, its generalized inverse function h^{-1} is decreasing on $(0, h(t_0)]$,

$h^{-1}(s) \to \infty$ as $s \downarrow 0$, and we have $h(h^{-1}(s)) = s$ for every $s \in (0, h(t_0)]$. Therefore, η is increasing on $[-\gamma^{-1} \log h(t_0), \infty)$ as it is a composition of two decreasing functions h^{-1} and $t \mapsto e^{-\gamma t}$, $\eta(t) \to \infty$ as $t \to \infty$, and $e^{\gamma t} h(\eta(t)) = e^{\gamma t} h(h^{-1}(e^{-\gamma t})) = e^{\gamma t} e^{-\gamma t} = 1$. This completes the proof. $\qquad\square$

In the following theorem, we will use the notation $\mathcal{M}^1(K)$ to denote all probability measures σ on $(M, \mathcal{B}(M))$ such that $\mathrm{supp}\,\sigma \subset K$. Recall that we already know from Theorem 3.4 that the finiteness of the heat content for large times implies that $\phi_0, \psi_0 \in L^1(M, \mu)$ and that the measures m, and m^*, defined in (3.1) are the only quasi-stationary measures on each $L^p(M, \mu)$, $p \in [1, \infty]$, of the semigroups $\{U_t : t \geqslant 0\}$, and $\{U_t^* : t \geqslant 0\}$, respectively.

Theorem 4.6. *Let M be a locally compact, but not compact, Polish space. Assume that (A0)–(A4) hold, let $\{K_t : t \geqslant t_0\}$ be an exhausting family of compact sets in M, and assume that $\mathcal{N} = \emptyset$. Set $\widetilde{t}_0 = \max\{0, -\gamma^{-1} \log h(t_0)\}$ and let η be the function defined in (4.4). If there is some $t_1 \geqslant t_0$ such that the heat content $Z(t_1) < \infty$, then the following assertions hold.*

a) *Both semigroups $\{U_t : t \geqslant 0\}$ and $\{U_t^* : t \geqslant 0\}$ are pGSD for the exhausting family of sets $\{K_{\eta(t)} : t \geqslant \widetilde{t}_0\}$.*

b) *Both semigroups $\{U_t : t \geqslant 0\}$ and $\{U_t^* : t \geqslant 0\}$ are exponentially progressively uniformly quasi-ergodic; that is, for every $p \in [1, \infty]$, there exists some $C > 0$, such that*

$$\sup_{\sigma \in \mathcal{M}^1(K_{\eta(t/2)})} \sup_{\substack{f \in L^p(M, \mu) \\ \|f\|_p \leqslant 1}} \left| \frac{\sigma(U_t f)}{\sigma(U_t \mathbb{1}_M)} - m(f) \right| \leqslant C e^{-(\gamma/2)t}, \quad t > 6t_1,$$

and

$$\sup_{\sigma \in \mathcal{M}^1(K_{\eta(t/2)})} \sup_{\substack{f \in L^p(M, \mu) \\ \|f\|_p \leqslant 1}} \left| \frac{\sigma(U_t^* f)}{\sigma(U_t^* \mathbb{1}_M)} - m^*(f) \right| \leqslant C e^{-(\gamma/2)t}, \quad t > 6t_1.$$

Proof. Recall that $\psi_0(x) \leqslant \|\psi_0\|_\infty e^{\lambda_0 t} U_t^* \mathbb{1}_M(x)$ and $\phi_0(x) \leqslant \|\phi_0\|_\infty e^{\lambda_0 t} U_t \mathbb{1}_M(x)$. Since $Z(t_1) < \infty$, this yields $\phi_0, \psi_0 \in L^1(M, \mu)$. Therefore, Lemma 3.3 (with $s = 0$ and $r = t_1$) and Lemma 4.5 give

$$\begin{aligned}
\left| e^{\lambda_0 t} U_t \mathbb{1}_M(x) - (1/\Lambda) \phi_0(x) \|\psi_0\|_1 \right| &\leqslant \int_M \left| e^{\lambda_0 t} u_t(x, y) - (1/\Lambda) \phi_0(x) \psi_0(y) \right| \mu(dy) \\
&\leqslant C e^{-\gamma(t - t_1)} Z(t_1) \\
&\leqslant C e^{\gamma t_1} Z(t_1) \frac{\phi_0(x)}{e^{\gamma t} \inf_{x \in K_{\eta(t)}} \phi_0(x)} \\
&\leqslant c_1 \phi_0(x),
\end{aligned}$$

for every $x \in K_{\eta(t)}$ and sufficiently large t; since the function η is increasing and $\eta(t) \to \infty$ as $t \to \infty$, the family $\{K_{\eta(t)} : t \geqslant \widetilde{t_0}\}$ is indeed exhausting. This gives

$$U_t \mathbb{1}_M(x) \leqslant (c_1 + (1/\Lambda)\|\psi_0\|_1)e^{-\lambda_0 t}\phi_0(x), \quad x \in K_{\eta(t)},$$

for sufficiently large t, showing that the semigroup $\{U_t : t \geqslant 0\}$ is pGSD. The pGSD property for the semigroup $\{U_t^* : t \geqslant 0\}$ follows with exactly the same argument. This completes the proof of a).

Part b) follows from Theorem 3.4.b) with $s = t/2$ and probability measures σ supported in $K_{\eta(t/2)}$, by an application of the pGSD property which we have already established in Part a). □

5. Progressive ground state domination and quasi-ergodicity

As we have already noticed in the previous section, in a non-compact space M, aGSD is stronger than the finiteness of heat content for large times. However, the finiteness of the heat content for large times implies pGSD with a certain exhausting family of sets. Recall that these properties – aGSD, pGSD, finite heat content – automatically hold in compact spaces. Consequently, we also have uniform quasi-ergodicity at an exponential time-rate; see Remark 4.3.e), Corollary 4.4, and the comments following the corollary.

On the other hand, see [26], pGSD is a general regularity property for compact non-aGSD semigroups, which can be studied on its own; in particular, pGSD is still true for a wide range of compact semigroups having infinite heat content for large times. In the present section, we follow this path, introducing the concept of progressive uniform quasi-ergodicity and studying its relation to pGSD.

Let $a, b \in (0,1)$ be arbitrary numbers such that $a + 2b = 1$. We show first that in a noncompact space M and under (**A4**), pGSD for the exhausting family $\{K_t : t \geqslant t_0\}$ implies progressive uniform quasi-ergodicity for the exhausting family $\{K_{at} : t \geqslant t_0/a\}$; the time-rate is given by

$$\kappa_b(t) := e^{-\gamma bt} + \sup_{x \in K_{bt}^c} U_{t_0}\mathbb{1}_M(x) + \sup_{x \in K_{bt}^c} U_{t_0}^*\mathbb{1}_M(x). \tag{5.1}$$

Note that $\kappa_b(t) \downarrow 0$ as $t \uparrow \infty$. Quite often (see the examples), κ_b decays exponentially at infinity.

We have to make sure that $\phi_0, \psi_0 \in L^1(M, \mu)$, which means that both measures m and m^* in (3.1) are well defined. If the space M is compact, this follows from $\mu(M) < \infty$ and $\phi_0, \psi_0 \in L^\infty(M, \mu)$ due to (**A3**). For locally compact (but non-compact) Polish spaces M, this is still true under aGSD; see Remark 4.3.c). Now we show that $\phi_0, \psi_0 \in L^1(M, \mu)$ holds if both semigroups $\{U_t : t \geqslant 0\}$ and $\{U_t^* : t \geqslant 0\}$ enjoy the pGSD property.

Lemma 5.1. *Let M be a locally compact, but not compact, Polish space and assume* (**A0**)–(**A3**). *If the semigroups $\{U_t : t \geqslant 0\}$ and $\{U_t^* : t \geqslant 0\}$ are pGSD, then $\phi_0, \psi_0 \in L^1(M, \mu)$.*

Proof. Suppose that pGSD holds for the exhausting family of compact sets $\{K_t : t \geqslant t_0\}$. We only show that $\phi_0 \in L^1(M, \mu)$. The proof for ψ_0 is similar. By Lemma 3.3 (with

$t_0 < s < t$ such that $t - s > 2t_0$ and $r = 0$) and pGSD of $\{U_t : t \geqslant 0\}$, we have for μ-almost all $x \in K_s$ and $y \in K_s \setminus \mathcal{N}$

$$
\begin{aligned}
e^{\lambda_0 t} u_t(x,y) &\geqslant (1/\Lambda)\phi_0(x)\psi_0(y) - Ce^{-\gamma(t-s)}U_s \mathbb{1}_M(x) \\
&\geqslant \left((1/\Lambda)\psi_0(y) - c_1 e^{-\gamma(t-s)-\lambda_0 s} \right)\phi_0(x).
\end{aligned}
$$

We integrate on both sides of the inequality in $x \in K_s$ and use pGSD of $\{U_t^* : t \geqslant 0\}$ to get

$$
\int_{K_s} \phi_0(x)\,\mu(dx) \leqslant \frac{e^{\lambda_0 t} U_t^* \mathbb{1}_M(y)}{(1/\Lambda)\psi_0(y) - c_1 e^{-\gamma(t-s)-\lambda_0 s}} \leqslant \frac{c_2 \psi_0(y)}{(1/\Lambda)\psi_0(y) - c_1 e^{-\gamma(t-s)-\lambda_0 s}},
$$

for sufficiently large t. Letting $t \to \infty$, we obtain

$$
\int_{K_s} \phi_0(x)\,\mu(dx) \leqslant c_2 \Lambda \quad \text{for all} \ \ s > t_0,
$$

and monotone convergence finally proves $\phi_0 \in L^1(M,\mu)$. $\qquad\square$

Theorem 5.2. *Let M be a locally compact, but not compact, Polish space. Assume that (A0)–(A4) hold and that both semigroups $\{U_t : t \geqslant 0\}$ and $\{U_t^* : t \geqslant 0\}$ are pGSD for the exhausting family of compact sets $\{K_t : t \geqslant t_0\}$. Then we have $\phi_0, \psi_0 \in L^1(M,\mu)$ and, for every $a,b \in (0,1)$ such that $a + 2b = 1$, both semigroups $\{U_t : t \geqslant 0\}$ and $\{U_t^* : t \geqslant 0\}$ are progressively uniformly quasi-ergodic for the exhausting family $\{K_{at} : t \geqslant t_0/a\}$ with time-rate $\kappa_b(t)$ given by (5.1); that is, for every $p \in [1,\infty]$, there exists some $C > 0$ such that for all $t > (4/(a \wedge b))t_0$,*

$$
\sup_{\substack{\sigma \in \mathcal{M}^1(K_{at}) \\ \operatorname{supp}\sigma \cap (M \setminus \mathcal{N}) \neq \emptyset}} \ \sup_{\substack{f \in L^p(M,\mu) \\ \|f\|_p \leqslant 1}} \left| \frac{\sigma(U_t f)}{\sigma(U_t \mathbb{1}_M)} - m(f) \right| \leqslant C\kappa_b(t),
$$

and

$$
\sup_{\substack{\sigma \in \mathcal{M}^1(K_{at}) \\ \operatorname{supp}\sigma \cap (M \setminus \mathcal{N}) \neq \emptyset}} \ \sup_{\substack{f \in L^p(M,\mu) \\ \|f\|_p \leqslant 1}} \left| \frac{\sigma(U_t^* f)}{\sigma(U_t^* \mathbb{1}_M)} - m^*(f) \right| \leqslant C\kappa_b(t).
$$

Proof. In view of Lemma 5.1 we have $\phi_0, \psi_0 \in L^1(M,\mu)$. We proceed as in the proof of Theorem 3.4. Since the proof for the dual semigroup is similar, we restrict our attention to $\{U_t : t \geqslant 0\}$. Fix $p \in [1,\infty]$. For every $f \in L^p(M,\mu)$, all $t > (4/(a \wedge b))t_0$ and $\sigma \in \mathcal{M}^1(K_{at})$ such that $\operatorname{supp}\sigma \cap (M \setminus \mathcal{N}) \neq \emptyset$, we have

$$
\begin{aligned}
\frac{\sigma(U_t f)}{\sigma(U_t \mathbb{1}_M)} - m(f) &= \frac{e^{\lambda_0 t}\sigma(U_t f) - m(f)e^{\lambda_0 t}\sigma(U_t \mathbb{1}_M)}{e^{\lambda_0 t}\sigma(U_t \mathbb{1}_M)} \\
&= \frac{e^{\lambda_0 t}\sigma(U_t f) - (1/\Lambda)\,\sigma(\phi_0)\int_M f(y)\psi_0(y)\,\mu(dy)}{e^{\lambda_0 t}\sigma(U_t \mathbb{1}_M)} \\
&\quad + m(f)\frac{(1/\Lambda)\,\sigma(\phi_0)\int_M \psi_0(y)\,\mu(dy) - e^{\lambda_0 t}\sigma(U_t \mathbb{1}_M)}{e^{\lambda_0 t}\sigma(U_t \mathbb{1}_M)}
\end{aligned} \tag{5.2}
$$

as well as

$$e^{\lambda_0 t} U_t f(x) - (1/\Lambda)\, \phi_0(x) \int_M f(y)\psi_0(y)\,\mu(dy)$$
$$= \int_M \left(e^{\lambda_0 t} u_t(x,y) - (1/\Lambda)\, \phi_0(x)\psi_0(y) \right) f(y)\,\mu(dy), \tag{5.3}$$

and

$$e^{\lambda_0 t} U_t \mathbb{1}_M(x) - (1/\Lambda)\, \phi_0(x) \int_M \psi_0(y)\,\mu(dy)$$
$$= \int_M \left(e^{\lambda_0 t} u_t(x,y) - (1/\Lambda)\, \phi_0(x)\psi_0(y) \right) \mu(dy). \tag{5.4}$$

We first estimate (5.3). With the Chapman–Kolmogorov equation and the eigenequation, we get

$$e^{\lambda_0 t} U_t f(x) - (1/\Lambda)\, \phi_0(x) \int_M f(y)\psi_0(y)\,\mu(dy)$$
$$= \int_M \left[\left(e^{\lambda_0 (t-2t_0)} u_{t-2t_0}(x,w) - (1/\Lambda)\, \phi_0(x)\psi_0(w) \right) e^{2\lambda_0 t_0} \int_M u_{2t_0}(w,y) f(y)\,\mu(dy) \right] \mu(dw)$$
$$= \int_{K_{bt}} \left(\ldots \right) e^{2\lambda_0 t_0} U_{2t_0} f(w)\,\mu(dw) + \int_{K_{bt}^c} \left(\ldots \right) e^{2\lambda_0 t_0} U_{2t_0} f(w)\,\mu(dw).$$

Since

$$|U_{2t_0} f(x)| \leqslant \|U_{t_0}\|_{p,\infty} \|f\|_p\, U_{t_0}\mathbb{1}_M(x), \quad x \in M,$$

we obtain

$$\left| e^{\lambda_0 t} U_t f(x) - (1/\Lambda)\, \phi_0(x) \int_M f(y)\psi_0(y)\,\mu(dy) \right|$$
$$\leqslant e^{2\lambda_0 t_0} \|U_{t_0}\|_{p,\infty} \|f\|_p \left(\|U_{t_0}\mathbb{1}_M\|_\infty I_1(t,x) + \sup_{w \in K_{bt}^c} U_{t_0}\mathbb{1}_M(w) I_2(t,x) \right),$$

where

$$I_1(t,x) := \int_{K_{bt}} \left| e^{\lambda_0 (t-2t_0)} u_{t-2t_0}(x,w) - (1/\Lambda)\, \phi_0(x)\psi_0(w) \right| \mu(dw),$$

and

$$I_2(t,x) := \int_{K_{bt}^c} \left| e^{\lambda_0 (t-2t_0)} u_{t-2t_0}(x,w) - (1/\Lambda)\, \phi_0(x)\psi_0(w) \right| \mu(dw).$$

By Lemma 3.3 (with $s = at$ and $r = bt$; in particular $t - 2t_0 - s - r = bt - 2t_0 > 2t_0$) and pGSD, we see that for all $x \in K_{at}$,

$$I_1(t,x) \leqslant e^{-\gamma(bt-2t_0)} e^{\lambda_0 at} U_{at}\mathbb{1}_M(x) e^{\lambda_0 bt} \int_{K_{bt}} U_{bt}^* \mathbb{1}_M(w)\,\mu(dw) \leqslant c_1(t_0) e^{-\gamma bt} \phi_0(x) \|\psi_0\|_1.$$

Again by pGSD, we get for $x \in K_{at} \subset K_{t-2t_0}$,

$$I_2(t,x) \leqslant e^{\lambda_0(t-2t_0)} U_{t-2t_0} \mathbb{1}_M(x) + (1/\Lambda)\,\phi_0(x)\,\|\psi_0\|_1 \leqslant (c_2 + (1/\Lambda)\,\|\psi_0\|_1)\phi_0(x).$$

Together this gives, for any measure $\sigma \in \mathcal{M}^1(K_{at})$ with $\operatorname{supp}\sigma \cap (M \setminus \mathcal{N}) \neq \emptyset$, the estimate

$$\left| e^{\lambda_0 t}\sigma(U_t f) - (1/\Lambda)\,\sigma(\phi_0) \int_M f(y)\psi_0(y)\,\mu(dy) \right|$$
$$\leqslant c_3 \left(e^{-\gamma b t} + \sup_{w \in K_{bt}^c} U_{t_0}\mathbb{1}_M(w) \right) \|f\|_p\,\sigma(\phi_0).$$

The estimate for (5.4) follows if we apply the above estimate for $p = \infty$ to $f = \mathbb{1}_M \in L^\infty(M,\mu)$; that is, we get for all $\sigma \in \mathcal{M}^1(K_{at})$, $\operatorname{supp}\sigma \cap (M \setminus \mathcal{N}) \neq \emptyset$,

$$\left| e^{\lambda_0 t}\sigma(U_t \mathbb{1}_M) - (1/\Lambda)\,\sigma(\phi_0)\,\|\psi_0\|_1 \right| \leqslant c_4 \left(e^{-\gamma b t} + \sup_{w \in K_{bt}^c} U_{t_0}\mathbb{1}_M(w) \right) \sigma(\phi_0).$$

It is important that the constants c_3 and c_4 do not depend on σ. Finally, using (5.2), (3.5) and the above estimates, we arrive at

$$\left| \frac{\sigma(U_t f)}{\sigma(U_t \mathbb{1}_M)} - m(f) \right| \leqslant c_4 \left(e^{-\gamma b t} + \sup_{w \in K_{bt}^c} U_{t_0}\mathbb{1}_M(w) \right) \|f\|_p,$$

which is valid for all measures $\sigma \in \mathcal{M}^1(K_{at})$ such that $\operatorname{supp}\sigma \cap (M \setminus \mathcal{N}) \neq \emptyset$. This completes the proof. □

Remark 5.3. In Theorem 5.2, we prove that pGSD with the exhausting family $\{K_t : t \geqslant t_0\}$ implies progressive quasi-ergodicity with $\{K_{at} : t \geqslant t_0/a\}$, for an arbitrary $a \in (0,1)$. We will see in Section 7.2 that this result is sharp in the sense that one cannot expect progressive quasi-ergodicity which is uniform for the original exhausting family $\{K_t : t \geqslant t_0\}$.

Our next theorem shows that pGSD is, in fact, equivalent to a version of progressive uniform quasi-ergodicity; cf. Corollary 4.4.

Theorem 5.4. *Let M be a locally compact, but not compact, Polish space. Assume that (A0)–(A4) hold. Then the following conditions are equivalent:*

a) *Both semigroups $\{U_t : t \geqslant 0\}$ and $\{U_t^* : t \geqslant 0\}$ are pGSD.*

b) *There exist probability measures $\rho, \rho^* \in \mathcal{M}_1(M)$ such that $\operatorname{supp}\rho \cap (M \setminus \mathcal{N}) \neq \emptyset$, $\operatorname{supp}\rho^* \cap (M \setminus \mathcal{N}) \neq \emptyset$, and an exhausting family of sets $\{K_t : t \geqslant t_0\}$ in M such that*

$$\sup_{x \in K_t \setminus \mathcal{N}} \sup_{\substack{f \in L^\infty(M,\mu) \\ \|f\|_\infty \leqslant 1}} \left| \frac{U_t f(x)}{U_t \mathbb{1}_M(x)} - \rho(f) \right| \to 0 \quad as \quad t \to \infty,$$

and

$$\sup_{\substack{y \in K_t \backslash \mathcal{N}}} \sup_{\substack{f \in L^\infty(M,\mu) \\ \|f\|_\infty \leqslant 1}} \left| \frac{U_t^* f(y)}{U_t^* \mathbb{1}_M(y)} - \rho^*(f) \right| \to 0 \quad as \quad t \to \infty.$$

Proof. The implication 'a) $\Rightarrow$ b)' follows directly from the previous theorem. In order to show the converse direction, we check that the semigroup $\{U_t : t \geqslant 0\}$ is pGSD. The argument for $\{U_t^* : t \geqslant 0\}$ is similar.

Since $\operatorname{supp} \rho \cap (M \setminus \mathcal{N}) \neq \emptyset$, we have $\rho(\phi_0) > 0$. Recall that $\phi_0 \in L^\infty(M,\mu)$. Because of b), there exists some $t_1 > t_0$ such that

$$\left| \rho(\phi_0) - \frac{U_t \phi_0(x)}{U_t \mathbb{1}_M(x)} \right| \leqslant \frac{1}{2} \rho(\phi_0), \quad t \geqslant t_1, \, x \in K_t.$$

Hence, we have

$$U_t \mathbb{1}_M(x) \leqslant \frac{2}{\rho(\phi_0)} U_t \phi_0(x) = \frac{2}{\rho(\phi_0)} e^{-\lambda_0 t} \phi_0(x), \quad t \geqslant t_1, \, x \in K_t.$$

This gives pGSD for the semigroup $\{U_t : t \geqslant 0\}$, and the proof is complete. $\qquad\square$

We already know that the finiteness of the heat content for large t implies the uniqueness of the quasi-stationary probability measures for both semigroups $\{U_t : t \geqslant 0\}$ and $\{U_t^* : t \geqslant 0\}$; see Theorem 3.4.c). We will now show a much stronger result, which says that the pGSD property and some additional information on the operator norm of U_t and U_t^* for large times, also imply the uniqueness of the quasi-stationary measures. The additional condition can be expressed as follows: there exists $t_2 > 0$ such that

$$\sup_{t \geqslant t_2} \left(e^{\lambda_0 t} \sup_{x \in M} \left(U_t \mathbb{1}(x) + U_t^* \mathbb{1}(x) \right) \right) < \infty. \tag{5.5}$$

Since

$$\sup_{x \in M} U_t \mathbb{1}(x) = \|U_t\|_{\infty,\infty} \quad \text{and} \quad \sup_{x \in M} U_t^* \mathbb{1}(x) = \|U_t^*\|_{\infty,\infty},$$

(5.5) is equivalent to the combination of the following two conditions (5.6) and (5.7):

$$\sup_{t \geqslant t_2} \frac{\|U_t\|_{\infty,\infty}}{\|U_t\|_{2,2}} < \infty, \quad \sup_{t \geqslant t_2} \frac{\|U_t^*\|_{\infty,\infty}}{\|U_t^*\|_{2,2}} < \infty; \tag{5.6}$$

$$\sup_{t \geqslant t_2} e^{\lambda_0 t} \|U_t\|_{2,2} < \infty \quad \text{(or, equivalently,} \quad \sup_{t \geqslant t_2} e^{\lambda_0 t} \|U_t^*\|_{2,2} < \infty). \tag{5.7}$$

If the semigroups $\{U_t : t \geqslant 0\}$ and $\{U_t^* : t \geqslant 0\}$ are pGSD for the exhausting family of compact sets $\{K_t : t \geqslant t_2\}$, the condition (5.5) is equivalent to

$$\sup_{t \geqslant t_2} \left(e^{\lambda_0 t} \sup_{x \in K_t^c} \left(U_t \mathbb{1}(x) + U_t^* \mathbb{1}(x) \right) \right) < \infty. \tag{5.8}$$

The next theorem shows that the uniqueness of the quasi-stationary measures extends to a large class of compact semigroups with infinite heat content.

Theorem 5.5. *Let M be a locally compact, but not compact, Polish space. Assume that (A0)–(A4) hold and that both semigroups $\{U_t : t \geqslant 0\}$ and $\{U_t^* : t \geqslant 0\}$ are pGSD and satisfy (5.5). Then the measures m and m^* defined in (3.1) are the unique quasi-stationary probability measures of $\{U_t : t \geqslant 0\}$ and $\{U_t^* : t \geqslant 0\}$, respectively, such that $m(M \setminus \mathcal{N}) \cdot m^*(M \setminus \mathcal{N}) > 0$.*

Proof. We give a proof for the measure m only; the argument for m^* is similar. Suppose that $\bar{m}$ is a quasi-stationary probability measure of $\{U_t : t \geqslant 0\}$ such that $\operatorname{supp} \bar{m} \cap (M \setminus \mathcal{N}) \neq \emptyset$. Following the argument from the proof of Theorem 3.4.c), it is enough to show that $m(f) = \bar{m}(f)$ for $f \in L^\infty(M, \mu)$.

For a compact set $K \subset M$ such that $\bar{m}(\mathbb{1}_K \phi_0) > 0$ (in particular, $\bar{m}(K) > 0$), we define

$$\sigma_K(dx) = \frac{\mathbb{1}_K(x) \bar{m}(dx)}{\bar{m}(K)}.$$

Fix $f \in L^\infty(M, \mu)$. We have

$$|m(f) - \bar{m}(f)| \leqslant \left| m(f) - \frac{\sigma_K(U_t f)}{\sigma_K(U_t \mathbb{1}_M)} \right| + \left| \frac{\sigma_K(U_t f)}{\sigma_K(U_t \mathbb{1}_M)} - \bar{m}(f) \right|.$$

Using the convergence assertion from Theorem 5.2 with the probability measure $\sigma = \sigma_K$, we see that the first term on the right-hand side vanishes as $t \to \infty$. We will now show that for every $\epsilon \in (0, 1)$, there exists a compact set $K \subset M$ such that the second term is less than ϵ, uniformly for large t.

Fix $\epsilon \in (0, 1)$. By (5.5),

$$c := \sup_{t \geqslant t_2} \left(e^{\lambda_0 t} \sup_{x \in M} U_t \mathbb{1}(x) \right) < \infty.$$

Since $\bar{m}$ is a finite measure, there exists a compact set $K = K(\epsilon) \subset M$ as above such that

$$\bar{m}(K^c) \leqslant \frac{\bar{m}(\phi_0)}{c \|\phi_0\|_\infty (1 + \bar{m}(f) + \|f\|_\infty)} \frac{\epsilon}{1 + \epsilon}. \tag{5.9}$$

The existence of such a set is a consequence of the σ-compactness of the space M. By the quasi-stationarity property of $\bar{m}$, cf. (3.2), we have

$$\bar{m}(\mathbb{1}_K U_t f) + \bar{m}(\mathbb{1}_{K^c} U_t f) = \bar{m}(f) \big(\bar{m}(\mathbb{1}_K U_t \mathbb{1}_M) + \bar{m}(\mathbb{1}_{K^c} U_t \mathbb{1}_M) \big),$$

which implies that

$$\left| \frac{\sigma_K(U_t f)}{\sigma_K(U_t \mathbb{1}_M)} - \bar{m}(f) \right| = \left| \frac{\bar{m}(\mathbb{1}_K U_t f)}{\bar{m}(\mathbb{1}_K U_t \mathbb{1}_M)} - \bar{m}(f) \right|$$

$$= \left| \frac{\bar{m}(f) \big(\bar{m}(\mathbb{1}_K U_t \mathbb{1}_M) + \bar{m}(\mathbb{1}_{K^c} U_t \mathbb{1}_M) \big) - \bar{m}(\mathbb{1}_{K^c} U_t f)}{\bar{m}(\mathbb{1}_K U_t \mathbb{1}_M)} - \bar{m}(f) \right|$$

$$= \left| \frac{\bar{m}(f)\bar{m}(\mathbb{1}_{K^c}U_t\mathbb{1}_M) - \bar{m}(\mathbb{1}_{K^c}U_tf)}{\bar{m}(\mathbb{1}_K U_t\mathbb{1}_M)} \right|$$

$$\leqslant (\bar{m}(f) + \|f\|_\infty) \frac{\bar{m}(\mathbb{1}_{K^c}U_t\mathbb{1}_M)}{\bar{m}(U_t\mathbb{1}_M) - \bar{m}(\mathbb{1}_{K^c}U_t\mathbb{1}_M)}$$

$$= (\bar{m}(f) + \|f\|_\infty) \frac{\frac{\bar{m}(\mathbb{1}_{K^c}U_t\mathbb{1}_M)}{\bar{m}(U_t\mathbb{1}_M)}}{1 - \frac{\bar{m}(\mathbb{1}_{K^c}U_t\mathbb{1}_M)}{\bar{m}(U_t\mathbb{1}_M)}}.$$

From (3.5), we infer that

$$\bar{m}(U_t\mathbb{1}_M) \geqslant e^{-\lambda_0 t} \frac{\bar{m}(\phi_0)}{\|\phi_0\|_\infty}, \quad t > 0.$$

Combining this inequality with the pGSD property and (5.9) yields

$$\frac{\bar{m}(\mathbb{1}_{K^c}U_t\mathbb{1}_M)}{\bar{m}(U_t\mathbb{1}_M)} \leqslant c \frac{\|\phi_0\|_\infty}{\bar{m}(\phi_0)} \bar{m}(K^c) \leqslant \frac{1}{1 + \bar{m}(f) + \|f\|_\infty} \frac{\epsilon}{1 + \epsilon}, \quad t \geqslant t_2.$$

Since the function $(0,1) \ni x \mapsto x/(1-x)$ is increasing, we finally get

$$|m(f) - \bar{m}(f)| \leqslant \limsup_{t\to\infty} \left| \frac{\sigma_K(U_tf)}{\sigma_K(U_t\mathbb{1}_M)} - \bar{m}(f) \right| \leqslant \epsilon,$$

which completes the proof. $\qquad\square$

6. Large time asymptotic behavior of compact semigroups

As a by-product of our investigations on quasi-ergodicity, we also obtain some statements on the asymptotic behavior of compact L^2-semigroups on L^p spaces. In this generality, such results seem to be new.

Corollary 6.1. *Let M be a locally compact Polish space and assume that (A0)–(A3) hold. If there exists some $t_1 \geqslant t_0$ such that the heat content is finite, that is,*

$$Z(t_1) < \infty,$$

then we have the following:

a) *For every $p \in [1,\infty]$, there exists $C > 0$ such that for every finite measure σ on M, and for all $t > 6t_1$ and $t_1 \leqslant s \leqslant t/2$, we have*

$$\sup_{\substack{f \in L^p(M,\mu) \\ \|f\|_p \leqslant 1}} \left| e^{\lambda_0 t} \sigma(U_tf) - \frac{1}{\Lambda} \sigma(\phi_0) \int_M f(y)\psi_0(y)\,\mu(dy) \right| \leqslant C e^{-\gamma(t-s)} e^{\lambda_0 s} \sigma(U_s\mathbb{1}_M),$$

and

$$\sup_{\substack{f \in L^p(M,\mu) \\ \|f\|_p \leqslant 1}} \left| e^{\lambda_0 t} \sigma(U_t^*f) - \frac{1}{\Lambda} \sigma(\psi_0) \int_M f(x)\phi_0(x)\,\mu(dx) \right| \leqslant C e^{-\gamma(t-s)} e^{\lambda_0 s} \sigma(U_s^*\mathbb{1}_M).$$

In particular, taking $\sigma = \delta_z$, $z \in M$, and $s = t_1$, we obtain uniform estimates

$$\sup_{x \in M} \sup_{\substack{f \in L^p(M, \mu) \\ \|f\|_p \leqslant 1}} \left| e^{\lambda_0 t} U_t f(x) - \frac{1}{\Lambda} \phi_0(x) \int_M f(y) \psi_0(y) \, \mu(dy) \right| \leqslant \widetilde{C} e^{-\gamma t},$$

and

$$\sup_{y \in M} \sup_{\substack{f \in L^p(M, \mu) \\ \|f\|_p \leqslant 1}} \left| e^{\lambda_0 t} U_t^* f(y) - \frac{1}{\Lambda} \psi_0(y) \int_M f(x) \phi_0(x) \, \mu(dx) \right| \leqslant \widetilde{C} e^{-\gamma t}.$$

b) *For every $p, r \in [1, \infty]$, there exists $C > 0$ such that for all $t > 6t_1$, we have*

$$\sup_{\substack{f \in L^p(M, \mu) \\ \|f\|_p \leqslant 1}} \left\| e^{\lambda_0 t} U_t f - \frac{1}{\Lambda} \phi_0 \int_M f(y) \psi_0(y) \, \mu(dy) \right\|_r \leqslant C e^{-\gamma t},$$

and

$$\sup_{\substack{f \in L^p(M, \mu) \\ \|f\|_p \leqslant 1}} \left\| e^{\lambda_0 t} U_t^* f - \frac{1}{\Lambda} \psi_0 \int_M f(x) \phi_0(x) \, \mu(dx) \right\|_r \leqslant C e^{-\gamma t}.$$

The above bounds follow from direct inspection of the estimates in the proof of Theorem 3.4. The cases $p = 1$ in Part a) and $p = 1$, $r = \infty$ in in Part b) are special – they follow directly from (**A2**), and the finiteness of the heat content is actually needed in that case.

Let $\{K_t : t \geqslant t_0\}$ be an exhausting family of sets in M. By considering the sequence of measures (σ_n) such that $\sigma_n(\cdot) := \mu(\cdot \cap K_n)$, $n \in \mathbb{N}$, and using Corollary 6.1.a), we obtain the following asymptotics of the heat content.

Corollary 6.2. *Let M be a locally compact Polish space and assume that (**A0**)–(**A3**) hold. If there exists some $t_1 \geqslant t_0$ such that the heat content is finite, that is,*

$$Z(t_1) < \infty,$$

then there is some $C > 0$ such that

$$\left| e^{\lambda_0 t} Z(t) - \frac{\|\phi_0\|_1 \|\psi_0\|_1}{\Lambda} \right| \leqslant C e^{-\gamma t}, \quad t > 6t_1.$$

If $Z(t) = \infty$ for large t's, we cannot expect the asymptotic estimates to be uniform as in Corollary 6.1.a). Still, we can prove a progressive version which is similar to Theorem 5.2. The following result can be obtained directly by inspection of the proof of that theorem.

Corollary 6.3. *Let M be a locally compact, but not compact, Polish space. Assume that (**A0**)–(**A4**) hold and that both semigroups $\{U_t : t \geqslant 0\}$ and $\{U_t^* : t \geqslant 0\}$ are pGSD for the exhausting family of compact sets $\{K_t : t \geqslant t_0\}$. Then for every $a, b \in (0,1)$ such that $a + 2b = 1$ and for every $p \in [1, \infty]$, there exists some $C > 0$ such that for all $t > (4/(a \wedge b)) t_0$,*

$$\sup_{\sigma \in \mathcal{M}^1(K_{at})} \sup_{\substack{f \in L^p(M, \mu) \\ \|f\|_p \leqslant 1}} \left| e^{\lambda_0 t} \sigma(U_t f) - \frac{1}{\Lambda} \sigma(\phi_0) \int_M f(y) \psi_0(y) \, \mu(dy) \right| \leqslant C \kappa_b(t),$$

and

$$\sup_{\sigma \in \mathcal{M}^1(K_{at})} \sup_{\substack{f \in L^p(M,\mu) \\ \|f\|_p \leqslant 1}} \left| e^{\lambda_0 t} \sigma(U_t^* f) - \frac{1}{\Lambda} \sigma(\psi_0) \int_M f(x) \phi_0(x) \, \mu(dx) \right| \leqslant C \kappa_b(t),$$

where the time-rate $\kappa_b(t)$ is given by (5.1). By considering $\sigma = \delta_z$, $z \in K_{at}$, we obtain progressive uniform estimate on the sets K_{at} when $t \to \infty$.

7. Examples and applications

7.1. Feynman–Kac semigroups of Feller processes

Suppose that (M,d) is a locally compact Polish space, and μ is a positive, locally finite measure with full topological support on the Borel sets $\mathcal{B}(M)$. We assume that the space (M,d) is *unbounded* (in particular, it is *not compact*) and the measure μ is *not finite*. Let $(X_t)_{t \geqslant 0}$ and $(\widehat{X}_t)_{t \geqslant 0}$ be Markov processes with values in M and the following transition semigroups:

$$P_t f(x) = \mathbb{E}^x f(X_t) \quad \text{and} \quad \widehat{P}_t f(y) = \widehat{\mathbb{E}}^y f(\widehat{X}_t), \quad f \in L^2(M,\mu);$$

Throughout this section, we assume that P_t and $\widehat{P}_t$ are contractions on $L^2(M,\mu)$ and that the semigroups $\{P_t : t \geqslant 0\}$, $\{\widehat{P}_t : t \geqslant 0\}$ have both the Feller and the strong Feller property (i.e., they are 'doubly Feller' in the sense of Chung [11]). Moreover, we assume that the corresponding Markov resolvents are in a weak duality relation with respect to the measure μ; see [12, Definition 13.1]. By [12, Proposition 13.6], we have

$$\int f(x) \cdot P_t g(x) \, \mu(dx) = \int \widehat{P}_t f(x) \cdot g(x) \, \mu(dx), \quad f, g \in L^2(M,\mu), \, t > 0.$$

This means that $\widehat{P}_t$ is the (functional analytic) adjoint operator P_t^* of P_t in the space $L^2(M,\mu)$. Finally, we assume that for every $t > 0$, the operators P_t and $\widehat{P}_t$ are positivity improving and bounded from $L^2(M,\mu)$ to $L^\infty(M,\mu)$.

In particular, the 'free' semigropus $\{P_t, t \geqslant 0\}$ and $\{\widehat{P}_t, t \geqslant 0\}$ are such that they satisfy our assumptions (**A1**)–(**A3**). We are now going to give sufficient criteria such that the corresponding Feynman–Kac semigroups $\{U_t, t \geqslant 0\}$ and $\{U_t^*, t \geqslant 0\}$ satisfy (**A0**)–(**A4**).

Let V be a locally bounded potential on M which is bounded below, that is, $V = V_+ - V_-$ with $V_+ \in L_{\mathrm{loc}}^\infty(M,\mu)$ and $V_- \in L^\infty(M,\mu)$, for which we define the *Feynman–Kac semigroup*

$$U_t f(x) = \mathbb{E}^x \left[e^{-\int_0^t V(X_s) \, ds} f(X_t) \right], \quad f \in L^2(M,\mu), \, t > 0.$$

By [12, Theorem 13.25], the weak dual or adjoint semigroup consists of operators

$$\widehat{U}_t f(x) = U_t^* f(x) = \widehat{\mathbb{E}}^x \left[e^{-\int_0^t V(\widehat{X}_s) \, ds} f(\widehat{X}_t) \right], \quad f \in L^2(M,\mu), \, t > 0.$$

We first verify the assumptions (**A1**)–(**A3**) and show that U_t and U_t^* are integral operators. If $(X_t)_{t \geqslant 0}$ is a stochastic process, we denote by $\tau_B := \inf \{t > 0 : X_t \in B^c\}$ the first exit time from the set B.

Lemma 7.1. *The following assertions hold.*

a) *The operators U_t, $t > 0$, are positivity improving.*

b) *Both semigroups $\{U_t : t \geqslant 0\}$ and $\{U_t^* : t \geqslant 0\}$ are Feller and strong Feller.*

c) *For every $t > 0$, the operators $U_t, U_t^* : L^2(M, \mu) \to L^\infty(M, \mu)$ are bounded. In particular, for every $t > 0$, there exists a measurable kernel $u_t(x, y)$ such that (2.1) holds, and the semigroups $\{U_t : t \geqslant 0\}$ and $\{\widehat{U}_t : t \geqslant 0\}$ are in duality with respect to the measure μ.*

Proof. a) Let $0 \not\equiv f \in L^2(M, \mu)$ be positive and $t, r > 0$. We have

$$U_t f(x) = \mathbb{E}^x \left[e^{-\int_0^t V(X_s)\,ds} f(X_t) \right] \geqslant e^{-t \left\| \mathbb{1}_{B_r(x)} V_+ \right\|_\infty} \mathbb{E}^x \left[f(X_t); t < \tau_{B_r(x)} \right].$$

The expectation appearing on the right-hand side tends to $\mathbb{E}^x f(X_t) = P_t f(x)$ as $r \to \infty$. Since P_t is positivity improving, and V is locally bounded, the expression on the right-hand side is strictly positive for μ-almost every x and sufficiently large $r > 0$. To see this, fix any x such that $P_t f(x) > 0$. Since $\mathbb{E}^x \left[f(X_t); t < \tau_{B_r(x)} \right]$ increases to $P_t f(x) = \mathbb{E}[f(X_t)]$ as $r \uparrow \infty$, there is some $r = r(x)$ such that $\mathbb{E}^x \left[f(X_t); t < \tau_{B_r(x)} \right] > 0$; hence, $U_t f(x) > 0$. This gives the assertion.

Part b) follows directly from Chung [11, Theorem 2]. The assumptions (a)–(c) of Chung's theorem can be easily verified as V is both bounded below and locally bounded.

We still have to prove c). First, observe that for every $t > 0$, the operators $U_t, U_t^* : L^2(M, \mu) \to L^\infty(M, \mu)$ are bounded. Indeed, for $f \in L^2(M, \mu)$ and $t > 0$, we have

$$U_t f(x) = \mathbb{E}^x \left[e^{-\int_0^t V(X_s)\,ds} f(X_t) \right] \leqslant e^{t \|V_-\|_\infty} \mathbb{E}^x f(X_t) = e^{t \|V_-\|_\infty} P_t f(x),$$

implying that $\|U_t\|_{2,\infty} \leqslant e^{t \|V_-\|_\infty} \|P_t\|_{2,\infty}$. In the same way, we get $\|U_t^*\|_{2,\infty} \leqslant e^{t \|V_-\|_\infty} \|P_t^*\|_{2,\infty}$. Combining this with part b) and the semigroup property, we see that for every $t > 0$, the operators U_t, U_t^* map $L^2(M, \mu)$ $\{P_t, t \geqslant 0\}$, $\left\{ \widehat{P}_t, t \geqslant 0 \right\}$ into $L^2(M, \mu) \cap C_b(M)$, and we can apply [21, Theorem 1.6] to show that U_t, U_t^*, $t > 0$, are integral operators. That is, there exist non-negative measurable kernels $u_t(x, y)$ and $u_t^*(x, y)$ such that

$$U_t f(x) = \int_M f(y) u_t(x, y) \mu(dy), \quad U_t^* f(x) = \int_M f(y) u_t^*(x, y) \mu(dy), \quad f \in L^2(M, \mu),\, t > 0.$$

Since the operators U_t and U_t^* are adjoint, we have $u_t^*(x, y) = u_t(y, x)$. $\qquad \square$

Now we give general sufficient conditions for the compactness of the operators U_t, U_t^*, $t > 0$ (cf. assumption (**A0**)) and for the finiteness of the heat content for large times. Recall that the heat content is defined as

$$Z(t) = \int_M U_t \mathbb{1}(x) \mu(dx) = \int_M U_t^* \mathbb{1}(y) \mu(dy), \quad t > 0.$$

Lemma 7.2. *For $x \in M$ and $t > 0$, we have*

$$e^{-t \cdot \operatorname{ess\,sup}_y \left(\mathbb{1}_{B_1(x)}(y) V_+(y) \right)} \mathbb{P}^x \left(t \leqslant \tau_{B_1(x)} \right) \leqslant U_t \mathbb{1}(x) \leqslant \frac{1}{t} \int_0^t P_s (e^{-t V(\cdot)})(x)\,ds, \qquad (7.1)$$

as well as similar estimates for $U_t^* \mathbb{1}(x)$. In particular,

$$\inf_{x \in M} \mathbb{P}^x(t \leqslant \tau_{B_1(x)}) \int_M e^{-t \cdot \mathrm{ess\ sup}_y (\mathbb{1}_{B_1(x)}(y) V_+(y))} \mu(dx) \leqslant Z(t) \leqslant \int_M e^{-tV(x)} \mu(dx), \quad (7.2)$$

for $t > 0$. Consequently, the following assertions hold.

a) *If*

$$\lim_{d(x,x_0) \to \infty} V(x) = \infty, \quad (7.3)$$

*then the operators U_t, U_t^*, $t > 0$, are compact; in particular, **(A0)** holds.*

b) *Fix $t > 0$. Then*

$$\int_M e^{-tV(x)} \mu(dx) < \infty \implies Z(t) < \infty.$$

However, if $\inf_{x \in M} \mathbb{P}^x(t \leqslant \tau_{B_1(x)}) > 0$, then

$$Z(t) < \infty \implies \int_M e^{-t \cdot \mathrm{ess\ sup}_y (\mathbb{1}_{B_1(x)}(y) V_+(y))} \mu(dx) < \infty.$$

Proof. Fix $t > 0$. For the proof of the upper bound in (7.1), it is enough to use Jensen's inequality and Tonelli's theorem

$$U_t \mathbb{1}(x) = \mathbb{E}^x \left[e^{-\int_0^t V(X_s)\, ds} \right] \leqslant \mathbb{E}^x \left[\frac{1}{t} \int_0^t e^{-tV(X_s)}\, ds \right] = \frac{1}{t} \int_0^t P_s(e^{-tV(\cdot)})(x)\, ds,$$

and the upper estimate in (7.2) follows by integration with respect to $\mu(dx)$ and duality. The lower bounds in (7.1) and (7.2) follow from the definitions. Observe that Part b) follows directly from the latter inequality.

To get Part a), we show **(A4)** for every $t_0 > 0$, and then we use Lemma 2.3. By (7.3), there exists a function $g \in C_\infty(M)$ such that

$$e^{-tV(x)} \leqslant g(x), \quad \text{for } \mu \text{ almost every } x \in M.^2 \quad (7.4)$$

Hence, by (7.1) and (7.4),

$$0 \leqslant U_t \mathbb{1}(x) \leqslant \frac{1}{t} \int_0^t P_s g(x)\, ds, \quad x \in M.$$

By the Feller property, the function $x \mapsto (1/t) \int_0^t P_s g(x)\, ds$ belongs to $C_\infty(M)$, which implies that **(A4)** holds for any $t \geqslant t_0$ and any $t_0 > 0$.

The proof of **(A4)** for adjoint operators U_t^* is similar. $\square$

In particular, Lemma 7.2.b) shows that the finiteness of the heat content is completely independent of the free process.

---

[2] Indeed: fix an arbitrary $x_0 \in M$, set $h(r) := \mathrm{ess\,inf}_{\{x \in M : d(x,x_0) \geqslant r \vee 0\}} V_+(x)$ for $r \in \mathbb{R}$, and use a continuous function $\phi \geqslant 0$ with support in $[0,1]$ and $\int_0^1 \phi(s)\, ds = 1$ to get $h_\phi(r) := h * \phi(r) \leqslant h(r+1)$, $r \in \mathbb{R}$. Then $V_+(x) \geqslant h(d(x,x_0)) \geqslant h_\phi(d(x,x_0) - 1)$, $x \in M$. Since t is fixed, $g(x) := \exp\left(t\left(\|V_-\|_\infty - h_\phi(d(x,x_0) - 1) \right) \right)$ is a continuous function which vanishes at infinity.

We will now discuss some concrete classes of Feller processes which are covered by the framework described above. In order to keep the exposition short, we give in each case standard references for the notation and further results.

7.1.1. Lévy processes. Standard references: Sato [49], Jacob [20, Vol. 1], Schilling [51]. Throughout this section, the reference measure μ is d-dimensional Lebesgue measure and $d(\cdot,\cdot)$ is the Euclidean distance.

A *Lévy process* $(X_t)_{t \geqslant 0}$ is a stochastic process on $\mathbb{R}^d$ with càdlàg (right-continuous with finite left limits) paths and stationary and independent increments. A Lévy process can be described in terms of its characteristic exponent $\psi(\xi) = -\log \mathbb{E}e^{i\xi \cdot X_1}$. It is known that ψ is uniquely determined by the Lévy–Khintchine formula

$$\psi(\xi) = -i\xi \cdot b + \frac{1}{2}\xi \cdot Q\xi + \int_{\mathbb{R}^d \setminus \{0\}} \left(1 - e^{i\xi \cdot y} + i\xi \cdot y \mathbb{1}_{(0,1)}(|y|)\right) \nu(dy), \quad \xi \in \mathbb{R}^d, \qquad (7.5)$$

or by the corresponding the Lévy triplet (b,Q,ν), which is given by $b \in \mathbb{R}^d$ (*drift term*), Q is a symmetric, positive semidefinite $d \times d$ matrix (*Gaussian covariance matrix*) and ν is a measure on $\mathbb{R}^d \setminus \{0\}$ such that $\int_{\mathbb{R}^d \setminus \{0\}} (1 \wedge |x|^2)\nu(dx) < \infty$ (*Lévy measure*).

Lévy processes are Markov processes, which are invariant under translations; that is, the transition semigroup is a convolution semigroup of the form

$$P_t f(x) = \mathbb{E}^x \left[f(X_t)\right] = \mathbb{E}^0 \left[f(X_t + x)\right] = f * \widetilde{\mu}_t(x),$$

where $\widetilde{\mu}_t(dy) = \mathbb{P}^0(X_t \in -dy)$ is the law of $-X_t$. The dual process $(\widehat{X}_t)_{t \geqslant 0}$ of a Lévy process $(X_t)_{t \geqslant 0}$ is given by $\widehat{X}_t = -X_t$, and its transition semigroup $\{\widehat{P}_t : t \geqslant 0\}$ is defined accordingly. The convolution structure of P_t shows that $(P_t)_{t \geqslant 0}$ is both a Feller semigroup on $C_\infty(\mathbb{R}^d)$ and an $L^p(\mathbb{R}^d, dx)$ Markov semigroup for any $p \in [1,\infty)$.

The process $(X_t)_{t \geqslant 0}$ is a strong Feller process, that is, P_t maps $L^\infty(\mathbb{R}^d, dx)$ into $C_b(\mathbb{R}^d)$, if and only if $\mu_t(dy) = p_t(y) \, dy$ is absolutely continuous w.r.t. Lebesgue measure; cf. Jacob [20, Vol. 1, Lemma 4.8.19, 4.8.20]. In particular, the dual process is strong Feller, too. A sufficient condition for the existence of the densities $p_t(x)$, $t > 0$ is that $Q \neq 0$ or that ν is infinite and absolutely continuous with respect to Lebesgue measure; see, for example, [49, Theorem 27.7], and see also the discussion in [32]. If, in addition, there is some $t_0 > 0$ such that $\int_{\mathbb{R}^d} e^{-t_0 \operatorname{Re}\psi(\xi)} \, d\xi < \infty$, then $\sup_{t \geqslant t_0} \sup_{x \in \mathbb{R}^d} p_t(x) < \infty$, thanks to the Fourier inversion formula. Consequently, the operators $P_t, \widehat{P}_t : L^2(\mathbb{R}^d, dx) \to L^\infty(\mathbb{R}^d, dx)$ are bounded for every $t \geqslant t_0$. In general, if the integral $\int_{\mathbb{R}^d} e^{-t_0 \psi(\xi)} \, d\xi$ is finite for every $t_0 > 0$, then we get the existence of bounded and jointly (in (t,x)) continuous densities for every $t > 0$; see [32, Lem. 2.1].

In order to see that P_t and $\widehat{P}_t$ are positivity improving, it is enough to show that $\mathbb{P}^0(X_t \in B) = \mathbb{P}^0(\widehat{X}_t \in -B) = \int_B p_t(x) \, dx > 0$ for every Borel set $B \subset \mathbb{R}^d$ with $\operatorname{Leb}(B) > 0$. Let us fix such a Borel set B. Using the Lévy–Khintchine formula, we can decompose a Lévy process into a sum of independent Lévy processes $X_t = W_t + Y_t$ whose triplets are $(b,Q,0)$ and $(0,0,\nu)$. If $\det(Q) \neq 0$, then we always have a nondegenerate Gaussian part, and it is clear that

$$\mathbb{P}^0(X_t \in B) = \int \mathbb{P}^0(W_t + y \in B)\, \mathbb{P}^0(Y_t \in dy) > 0$$

since $\mathbb{P}^0(W_t + y \in B) > 0$ for all $y \in \mathbb{R}^d$.

Let us assume that $\det(Q) = 0$. If the Lévy measure ν satisfies $\nu(dy) \geqslant h(y)\, dy$ with a density $h(y)$ which is strictly positive in some neighborhood of $0 \in \mathbb{R}^d$, say $B_\epsilon(0)$, then we can decompose $X_t = Z_t + C_t$ into two independent Lévy processes whose triplets are $(b, Q, \nu(dy) - h(y)\, dy)$ and $(0, 0, h(y)\, dy)$. Since C_t is a compound Poisson process, we know from [51, Cor. 3.5] that

$$\mathbb{P}^0(C_t \in B) = e^{-ht}\delta_0(B) + e^{-ht}\sum_{k=1}^{\infty} \frac{t^k}{k!}\int_B h^{*k}(x)\, dx, \quad h = \int_{\mathbb{R}^d} h(x)\, dx.$$

Since the k-fold convolution $h^{*k}(x)$ is strictly positive in $B_{\epsilon k}(0)$, we conclude that $\mathbb{P}^0(C_t \in B + z) > 0$ for any $z \in \mathbb{R}^d$; hence, $\mathbb{P}^0(X_t \in B) = \int \mathbb{P}^0(C_t + z \in B)\, \mathbb{P}^0(Z_t \in dz) > 0$. With a little more effort, the above argument even shows that the density of X_t satisfies $p_t(x) > 0$ for all $t > 0$ and Lebesgue a.a. $x \in \mathbb{R}^d$.

It remains to check the conditions for compactness and the finiteness of the heat content stated in Lemma 7.2. Let $x_0 = 0$. Choosing the potential V in such a way that $V(x) \to \infty$ as $|x| \to \infty$, we obtain that the Feynman–Kac semigroup operators $U_t, \widehat{U}_t, t > 0$, are compact operators on $L^2(M, \mu)$. Moreover, if we know that

$$\liminf_{|x| \to \infty} \frac{V(x)}{\log|x|} > 0,$$

then there is $t > 0$ such that $Z(t) < \infty$.

7.1.2. Levy-type processes. Standard references: Jacob [20, Vol. 1], Böttcher et al. [5]. A Lévy-type or Feller process is a Markov process X_t with values in $\mathbb{R}^d$ which behaves locally like a Lévy process. In particular, the transition semigroup $P_t u(x) = \mathbb{E}^x u(X_t)$ is a Feller semigroup, and one can show that the infinitesimal generator is a pseudo-differential operator of the form $Au(x) = -F_{\xi \to x}^{-1}(p(x,\xi)\mathcal{F}u(\xi))$ ($\mathcal{F}$ denotes the Fourier transform, u is a sufficiently regular function – for example, a test function $u \in C_c^\infty(\mathbb{R}^d)$) whose symbol $p(x,\xi)$ is, for fixed x, the characteristic exponent of a Lévy process. Therefore, $p(x,\xi)$ is given by a Lévy–Khintchine formula (7.5) where the Lévy-triplet depends on x (i.e. $(b, Q, \nu(dy)) = (b(x), Q(x), \nu(x, dy))$). Thus, A is a 'Lévy generator with variable coefficients'. A typical example are stable-like generators of the form $Au(x) = (-\Delta)^{\alpha(x)}u(x)$ where $\alpha : \mathbb{R}^d \to [0, 2]$ is a variable order of differentiability and the symbol is of the form $p(x,\xi) = |\xi|^{\alpha(x)}$

It is a major problem to construct a Lévy-type process given an admissible symbol $p(x,\xi)$ or, which is the same, an x-dependent Lévy triplet. A thorough discussion is in Jacob [20, Vol. 3] and Böttcher et al. [5]; probably the most general results to-date are due to Kühn [34] and Knopova et al. [33].

In what follows, we indicate some conditions, under which the assumptions needed in Section 7.1 are satisfied. Our main intention is not to cover the most general situation, but to illustrate that within the class of Lévy-type processes, there are many examples to be found. The following assumptions on the symbol are frequently used:

$$C_1|\xi|^{\alpha_0} \leqslant \mathrm{Re}\,p(x,\xi) \qquad \forall x \in \mathbb{R}^d,\ |\xi| \geqslant 1, \qquad \text{(minimal growth)} \qquad (7.6)$$

$$|p(x,\xi) - p(y,\xi)| \leqslant C_2|x-y|^\gamma|\xi|^\alpha \qquad \forall x \in \mathbb{R}^d,\ |\xi| \geqslant 1, \qquad \text{(Hölder condition)} \qquad (7.7)$$

$$|p(x,\xi)| \leqslant C_3(1+|x|^\gamma)|\xi|^\beta \qquad \forall x \in \mathbb{R}^d,\ |\xi| \leqslant 1, \qquad \text{(growth of coefficients)} \qquad (7.8)$$

where C_1, C_2, C_3 are constants and $\alpha_0, \alpha, \beta \in (0,2]$ and $\gamma \in (0,\beta)$ are suitable exponents.

Assume that $p(x,\xi)$ is the symbol of a Lévy-type process. If either $\alpha \geqslant \gamma > \frac{\alpha}{\alpha_0}\left(\alpha - \alpha_0 + \frac{1}{2}\right)$ or $\alpha \leqslant \gamma < \frac{2}{\alpha_0} - \frac{1}{2}$, then X_t has a transition density $p_t(x,y)$ such that $\sup_x \int p_t^2(x,y)\,dy < \infty$ (i.e., (**A3**) holds for the free semigroup P_t; see [34, Thm. 2.14]).

The existence of a Lévy-type process with a given symbol $p(x,\xi)$ is a much more delicate issue. Here, we content ourselves to mentioning that in the stable-like case $p(x,\xi) = |\xi|^{\alpha(x)}$ where $\alpha : \mathbb{R}^d \to [a_1,a_2] \subset (0,2)$ is Hölder continuous, the existence of a stable-like process with a continuous density $(t,x,y) \mapsto p_t(x,y) > 0$ has been established in [34, Thm. 3.8, Thm. 3.11] as well as in [33, Section 3]. In particular, we get Feller and strong Feller processes. It is also clear that stable-like symbols satisfy the assumptions (7.6)–(7.8). We would like to mention that the methods presented in these papers go way beyond the stable-like case; [34] covers symbols of the type $p(x,\xi) = \psi_{a(x)}(|\xi|)$ where $z \mapsto \psi_a(z)$ has a holomorphic extension on a bow-tie shaped domain of the complex plane (and satisfies certain polynomial boundedness assumptions), while [33] are mainly interested in non-rotationally symmetric symbols. Therefore, (**A1**) and (**A2**) are also satisfied.

In general, it is very difficult to determine the symbol of the (formal) adjoint to A. Building on [53], the paper [54] contains the general form of the symbol of the adjoint operator for a wide class of symbols (Section 3) and a fully-worked out discussion of the case $p(x,\xi) = |\xi|^{\alpha(x)}$. If $\alpha(x)$ is Hölder, takes values in $[a_1,a_2] \subset (0,2)$ and satisfies (7.6)–(7.8), then so does the adjoint symbol $p^*(x,\xi)$ which is given by a Lévy–Khintchine formula with triplet $(0,0,w(x)|x-y|^{-d-\alpha(x)}\,dy)$ for the weight $w(x) = \alpha(x)2^{\alpha(x)-1}\Gamma\left(\frac{1}{2}\alpha(x) + \frac{1}{2}d\right)/\Gamma\left(1 - \frac{1}{2}\alpha(x)\right)$. This means that the assumptions (**A1**)–(**A3**) also hold for the adjoint free semigroup $\widehat{P}_t$ in this case. Of course, other symbols require case-by-case investigations (e.g., based on the above-mentioned literature).

7.1.3. Continuous-time Markov chains on a countably infinite uniformly discrete metric space.

Standard references: Murugan and Saloff–Coste [46, 47] and Cygan, Kaleta and Sliwiński [14]. Let (M,d) be a countably infinite, unbounded, complete metric space, which is also *uniformly discrete*; this means that there exists a number $a > 0$ such for any two distinct points $x,y \in M$, one has $d(x,y) \geqslant a$; see [46, 47]. The underlying measure $\mu : M \to (0,\infty)$ is assumed to be a *Frostman measure* with exponent $d_M > 0$ (i.e., there is a constant $c > 0$ such that $\mu(B_r(x)) \leqslant cr^{d_M}$ for any $x \in M$ and $r > 0$).

In particular, we have for all $\gamma > d_M$,

$$\int_{B_1(x_0)^c} d(x,x_0)^{-\gamma}\,\mu(dx) = \sum_{n=0}^\infty \int_{2^n \leqslant d(x,x_0) < 2^{n+1}} d(x,x_0)^{-\gamma}\,\mu(dx)$$
$$\leqslant \sum_{n=0}^\infty 2^{-\gamma n}\mu\big(B_{2^{n+1}}(x_0)\big) \leqslant c2^{d_M}\sum_{n=0}^\infty 2^{-(\gamma-d_M)n} < \infty.$$

Under these assumptions, we can give a sufficient condition for the integrability of the function $x \mapsto e^{-tV(x)}$ which is needed for the finiteness of the heat content; see Lemma 7.2.b). Indeed, if there exist $c_0, R > 0$ and $x_0 \in M$ such that

$$\frac{V(x)}{\log d(x, x_0)} \geqslant c_0 > 0, \quad d(x, x_0) \geqslant R,$$

then for every $t > d_M/c_0$, we have

$$\int_M e^{-tV(x)} \mu(dx) \leqslant e^{\|V_-\|_\infty} \mu(B_R(x_0)) + \int_{B_R(x_0)^c} d(x, x_0)^{-c_0 t} \mu(dx) < \infty.$$

Consider two probability kernels $Q, \widehat{Q} : M \times M \to [0,1]$, $\sum_{y \in M} Q(x,y) = 1 = \sum_{y \in M} \widehat{Q}(x,y)$, $x \in M$, which are connected via the duality relation with respect to the measure μ – that is,

$$\mu(x)Q(x,y) = \mu(y)\widehat{Q}(y,x), \quad x, y \in M. \tag{7.9}$$

Thus, there are two time-homogeneous Markov chains $\{Z_n : n \in \mathbb{N}_0\}$, $\{\widehat{Z}_n : n \in \mathbb{N}_0\}$ with values in M and one-step transition probabilities given by Q and $\widehat{Q}$, respectively. We denote by $\mathbb{P}^x$, $\widehat{\mathbb{P}}^x$ the laws of the chains starting at $x \in M$ – that is,

$$\mathbb{P}^x(Z_n = y) = \mathbb{P}(Z_n = y \mid Z_0 = x) \quad \text{and} \quad \widehat{\mathbb{P}}^x(\widehat{Z}_n = y) = \mathbb{P}(\widehat{Z}_n = y \mid \widehat{Z}_0 = x).$$

We have

$$\mathbb{P}^x(Z_n = y) = Q_n(x,y) \quad \text{and} \quad \widehat{\mathbb{P}}^x(\widehat{Z}_n = y) = \widehat{Q}_n(x,y),$$

where $Q_0(x,y) = \mathbb{1}_{\{x\}}(y)$, $Q_1(x,y) = Q(x,y)$ and $Q_{n+1}(x,y) = \sum_{z \in M} Q_n(x,z)Q_1(z,y)$, $n \geqslant 1$, and $\widehat{Q}_n$'s are defined accordingly. The densities with respect to μ are given by

$$q_n(x,y) = \frac{\mathbb{P}^x(Z_n = y)}{\mu(y)} = \frac{Q_n(x,y)}{\mu(y)} \quad \text{and} \quad \widehat{q}_n(x,y) = \frac{\widehat{\mathbb{P}}^x(\widehat{Z}_n = y)}{\mu(y)} = \frac{\widehat{Q}_n(x,y)}{\mu(y)}.$$

Clearly, (7.9) extends to $\mu(x)Q_n(x,y) = \mu(y)\widehat{Q}_n(y,x)$, $x, y \in M$, $n \in \mathbb{N}$, and we have $\widehat{q}_n(x,y) = q_n(y,x)$. The corresponding n-step transition operator is

$$Q_n f(x) = \sum_{y \in M} Q_n(x,y)f(y) = \sum_{y \in M} q_n(x,y)f(y)\mu(y)$$

whenever the series is finite; $\widehat{Q}_n f$ is defined in a similar way.

The main objects are the continuous-time chains $\{X_t : t \geqslant 0\}$ and $\{\widehat{X}_t : t \geqslant 0\}$ defined by $X_t := Z_{N_t}$, $\widehat{X}_t := \widehat{Z}_{N_t}$, respectively, where $\{N_t : t \geqslant 0\}$ is an independent Poisson process with parameter $\lambda = 1$. Clearly, we have $\mathbb{P}^x(X_t \in A) = \sum_{y \in A} P_t(x,y)$, $A \subset M$, where

$$P_t(x,y) = e^{-t} \sum_{n=0}^\infty \frac{t^n}{n!} Q_n(x,y), \quad t > 0, \ x \in M.$$

The transition semigroup $\{P_t : t \geqslant 0\}$ of this process is defined as

$$P_t f(x) = \sum_{y \in M} f(y) P_t(x,y) = \sum_{y \in M} f(y) p_t(x,y) \mu(y), \quad \text{with} \quad p_t(x,y) := \frac{P_t(x,y)}{\mu(y)},$$

for all admissible functions f on M. The transition probabilities $\widehat{P}_t(x,y)$, kernels $\widehat{p}_t(x,y)$ and the transition semigroup $\{\widehat{P}_t : t \geqslant 0\}$ of the process $\{\widehat{X}_t : t \geqslant 0\}$ are defined accordingly. By the Cauchy–Schwarz inequality, we can easily see that all operators P_t and $\widehat{P}_t$ are contractions on $L^2(M,\mu)$; again, we have $\widehat{p}_t(x,y) = p_t(y,x)$, which means that the weak duality relation, which was described at the beginning of Section 7.1, is in force. Moreover, it is straightforward to see that both transition semigroups are doubly Feller. In particular, the strong Feller property follows from the fact that (M,d) is uniformly discrete. One can also check that the ultracontractivity of these semigroups is equivalent to the boundedness of the kernel $q_1(x,y)$ (e.g., if $\inf_{x \in M} \mu(x) > 0$). The positivity improving property follows from the strict positivity of the kernels $P_t(x,y)$ and $\widehat{P}_t(x,y)$. This is the case if the chains $\{Z_n : n \in \mathbb{N}_0\}$, $\{\widehat{Z}_n : n \in \mathbb{N}_0\}$ are irreducible (i.e., for every $x,y \in M$, there is $n = n(x,y) \in \mathbb{N}$ such that $P_n(x,y), \widehat{P}_n(x,y) > 0$).

We can now explicitly estimate the space-rates of the quasi-ergodic behavior of the Feynman–Kac semigroups with confining potentials (see Corollary 3.5). These rates take the form $U_{t_1} \mathbb{1}_M(x)/\phi_0(x)$ and $U_{t_1}^* \mathbb{1}_M(x)/\psi_0(x)$, for some $t_1 > 0$. The functions $U_{t_1} \mathbb{1}_M(x)$ and $U_{t_1}^* \mathbb{1}_M(x)$ can be estimated by a constant or by using the upper bound in (7.1). Sharper estimates for a large class of processes will be provided in [13]. However, sharp lower estimates for the ground states for a fairly general class of processes and rather general confining potentials can be found in [14].

If we have good lower estimates for the ground states, then we can apply Lemma 4.5 and Theorem 4.6 to derive the progressive GSD and progressive exponential quasi-ergodicity from the finiteness of the heat content for large times. Such estimates can, for example, be found in [14].

7.1.4. Continuous-time simple random walk on an infinite graph. Standard reference: Barlow [4]. This is a special case of the example in the previous section: Let M be a countably infinite set and $\Gamma = (M,E)$ be a connected and locally finite graph over M. We equip M with its geodesic metric $d : M \times M \to \mathbb{N}_0$; the underlying measure and the transition probability of a simple random walk on the graph is defined in terms of a conductance network on Γ. The Frostman property of the measure $\mu = (\mu_x)_{x \in M}$ is, for example, guaranteed by the following condition: there is a constant $c_1 > 0$ such that $\mu_x \leqslant c_1$, $x \in M$, and the cardinality of a geodesic ball $B_n(x)$ is controlled by a power-type function (i.e., $\sup_{x \in M} \sup_{n \in \mathbb{N}} n^{-d_M} |B_n(x)| < \infty$ for some $d_M > 0$).

7.1.5. Brownian motion on simple nested fractals. Standard references: Lindstrøm [41], Kusuoka [37], Fukushima [19], Kumagai [35]. Let $M \subset \mathbb{R}^d$, $d \geqslant 1$, be an (unbounded) *simple nested fractal* and let $(X_t)_{t \geqslant 0}$ be a Brownian motion with the state space M. We refer the reader to [41, 37, 19] for the definition of a simple nested fractal and various constructions of Brownian motion thereon. The set M inherits the Euclidean topology from $\mathbb{R}^d$, and the measure μ is the Hausdorff measure on M (in

particular, μ is a d_M-Frostman measure, where d_M is the Hausdorff dimension of M). It is known that $(X_t)_{t\geqslant 0}$ is a Markov process with symmetric, jointly continuous and bounded transition densities that satisfy sub-Gaussian estimates [35, Section 5]. Consequently, the transition semigroup of the process $(X_t)_{t\geqslant 0}$ is a doubly Feller, ultracontractive and positivity improving semigroup.

7.1.6. Sobordinate processes. Standard references: Sato [49], Schilling, Song and Vondraček [52]. Another class of processes which is covered by our results are subordinate processes. Recall that a subordinator is an increasing Lévy process with values in $[0,\infty)$, starting at 0. It is uniquely determined by its Laplace transform: we have $\mathbb{E}e^{-\lambda S_t} = e^{-t\phi(\lambda)}$, $t > 0$, where the exponent $\phi : (0,\infty) \to [0,\infty)$ is a Bernstein function of the form

$$\phi(\lambda) = b\lambda + \int_0^\infty (1 - e^{-\lambda u})\,\rho(du).$$

Here, $b \geqslant 0$ is the drift term and ρ is a measure on $(0,\infty)$ such that $\int_0^\infty (1 \wedge u)\rho(du) < \infty$ (Lévy measure).

Let $\{X_t : t \geqslant 0\}$ and $\{\widehat{X}_t : t \geqslant 0\}$ be Markov processes as in Section 7.1, and let $\{S_t : t \geqslant 0\}$ be a subordinator which is independent of $\{X_t : t \geqslant 0\}$ and $\{\widehat{X}_t : t \geqslant 0\}$. We define the subordinate processes $\{X_t^\phi : t \geqslant 0\}$, $\{\widehat{X}_t^\phi : t \geqslant 0\}$ by the formulas $X_t^\phi := X_{S_t}$, $\widehat{X}_t^\phi := \widehat{X}_{S_t}$, respectively. By independence, their transition semigroups $\{P_t^\phi : t \geqslant 0\}$ and $\{\widehat{P}_t^\phi : t \geqslant 0\}$ are given by

$$P_t^\phi f(x) = \int_{[0,\infty)} P_s f(x)\,\mathbb{P}(S_t \in ds), \quad \widehat{P}_t^\phi f(x) = \int_{[0,\infty)} \widehat{P}_s f(x)\,\mathbb{P}(S_t \in ds),$$

for $f \in L^2(M,\mu)$ or $f \in L^\infty(M,\mu)$. It is obvious that these formulas define contractions on $f \in L^2(M,\mu)$, and the weak duality relation described in Section 7.1 holds. Moreover, these semigroups inherit the Feller property, the strong Feller property (provided that S_t is not a compound Poisson subordinator), and the positivity improving property from the semigroups $\{P_t : t \geqslant 0\}$ and $\{\widehat{P}_t : t \geqslant 0\}$. The only regularity property that requires an additional assumption is ultracontractivity: Suppose that $P_t, \widehat{P}_t$ are bounded operators from $L^2(M,\mu)$ to $L^\infty(M,\mu)$ for every $t > 0$. By Remark 2.1, this is equivalent to the boundedness from $L^1(M,\mu)$ to $L^\infty(M,\mu)$, for any $t > 0$. Therefore, ultracontractivity of the subordinate semigroup and its dual is ensured by the assumption that

$$\left\| P_t^\phi \right\|_{1,\infty} \leqslant \int_{[0,\infty)} \|P_s\|_{1,\infty}\,\mathbb{P}(S_t \in ds) < \infty, \quad t > 0.$$

7.2. Progressive uniform exponential quasi-ergodicity for the harmonic oscillator

Let $H = -\Delta + |x|^2$ be the Schrödinger operator acting in $L^2(\mathbb{R}^d, dx)$, and denote by $U_t = e^{-tH}$, $t \geqslant 0$, the operators of the corresponding Schrödinger semigroup. It is well known that, for every $t > 0$, U_t is a compact, self-adjoint and positivity-improving integral operator with the kernel

$$u_t(x,y) = (2\pi\sinh(2t))^{-d/2}\exp\left(-\frac{1}{4}\left(\tanh(t)|x+y|^2 + \coth(t)|x-y|^2\right)\right) \qquad (7.10)$$

(see, for example, [60, (1.4)]. In particular, the assumptions (**A0**)–(**A3**) are satisfied. The ground state eigenfunction and eigenvalue are given by

$$\phi_0(x) = \pi^{-d/4} e^{-\frac{|x|^2}{2}} \quad \text{and} \quad \lambda_0 = d, \tag{7.11}$$

and (see [60, Proposition 3.3][3])

$$U_t \mathbb{1}_{\mathbb{R}^d}(x) = (\cosh(2t))^{-d/2} \exp\left(-\frac{|x|^2}{2 \coth(2t)}\right), \quad t > 0. \tag{7.12}$$

It is easy to check that the semigroup $\{U_t : t \geqslant 0\}$ is not aGSD; cf. [24, Example 4.4]; in particular, using Corollary 4.4, it is not uniformly quasi-ergodic (see also the discussion in [31, Section 4.3]). We will show that our present results can be directly used to describe the true quasi-ergodic properties of this semigroup.

First of all, observe that (7.12) shows that the heat content $Z(t) = \int_{\mathbb{R}^d} U_t \mathbb{1}(x) dx$ is finite for every $t > 0$, so that our results from Section 3 apply. In particular, Theorem 3.4 ensures that the semigroup $\{U_t : t \geqslant 0\}$ is exponentially quasi-ergodic for every finite initial distribution, and the limiting measure

$$m(dx) = \frac{\phi_0(x)}{\|\phi_0\|_1} \, dx = \frac{1}{(2\pi)^{d/2}} e^{-\frac{|x|^2}{2}} \, dx$$

is the only quasi-stationary measure of $\{U_t : t \geqslant 0\}$. By Corollary 3.5, we also get a pointwise version with explicit space-rate

$$\frac{U_1 \mathbb{1}_{\mathbb{R}^d}(x)}{\phi_0(x)} = \left(\frac{\sqrt{\pi}}{\cosh 2}\right)^{d/2} \exp\left(-\frac{|x|^2}{e^4 - 1}\right),$$

which is uniform on compact sets.

We will show that the quasi-ergodicity is progressively uniform. Observe that the semigroup $\{U_t : t \geqslant 0\}$ is pGSD; cf. Definition 4.2.b).

Proposition 7.3. *Let $x \in \mathbb{R}^d$, $t > 0$ and $C > 0$. Then*

$$U_t \mathbb{1}_{\mathbb{R}^d}(x) \leqslant C e^{-\lambda_0 t} \phi_0(x) \quad \Longleftrightarrow$$

$$|x| \leqslant \sqrt{(e^{4t} + 1)\left(\log C - \frac{d}{2}\log(2\sqrt{\pi}) + \frac{d}{2}\log(1 + e^{-4t})\right)}.$$

In particular, if $C < (2\sqrt{\pi})^{d/2}$, the right-hand side is void for large t, and if $C = (2\sqrt{\pi})^{d/2}$, then the inequality on the right-hand side defines a bounded set of x for large t's.

Proof. Because of (7.11) and (7.12), the inequality on the left-hand side reads

$$\exp\left[|x|^2\left(\frac{e^{-2t}}{e^{2t} + e^{-2t}}\right)\right] \leqslant C\pi^{-\frac{d}{4}}\left(\frac{e^{2t} + e^{-2t}}{2}\right)^{\frac{d}{2}} e^{-dt} = C(2\sqrt{\pi})^{-\frac{d}{2}}(1 + e^{-4t})^{\frac{d}{2}},$$

---

[3]The statement of the result in this reference contains a minor typographical error: the factor 2π is missing before $\cosh(2t)$; the proof is, however, correct.

which can be equivalently rewritten as

$$|x|^2 \leqslant \left(1+e^{4t}\right)\left(\log C - \frac{d}{2}\log(2\sqrt{\pi}) + \frac{d}{2}\log(1+e^{-4t})\right). \qquad \square$$

Assume now that $C > (2\sqrt{\pi})^{d/2}$ and write

$$\rho(t) := Ke^{2t}\sqrt{(1+e^{-4t})\left(1+\frac{d}{2K^2}\log(1+e^{-4t})\right)}, \quad t \geqslant 1,$$

where $K = \sqrt{\log\left(C/(2\sqrt{\pi})^{d/2}\right)}$. Clearly, we have $\rho(t) = Ke^{2t}(1+o(1))$, as $t \to \infty$. It follows from Proposition 7.3 that the semigroup $\{U_t : t \geqslant 0\}$ is pGSD with the exhausting family $\{K_t : t \geqslant 1\}$ where $K_t = \overline{B}_{\rho(t)}(0)$; that is,

$$U_t \mathbb{1}_{\mathbb{R}^d}(x) \leqslant Ce^{-\lambda_0 t}\phi_0(x), \quad \text{for} \quad |x| \leqslant \rho(t), \ t \geqslant 1. \tag{7.13}$$

Therefore, we can apply Theorem 5.2 to show that the progressive exponential uniform quasi-ergodicity holds on any space L^p; it is easily seen from (7.12) that the assumption (**A4**) holds for every $t_0 > 0$. More precisely, for every $a \in (0,1)$ and $p \in [1,\infty]$, there are constants $C_1, C_2 > 0$ such that

$$\sup_{\sigma\in\mathcal{M}^1\left(\overline{B}_{\rho(at)}(0)\right)} \sup_{\substack{f\in L^p(M,\mu) \\ \|f\|_p \leqslant 1}} \left|\frac{\sigma(U_t f)}{\sigma(U_t \mathbb{1}_{\mathbb{R}^d})} - m(f)\right| \leqslant C_1 e^{-C_2 t}, \tag{7.14}$$

for sufficiently large $t > 1$.

It is interesting to note that one can use (7.10) and (7.12) to verify the following: for a family of measures $\{\sigma_t\}_{t\geqslant 1}$ given by $\sigma_t = \delta_{x_t}$ with $x_t = (e^{2t}x_0)/2$ where $x_0 \in \mathbb{R}^d$ is fixed and for $f = \mathbb{1}_K$ where K is a Borel set of finite Lebesgue measure, we have

$$\frac{U_t f(x_t)}{U_t \mathbb{1}_{\mathbb{R}^d}(x_t)} = \frac{\sigma_t(U_t f)}{\sigma_t(U_t \mathbb{1}_{\mathbb{R}^d})} \xrightarrow{t\to\infty} \frac{1}{(2\pi)^{d/2}} \int_K e^{\frac{(y-x_0)^2}{2}} \, dy. \tag{7.15}$$

This was first noticed in [31, p. 135] for $d = 1$; the aim was to show that the quasi-ergodicity property for the harmonic oscillator is not uniform in $\mathbb{R}$ (note that the authors of that paper work with a Schrödinger operator with rescaled kinetic term; that is, $Hf = (1/2)f'' + |x|^2 f$).

We can now see more. Indeed, (7.15) shows that the convergence in (7.14) cannot be uniform in the sets K_t that come from the pGSD property (7.13) – one can choose x_0 is such a way that $x_t \in K_t$ for sufficiently large t. This means that Theorem 5.2 is optimal in the sense that, in general, the pGSD property with the exhausting family $\{K_t : t \geqslant t_0\}$ implies the progressive exponential quasi-ergodicity with $\{K_{at} : t \geqslant t_0/\theta\}$ only for $a < 1$; cf. Remark 5.3.

7.3. Semigroups of non-local Schrödinger operators in the DJP setting

One of the main motivations for this paper was to describe the quasi-ergodic properties of the Schrödinger semigroups studied in a recent paper [26]. These are evolution semigroups that belong to non-local Schrödinger operators of the form $H = -L + V$, where L is the

generator of a symmetric Lévy process $\{X_t : t \geqslant 0\}$ (so-called *Lévy operator*) with the *direct jump property* (DJP in short) in $\mathbb{R}^d$, $d \geqslant 1$, and V is a sufficiently regular confining potential. Working in that generality, we identified in [26] a new regularity property of compact semigroups in L^2, that we called *progressive intrinsic ultracontractivity* (pIUC in short). It means that the regularity of the semigroup improves as $t \to \infty$. We now want to understand what is the impact of pIUC on the quasi-ergodic properties of the semigroup.

The class of processes that we consider in this section consists of symmetric (self-dual) and strong Feller Lévy processes (cf. Section 7.1.1, also for notation). In addition, we have to impose some extra regularity conditions on the density $p_t(x)$ and the Lévy measure ν (Assumptions (A1)–(A2) in [26]; we do not give details here). The key property is the following: we assume that the Lévy measure $\nu(dx)$ has a density $\nu(x)\,dx$ and that there exists a decreasing 'profile' function $f : (0,\infty) \to (0,\infty)$ such that $\nu(x) \asymp f(|x|)$, $x \neq 0$, and

$$f_1 * f_1(x) \leqslant c f_1(x), \quad x \in \mathbb{R}^d, \tag{7.16}$$

for a constant $c > 0$, where $f_1 := f \wedge 1$. This condition has a very suggestive probabilistic interpretation, and therefore, it is called the *direct jump property* (DJP). Note that this class contains many examples of jump Lévy processes and the corresponding Lévy operators which play an important role in various applications. For example, it includes the

- *fractional Laplace operator* $L = -(-\Delta)^{\alpha/2}$, $\alpha \in (0,2)$ (being the generator of the *isotropic α-stable process*); here, $\nu(x) = c_{d,\alpha}|x|^{-d-\alpha}$;
- *relativistic operator* $L = -(-\Delta + m^{2/\alpha})^{\alpha/2} + m$, $\alpha \in (0,2)$, $m > 0$ (being the generator of the *isotropic relativistic α-stable process*); here, $\nu(x) \asymp e^{-m^{1/\alpha}|x|}(1 \wedge |x|)^{-\frac{d+\alpha-1}{2}}|x|^{-\frac{d+\alpha+1}{2}}$.

We assume that $V \in L^\infty_{\text{loc}}(\mathbb{R}^d, dx)$ is a confining potential (i.e., $V(x) \to \infty$ as $|x| \to \infty$) such that there is a an increasing profile $g : [0,\infty) \to (0,\infty)$, growing at infinity not faster than an exponential function, and there are constants $c_1, c_2 \geqslant 1$ such that $c_1^{-1}g(|x|) \leqslant V(x) \leqslant c_1 g(|x|)$, for all $|x| \geqslant c_2$; see the detailed statement in [26, Assumption (A3)].

The Schrödinger operator $H = -L + V$ is well defined, bounded below and self-adjoint in $L^2(\mathbb{R}^d, dx)$. We denote by $U_t = e^{-tH}$, $t \geqslant 0$, the operators of the corresponding evolution semigroup. They are known to have the following probabilistic Feynman–Kac representation:

$$U_t f(x) = \mathbb{E}^x \left[e^{-\int_0^t V(X_s)\,ds} f(X_t) \right], \quad f \in L^2(\mathbb{R}^d, dx),\ t > 0. \tag{7.17}$$

The operators U_t are self-adjoint, and by [26, Lemma 2.3] (or Lemma 7.1) and Lemma 7.2.a), all regularity assumptions (**A0**)–(**A3**) are satisfied and (**A4**) follows from the proof of Lemma 7.2.a). Our standard reference for self-adjoint Schrödinger operators and the Feynman–Kac semigroups is the monograph [17].

We will now show how our present results apply to these semigroups. The discussion will be divided into three parts.

(1) *Heat content and exponential quasi-ergodicity.* First of all, due to Lemma 7.2.b), we know that

$$\liminf_{|x|\to\infty} \frac{V(x)}{\log|x|} > 0 \qquad \Longleftrightarrow \qquad \exists t_1 > 0 : Z(t_1) < \infty, \tag{7.18}$$

and our results in Section 3 apply. More precisely, by Theorem 3.4, the semigroup $\{U_t : t \geqslant 0\}$ is exponentially quasi-ergodic for every finite initial distribution, and the limiting measure

$$m(dx) = \frac{\phi_0(x)}{\|\phi_0\|_1}\, dx,$$

where $\phi_0(x)$ is the ground state of the operator H, is the only quasi-stationary measure of $\{U_t : t \geqslant 0\}$. Furthermore, in Corollary 3.5, we obtain a pointwise version with the space-rate $U_{t_1}\mathbb{1}_{\mathbb{R}^d}(x)/\phi_0(x)$ which is uniform on compact sets. By [25, Corollary 2.2], there is a constant $c_3 > 0$ such that

$$\phi_0(x) \geqslant c_6 \left(1 \wedge \frac{\nu(x)}{V(x)}\right), \quad x \in \mathbb{R}^d$$

(this remains true for potentials without radial profiles). However, we always have

$$U_{t_1}\mathbb{1}_{\mathbb{R}^d}(x) \leqslant e^{\|V_-\|_\infty t_1}, \quad x \in \mathbb{R}^d,$$

which leads to the following explicit estimate of the rate

$$\frac{U_{t_1}\mathbb{1}_{\mathbb{R}^d}(x)}{\phi_0(x)} \leqslant c_3 \frac{V(x)}{\nu(x)}, \quad x \in \mathbb{R}^d. \tag{7.19}$$

The function on the right-hand side tends to infinity as $|x| \to \infty$, but it is bounded on compact sets. Better estimates for $U_{t_1}\mathbb{1}_{\mathbb{R}^d}(x)$ can be found in [26, Theorem 4.8]. Observe that the sufficient condition in (7.18) does not depend on ν.

(2) *aGSD regime and uniform exponential quasi-ergodicity.* By [25, Theorem 2.6],

$$\liminf_{|x|\to\infty} \frac{V(x)}{|\log\nu(x)|} > 0 \iff \exists t_1, c_4 > 0 \ \forall x \in \mathbb{R}^d : U_{t_1}\mathbb{1}_{\mathbb{R}^d}(x) \leqslant c_4\phi_0(x). \tag{7.20}$$

Recall that the property on the right-hand side is called aGSD; see Definition 4.2.a). If we are in the aGSD regime – that is, if the growth of the potential is sufficiently large with respect to $|\log\nu(x)|$ – then the exponential quasi-ergodicity in Corollary 3.5 discussed above becomes uniform on the full space $\mathbb{R}^d$. This is exactly what we saw in Corollary 4.4.a). The same conclusion follows from [31, Theorem 1, Corollary 2]. However, now we get much more. Combining Corollary 4.4 and (7.20), we get that the uniform quasi-ergodicity of the semigroup $\{U_t : t \geqslant 0\}$ is equivalent to the condition

$$\liminf_{|x|\to\infty} \frac{V(x)}{|\log\nu(x)|} > 0.$$

If $\nu(x)$ has polynomial decay at infinity (as is, for example, the case for the fractional Laplacian), then $|\log\nu(x)| \asymp \log|x|$, for $|x|$ large enough, and the properties in (7.18) and (7.20) coincide. However, if ν decays at infinity faster than any polynomial, then (7.20)

is more restrictive than (7.18). For example, if $\nu(x)$ is exponential at infinity (e.g., for relativistic operators), then $|\log\nu(x)| \asymp |x|$, for large $|x|$.

(3) *pGSD and progressive uniform quasi-ergodicity.* Finally, we characterize the quasi-ergodic behavior of the Schrödinger semigroups $\{U_t : t \geqslant 0\}$ with confining potentials as above which do not satisfy (7.20), no matter how slow the potential may grow at infinity; this brings us out of the aGSD regime. Note that this problem was completely open. Since we want to be here as sharp as possible, we have to monitor precisely the threshold between the aGSD and the non-aGSD regime. For this reason, we follow [26, Assumption (A4)] and impose an additional technical condition on the profile g of the potential V. More precisely, we require that g depends on the profile f of the Lévy density ν in a sufficiently regular way: we consider $R_0 > 0$ such that $f(R_0) < 1$ and assume that $g(r) = h(|\log f(r)|)$, $r \geqslant R_0$, for an increasing function $h : [|\log f(R_0)|, \infty) \to (0, \infty)$ such that $h(s)/s$ is monotone. Under this assumption, the class of Schrödinger semigroups $\{U_t : t \geqslant 0\}$ with Lévy densities ν and potentials V is divided into two classes:

a) **(aGSD regime):** $\liminf_{|x|\to\infty} \frac{V(x)}{|\log\nu(x)|} > 0$, i.e. (7.20) holds;

b) **(non-aGSD regime):** $\lim_{|x|\to\infty} \frac{V(x)}{|\log\nu(x)|} = 0$;

cf. [26, Remark 5.1]. We already know from Part (2) that in the aGSD regime, the Schrödinger semigroup is uniformly exponentially quasi-ergodic, and, in fact, we have the equivalence of these two properties.

The rest of this section will be devoted to the analysis of the quasi-ergodic behavior in the non-aGSD regime. Of course, if (7.18) holds, then we know from Part (1) above that we have the exponential quasi-ergodicity with semi-explicit rate, which is uniform on compact sets. With [26, Corollary 5.6 b)], we can now give even better estimates for the rate (7.19)

$$\frac{U_{t_1}\mathbb{1}_{\mathbb{R}^d}(x)}{\phi_0(x)} \leqslant c_5 \left(1 \vee \frac{e^{-c_5 V(x)}}{\nu(x)}\right), \quad x \in \mathbb{R}^d,$$

for some $c_4, c_5 > 0$. Our theorems from Section 5 now yield much stronger results. In the paper [26], we identify a new large-time regularity property of compact semigroups, which we call progressive intrinsic ultracontractivity (pIUC). This property holds for semigroups considered here. In [26, Corollary 5.6 b)], we show that pIUC implies the following two-sided sharp estimates: there exist $\theta \in (0,1)$, $t_2 > 0$ and an increasing function $\rho : [t_2, \infty) \to (0, \infty)$, with $\lim_{t\to\infty} \rho(t) = \infty$, such that

$$U_t\mathbb{1}_{\mathbb{R}^d}(x) \asymp e^{-\lambda_0 t}\phi_0(x) \asymp e^{-\lambda_0 t}\left(1 \wedge \frac{\nu(x)}{V(x)}\right), \quad |x| \leqslant \rho(\theta t),\ t \geqslant t_2/\theta \qquad (7.21)$$

(note that the comparison constants are uniform in x and t!). The function ρ is the right-continuous generalized inverse function of $r \mapsto |\log f(r)|/g(r)$; see [26, Lemma 5.2]. Note that (7.21) is the *progressive ground state domination* (pGSD) property; cf. Definition 4.2.b).

Therefore, we see that $\{U_t : t \geqslant 0\}$ is pGSD with the exhausting family $\{K_t : t \geqslant t_2/\theta\}$ where $K_t = \overline{B}_{\rho(\theta t)}(0)$, and we can apply Theorem 5.2 to show that the progressive uniform

quasi-ergodicity on any space L^p holds. We obtain that there exist $\gamma, t_0 > 0$ such that for every $a, b \in (0,1)$ such that $a + 2b = 1$ and $p \in [1, \infty]$, there is a constant $c_6 > 0$ such that

$$\sup_{\sigma \in \mathcal{M}^1\left(\overline{B}_{\rho(a\theta t)}(0)\right)} \sup_{\substack{f \in L^p(M, \mu) \\ \|f\|_p \leqslant 1}} \left| \frac{\sigma(U_t f)}{\sigma(U_t \mathbb{1}_{\mathbb{R}^d})} - m(f) \right| \leqslant c_6 \kappa_b(t), \quad t > (4/(a \wedge b)) t_0,$$

where

$$\kappa_b(t) := e^{-\gamma b t} + \sup_{|x| \geqslant \rho(b\theta t)} U_{t_0} \mathbb{1}_{\mathbb{R}^d}(x).$$

Because of Theorem 5.5, the measure m is the only quasi-stationary distribution of the Schrödinger semigroup $\{U_t : t \geqslant 0\}$ – no matter how slowly V grows at infinity. We need to verify only the condition (5.5). To this end, we observe that by [26, Corollary 5.6 b)], there are constants $c_7, c_8 > 0$ such that

$$e^{\lambda_0 t} U_t \mathbb{1}_{\mathbb{R}^d}(x) \leqslant c_7 e^{-(c_8 V(x) - \lambda_0) t}, \quad |x| \geqslant \rho(\theta t),$$

for sufficiently large values of t. Since $V(x) \to \infty$ as $|x| \to \infty$ and $\rho(t) \to \infty$ as $t \to \infty$, this implies (5.8), which is equivalent to (5.5).

Example 7.4. Let us illustrate these results for two specific classes of Lévy measures and potentials. See Section 7.1.1 for the notation.

  a) **(polynomial Lévy densities**; cf. [26, Section 5.4, Example 5.7]): let

$$\nu(x) \asymp |x|^{-d-\alpha}(e \vee |x|)^{-\delta}, \quad \alpha \in (0,2), \ \delta \geqslant 0,$$

and let $V(x) = (1 \vee \log|x|)^\beta$, $\beta > 0$. Then we have the following:
  - if $\beta \geqslant 1$, then the corresponding Schrödinger semigroup is in the aGSD regime and exponential uniform quasi-ergodicity holds; see Part (2) above;
  - if $\beta \in (0,1)$, then the semigroup is in the non-aGSD regime; it is pGSD and progressive uniform quasi-ergodicity holds; see Part (3). We have

$$\rho(t) = \exp\left(t^{1/(1-\beta)}\right) \quad \text{and} \quad \kappa_b(t) \asymp \exp\left(-c_9\left(t \wedge t^{\frac{\beta}{1-\beta}}\right)\right), \tag{7.22}$$

    for large values of t, with different constants c_9 in the lower and in the upper bound. The rate $\kappa_b(t)$ is exponential for $\beta \geqslant 1/2$, and subexponential (stretched-exponential) for $\beta < 1/2$.
  b) **(exponential Lévy densities**; cf. [26, Section 5.5, Example 5.10]): let

$$\nu(x) \asymp e^{-m|x|}(1 \wedge |x|)^{-d-\alpha}(1 \vee |x|)^{-\delta}, \quad m > 0, \ \alpha \in (0,2), \ \delta > (d+1)/2,$$

and let $V(x) = (1 \vee |x|)^\beta$, $\beta > 0$. Then we have the following:
  - if $\beta \geqslant 1$, then the corresponding Schrödinger semigroup is in the aGSD regime and exponential uniform quasi-ergodicity holds; see Part (2) above;
  - if $\beta \in (0,1)$, then the semigroup is in the non-aGSD regime; it is pGSD and progressive uniform quasi-ergodicity holds; see Part (3). We have

$$\rho(t) = t^{1/(1-\beta)}$$

    for large values of t, but the rate $\kappa_b(t)$ takes the same form as that in (7.22).

Theorem 4.6 also shows that any semigroup $\{U_t : t \geqslant 0\}$ with finite heat content (for large times) is automatically pGSD; moreover, progressive uniform quasi-ergodicity holds with exponential time-rate. For polynomial Lévy densities, this result is not interesting as we already know that in this case, (7.18) is equivalent to (7.20). However, for faster decays, the assertions of the theorem are nontrivial. For instance, in the exponential case, one can check that Theorem 4.6 implies pGSD and progressive uniform quasi-ergodicity with the exhausting family $\{K_t : t \geqslant t_0\}$ where $K_t = \overline{B}_{ct}(0)$, for a constant $c > 0$.

Competing interest. The authors have no competing interests to declare that are relevant to the content of this article.

References

[1] D. BAKRY, I. GENTIL AND M. LEDOUX, *Analysis and Geometry of Markov Diffusion Operators*, vol. 348. Grundlehren Math. Wiss. (Berlin: Springer, 2013).

[2] R. BAÑUELOS, Intrinsic ultracontractivity and eigenfunction estimates for Schrödinger operators. *J. Funct. Anal.* **100** (1991), 181–206.

[3] R. BAÑUELOS AND M. VAN DEN BERG, Dirichlet eigenfunctions for horn-shaped regions and Laplacians on cross sections. *J. London Math. Soc. (2)* 53 (1996), 503–511.

[4] M. BARLOW, *Random Walks and Heat Kernels on Graphs*, vol. 458. London Math. Soc. Lecture Note Ser. (Cambridge: Cambridge University Press, 2017).

[5] B. BÖTTCHER, R. SCHILLING AND J. WANG *Lévy-Type Processes: Construction, Approximation and Sample Path Properties*, vol. 2099. *Lecture Notes in Math.* (Cham: Springer, 2013).

[6] R. CARMONA, W.C. MASTERS AND B. SIMON, Relativistic Schrödinger operators: asymptotic behaviour of the eigenfunctions. *J. Funct. Anal.* **91** (1990), 117–142.

[7] I. CHAVEL AND L. KARP, Large time behavior of the heat kernel: the parabolic λ-potential alternative. *Comment. Math. Helv.* **66** (1991), 541–556.

[8] X. CHEN, P. KIM AND J. WANG, Intrinsic ultracontractivity and ground state estimates of non-local Dirichlet forms on unbounded open sets. *Comm. Math. Phys.* **366** (2019), 67–117.

[9] X. CHEN, P. KIM AND J. WANG, Two-sided Dirichlet heat kernel estimates of symmetric stable processes on horn-shaped regions. *Math. Ann.* **384** (2022), 1–46.

[10] X. CHEN AND J. WANG, Intrinsic ultracontractivity of Feynman–Kac semigroups for symmetric jump processes. *J. Funct. Anal.* **270** (2016), 4152–4195.

[11] K. L. CHUNG, Doubly-Feller processes with multiplicative functionals. In Çinlar et al. (eds), *Seminar on Stochastic Processes*, vol. 12. Progr. Probab. Stat. (Boston: Birkhäuser, 1986), 63–78.

[12]   K. L. Chung and J. B. Walsh, *Markov Processes, Brownian Motion, and Time Symmetry*, vol. 249, 2nd ed. Grundlehren Math. Wiss. (New York: Springer, 2005).

[13]   W. Cygan, K. Kaleta, R. L. Schilling and M. Śliwiński, Feynman–Kac semigroups for continuous-time Markov chains on countably infinite spaces. In progress.

[14]   W. Cygan, K. Kaleta and M. Śliwiński, Decay of harmonic functions for discrete time Feynman–Kac operators with confining potentials. *ALEA Lat. Am. J. Probab. Math. Stat.* **19** (2022), 1071–1101.

[15]   E. B. Davies, *Heat Kernels and Spectral Theory*, vol. 92. Cambridge Tracts in Math. (Cambridge: Cambridge University Press, 1990).

[16]   E. B. Davies and B. Simon, Ultracontractivity and the heat kernel for Schrödinger operators and Dirichlet Laplacians. *J. Funct. Anal.* **59** (1984), 335–395.

[17]   M. Demuth and J. A. van Casteren, *Stochastic Spectral Theory for Self-adjoint Feller Operators. A Functional Analysis Approach.* Probab. Appl. (Basel: Birkhäuser, 2000).

[18]   R. Engelking, *General Topology. Revised and Completed Edition.* (Berlin: Heldermann Verlag, 1989). (1st ed., PWN, Warsaw 1976).

[19]   M. Fukushima, Dirichlet forms, diffusion processes, and spectral dimensions for nested fractals. In *Ideas and Methods in Stochastic Analysis*, (Stochastics and Applications. Cambridge: Cambridge University Press, 1992), 151–161.

[20]   N. Jacob, *Pseudo-Differential Operators and Markov Processes: Markov Processes and Applications 1,2,3.* (London: Imperial College Press, 2001–2005).

[21]   N. Jacob and R. L. Schilling, Towards an Lp-potential theory for sub-Markovian semigroups: kernels and capacities. *Acta Math. Sinica* **22** (2006), 1227–1250.

[22]   N. Jacob and F. Y. Wang, Higher order eigenvalues for non-local Schrödinger operators. *Commun. Pure Appl. Anal.* **17** (2018), 191–208.

[23]   K. Kaleta and T. Kulczycki, Intrinsic ultracontractivity for Schrödinger operators based on fractional Laplacians. *Potential Anal.* **33** (2010), 313–339.

[24]   K. Kaleta, M. Kwaśnicki and J. Lőrinczi, Contractivity and ground state domination properties for non-local Schrödinger operators. *J. Spectr. Theory* **8** (2018), 165–189.

[25]   K. Kaleta and J. Lőrinczi, Pointwise eigenfunction estimates and intrinsic ultracontractivity-type properties of Feynman-Kac semigroups for a class of Lévy processes. *Ann. Probab.* **43** (2015), 1350–1398.

[26]   K. Kaleta and R. L. Schilling, Progressive intrinsic ultracontractivity and heat kernel estimates for non-local Schrödinger operators. *J. Funct. Anal.* **278**, 108606 (2020), 69 pp.

[27]   M. Keller, D. Lenz, H. Vogt and R. Wojciechowski, Note on basic features of large time behaviour of heat kernels. *J. Reine Angew. Math.* **2015** (2015), 73–95.

[28]   P. Kim and R. Song, Intrinsic ultracontractivity of non-symmetric diffusion semigroups in bounded domains. *Tôhoku Math. J.* **60** (2008), 527–547.

[29]   P. Kim and R. Song, Intrinsic ultracontractivity of non-symmetric diffusions with measure-valued drifts and potentials. *Ann. Probab.* **36** (2008), 1904–1945.

[30]   P. Kim and R. Song, Intrinsic ultracontractivity for non-symmetric Lévy processes. *Forum Math.* **21** (2009), 43–66.

[31]   R. Knobloch and L. Partzsch, Uniform conditional ergodicity and intrinsic ultracontractivity. *Potential Anal.* **33** (2010), 107–136.

[32]   V. Knopova and R. L. Schilling, A note on the existence of transition probability densities for Lévy processes. *Forum Math.* **25** (2013), 125–149.

[33]   V. Knopova, A. Kulik and R. L. Schilling, Construction and heat kernel estimates of general stable-like Markov processes. *Dissertationes Math.* **569** (2021), 86 pp.

[34]   F. Kühn, *Lévy-Type Processes: Moments, Construction and Heat Kernel Estimates*, vol. 2187. Lecture Notes in Math. (Berlin: Springer, 2017).

[35] T. KUMAGAI, Estimates of transition densities for Brownian motion on nested fractals. *Probab. Theory Related Fields* **96** (1993), 205–224.

[36] T. KULCZYCKI AND B. SIUDEJA, Intrinsic ultracontractivity of the Feynman-Kac semi-group for relativistic stable processes. *Trans. Amer. Math. Soc.* **358** (2006), 5025–5057.

[37] S. KUSUOKA, Lecture on diffusion processes on nested fractals. In *Statistical Mechanics and Fractals*, vol. 1567. Springer Lecture Notes in Math. (Berlin: Springer, 1993).

[38] M. KWAŚNICKI, Intrinsic ultracontractivity for stable semigroups on unbounded open sets. *Potential Anal.* **31** (2009), 57–77.

[39] M. KWAŚNICKI AND J. MUCHA, Extension technique for complete Bernstein functions of the Laplace operator. *J. Evol. Equ.* **18** (2018), 1341–1379.

[40] D. LENZ, P. STOLLMANN AND D. WINGERT, Compactness of Schrödinger semigroups. *Math. Nachr.* **283** (2010), 94–103.

[41] T. LINDSTRØM, *Brownian motion on nested fractals.* (Mem. Amer. Math. Soc, 1990), 83.

[42] G. METAFUNE, D. PALLARA AND A. RHANDI, Kernel estimates for Schrödinger operators. *J. Evol. Equ.* **6** (2006), 433–457.

[43] G. METAFUNE AND C. SPINA, Kernel estimates for a class of Schrödinger semigroups. *J. Evol. Equ.* **7** (2007), 719–742.

[44] EL. M. OUHABAZ AND A. RHANDI, Kernel and eigenfunction estimates for some second order elliptic operators. *J. Math. Anal. Appl.* **387** (2012), 799–806.

[45] EL. M. OUHABAZ, Sharp Gaussian bounds and Lp-growth of semigroups associated with elliptic and Schrödinger operators. *Proc. Amer. Math. Soc.* **134** (2006), 3567–3575.

[46] M. MURUGAN AND L. SALOFF-COSTE, Transition probability estimates for long range random walks. *New York J. Math.* **21** (2015), 723–757.

[47] M. MURUGAN AND L. SALOFF-COSTE, Heat kernel estimates for anomalous heavy-tailed random walks. *Ann. Inst. Henri Poincaré Probab. Stat.* **55** (2019), 697–719.

[48] R. G. PINSKY, The lifetimes of conditioned diffusion processes. *Ann. Inst. Henri Poincaré Probab. Stat.* **26** (1990), 87–99.

[49] K. SATO, *Lévy Processes and Infinitely Divisible Distributions*, vol. 68, revised ed. *Cambridge Studies Adv. Math.* (Cambridge: Cambridge University Press, 2013).

[50] H. H. SCHAEFER, *Banach Lattices and Positive Operators*, vol. 215. Grundlehren Math. Wiss. (New York: Springer, 1974).

[51] R. SCHILLING, An introduction to Lévy and Feller processes. In Khoshnevisan, D and Schilling, R (eds), *From Lévy-Type Processes to Parabolic SPDEs*. (Cham: Birkhäuser, 2016).

[52] R. SCHILLING, R. SONG AND Z. VONDRAČEK, *Bernstein Functions*, vol. 37, 2nd ed. de Gruyter Studies in Math. (Berlin: Walter de Gruyter, 2012).

[53] R. L. SCHILLING AND T. UEMURA, On the Feller property of Dirichlet forms generated by pseudo differential operators. *Tôhoku J. Math.* **59** (2007), 401–422.

[54] R. L. SCHILLING AND J. WANG, Lower bounded semi-Dirichlet forms associated with Lévy type operators. In Chen, ZQ, Jacob, N, Takeda, M and Uemura, T (eds), *Festschrift Masatoshi Fukushima*. In Honor of Masatoshi Fukushima's Sanju. (New Jersey, World Scientific, 2015), 507–526.

[55] Y. SHIOZAWA AND J. WANG, Compactness of semigroups generated by symmetric non-local Dirichlet forms with unbounded coefficients. *Potential Anal.* **58** (2023), 373–392.

[56] A. SIKORA, On-diagonal estimates on Schrödinger semigroup kernels and reduced heat kernels. *Commun. Math. Phys.* **188** (1997), 233–249.

[57] B. SIMON, Pointwise bounds on eigenfunctions and wave packets in N-body quantum systems. III. *Trans. Amer. Math. Soc.*, (1975), 317–329.

[58] B. SIMON, Schrödinger semigroups. *Bull. Amer. Math. Soc. (N.S.)* **7** (1982), 447–526.

[59] B. SIMON, Large time behavior of the heat kernel: On a theorem of Chavel and Karp. *Proc. Amer. Math. Soc.* **118** (1993), 513–514.

[60] K. STEMPAK AND J. L. TORREA, BMO results for operators associated to Hermite expansions. *Illinois J. Math.* **49** (2005), 1111–1131.

[61] M. TAKEDA, Y. TAWARA AND K. TSUCHIDA, Compactness of Markov and Schrödinger semigroups: a probabilistic approach. *Osaka J. Math.* **54** (2017), 517–532.

[62] M. TAKEDA, Existence and uniqueness of quasi-stationary distributions for symmetric Markov processes with tightness property. *J. Theoret. Probab.* **32** (2019), 2006–2019.

[63] F. Y. WANG AND J. L. WU, Compactness of Schrödinger semigroups with unbounded below potentials. *Bull. Sci. Math.* **132** (2008), 679–689.

[64] J. WANG, Compactness and density estimates for weighted fractional heat semigroups. *J. Theoret. Probab.* **32** (2019), 2066–2087.

[65] J. ZHANG, S. LI AND R. SONG, Quasi-stationarity and quasi-ergodicity of general Markov processes. *Sci. China Ser. A* **57** (2014), 2013–2024.