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REVERSIBLE SYMMETRIC PERIODIC SOLUTIONS IN SPATIAL PERTURBED SYSTEMS

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Abstract. We study the existence of several families of reversible-symmetric periodic solutions in a three-dimensional system of differential equations that admits a reversible symmetry and includes differentiable perturbations depending on a small parameter ε, where the periodic solutions arise as continuations of circular solutions from the unperturbed system. We essentially impose symmetric constraints on the initial conditions and make use of the Poincaré continuation method. Both fixed-period and variable-period reversible-symmetric solutions are obtained. We provide sufficient conditions for their existence, expressed in terms of the perturbation functions. In addition, we compute the characteristic multipliers of these families of reversible-symmetric periodic solutions. We also compare different types of reversible symmetries with results from the averaging method. Several examples illustrating our results are presented.

1. Introduction

The objective of this paper is to demonstrate the existence of different families of reversible symmetric periodic solutions in close approximation to circular periodic solutions of some three-dimensional unperturbed ordinary differential equations (ODEs). In more precise terms, it is assumed that the ordinary differential equations are analytic and are symmetric with respect to some reversible symmetry, called S. That is, we consider the system

$$\dot{x} = X(x, \varepsilon) \tag{1}$$

where $X : U \times \mathbb{R} \subset \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ (U is an open set of $\mathbb{R}^3$) is an analytic function in both variables, ε is assumed to be a small parameter. Let $S : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation in involution, that is, $S^2 = I$. A vector field X is said S-reversible symmetric if $X \circ S = -S \circ X$ for all $x \in U$, with ε fixed; that is, $X(Sx, \varepsilon) = -S(X(x, \varepsilon))$. Additionally, one solution $\mathbf{x}(t, \varepsilon)$ of the system (1) is called S-reversible symmetric if $\mathbf{x}(t, \varepsilon) = S(\mathbf{x}(-t, \varepsilon))$ (see [21]).

In order to make this work self-contained, we recall the following basic but essential result for our purposes.

Lemma 1.1. *Let $S : \mathbb{R}^3 \to \mathbb{R}^3$ be a reversible symmetry of the vector field (1). We denote the set of the fixed points of the symmetry S as $\mathscr{L}$:=Fix$(S) = \{\xi \in \mathbb{R}^3 : S(\xi) = \xi\}$. Assume that $\mathbf{x}(t, \xi, \varepsilon)$ is solution of (1) such that:*

1) $\xi \in \mathscr{L}$,
2) $\mathbf{x}(\tau/2, \xi, \varepsilon) \in \mathscr{L}$,

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then $\mathbf{x}(t,\xi,\varepsilon)$ *is a* τ*-periodic solution of* (1)*. Moreover, this solution is* S*-reversible symmetric, that is,* $\mathbf{x}(t) = S(\mathbf{x}(-t))$*.*

The proof of this lemma can be found in [21].

The existence of periodic solutions has been the subject of study by many authors, who have employed a variety of approaches. It is important to recall the relevant literature on this topic, including the works of Poincaré [22], J. Hale [10, 11], Farkas [7], Sanders et al. [23], and Verhulst [24], among others. For example, in [4], Duistermaat considers ordinary differential equations of the form $\dot{x} = f(t,x,\omega,\varepsilon)$, with $x \in \mathbb{R}^n$, where f is periodic in t with period ω. This study addresses the existence of periodic solutions as continuations of a known periodic solution of the unperturbed system $\dot{x} = f(t,x,\omega_0,\varepsilon_0)$ with initial condition ξ_0. In another paper [5], the same problem is considered, but under the assumption of the existence of a first integral. In both cases, the existence of periodic solutions is reduced to finding appropriate initial conditions ξ of the unperturbed system, or equivalently, to solving the periodicity equation $\phi(\xi,\omega,\varepsilon) = x(\omega,\xi,\omega,\varepsilon) - \xi = 0$. In this way, one obtains a periodic solution with initial condition ξ and period ω. The authors study the case in which the rank of $D_\xi\phi(\omega_0,\xi_0,\omega_0,\varepsilon_0)$ is maximal. However, in the autonomous case, it is known that this rank is never maximal, and thus these results cannot be directly applied. The work [5] also considers the case in which the dimension of $\ker D_\xi\phi(\omega_0,\xi_0,\omega_0,\varepsilon_0)$ is one. Lamb and Roberts [18] provide a comprehensive overview of the role of time-reversal symmetry in dynamical systems, with particular emphasis on its connections to Hamiltonian and equivariant structures. Their work surveys major developments in the theory of reversible systems, including symmetric periodic orbits, local bifurcations, homoclinic phenomena, and renormalization techniques. They also explore various contexts, both in physics and mathematics, where time-reversible dynamics naturally arise. The study includes an extensive bibliography, making it a valuable resource for further exploration of time-reversal symmetry in dynamical systems.

In this work, we present results aimed at obtaining reversible symmetric periodic solutions as a consequence of imposing symmetry on the vector field of the ordinary differential equations. In the *Concluding Remarks* section, we analyze each reversible symmetry considered in this paper, comparing two different approaches: the method developed herein and the Averaging Method. We highlight the differences between these methods and discuss the type of information each provides regarding the existence of periodic solutions.

In this paper, we consider the following family of ODEs in $\mathbb{R}^3$ of the form

$$\begin{aligned}
\dot{x}_1 &= -x_2 + \varepsilon^\alpha f_1(x_1,x_2,x_3) + \varepsilon^{\alpha+\ell} R_1(x_1,x_2,x_3,\varepsilon),\\
\dot{x}_2 &= x_1 + \varepsilon^\alpha g_1(x_1,x_2,x_3) + \varepsilon^{\alpha+\ell} R_2(x_1,x_2,x_3,\varepsilon),\\
\dot{x}_3 &= \phantom{-x_1 +{}} \varepsilon^\alpha h_1(x_1,x_2,x_3) + \varepsilon^{\alpha+\ell} R_3(x_1,x_2,x_3,\varepsilon),
\end{aligned} \tag{2}$$

where $\alpha, \ell \in \mathbb{N}$. Here, the dot denotes the derivative with respect to the independent real variable t, and ε is a small parameter. In these differential systems, the functions f_1, g_1, h_1, and R_i are analytic functions in the Cartesian variables, and moreover, R_i are also analytic in ε. As we will see throughout this paper, these conditions can be weakened by requiring the functions to belong to the appropriate class C^k. Of course, in the spatial case, the origin is an equilibrium point and $\dim W^c(0,0,0) = 3$; in particular, for $\varepsilon = 0$, the xy-plane is foliated by periodic solutions of fixed period 2π.

There are many works in the literature that address the problem of symmetries and their implications in ordinary differential equations (ODEs). In particular, we are interested in works concerning reversible or time-reversible symmetries, where conditions for the existence of symmetric periodic solutions are established.

For example, in [13], the authors consider the system $\dot{x} = f(t,x)$, where $x \in \mathbb{R}^n$ and f is T-periodic in t. It is assumed that $f(-t,Qx) = -Qf(t,x)$ for all t and x, for a certain constant matrix Q. In that work, the authors prove a simple theorem that characterizes the existence of periodic solutions of the system possessing the symmetry property induced by Q. This result is similar to that given in our Lemma 1.1. Moreover, in the two-dimensional case, periodic solutions are constructed using elementary geometrical and analytical arguments together with the presence of reversible symmetries.

In [8], the authors consider a nonlinear ODE of the form $\dot{x} = \varepsilon f(t,x)$, where $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, and $\varepsilon \in \mathbb{R}$ is a small parameter. It is assumed that f satisfies the following symmetry: there exist a matrix A and a function $\psi \in C^1(\mathbb{R},\mathbb{R})$ such that $Af(t,x) = \psi'(t)f(\psi(t),Ax)$. For this class of weakly nonlinear ODEs, the authors prove the existence of a unique symmetric solution and provide information about its stability.

In [3], the authors consider a nonlinear ODE of the form $\dot{x} = \varepsilon f(t,x,\mu)$, where $x \in \mathbb{R}^n$, $\mu \in \mathbb{R}^k$, $t \in \mathbb{R}$, and $\varepsilon \in \mathbb{R}$ is a small parameter. The symmetry condition assumed is $Af(t,x,\mu) = -f(-t-\tau,Ax,\mu)$, where A is a constant matrix, $\tau \in \mathbb{R}$ is fixed, and f is τ-periodic in t. The authors establish two results about the existence of periodic symmetric solutions for this class of weakly nonlinear ODEs, under the assumption that the initial condition satisfies $x_0 \in \ker(I-A)$. They analyze the cases when $\ker(I-A) = \{0\}$ and $\ker(I-A) \neq \{0\}$, considering parametric dependence and time-reversal symmetries. Furthermore, they rigorously analyze the local asymptotic behavior of these solutions. It is worth noting that these previous works consider spatially non-autonomous ODEs, unlike the approach presented in this paper.

In [17], the authors introduce a general reduction method to study periodic solutions near equilibrium points in autonomous reversible systems. They consider the autonomous differential equation $\dot{x} = f(x,\lambda)$, where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a smooth vector field satisfying $f(0,\lambda) = 0$ for all $\lambda \in \mathbb{R}^m$. For some $\lambda_0 \in \mathbb{R}^m$, the linear operator $A_0 := D_x f(0,\lambda_0) \in \mathscr{L}(\mathbb{R}^n)$ is nonsingular. The system is assumed to be reversible, that is, there exists a linear involution $R_0 \in \mathscr{L}(\mathbb{R}^n)$ (i.e., $R_0^2 = I$) such that $f(R_0x,\lambda) = -R_0 f(x,\lambda)$. No restrictions are imposed on the linearization at the equilibrium, allowing for higher multiplicities and all types of resonances. The problem is reduced to one for a lower-dimensional system, which is either conservative or reversible and also possesses an additional symmetry. The proof combines normal forms and the Lyapunov-Schmidt reduction method.

There are also several studies in the literature where the proposed system undergoes a zero-Hopf bifurcation. Examples include predator-prey models and Chua's system, a classical model in electronic circuits. Some references include [6, 19, 25, 9, 12, 15, 1, 2]. In these works, the existence of periodic solutions is proven as a bifurcation from an equilibrium

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point with eigenvalues $\pm i$ and 0. The existence of such periodic solutions, typically with variable period, is often established using Averaging theory and cylindrical coordinates.

In our problem, the unperturbed system (2) also admits a zero-Hopf bifurcation, i.e., its linearization has eigenvalues $\pm i$ and 0. Therefore, the continuation of periodic solutions in the perturbed system is studied using symmetry conditions to obtain symmetric periodic solutions.

In [16], Kassa, Llibre, and Makhlouf provide necessary and sufficient conditions for the existence of two or more limit cycles bifurcating from a zero-Hopf equilibrium in the following three-dimensional Lipschitzian differential system:

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = -a|x| - y + 3y^3 - xz - b, \tag{3}$$

when $a = b = 0$. Although the system is not initially in the form of (2), the authors apply changes of coordinates and introduce a small parameter by setting $a = \varepsilon$, $b = \varepsilon^2\beta$, and rescaling $(x, y, z) = \varepsilon(X, Y, Z)$. Then they apply the transformation $(X, Y, Z) = (u + w, -v, -u)$ to rewrite the system in a form resembling ours. Finally, they use cylindrical coordinates and take θ as a new independent variable to compute the averaged function. By applying averaging theory, they prove the existence and uniqueness of a T-periodic solution based on the analysis of two equations. In contrast, our method often requires analyzing only one equation due to the presence of reversible symmetries.

The main objective of this paper is to carry out an analytical study that provides sufficient conditions for the existence of several families of reversible symmetric periodic solutions of the spatial system (2), close to the circular solutions of the corresponding unperturbed differentiable system. We assume that the perturbed system (2) possesses a reversible symmetry, specifically reflections with respect to the coordinate axes or coordinate planes. More precisely, we aim to determine under which conditions the circular orbits of the system

$$\dot{x}_1 = -x_2, \quad \dot{x}_2 = x_1, \quad \dot{x}_3 = 0$$

can be continued as periodic orbits of the full system. To carry out our work, we introduce the change to cylindrical coordinates:

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta, \quad x_3 = z,$$

then the system (2) takes the form

$$\begin{aligned}
\dot{r} &= \varepsilon^\alpha \left[\tilde{f}_1(r,\theta,z)\cos\theta + \tilde{g}_1(r,\theta,z)\sin\theta\right] + \mathcal{O}(\varepsilon^{\alpha+1}),\\
\dot{\theta} &= 1 + \frac{\varepsilon^\alpha}{r}\left[\tilde{g}_1(r,\theta,z)\cos\theta - \tilde{f}_1(r,\theta,z)\sin\theta\right] + \mathcal{O}(\varepsilon^{\alpha+1}),\\
\dot{z} &= \varepsilon^\alpha \tilde{h}_1(r,\theta,z) + \mathcal{O}(\varepsilon^{\alpha+1}),
\end{aligned} \tag{4}$$

where $\tilde{f}_1(r,\theta,z) = f_1(r\cos\theta, r\sin\theta, z)$, $\tilde{g}_1(r,\theta,z) = g_1(r\cos\theta, r\sin\theta, z)$, and $\tilde{h}_1(r,\theta,z) = h_1(r\cos\theta, r\sin\theta, z)$.

We write the previous system in the reduced form:

$$\begin{aligned}
\dot{r} &= \varepsilon^\alpha F_1(r,\theta,z) + \mathcal{O}(\varepsilon^{\alpha+1}),\\
\dot{\theta} &= 1 + \varepsilon^\alpha G_1(r,\theta,z) + \mathcal{O}(\varepsilon^{\alpha+1}),\\
\dot{z} &= \varepsilon^\alpha K_1(r,\theta,z) + \mathcal{O}(\varepsilon^{\alpha+1}),
\end{aligned} \tag{5}$$

where the functions F_1, G_1, and K_1 are given by

$$
\begin{aligned}
F_1(r,\theta,z) &= \tilde{f}_1(r,\theta,z)\cos\theta + \tilde{g}_1(r,\theta,z)\sin\theta,\\
G_1(r,\theta,z) &= \frac{1}{r}\left[\tilde{g}_1(r,\theta,z)\cos\theta - \tilde{f}_1(r,\theta,z)\sin\theta\right],\\
K_1(r,\theta,z) &= \tilde{h}_1(r,\theta,z).
\end{aligned}
\tag{6}
$$

Remark 1. *For simplicity, and to avoid complications arising from a possible singularity at $r = 0$ in system (5), we consider only the case where $r > 0$.*

We are now ready to state our first result for the spatial case, which provides sufficient conditions for the existence of T_j-reversible symmetric periodic solutions of (5), both with fixed period 2π and with a period close to 2π (but not necessarily fixed). We consider reversible symmetries such that one maps $x_1 \mapsto -x_1$ and the other maps $x_2 \mapsto -x_2$.

Theorem 1.1 (T_j-reversible symmetric periodic solutions)**.** *Let $\delta_1 = 1$ and $\delta_2 = 0$. Assume that the system (2) admits a reversible symmetry of the form*

$$T_j(x_1,x_2,x_3) = \left((-1)^j x_1,\ (-1)^{j+1}x_2,\ x_3\right), \quad j = 1,2.$$

Let $(r_0(t),\theta_0(t),z_0(t)) = \left(r_0+\delta r,\ t+\delta_j\frac{\pi}{2},\ z_0+\delta z\right)$ be a T_j-symmetric and 2π-periodic solution of the unperturbed system (5), of radius $r_0+\delta r$ and lying in the plane $z = z_0+\delta z$ with δr and δz sufficiently small. Suppose that there exist r_0 and z_0 such that the following conditions are satisfied:

a) $\displaystyle\int_0^\pi \left[\tilde{g}_1\left(r_0, s+\delta_j\frac{\pi}{2}, z_0\right)\cos\left(s+\delta_j\frac{\pi}{2}\right) - \tilde{f}_1\left(r_0, s+\delta_j\frac{\pi}{2}, z_0\right)\sin\left(s+\delta_j\frac{\pi}{2}\right)\right] ds = 0,$

b) $\displaystyle\int_0^\pi \left[\frac{\partial\tilde{g}_1}{\partial r}\left(r_0, s+\delta_j\frac{\pi}{2}, z_0\right)\cos\left(s+\delta_j\frac{\pi}{2}\right) - \frac{\partial\tilde{f}_1}{\partial r}\left(r_0, s+\delta_j\frac{\pi}{2}, z_0\right)\sin\left(s+\delta_j\frac{\pi}{2}\right)\right] ds \neq 0,$

c) $\displaystyle\int_0^\pi \left[\frac{\partial\tilde{g}_1}{\partial z}\left(r_0, s+\delta_j\frac{\pi}{2}, z_0\right)\cos\left(s+\delta_j\frac{\pi}{2}\right) - \frac{\partial\tilde{f}_1}{\partial z}\left(r_0, s+\delta_j\frac{\pi}{2}, z_0\right)\sin\left(s+\delta_j\frac{\pi}{2}\right)\right] ds \neq 0.$

Then the following holds:

i) ***(Fixed period and two parameters** $(\delta z,\varepsilon)$**)** If conditions a) and b) hold for $j = 1$ or $j = 2$, and ε is sufficiently small, then there exists a unique family of initial conditions $(r(\varepsilon),\theta(\varepsilon),z(\varepsilon)) = \left(r_0+\delta r(\delta z,\varepsilon),\ \delta_j\frac{\pi}{2},\ z_0+\delta z\right)$, depending on two parameters, with $\delta r(0,0) = 0$, such that each of them defines a family of T_j-reversible symmetric periodic solutions (for $j = 1$ or $j = 2$) of the full system (2) with fixed period $T = 2\pi$. Moreover, these solutions are close to the circular solution of radius r_0 and near the plane $z = z_0$.*

ii) ***(Fixed period and two parameters** $(\delta r,\varepsilon)$**)** If conditions a) and c) hold for $j = 1$ or $j = 2$, and δr and ε are sufficiently small, then there exists a unique family of initial conditions $(r(\varepsilon),\theta(\varepsilon),z(\varepsilon)) = \left(r_0+\delta r,\ \delta_j\frac{\pi}{2},\ z_0+\delta z(\delta r,\varepsilon)\right)$, depending on two parameters, with $\delta z(0,0) = 0$, such that each of them defines a family of T_j-reversible symmetric periodic solutions (for $j = 1$ or $j = 2$) of the full system (2) with fixed period $T = 2\pi$. Moreover, these solutions are close to the circular solution of radius r_0 and near the plane $z = z_0$.*

iii) ***(Variable period and three parameters** $(\delta r,\delta z,\varepsilon)$**)** If condition a) holds for $j = 1$ or $j = 2$, and δr, δz, and ε are sufficiently small, then there exists a unique*

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differentiable function $\tau(\delta r, \delta z, \varepsilon)$ *with* $\tau(0,0,0) = \pi$*, such that each initial condition* $(r_0 + \delta r,\ \delta_j \frac{\pi}{2},\ z_0 + \delta z)$ *defines a family of* T_j*-reversible symmetric periodic solutions of the full system* (2)*, depending on three parameters* $(\delta r, \delta z, \varepsilon)$*, with period close to* $2\tau(\delta r, \delta z, \varepsilon) = 2\pi + \mathcal{O}(\varepsilon^\alpha)$*. Moreover, these solutions are close to the circular solution of radius* r_0 *and near the plane* $z = z_0$*.*

Furthermore, the characteristic multipliers of the T_1*-symmetric and* T_2*-symmetric periodic solutions* $\varphi(t, (r(\varepsilon), \theta(\varepsilon), z(\varepsilon)), \varepsilon)$ *are, respectively* $1,\ 1+\varepsilon^\alpha\lambda_1+\mathcal{O}(\varepsilon^{\alpha+1}),\ 1+\varepsilon^\alpha\lambda_2+\mathcal{O}(\varepsilon^{\alpha+1})$ *and* $1,\ 1+\varepsilon^\alpha\lambda_3+\mathcal{O}(\varepsilon^{\alpha+1}),\ 1+\varepsilon^\alpha\lambda_4+\mathcal{O}(\varepsilon^{\alpha+1})$*.*

In Section 2, we present the proof of this theorem. Note that the conditions of the previous theorem involve only the functions f_1 and g_1, however h_1 must have the reversible symmetry condition. We include in Theorem 1.1 case *iii*), that is, the existence of periodic solutions with a period close to 2π but not fixed, because it requires fewer restrictions. Our second result assumes the existence of the reversible symmetry $T_3(x_1, x_2, x_3) = (x_1, x_2, -x_3)$, that is, reflection with respect to the plane x_1x_2.

Theorem 1.2 (T_3-reversible symmetric periodic solutions)**.** *Assume that the 96system* (2) *possesses the reversible symmetry* $T_3(x_1, x_2, x_3) = (x_1, x_2, -x_3)$*. Let* $(r_0(t), \theta_0(t), z_0(t)) = (r_0 + \delta r, t + \theta_0, 0)$ *(with* $\theta_0 \in [0, 2\pi)$ *fixed) be a* T_3*-symmetric and* 2π*-periodic solution of the unperturbed system in* (5)*, of radius* $r_0 + \delta r$ *with* δr *sufficiently small, and lying on the plane* $z = 0$*. Assume that there exist* (r_0, θ_0) *such that the following conditions are satisfied:*

a) $\displaystyle\int_0^\pi \tilde{h}_1(r_0, s + \theta_0, 0)\, ds = 0,$

b) $\displaystyle\int_0^\pi \frac{\partial \tilde{h}_1}{\partial r}(r_0, s + \theta_0, 0)\, ds \neq 0,$

c) $\tilde{h}_1(r_0, \pi + \theta_0, 0) \neq 0.$

We have the following:

i) ***(Fixed period and one parameter*** ε***)*** *If conditions a) and b) hold and* ε *is sufficiently small, then there exists a unique family of initial conditions* $(r(\varepsilon), \theta(\varepsilon), z(\varepsilon)) = (r_0 + \delta r(\varepsilon), \theta_0, 0)$ *depending on one parameter, with* $\delta r(0) = 0$*, such that each of them generates a family of* T_3*-reversible symmetric periodic solutions of the complete system* (2) *with fixed period* $T = 2\pi$*. Moreover, these solutions are close to the circular solution of radius* r_0 *and near the plane* $z = 0$*.*

ii) ***(Variable period and two parameters*** $(\delta r, \varepsilon)$***)*** *If conditions a) and c) hold and both* δr *and* ε *are sufficiently small, then there exists a unique differentiable function* $\tau(\delta r, \varepsilon)$ *with* $\tau(0,0) = \pi$*, such that any initial condition* $(r_0 + \delta r, \theta_0, 0)$ *generates a family of* T_3*-reversible symmetric periodic solutions of the complete system* (2)*, depending on the two parameters* $(\delta r, \varepsilon)$*, with period close to* $2\tau(\delta r, \varepsilon) = 2\pi + \mathcal{O}(\varepsilon^\alpha)$*. Moreover, they are close to a circular solution of radius* r_0 *and near the plane* $z = 0$*.*

Furthermore, the characteristic multipliers of the periodic solution $\varphi(t, (r(\varepsilon), \theta(\varepsilon), z(\varepsilon)), \varepsilon)$ *are* $1, 1+\varepsilon^\alpha\lambda_1+\mathcal{O}(\varepsilon^{\alpha+1}), 1+\varepsilon^\alpha\lambda_2+\mathcal{O}(\varepsilon^{\alpha+1})$*.*

Note that in the above theorem the conditions involve only the function h_1. The figure 1 shows the existence of the curve of initial conditions that give rise T_3-reversible symmetric periodic solution of period fixed close to 2π and it is on the $z = 0$ plane.

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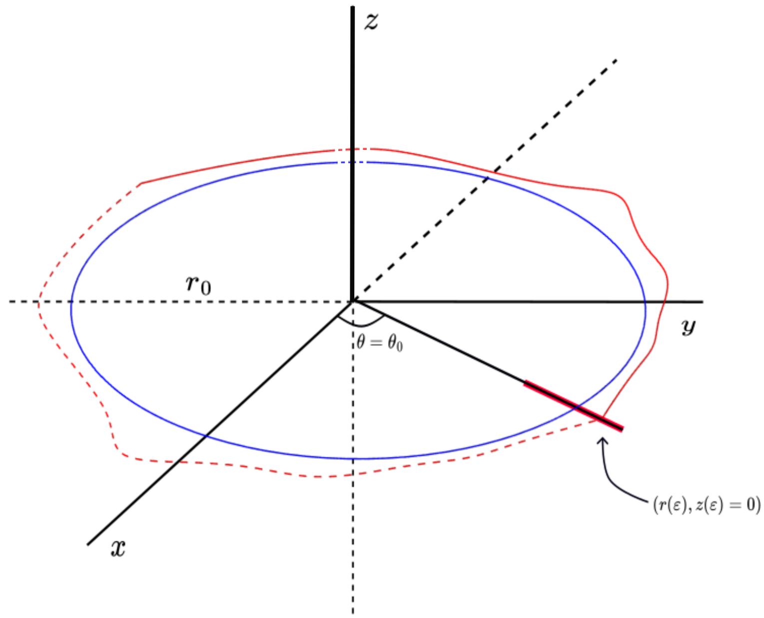

FIGURE 1. T_3-reversible symmetric periodic solution generated by a family of initial conditions $(r(\varepsilon), z(\varepsilon) = 0)$ parameterized by ε, obtained by Theorem 1.2 when the field (2) is under the reversible symmetry T_3 and $\theta = \theta_0$.

The next result concerns another type of reversible symmetry that changes two variables.

Theorem 1.3 (T_{j+4}-reversible symmetric periodic solutions)**.** *Let $\delta_1 = 1$ and $\delta_2 = 0$. Assume that the system (2) admits the reversible symmetry*

$$T_{j+4}(x_1, x_2, x_3) = ((-1)^j x_1, (-1)^{j+1} x_2, -x_3), \quad j = 1, 2.$$

Let $(r_0(t), \theta_0(t), z_0(t)) = (r_0 + \delta r, t + \delta_j \frac{\pi}{2}, \delta z)$ be a 2π-periodic solution of the unperturbed system in (5) with δr and δz sufficiently small. Suppose that there exists $r_0 > 0$ such that the following conditions are satisfied:

a) $\displaystyle\int_0^\pi \tilde{h}_1 \left(r_0, s + \delta_j \frac{\pi}{2}, 0\right) ds = 0,$

b) $\displaystyle\int_0^\pi \frac{\partial \tilde{h}_1}{\partial r} \left(r_0, s + \delta_j \frac{\pi}{2}, 0\right) ds - \tilde{h}_1 \left(r_0, \pi + \delta_j \frac{\pi}{2}, 0\right) \cdot I \neq 0,$

where

$$I = \int_0^\pi \left[\frac{\partial \tilde{g}_1}{\partial r} \left(r_0, s + \delta_j \frac{\pi}{2}, 0\right) \cos \left(s + \delta_j \frac{\pi}{2}\right) - \frac{\partial \tilde{f}_1}{\partial r} \left(r_0, s + \delta_j \frac{\pi}{2}, 0\right) \sin \left(s + \delta_j \frac{\pi}{2}\right)\right] ds.$$

If conditions a) and b) are satisfied, and δr and ε are sufficiently small, then there exists a unique one-parameter family of initial conditions $(r_0 + \delta r(\varepsilon), \delta_j \frac{\pi}{2}, 0)$, with $\delta r(0) = 0$, and a differentiable function $\tau(\varepsilon)$, such that each of them gives rise to a T_{j+4}-reversible symmetric periodic solution of the complete system (2) with period $2\tau(\varepsilon) = 2\pi + \mathcal{O}(\varepsilon^\alpha)$. Moreover, these solutions are close to a circular solution of radius r_0 and lie on the x_1x_2-plane.

The proof of this result can be found in Section 4. Observe that the above theorem involves the functions f_1, g_1, and h_1. Furthermore, we only obtained periodic solutions with variable period.

2. Proof of Theorem 1.1

Consider the system in $\mathbb{R}^3$ given by (2), and assume that it is T_j-reversible symmetric for $j = 1$ or $j = 2$. Note that when $j = 1$, the symmetry T_1 is a reflection with respect to the x_2x_3-plane, and when $j = 2$, T_2 is a reflection with respect to the x_1x_3-plane. The set of fixed points for $j = 1$ or $j = 2$ is given by $\mathcal{L}_j := \mathrm{Fix}(T_j) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_j = 0\}$. In this case, we have that $\dim(\mathcal{L}_j) = 2$.

From the reversible symmetry condition, we have that for $j = 1$,

$$f_1(-x_1, x_2, x_3) = f_1(x_1, x_2, x_3),\ g_1(-x_1, x_2, x_3) = -g_1(x_1, x_2, x_3),\ h_1(-x_1, x_2, x_3) = -h_1(x_1, x_2, x_3),$$

that is, the functions g_1 and h_1 are odd in the variable x_1, and f_1 is even in x_1. For $j = 2$, we obtain

$$f_1(x_1, -x_2, x_3) = -f_1(x_1, x_2, x_3),\ \ g_1(x_1, -x_2, x_3) = g_1(x_1, x_2, x_3),\ \ h_1(x_1, -x_2, x_3) = -h_1(x_1, x_2, x_3),$$

that is, the functions f_1 and h_1 are odd in x_2, and g_1 is even in x_2.

Our next step is to obtain an approximation of the solutions of the perturbed system (5). For this, we suppose the solution has the form

$$\begin{aligned} r(t, \varepsilon) &= r_0(t) + \varepsilon^\alpha r_1(t) + \mathcal{O}(\varepsilon^{\alpha+1}), \\ \theta(t, \varepsilon) &= \theta_0(t) + \varepsilon^\alpha \theta_1(t) + \mathcal{O}(\varepsilon^{\alpha+1}), \\ z(t, \varepsilon) &= z_0(t) + \varepsilon^\alpha z_1(t) + \mathcal{O}(\varepsilon^{\alpha+1}). \end{aligned} \tag{7}$$

It can be verified that the variational equations in this case are

$$\begin{aligned} \dot{r}_0 &= 0, \\ \dot{\theta}_0 &= 1, \\ \dot{z}_0 &= 0, \\ \dot{r}_1 &= F_1(r_0(t), \theta_0(t), z_0(t)), \\ \dot{\theta}_1 &= G_1(r_0(t), \theta_0(t), z_0(t)), \\ \dot{z}_1 &= K_1(r_0(t), \theta_0(t), z_0(t)). \end{aligned} \tag{8}$$

Considering that the initial condition of the solution is of the form (r_0, θ_0, z_0), and solving the equations in (8), we obtain:

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$$
(9)\qquad
\begin{aligned}
r_0(t) &= r_0,\\
\theta_0(t) &= t+\theta_0,\\
z_0(t) &= z_0,\\
r_1(t) &= \int_0^t \Big[\tilde f_1(r_0,s+\theta_0,z_0)\cos(s+\theta_0)+\tilde g_1(r_0,s+\theta_0,z_0)\sin(s+\theta_0)\Big]\,ds,\\
\theta_1(t) &= \frac{1}{r_0}\int_0^t \Big[\tilde g_1(r_0,s+\theta_0,z_0)\cos(s+\theta_0)-\tilde f_1(r_0,s+\theta_0,z_0)\sin(s+\theta_0)\Big]\,ds,\\
z_1(t) &= \int_0^t \tilde h_1(r_0,s+\theta_0,z_0)\,ds.
\end{aligned}
$$

The characterization of the points in the set $\mathcal{L}_j$ in cylindrical coordinates is given by $\theta=\delta_j\frac{\pi}{2}\pmod{\pi}$ for $j=1,2$, respectively, for each symmetry T_1 and T_2. Thus, the solution of the unperturbed system has the form $(r_0(t),\theta_0(t),z_0(t))=(r_0,t+\delta_j\frac{\pi}{2},z_0)$, with initial condition $(r_0,\delta_j\frac{\pi}{2},z_0)$, and it is a 2π-periodic circular solution of radius r_0 on the plane $z=z_0$, with $(r_0,\delta_j\frac{\pi}{2},z_0)\in\mathcal{L}_j$ for $j=1,2$.

Lemma 2.1. *Let $j=1$ or $j=2$ such that $\delta_1=1$ and $\delta_2=0$.*
A solution $(r(t,(r_0,\theta_0,z_0),\varepsilon),\theta(t,(r_0,\theta_0,z_0),\varepsilon),z(t,(r_0,\theta_0,z_0),\varepsilon))$ of the perturbed system (4) is 2π-periodic and T_j-reversible ($j=1$ or $j=2$) symmetric if we take $\theta_0=\delta_j\frac{\pi}{2}$ and it satisfies the condition

$$\theta(\pi,(r_0,\delta_j\tfrac{\pi}{2},z_0),\varepsilon)=\delta_j\tfrac{\pi}{2}\ (mod\ \pi),$$

where $(r_0,\delta_j\dfrac{\pi}{2},z_0)\in\mathcal{L}_j$ is the initial condition.

Now, from Lemma 2.1, since $\theta_0(\pi)=\pi+\delta_j\frac{\pi}{2}$, the first-order approximation $\theta_1(t,(r_0,\theta_0,z_0))$ at $t=\pi$ takes the form

$$\theta_1\left(\pi,\left(r_0,\delta_j\tfrac{\pi}{2},z_0\right)\right)=\frac{1}{r_0}\int_0^\pi \left[\tilde g_1\left(r_0,s+\delta_j\tfrac{\pi}{2},z_0\right)\cos\left(s+\delta_j\tfrac{\pi}{2}\right)-\tilde f_1\left(r_0,s+\delta_j\tfrac{\pi}{2},z_0\right)\sin\left(s+\delta_j\tfrac{\pi}{2}\right)\right]ds,$$

then, the symmetry condition from Lemma 2.1 takes the form

$$
(10)\qquad
\begin{aligned}
\theta\left(\pi,\left(r_0,\delta_j\tfrac{\pi}{2},z_0\right),\varepsilon\right) &= \pi+\delta_j\tfrac{\pi}{2}+\tfrac{\varepsilon^\alpha}{r_0}\int_0^\pi \left[\tilde g_1\left(r_0,s+\delta_j\tfrac{\pi}{2},z_0\right)\cos\left(s+\delta_j\tfrac{\pi}{2}\right)\right.\\
&\quad \left.-\tilde f_1\left(r_0,s+\delta_j\tfrac{\pi}{2},z_0\right)\sin\left(s+\delta_j\tfrac{\pi}{2}\right)\right]ds+\mathcal{O}(\varepsilon^{\alpha+1}).
\end{aligned}
$$

Since $\theta(\pi,(r_0,\delta_j\frac{\pi}{2},z_0),\varepsilon)=\delta_j\frac{\pi}{2}$ $(mod\ \pi)$, we consider

$$\theta\left(\pi,\left(r_0,\delta_j\tfrac{\pi}{2},z_0\right),\varepsilon\right)=\pi+\delta_j\tfrac{\pi}{2},$$

so equation (10) is equivalent to solving

$$(11)\quad \int_0^\pi \left[\tilde g_1\left(r_0,s+\delta_j\tfrac{\pi}{2},z_0\right)\cos\left(s+\delta_j\tfrac{\pi}{2}\right)-\tilde f_1\left(r_0,s+\delta_j\tfrac{\pi}{2},z_0\right)\sin\left(s+\delta_j\tfrac{\pi}{2}\right)\right]ds+\mathcal{O}(\varepsilon)=0.$$

To obtain families of T_j-reversible symmetric periodic solutions of fixed period $T=2\pi$ (or variable period close to 2π), we perturb the initial condition by replacing r_0 and z_0 with $r_0+\delta r$ and $z_0+\delta z$, respectively, where δr and δz are small. Clearly, for $j=1$ or $j=2$, we have $(r_0+\delta r,\delta_j\frac{\pi}{2},z_0+\delta z)\in\mathcal{L}_j$. Thus, the initial radius and height of the T_j-symmetric

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circular solution are now $r_0 + \delta r$ and $z_0 + \delta z$, respectively. For future computations, we introduce time τ as an additional independent variable. Then, equation (10) becomes

$$
\begin{aligned}
\mathcal{F}(\tau, \delta r, \delta z, \varepsilon) \; &= \tau - \pi + \tfrac{\varepsilon^\alpha}{r_0+\delta r} \int_0^\tau \Big[\tilde{g}_1 \left(r_0 + \delta r, s + \delta_j \tfrac{\pi}{2}, z_0 + \delta z\right) \cos\left(s + \delta_j \tfrac{\pi}{2}\right) \\
&\quad - \tilde{f}_1 \left(r_0 + \delta r, s + \delta_j \tfrac{\pi}{2}, z_0 + \delta z\right) \sin\left(s + \delta_j \tfrac{\pi}{2}\right) \Big] ds,
\end{aligned}
\tag{12}
$$

for $\varepsilon \neq 0$, with $\mathcal{F} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

In the case of fixed period, we set $\tau = \pi$, and then we define a new function on the three variables $(\delta r, \delta z, \varepsilon)$ as

$$\overline{\mathcal{F}}(\delta r, \delta z, \varepsilon) = \int_0^\pi \left[\tilde{g}_1 \left(r_0 + \delta r, s + \delta_j \tfrac{\pi}{2}, z_0 + \delta z\right) \cos\left(s + \delta_j \tfrac{\pi}{2}\right) - \tilde{f}_1 \left(r_0 + \delta r, s + \delta_j \tfrac{\pi}{2}, z_0 + \delta z\right) \sin\left(s + \delta_j \tfrac{\pi}{2}\right) \right] ds,$$

with $\overline{\mathcal{F}} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

Note that under hypothesis (a), we have $\overline{\mathcal{F}}(0,0,0) = 0$, since $\mathcal{F}(0,0,0,0) = 0$. To prove item (i), we compute the derivative:

$$\frac{\partial \overline{\mathcal{F}}}{\partial \delta r}(\delta r, \delta z, \varepsilon) = \int_0^\pi \left[\frac{\partial \tilde{g}_1}{\partial r} \left(r_0 + \delta r, s + \delta_j \tfrac{\pi}{2}, z_0 + \delta z\right) \cos\left(s + \delta_j \tfrac{\pi}{2}\right) - \frac{\partial \tilde{f}_1}{\partial r} \left(r_0 + \delta r, s + \delta_j \tfrac{\pi}{2}, z_0 + \delta z\right) \sin\left(s + \delta_j \tfrac{\pi}{2}\right) \right] ds.$$

Evaluating at $\delta r = 0$, $\delta z = 0$, and $\varepsilon = 0$, hypothesis (b) implies that

$$\frac{\partial \overline{\mathcal{F}}}{\partial \delta r}(0,0,0) \neq 0,$$

and the Implicit Function Theorem ensures the result of item (i). The proof of item (ii) is analogous, taking the derivative with respect to δz and using hypothesis (c).

To obtain T_j-reversible symmetric periodic solutions (for $j = 1$ or $j = 2$) of variable period close to $T = 2\pi$, we treat time τ as an independent variable. Then it is enough to observe that

$$\frac{\partial \mathcal{F}}{\partial \tau}(\pi, 0, 0, 0) = 1,$$

and by the Implicit Function Theorem, item (iii) follows.

In order to compute the characteristic multiplier associated with the family of periodic solutions $\varphi(t, (r_0 + \delta r, \delta_j \frac{\pi}{2}, z_0 + \delta z), \varepsilon) = (r(t, (r_0 + \delta r, \delta_j \frac{\pi}{2}, z_0 + \delta z), \varepsilon), \theta(t, (r_0 + \delta r, \delta_j \frac{\pi}{2}, z_0 + \delta z), \varepsilon), z(t, (r_0 + \delta r, \delta_j \frac{\pi}{2}, z_0 + \delta z), \varepsilon))$ obtained previously, we consider the cross section $\Sigma = \{\theta = \delta_j \frac{\pi}{2}\}$ for the system (5). Thus, to study the stability of the $T(\varepsilon)$-periodic solutions $\varphi(t, (r_0 + \delta r, \delta_j \frac{\pi}{2}, z_0 + \delta z), \varepsilon)$ by means of the Poincaré map, we take this cross section as the domain of the return map. We take the map as

$$P : \Sigma \to \Sigma, \quad (r, z) \mapsto P(T(\varepsilon), r, z, \varepsilon) = P_\varepsilon(T(\varepsilon), r, z),$$

where $T(\varepsilon) = 2\pi + \mathcal{O}(\varepsilon)$. Since

$$r(T(\varepsilon), (r, \delta_j \tfrac{\pi}{2}, z), \varepsilon) = r + \varepsilon^\alpha r_1(T(\varepsilon), (r, \delta_j \tfrac{\pi}{2}, z)) + \mathcal{O}(\varepsilon^{\alpha+1}), \tag{13}$$

$$z(T(\varepsilon), (r, \delta_j \tfrac{\pi}{2}, z), \varepsilon) = z + \varepsilon^\alpha z_1(T(\varepsilon), (r, \delta_j \tfrac{\pi}{2}, z)) + \mathcal{O}(\varepsilon^{\alpha+1}), \tag{14}$$

the Poincaré map is

$$P_\varepsilon(r, z) = (r, z) + \varepsilon^\alpha \left(r_1(T(\varepsilon), (r, \delta_j \tfrac{\pi}{2}, z)), z_1(T(\varepsilon), (r, \delta_j \tfrac{\pi}{2}, z)) \right) + \mathcal{O}(\varepsilon^{\alpha+1}).$$

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The derivative of P_ε is

$$D_{(r,z)}P_\varepsilon(r,z) = \mathcal{I} + \varepsilon^\alpha A_j + \mathcal{O}(\varepsilon^{\alpha+1}),$$

where

$$A_j = \begin{pmatrix} \int_0^{2\pi}\left[\frac{\partial \tilde f_1}{\partial r}\cos\left(s+\delta_j\frac{\pi}{2}\right) + \frac{\partial \tilde g_1}{\partial r}\sin\left(s+\delta_j\frac{\pi}{2}\right)\right]ds & \int_0^{2\pi}\left[\frac{\partial \tilde f_1}{\partial z}\cos\left(s+\delta_j\frac{\pi}{2}\right) + \frac{\partial \tilde g_1}{\partial z}\sin\left(s+\delta_j\frac{\pi}{2}\right)\right]ds \\ \int_0^{2\pi}\frac{\partial \tilde h_1}{\partial r}ds & \int_0^{2\pi}\frac{\partial \tilde h_1}{\partial z}ds \end{pmatrix}.$$

and each function in the integrands are evaluated at the initial conditions $(r_0, s+\delta_j\frac{\pi}{2}, z_0)$ for $j=1$ or $j=2$. For the symmetry T_2, the integrands of the integrals in the matrix A_2 are odd in s, so A_2 is the zero matrix when is evaluate at $\delta r = 0$, $\delta z = 0$ and $\varepsilon = 0$ and the derivate of P_ε is

$$D_{(r,z)}P_\varepsilon(r,z) = \mathcal{I} + \mathcal{O}(\varepsilon^{\alpha+1}),$$

which applies to all families of T_2-symmetric periodic solutions in items (i), (ii), and (iii).

However, for T_1, the parity of $\tilde f_1$, $\tilde g_1$, and $\tilde h_1$ with respect to s cannot be determined, so the matrix A_1 may have nonzero entries. Therefore, the characteristic multipliers (i.e., eigenvalues of DP_ε) can be calculate and may provide information about the stability of the associated symmetric periodic solution. This concludes the proof of Theorem 1.1.

□

Remark 2. *The eigenvalues of the first approximation do not provide any information about the stability of the symmetric periodic solutions for any arbitrary functions possessing the T_2-reversible symmetry.*

We present the following example where study the symmetry T_1.

Example 2.1. *Consider the system (2) with the following functions,*

$$\begin{aligned} f_1(x_1,x_2,x_3) &= a_{200}x_1^2 + a_{210}x_1^2x_2 + a_{201}x_1^2x_3 + a_{012}x_2x_3^2 + a_{030}x_2^3 + a_{400}x_1^4, \\ g_1(x_1,x_2,x_3) &= b_{101}x_1x_3 + b_{300}x_1^3 + b_{120}x_1x_2^2, \\ h_1(x_1,x_2,x_3) &= c_{110}x_1x_2 + c_{101}x_1x_3 + c_{301}x^3z, \end{aligned} \tag{15}$$

where a_{ij}, b_{ij} and c_{ij} are arbitrary real numbers, and R_1, R_2, R_3 arbitrary functions that have the reversible symmetry T_1. Clearly this system is T_1-reversible. The previous functions (15) in cylindrical coordinates takes the form as follows

$$\begin{aligned} \tilde f_1(r,\theta,z) &= a_{200}r^2\cos^2\theta + a_{210}r^3\sin\theta\cos^2\theta + a_{201}r^2z\cos^2\theta + a_{012}rz^2\sin\theta + a_{030}r^3\sin^3\theta + a_{400}r^4\cos^4\theta, \\ \tilde g_1(r,\theta,z) &= b_{101}rz\cos\theta + b_{300}r^3\cos^3\theta + b_{120}r^3\sin^2\theta\cos\theta, \\ \tilde h_1(r,\theta,z) &= c_{110}r^2\sin\theta\cos\theta + c_{101}rz\cos\theta + c_{301}r^3z\cos^3\theta. \end{aligned}$$

We note that the equation in item a) of Theorem 1.1 reduces to

$$\int_0^\pi \left[\tilde g_1\left(r_0, s+\frac{\pi}{2}, z_0\right)\sin s + \tilde f_1\left(r_0, s+\frac{\pi}{2}, z_0\right)\cos s\right]ds = -\frac{4}{15}r^2\left(5a_{200} + 5a_{201}z + 4a_{400}r^2\right) = 0. \tag{16}$$

So, to apply item i) of Theorem 1.1, we need to solve (16) for r_0, whose solution is

$$r_0 = \frac{\sqrt{5}}{2}\sqrt{\frac{-a_{200} - a_{201}z}{a_{400}}},$$

whether

$$a_{400}(-a_{200} - a_{201}z) > 0.$$

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Now, the condition of no-degeneration, that is, item b) of Theorem 1.1 becomes

$$\frac{\sqrt{(-a_{200}-a_{201}z)^3}}{a_{400}} \neq 0.$$

On the other hand, to apply the second part of Theorem 1.1, we need to solve (16) with respect to z_0. In this case, we find a solution:

$$z = -\frac{5a_{200}+4a_{400}r^2}{5a_{201}},$$

whether $a_{201} \neq 0$. Condition c) of Theorem 1.1 becomes

$$4a_{201}r^2 \neq 0,$$

In conclusion, by Theorem 1.1 we have the following result.

Theorem 2.1. *Consider the spatial system (2), where the functions $f_1(x_1,x_2,x_3)$, $g_1(x_1,x_2,x_3)$, and $h_1(x_1,x_2,x_3)$ are given by (15), and let R_1, R_2, and R_3 be arbitrary functions that preserve the T_1-reversible symmetry. Then, we have the following results:*

i) ***(Fixed period)*** *Let z_0 be fixed such that $a_{400}(a_{200}+a_{201}z_0) < 0$, and let ε be sufficiently small. Then there exists a unique two-parameter family of initial conditions of the form $(r(\varepsilon),\theta(\varepsilon),z(\varepsilon)) = (r_0+\delta r(\varepsilon,\delta z),\frac{\pi}{2},z_0)$, with $\delta r(0,0)=0$, such that each initial condition gives rise to a T_1-reversible symmetric periodic solution of the system (4) with fixed period $T=2\pi$. Moreover, these solutions are close to the circular solution of radius $r_0 = \frac{\sqrt{5}}{2}\sqrt{\frac{-a_{200}-a_{201}z_0}{a_{400}}}$ and lie near the plane $z=z_0$.*

ii) ***(Fixed period)*** *Let $a_{201} \neq 0$, and let ε be sufficiently small. Then there exists a two-parameter family of initial conditions of the form $(r(\varepsilon),\theta(\varepsilon),z(\varepsilon)) = (r_0,\frac{\pi}{2},z_0+\delta z(\varepsilon,\delta r))$, with $\delta z(0,0)=0$, such that each initial condition yields a T_1-reversible symmetric periodic solution of the system (4) with fixed period $T=2\pi$. Moreover, these solutions are close to the circular solution of radius r_0 and near the plane $z_0 = -\frac{5a_{200}+4a_{400}r_0^2}{5a_{201}}$.*

Now, we present an other example for the case of the reversible symmetry T_2.

Example 2.2. *Consider the system (2) with the following functions,*

$$\begin{aligned} f_1(x_1,x_2,x_3) &= a_{210}x_1^2x_2 + a_{030}x_2^3 + a_{012}x_2x_3^2, \\ g_1(x_1,x_2,x_3) &= b_{120}x_1x_2^2 + b_{101}x_1x_3 + b_{002}x_3^2, \\ h_1(x_1,x_2,x_3) &= c_{110}x_1x_2 + c_{011}x_2x_3, \end{aligned} \tag{17}$$

where a_{ij}, b_{ij} and c_{ij} are arbitrary real numbers, and R_1,R_2,R_3 arbitrary functions that have the reversible symmetry T_2. Clearly this system is T_2-reversible. Moreover, the previous functions (17) in cylindrical coordinates takes the form

$$\begin{aligned} \tilde{f}_1(r,\theta,z) &= a_{210}r^3\cos^2\theta\sin\theta + a_{030}r^3\sin^3\theta + a_{102}rz^2\cos\theta, \\ \tilde{g}_1(r,\theta,z) &= b_{120}r^3z\cos\theta\sin^2\theta + b_{101}rz\cos\theta + b_{002}z^2, \\ \tilde{h}_1(r,\theta,z) &= c_{110}r^2\cos\theta\sin\theta + c_{011}rz\sin\theta. \end{aligned}$$

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Next, we point out that the equation in the item a) of Theorem 1.1 becomes

$$\int_0^{\pi}\left[\tilde{g}_1(r_0,s,z_0)\cos s-\tilde{f}_1(r_0,s,z_0)\sin s\right]ds=\frac{1}{8}\pi r_0\left[4z_0(b_{101}-a_{012}z_0)+r_0^2(-3a_{030}-a_{210}+b_{120})\right]=0. \tag{18}$$

To apply item i) of Theorem 1.1, we need to solve (18) for r_0, whose solution is

$$r_0=2\sqrt{\frac{(b_{101}-a_{012}z_0)z_0}{3a_{030}+a_{210}-b_{120}}},$$

whether

$$\frac{(b_{101}-a_{012}z_0)z_0}{3a_{030}+a_{210}-b_{120}}>0.$$

Condition b) of Theorem 1.1 becomes

$$z_0(-b_{101}+a_{012}z_0)\neq 0.$$

On the other hand, if we want to apply the second part of Theorem 1.1, we have to solve (18) with respect to z_0, so we find two solutions

$$z_{0\pm}=\frac{b_{101}\pm\sqrt{b_{101}^2+a_{102}(-3a_{030}-a_{210}+b_{120})r_0^2}}{2a_{012}},$$

whether $b_{101}^2+a_{102}(-3a_{030}-a_{210}+b_{120})r_0^2>0$. Condition c) of Theorem 1.1 becomes

$$\pm\frac{\pi r_0}{2}\sqrt{b_{101}^2+a_{102}(-3a_{030}-a_{210}+b_{120})r_0^2}\neq 0,$$

In conclusion, by Theorem 1.1 we have the following result.

Theorem 2.2. *Consider the spatial system (2), where the functions $f_1(x_1,x_2,x_3)$, $g_1(x_1,x_2,x_3)$, and $h_1(x_1,x_2,x_3)$ are as given in (17), and let R_1, R_2, and R_3 be arbitrary functions that preserve the T_2-reversible symmetry. Then, we have the following results:*

i) ***(Fixed period)*** *Let z_0 be fixed such that $\frac{(b_{101}-a_{012}z_0)z_0}{3a_{030}+a_{210}-b_{120}}>0$, and let ε be sufficiently small. Then there exists a unique two-parameter family of initial conditions of the form $(r(\varepsilon),\theta(\varepsilon),z(\varepsilon))=(r_0+\delta r(\varepsilon),0,z_0)$, with $\delta r(0,0)=0$, such that each initial condition gives rise to a T_2-reversible symmetric periodic solution of the system (4) with fixed period $T=2\pi$. Moreover, these solutions are close to the circular solution of radius $r_0=2\sqrt{\frac{(b_{101}-a_{012}z_0)z_0}{3a_{030}+a_{210}-b_{120}}}$, and lie near the plane $z=z_0$.*

ii) ***(Fixed period)*** *Let r_0 be fixed such that $b_{101}^2+a_{102}(-3a_{030}-a_{210}+b_{120})r_0^2>0, a_{012}\neq 0$, and let ε be sufficiently small. Then there exist two families of initial conditions of the form $(r(\varepsilon),\theta(\varepsilon),z(\varepsilon))=(r_0,0,z_{0\pm}+\delta z(\varepsilon))$, with $\delta z(0,0)=0$, such that each initial condition gives rise to a T_2-reversible symmetric periodic solution of the system (4) with fixed period $T=2\pi$. Moreover, these solutions are close to the circular solution of radius r_0 and near the plane $z=z_{0\pm}=\frac{b_{101}\pm\sqrt{b_{101}^2+a_{102}(-3a_{030}-a_{210}+b_{120})r_0^2}}{2a_{012}}$.*

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3. Proof of Theorem 1.2

Consider the system in $\mathbb{R}^3$ given by (2), and assume that it is T_3-reversible symmetric, where $T_3(x_1, x_2, x_3) = (x_1, x_2, -x_3)$. Note that T_3 represents a reflection with respect to the x_1x_2-plane. The set of fixed points under T_3 is given by

$$\mathcal{L}_3 := \text{Fix}(T_3) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = 0\},$$

which is a plane of dimension $\dim(\mathcal{L}_3) = 2$. From the T_3-reversibility condition, we obtain the following identities:

$$f_1(x_1, x_2, -x_3) = -f_1(x_1, x_2, x_3), g_1(x_1, x_2, -x_3) = -g_1(x_1, x_2, x_3), h_1(x_1, x_2, -x_3) = h_1(x_1, x_2, x_3).$$

Therefore, the functions f_1 and g_1 are odd in the variable x_3, while h_1 is even in x_3.

The characterization of the points in the set $\mathcal{L}_3$ in cylindrical coordinates is given by $z = 0$. Moreover, the solution of the unperturbed system has the form $(r_0(t), \theta_0(t), z_0(t)) = (r_0, t+\theta_0, 0)$, with initial condition $(r_0, \theta_0, 0) \in \mathcal{L}_3$. Thus, it is a 2π-periodic circular solution of radius r_0 of the unperturbed system, lying in the plane $z = 0$.

Lemma 3.1. *A solution $(r(t,(r_0,\theta_0,z_0),\varepsilon),\ \theta(t,(r_0,\theta_0,z_0),\varepsilon),\ z(t,(r_0,\theta_0,z_0),\varepsilon))$ of the perturbed system (4) is 2π-periodic and T_3-reversible symmetric if we take $z_0 = 0$ and it satisfies the condition $z(\pi,(r_0,\theta_0,0),\varepsilon) = 0$, where $(r_0,\theta_0,0) \in \mathcal{L}_3$ is the initial condition.*

Since $z_0 = 0$, then by (9), the first-order approximation z_1 becomes

$$z_1(t,(r_0,\theta_0,0)) = \int_0^t \tilde{h}_1(r_0, s+\theta_0, 0)\, ds, \tag{19}$$

and the condition in Lemma 3.1 reduces to

$$z(\pi,(r_0,\theta_0,0),\varepsilon) = \int_0^\pi \tilde{h}_1(r_0, s+\theta_0, 0)\, ds. \tag{20}$$

To obtain families of T_3-reversible symmetric periodic solutions with fixed period $T = 2\pi$ (or not fixed but close to 2π), we modify the initial condition by perturbing r_0 to $r_0 + \delta r$, where δr is a small parameter. That is, we consider a perturbation of the initial radius of the T_3-symmetric circular solution of the unperturbed system. Then, equation (19) becomes

$$\mathcal{F}(\delta r, \varepsilon) = \int_0^\pi \tilde{h}_1(r_0 + \delta r, s+\theta_0, 0)\, ds + \mathcal{O}(\varepsilon), \tag{21}$$

where, in this case, $\mathcal{F} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

Now, by hypothesis $a)$ we have that $\mathcal{F}(0,0) = 0$. In order to prove item $i)$, we compute the following derivative:

$$\frac{\partial \mathcal{F}}{\partial \delta r}(\delta r, \varepsilon) = \int_0^\pi \frac{\partial \tilde{h}_1}{\partial r}(r_0 + \delta r, s+\theta_0, 0)\, ds.$$

Evaluating at $\delta r = 0$ and $\varepsilon = 0$, by hypothesis $b)$ we get

$$\frac{\partial \mathcal{F}}{\partial \delta r}(0,0) \neq 0,$$

thus, by the Implicit Function Theorem, the proof of item $i)$ follows.

For the case of variable period, we consider the function

$$\overline{\mathcal{F}}(\tau, \delta r, \varepsilon) = \int_0^{\tau} \tilde{h}_1(r_0 + \delta r, s + \theta_0, 0)\, ds + \mathcal{O}(\varepsilon). \tag{22}$$

By item a), we have $\overline{\mathcal{F}}(\pi, 0, 0) = 0$, and by item c),

$$\frac{\partial \overline{\mathcal{F}}}{\partial \tau}(\pi, 0, 0) = \tilde{h}_1(r_0, \pi + \theta_0, 0) \neq 0,$$

thus, again by the Implicit Function Theorem, the proof of item ii) follows.

In order to compute the characteristic multipliers associated the family of the previous periodic solutions $\varphi(t, (r_0+\delta r, \theta_0, 0), \varepsilon) = (r(t, (r_0+\delta r, \theta_0, 0), \varepsilon), \theta(t, (r_0+\delta r, \theta_0, 0), \varepsilon), z(t, (r_0+\delta r, \theta_0, 0), \varepsilon))$, we consider the cross section $\Sigma = \{\theta = 0\}$ for the system (5). Thus, to study the stability of the $T(\varepsilon)$-periodic solutions $\varphi(t, (r_0 + \delta r, 0, 0), \varepsilon)$ by considering the Poincaré map, we take

$$P_\varepsilon : \Sigma \to \Sigma, \quad (r, z) \mapsto P_\varepsilon(T(\varepsilon), r, z) = \left(r(T(\varepsilon), (r, 0, z), \varepsilon),\ z(T(\varepsilon), (r, 0, z), \varepsilon)\right), \tag{23}$$

where $T(\varepsilon) = 2\pi + \mathcal{O}(\varepsilon)$.

We expand the flow components as:

$$r(T(\varepsilon), (r, 0, 0), \varepsilon) = r + \varepsilon^{\alpha} r_1(T(\varepsilon), (r, 0, 0)) + \mathcal{O}(\varepsilon^{\alpha+1}), \tag{24}$$

$$z(T(\varepsilon), (r, 0, 0), \varepsilon) = z + \varepsilon^{\alpha} z_1(T(\varepsilon), (r, 0, 0)) + \mathcal{O}(\varepsilon^{\alpha+1}). \tag{25}$$

Thus, the Poincaré map on Σ becomes:

$$P_\varepsilon(r, z) = (r, z) + \varepsilon^{\alpha} \left(r_1(T(\varepsilon), (r, s, 0)), z_1(T(\varepsilon), (r, s, 0))\right) + \mathcal{O}(\varepsilon^{\alpha+1}). \tag{26}$$

In particular, the derivative of P_ε with respect to (r, z) takes the form:

$$D_{(r,z)} P_\varepsilon(r, z) = \mathcal{I} + \varepsilon^{\alpha} A + \mathcal{O}(\varepsilon^{\alpha+1}), \tag{27}$$

where

$$A_j = \begin{pmatrix} \displaystyle\int_0^{2\pi} \left[\frac{\partial \tilde{f}_1}{\partial r} \cos(s + \theta_0) + \frac{\partial \tilde{g}_1}{\partial r} \sin(s + \theta_0)\right] ds & \displaystyle\int_0^{2\pi} \left[\frac{\partial \tilde{f}_1}{\partial z} \cos(s + \theta_0) + \frac{\partial \tilde{g}_1}{\partial z} \sin(s + \theta_0)\right] ds \\ \displaystyle\int_0^{2\pi} \frac{\partial \tilde{h}_1}{\partial r} ds & \displaystyle\int_0^{2\pi} \frac{\partial \tilde{h}_1}{\partial z} ds \end{pmatrix}.$$

and each function in the integrands are evaluated at the initial condition $(r_0, s + \theta_0, 0)$. According to our calculations, the parity of the functions $\tilde{f}_1$, $\tilde{g}_1$, and $\tilde{h}_1$ with respect to the variable s cannot be determined when evaluated at $\delta r = 0$ and $\varepsilon = 0$ under the symmetry T_3. Therefore, the entries a_{ij} of the matrix A are not necessarily zero. This enables the analysis of the characteristic multipliers, which correspond to the eigenvalues of A, providing information about the stability of the associated symmetric periodic solution. This concludes the proof of Theorem 1.2. □

Now, we are going to show an example where we can apply Theorem 1.2.

Example 3.1. *Consider the system* (2) *with the following functions,*

$$
\begin{aligned}
f_1(x_1,x_2,x_3) =& a_{111}x_1x_2x_3 + a_{011}x_2x_3 + a_{101}x_1x_3,\\
g_1(x_1,x_2,x_3) =& b_{111}x_1x_2x_3 + b_{011}x_2x_3 + b_{101}x_1x_3,\\
h_1(x_1,x_2,x_3) =& c_{200}x_1^2 + c_{110}x_1x_2 + c_{002}x_3^2 + c_{210}x_1^2x_2 + c_{320}x_1^2x_2^2,
\end{aligned} \tag{28}
$$

where a_{ij}, b_{ij} *and* c_{ij} *are arbitrary real numbers, and* R_1, R_2, R_3 *arbitrary but satisfying the reversible symmetry* T_3*. Clearly this system is* T_3*-reversible. Moreover, the previous functions* (28) *in cylindrical coordinates take the form*

$$
\begin{aligned}
\tilde{f}_1(r,\theta,z) =& a_{111}r^2z\cos\theta\sin\theta + a_{011}rz\sin\theta + a_{101}rz\cos\theta,\\
\tilde{g}_1(r,\theta,z) =& b_{111}r^2z\cos\theta\sin\theta + b_{011}rz\sin\theta + b_{101}rz\cos\theta,\\
\tilde{h}_1(r,\theta,z) =& c_{200}r^2\cos^2\theta + c_{110}r^2\sin\theta\cos\theta + c_{002}z^2 + c_{210}r^3\sin\theta\cos^2\theta + c_{320}r^4\sin^2\theta\cos^2\theta.
\end{aligned}
$$

We note that the equation in item a) of Theorem 1.1 *reduces to*

$$
\int_0^\pi \tilde{h}_1(r_0, s+\theta_0, 0)ds = \frac{r_0^2}{24}\left[3\pi\left(4c_{200} + c_{320}r^2\right) + 16c_{210}r_0\cos^3\theta_0\right] = 0. \tag{29}
$$

To apply item i) of Theorem 1.2*, we solve the equation for* r_0 *and for* $c_{320} \neq 0$*, we obtain two possible roots, namely*

$$
r_{0\pm} = \frac{-8c_{210}\cos^3\theta_0 \pm 2\sqrt{16c_{210}^2\cos^6\theta_0 - 9\pi^2c_{200}c_{320}}}{3\pi c_{320}}. \tag{30}
$$

Now, we are going to analyze different situations according the number of solutions

Two distinct positive roots: *This occurs when the discriminant is positive, i.e.,*

$$
16c_{210}^2\cos^6\theta_0 - 9\pi^2c_{200}c_{320} > 0,
$$

and in addition one of the conditions below is satisfied

(i) $\theta_0 \in (\pi/2, 3\pi/2), \quad c_{320} > 0, \quad c_{210} > 0, \quad 0 < c_{200} < \dfrac{16c_{210}^2\cos^6\theta_0}{9\pi^2c_{320}},$

(ii) $\theta_0 \in [0,\pi/2) \cup (3\pi/2, 2\pi], \quad c_{320} > 0, \quad c_{210} < 0, \quad 0 < c_{200} < \dfrac{16c_{210}^2\cos^6\theta_0}{9\pi^2c_{320}}.$

The previous set will be called Ω_1 *of parameters.*

Unique positive root: *We obtain one root when* $c_{320} < 0,\ c_{200} > 0$ *or* $c_{320} > 0,\ c_{200} < 0$*. Or when occurs that,*

$$
cos^6\theta_0 = \frac{9\pi^2c_{320}c_{200}}{16c_{210}^2},
$$

and one of the following conditions holds:

(i) $\theta_0 \in (\pi/2, 3\pi/2), \quad c_{210} > 0, \quad c_{320} < 0,$ *or* $\quad c_{210} < 0, \quad c_{320} > 0.$
(ii) $\theta_0 \in [0,\pi/2) \cup (3\pi/2, 2\pi], \quad c_{210} > 0, \quad c_{320} < 0,$ *or* $\quad c_{210} < 0, \quad c_{320} > 0.$

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The previous set of parameters will be called Ω_2 .

Now, the computation of condition in item b) of Theorem 1.2 reduces to

$$\int_0^\pi \frac{\partial \tilde{h}_1}{\partial r}(r_0, s+\theta_0, 0)ds = \frac{A_\pm B_\pm}{27\pi^2 c_{320}^2}, \tag{31}$$

where

$$\begin{aligned} A_\pm &= -8c_{210}\cos^3\theta_0 \pm 2\sqrt{16c_{210}^2\cos^6\theta_0 - 9\pi^2 c_{200}c_{320}},\\ B_\pm &= 9\pi^2 c_{200}c_{320} - 16c_{210}^2\cos^6\theta_0 \mp 4c_{210}\cos^3\theta_0\sqrt{16c_{210}^2\cos^6\theta_0 - 9\pi^2 c_{200}c_{320}}. \end{aligned}$$

It is verified that in the regions Ω_1 and Ω_2 that $A_\pm \neq 0$ and $B_\pm \neq 0$. Therefore, conditions in the items a) and b) are satisfied.

For the other case, the computation of the condition in item c) of Theorem 1.2 reduces to

$$\tilde{h}_1(r_0, \pi+\theta_0, 0) = r_0^2\cos\theta_0\left[c_{110}\sin\theta_0 + \cos\theta_0\left(c_{200} - c_{210}r_0\sin\theta_0 + c_{320}r_0^2\sin^2\theta_0\right)\right]. \tag{32}$$

Now, if $r_0 = r_{0\pm}$ as in (30), on the regions Ω_1 and Ω_2, the condition

$$\tilde{h}_1(r_{0\pm}, \pi+\theta_0, 0) = A_+B_1 \text{ or } A_-B_2 \neq 0,$$

is always satisfied if $B_i \neq 0$, where

$$\begin{aligned} A_\pm &= 4\frac{\left(\sqrt{16c_{210}^2\cos^6\theta_0 - 9\pi^2 c_{200}c_{320}} \pm 4c_{210}\cos^3\theta_0\right)^2\cos\theta_0}{9\pi^2 c_{320}^2},\\ B_1 &= c_{110}\sin\theta_0 + \frac{\cos\theta_0}{9\pi^2 c_{320}}\Bigg(4\sin^2\theta_0\left(\sqrt{16c_{210}^2\cos^6\theta_0 - 9\pi^2 c_{200}c_{320}} + 4c_{210}\cos^3\theta_0\right)^2\\ &\quad + 6\pi c_{210}\sin\theta_0\cdot\left(\sqrt{16c_{210}^2\cos^6\theta_0 - 9\pi^2 c_{200}c_{320}} + 4c_{210}\cos^3\theta_0\right) + 9\pi^2 c_{200}c_{320}\Bigg),\\ B_2 &= c_{110}\sin\theta_0 + \cos\theta_0\Bigg(\frac{4\sin^2\theta_0\left(\sqrt{16c_{210}^2\cos^6\theta_0 - 9\pi^2 c_{200}c_{320}} - 4c_{210}\cos^3\theta_0\right)^2}{9\pi^2 c_{320}}\\ &\quad - \frac{2c_{210}\sin\theta_0\cdot\left(\sqrt{16c_{210}^2\cos^6\theta_0 - 9\pi^2 c_{200}c_{320}} - 4c_{210}\cos^3\theta_0\right)}{3\pi c_{320}} + c_{200}\Bigg). \end{aligned}$$

In conclusion, by Theorem 1.2 we obtain the following result.

Theorem 3.1. *Consider the spatial system (2), where $f_1(x_1,x_2,x_3)$, $g_1(x_1,x_2,x_3)$, and $h_1(x_1,x_2,x_3)$ are as in (28), and let R_1, R_2, R_3 be arbitrary functions possessing the symmetry T_3. Then, we have the following:*

i) ***(Fixed period)*** *Let $z = 0$ such that condition (31) is satisfied, and let ε be sufficiently small. If $(c_{200}, c_{210}, c_{320}, \theta_0)$ satisfy the conditions of the set Ω_1, then there exist two one-parameter families of initial conditions given by $(r(\varepsilon), \theta(\varepsilon), z(\varepsilon)) = (r_\pm + \delta r_i(\varepsilon), \theta_0, 0)$, $\delta r_i(0) = 0$, such that they generate two T_3-reversible symmetric*

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periodic solutions of system (4) *with fixed period* $T = 2\pi$. *Moreover, these solutions are close to the circular ones with radius* $r_{0\pm} = \frac{2\left(-4c_{210}\cos^3\theta_0 \mp \sqrt{16c_{210}^2\cos^6\theta_0 - 9\pi^2 c_{200}c_{320}}\right)}{3c_{320}\pi}$, *and lie close to the plane* $z = 0$.

ii) ***(Fixed period)*** *Let* $z = 0$ *such that condition* (31) *is satisfied, and let* ε *be sufficiently small. If* $(c_{200}, c_{210}, c_{320}, \theta_0)$ *satisfy the conditions of the set* Ω_2, *then there exists a one-parameter family of initial conditions* $(r(\varepsilon), \theta(\varepsilon), z(\varepsilon)) = (r_0+\delta r(\varepsilon), \theta_0, 0), \delta r(0) = 0$, *that generates a* T_3*-reversible symmetric periodic solution of system* (4) *with fixed period* $T = 2\pi$. *Moreover, this solution is close to the circular one with radius* $r_{0\pm} = \frac{2\left(-4c_{210}\cos^3\theta_0 \mp \sqrt{16c_{210}^2\cos^6\theta_0 - 9\pi^2 c_{200}c_{320}}\right)}{3c_{320}\pi}$, *and close to the plane* $z = 0$.

iii) ***(Variable period)*** *Let* ε *be sufficiently small and suppose* $(c_{200}, c_{210}, c_{320}, \theta_0)$ *satisfy the conditions of the set* $\Omega_3 := \Omega_1 \cup \Omega_2$. *If* $B_i \neq 0$, *then there exists a differentiable function* $\tau(\delta r, \varepsilon)$, *such that* $\tau(0,0) = \pi$, *and the full system* (2) *admits a two-parameter family of* T_3*-reversible symmetric periodic solutions parametrized by* $(\delta r, \varepsilon)$, *with period close to* $2\tau(\delta r, \varepsilon) = 2\pi + \mathcal{O}(\varepsilon^\alpha)$. *Moreover, these solutions are close to a circular solution with radius* $r_{0\pm}$ *and close to the plane* $z = 0$.

4. Proof of Theorem 1.3

Consider the system in $\mathbb{R}^3$ given by (2) and assume that the system admits the reversible symmetry T_{j+4} for $j = 1$ or $j = 2$. Consider the set of fixed points $\mathcal{L}_{j+4} = \{x_j = 0, x_3 = 0\}$ with $dim(\mathcal{L}_{j+4}) = 1$. In cylindrical coordinates, the set $\mathcal{L}_{j+4}$ is characterized by $\theta \equiv \delta_j \frac{\pi}{2}$ (mod π) and $z = 0$. Note that this set has two conditions, which is an important difference with respect to previous cases. Due to the reversible symmetry, the functions of the vector field satisfy the following properties: (a) for the symmetry T_5, the functions f_1 and h_1 are even in the variables (x_1, x_3), while g_1 is odd in (x_1, x_3); (b) for the symmetry T_6, the function f_1 is odd in the variables (x_2, x_3), and g_1 and h_1 are even in (x_2, x_3). In cylindrical coordinates, we have the following characterization.

Lemma 4.1. *One solution* $(r(t,(r_0,\theta_0,z_0),\varepsilon), \theta(t,(r_0,\theta_0,z_0),\varepsilon), z(t,(r_0,\theta_0,z_0),\varepsilon))$ *of the perturbed system* (4) *is* 2π*-periodic and* T_{j+4}*-reversible symmetric if it is satisfied the system*

$$\theta\left(\pi, \left(r_0, \delta_j \frac{\pi}{2}, 0\right), \varepsilon\right) = \delta_j \frac{\pi}{2} \ (mod\ \pi),$$
$$z\left(\pi, \left(r_0, \delta_j \frac{\pi}{2}, 0\right), \varepsilon\right) = 0, \tag{33}$$

where $(r_0, \delta_j \frac{\pi}{2}, 0) \in \mathcal{L}_{j+4}$ *is the initial condition for* $j = 1$ *and* $j = 2$, *respectively.*

Next, the system of equations in (33) takes the form

(34)
$$\theta\left(\pi, \left(r_0, \delta_j \frac{\pi}{2}, 0\right), \varepsilon\right) = \pi + \delta_j \frac{\pi}{2} + \frac{\varepsilon^\alpha}{r_0} \int_0^\pi \left[\tilde{g}_1\left(r_0, s + \delta_j \frac{\pi}{2}, 0\right) \cos\left(s + \delta_j \frac{\pi}{2}\right) - \tilde{f}_1\left(r_0, s + \delta_j \frac{\pi}{2}, 0\right) \sin\left(s + \delta_j \frac{\pi}{2}\right)\right] ds + \mathcal{O}\left(\varepsilon^{\alpha+1}\right) = \delta_j \frac{\pi}{2} \ (mod\ \pi),$$
$$z\left(\pi, \left(r_0, \delta_j \frac{\pi}{2}, 0\right), \varepsilon\right) = \varepsilon^\alpha \int_0^\pi \tilde{h}_1\left(r_0, s + \delta_j \frac{\pi}{2}, 0\right) ds + \mathcal{O}(\varepsilon^{\alpha+1}) = 0,$$

and here we will consider the symmetry condition as $\theta(\pi,(r_0,\delta_j\frac{\pi}{2},0),\varepsilon)=\pi+\delta_j\frac{\pi}{2}$ and $z(\pi,(r_0,\delta_j\frac{\pi}{2},0),\varepsilon)=0$. In order to obtain T_{j+4}-reversible symmetric periodic solutions with period close to $T=2\pi$, we modify the initial condition $(r_0,\delta_j\frac{\pi}{2},0)$ by taking $(r_0+\delta r,\delta_j\frac{\pi}{2},0)\in\mathcal{L}_{j+4}$ for $j=1,2$. Thus, the T_{j+4}-reversible symmetric periodic solutions of the unperturbed system with initial condition $(r_0+\delta r,\delta_j\frac{\pi}{2},0)$ are $(r_0(t),\theta_0(t),z_0(t))=(r_0+\delta r,\delta_j\frac{\pi}{2},0)$. Moreover, for this symmetry, since there are two equations in (34) and two variables $(\delta r,\varepsilon)$, it is necessary to include the time τ as an independent variable. Thus, the system (34) admits the following equivalent form

$$\begin{aligned}\mathcal{F}_1(\tau,\delta r,\varepsilon) &= \tau-\pi+\tfrac{\varepsilon^\alpha}{r_0+\delta r}\int_0^\tau\Big[\tilde{g}_1\Big(r_0+\delta r,s+\delta_j\frac{\pi}{2},0\Big)\cos\Big(s+\delta_j\frac{\pi}{2}\Big)-\\ &\qquad \tilde{f}_1\big(r_0+\delta r,s+\delta_j\tfrac{\pi}{2},0\big)\sin\big(s+\delta_j\tfrac{\pi}{2}\big)\Big]ds+\mathcal{O}(\varepsilon^{\alpha+1}),\\ \mathcal{F}_2(\tau,\delta r,\varepsilon) &= \int_0^\tau\tilde{h}_1\Big(r_0+\delta r,s+\delta_j\frac{\pi}{2},0\Big)ds+\mathcal{O}(\varepsilon).\end{aligned} \tag{35}$$

Note that the function $\mathcal{F}=(\mathcal{F}_1,\mathcal{F}_2):\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}^2$. Thus by the hypotheses $a)$ and $b)$, we get that $\mathcal{F}(\pi,0,0)=(0,0)$. On other hand,

$$\frac{\partial(\mathcal{F}_1,\mathcal{F}_2)}{\partial(\tau,\delta r)}=\begin{pmatrix}1 & \displaystyle\int_0^\pi\left[\frac{\partial\tilde{g}_1}{\partial r}\cos\Big(s+\delta_j\frac{\pi}{2}\Big)-\frac{\partial\tilde{f}_1}{\partial r}\sin\Big(s+\delta_j\frac{\pi}{2}\Big)\right]ds\\ \tilde{h}_1(r_0,\pi+\delta_j\frac{\pi}{2},0) & \displaystyle\int_0^\pi\frac{\partial\tilde{h}_1}{\partial r}ds\end{pmatrix},$$

which is non-singular at $\tau=\pi,\delta r=0,\varepsilon=0$, by hypothesis c) for $j=1$ or $j=2$. Therefore, the proof of the theorem follows from the Implicit Function Theorem. □

Example 4.1. *Consider the system (2) with the following functions*

$$\begin{aligned}f_1(x_1,x_2,x_3) &= a_{222}x_1^2x_2^2x_3^2+a_{202}x_1^2x_3^2+a_{022}x_2^2x_3^2,\\ g_1(x_1,x_2,x_3) &= b_{110}x_1x_2+b_{011}x_2x_3+b_{300}x_1^3,\\ h_1(x_1,x_2,x_3) &= c_{220}x_1^2x_2^2+c_{202}x_1^2x_3^2+c_{022}x_2^2x_3^2+c_{240}x_1^2x_2^4,\end{aligned} \tag{36}$$

where a_{ij},b_{ij} and c_{ij} are arbitrary real numbers, and R_1,R_2,R_3 are arbitrary functions that have the reversible symmetry T_5. Clearly this system is T_5-reversible. Moreover, the previous functions (36) in cylindrical coordinates takes the form

$$\begin{aligned}\tilde{f}_1(r,\theta,z) &= a_{222}r^4z^2\cos^2\theta\sin^2\theta+a_{202}r^2z^2\cos^2\theta+a_{022}r^2z^2\sin^2\theta,\\ \tilde{g}_1(r,\theta,z) &= b_{110}r^2\cos\theta\sin\theta+b_{011}rz\sin\theta+b_{300}r^3\cos^3\theta,\\ \tilde{h}_1(r,\theta,z) &= c_{220}r^4\cos^2\theta\sin^2\theta+c_{202}r^2z^2\cos^2\theta+c_{022}r^2z^2\sin^2\theta+c_{240}r^6\cos^2\theta\sin^4\theta.\end{aligned}$$

Next, we point out that the integral in the item a) of Theorem 1.3 becomes

$$\int_0^\pi\tilde{h}_1(r_0,s+\pi/2,0)ds=\frac{\pi r^4}{16}\left(2c_{220}+c_{240}r^2\right). \tag{37}$$

In order to apply item i) of Theorem 1.3, we need to solve the equation

$$\frac{\pi r^4}{16}\left(2c_{220}+c_{240}r^2\right)=0.$$

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Then, there is a unique $r_0 > 0$

$$r_0 = \sqrt{2}\sqrt{-\frac{c_{220}}{c_{240}}},$$

whether $c_{220}c_{240} < 0$. *The calculation of condition b) of Theorem 1.3 becomes*

$$\frac{\pi c_{220}^{5/2}}{\sqrt{2}c_{240}^{3/2}}. \tag{38}$$

In conclusion, by Theorem 1.3 we have the following result.

Theorem 4.1. *Let the spatial system (2) with* $f_1(x_1, x_2, x_3)$, $g_1(x_1, x_2, x_3)$, *and* $h_1(x_1, x_2, x_3)$ *as in (36), and let* R_1, R_2, R_3 *be arbitrary functions possessing the symmetry* T_5. *Assume that* $c_{220}c_{240} < 0$. *Then, there exists a unique one-parameter family of initial conditions* $(\delta r(\varepsilon), \frac{\pi}{2}, 0)$ *with* $\delta r(0) = 0$ *and a differentiable function* $\tau(\varepsilon)$ *such that each initial condition gives a* T_5*-reversible symmetric periodic solution of the full system (2), with period* $2\tau(\varepsilon) = 2\pi + \mathcal{O}(\varepsilon^\alpha)$. *Moreover, these solutions are close to a circular solution with radius* $r_0 = \sqrt{2}\sqrt{\frac{c_{220}}{c_{240}}}$ *and lie on the* x_1x_2*-plane.*

5. Concluding remarks

In this work we have considered the following analytic (or C^k- differentiable) family of ODE in $\mathbb{R}^3$ of the form

$$\begin{aligned} \dot{x_1} &= -x_2 + \varepsilon^\alpha f_1(x_1, x_2, x_3) + \varepsilon^{\alpha+1} R_1(x_1, x_2, x_3, \varepsilon), \\ \dot{x_2} &= \quad x_1 + \varepsilon^\alpha g_1(x_1, x_2, x_3) + \varepsilon^{\alpha+1} R_2(x_1, x_2, x_3, \varepsilon), \\ \dot{x_3} &= \qquad \varepsilon^\alpha h_1(x_1, x_2, x_3) + +\varepsilon^{\alpha+1} R_3(x_1, x_2, x_3, \varepsilon), \end{aligned} \tag{39}$$

where $\alpha \in \mathbb{N}$. Here ε is a small parameter. Note that the unperturbed system is

$$\dot{x}_1 = -x_2,\ \dot{x}_2 = x_1,\ \dot{x}_3 = 0,$$

which is foliated by circular solutions of fixed period 2π on each plane $x_3 = x_{30}$.

For our approach, we introduce cylindrical coordinates $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, $x_3 = x_3$, then the system (39) takes the form

$$\begin{aligned} \dot{r} &= \varepsilon^\alpha F_1(r, \theta, z) + \varepsilon^{\alpha+1} F_2(r, \theta, z) + \mathcal{O}(\varepsilon^{\alpha+2}) \\ \dot{\theta} &= 1 + \varepsilon^\alpha G_1(r, \theta, z) + \varepsilon^{\alpha+1} G_2(r, \theta, z) + \mathcal{O}(\varepsilon^{\alpha+2}) \\ \dot{z} &= \varepsilon^\alpha K_1(r, \theta, z) + \varepsilon^{\alpha+1} K_2(r, \theta, z) + \mathcal{O}(\varepsilon^{\alpha+2}), \end{aligned} \tag{40}$$

where

$$\begin{aligned} F_1(r, \theta, z) &= \tilde{f}_1(r, \theta, z)\cos\theta + \tilde{g}_1(r, \theta, z)\sin\theta \\ G_1(r, \theta, z) &= \frac{1}{r}\left[\tilde{g}_1(r, \theta, z)\cos\theta - \tilde{f}_1(r, \theta, z)\sin\theta\right], \\ K_1(r, \theta, z) &= \tilde{h}_1(r, \theta, z), \end{aligned} \tag{41}$$

with the functions $\tilde{f}_1(r, \theta, z) = f_1(r\cos\theta, r\sin\theta, z)$, $\tilde{g}_1(r, \theta, z) = g_1(r\cos\theta, r\sin\theta, z)$ and $\tilde{h}_1(r, \theta, z) = h_1(r\cos\theta, r\sin\theta, z)$.

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In this work, we combine the reversible-discrete symmetries T of the system (39) and the Poincaré continuation method, strongly using the first approximation of the solutions of the full ordinary differential system given by a variational system. The main contribution of this paper is to provide sufficient conditions for the existence of different families of reversible-symmetric periodic solutions for spatial perturbations 39 using cylindrical variables. Furthermore, we provide information about the characteristic multipliers of these solutions. We prove different theorems that give sufficient conditions for the existence of two types of families of initial conditions that produce reversible-symmetric periodic solutions for the system (39). The first type corresponds to reversible-symmetric periodic solutions with the same period as the circular orbit of the unperturbed system. The second type corresponds to reversible-symmetric periodic solutions with variable period, but close to that of the unperturbed circular orbit.

We point out that the periodicity equations (conditions for the existence of reversible-symmetric periodic solutions) depend on the dimension of the fixed point set S. Since S can be a reflection with respect to the plane $x = 0$, or $y = 0$, or $z = 0$, or a reflection with respect to the straight lines $x = y = 0$, $x = z = 0$, or $y = z = 0$, it follows that S) can be 1 or 2. In the case $\dim(S) = 2$ there is only one condition to be satisfied, while in the case $\dim(S) = 1$ two conditions must be satisfied.

The importance of stating many theorems as results in this work lies in the different cases obtained by imposing a reversible symmetry on the system (39) in order to find symmetric periodic solutions, which is the main objective of this work. Hence, the relevance lies in the set of fixed points for each symmetry and their corresponding periodicity equations, as well as in the conditions required to apply the Implicit Function Theorem. We would like to highlight the impossibility of applying our method for certain symmetries. For the reversible symmetry $T_4(x_1, x_2, x_3) = (-x_1, -x_2, x_3)$, we have $\mathcal{L}_4 = \text{Fix}(T_4) : x_1 = x_2 = 0$. In cylindrical coordinates, a solution with $r > 0$ would be T_4-reversible symmetric if $\theta_0 = 0$ and $\theta_0 = \pi/2 \pmod{\pi}$ simultaneously, which is impossible.

5.1. **A comparison with the Averaging method.** In order to complement our study, we analyze the possibility of using the Averaging method. We conclude that it is not possible to use this method in some cases due to the degeneracy of the averaging function at its zeros.

We start our analysis by considering the variable θ as a new time parameter, so the system (4) assumes the form

$$\begin{aligned}
\frac{dr}{d\theta} &= \frac{\varepsilon^\alpha F_1(r,\theta,z) + \varepsilon^{\alpha+1} F_2(r,\theta,z) + \mathcal{O}(\varepsilon^{\alpha+2})}{1 + \varepsilon^\alpha G_1(r,\theta,z) + \varepsilon^{\alpha+1} G_2(r,\theta,z) + \mathcal{O}(\varepsilon^{\alpha+2})}, \\
\frac{dz}{d\theta} &= \frac{\varepsilon^\alpha K_1(r,\theta,z) + \varepsilon^{\alpha+1} K_2(r,\theta,z) + \mathcal{O}(\varepsilon^{\alpha+2})}{1 + \varepsilon^\alpha G_1(r,\theta,z) + \varepsilon^{\alpha+1} G_2(r,\theta,z) + \mathcal{O}(\varepsilon^{\alpha+2})}.
\end{aligned} \tag{42}$$

Making the respective Taylor series approximation and $\varepsilon = 0$, we obtain the new equations

$$\begin{aligned}
\frac{dr}{d\theta} &= \varepsilon^\alpha F_1(r,\theta,z) + \varepsilon^{\alpha+1}\left[F_2(r,\theta,z) - F_1(r,\theta,z)G_1(r,\theta,z)\right] + \mathcal{O}(\varepsilon^{\alpha+2}), \\
\frac{dz}{d\theta} &= \varepsilon^\alpha K_1(r,\theta,z) + \varepsilon^{\alpha+1}\left[K_2(r,\theta,z) - K_1(r,\theta,z)G_1(r,\theta,z)\right] + \mathcal{O}(\varepsilon^{\alpha+2}),
\end{aligned} \tag{43}$$

where F_1, G_1 and K_1 are as (41) and $F_2 = \tilde{f}_2(r,\theta,z)\cos\theta + \tilde{g}_2(r,\theta,z)\sin\theta$ and $K_2 = \tilde{h}_2(r,\theta,z)$ where $\tilde{f}_2(r,\theta,z) = f_2(r\cos\theta, r\sin\theta, z), \tilde{g}_2(r,\theta,z) = g_2(r\cos\theta, r\sin\theta, z)$ and $\tilde{h}_2(r,\theta,z) = h_2(r\cos\theta, r\sin\theta, z)$. Moreover, $\tilde{f}_2, \tilde{h}_2$ are odd functions in θ and $\tilde{h}_2$ is even in θ. Now, we obtain the averaged functions for the first order,

$$\overline{F}_1(r,z) = \int_0^{2\pi} F_1(r,z,\theta)d\theta, \quad \overline{K}_1(r,z) = \int_0^{2\pi} K_1(r,z,\theta)d\theta. \tag{44}$$

We call the attention that in order to apply the Averaging theorem of first order, we need initially to find (r_0, z_0) such that,

$$\overline{F}_1(r_0,z_0) = 0, \quad \overline{K}_1(r_0,z_0) = 0, \tag{45}$$

i.e., we need to solve a system of two equations. In particular, in order to find (r_0, z_0) for this,method note that we need solve a system of two equations, this is an important difference with our approach in this work, because in some cases of our symmetries we need to solve only one equation.

It is important to emphasize that the existence of family of periodic solutions bifurcates from the point (r_0, z_0). We need to verify the additional condition of non-degeneration

$$\det \begin{pmatrix} \dfrac{\partial \overline{K}_1}{\partial r} & \dfrac{\partial \overline{K}_1}{\partial \theta} \\ \dfrac{\partial \overline{F}_1}{\partial r} & \dfrac{\partial \overline{F}_1}{\partial \theta} \end{pmatrix}\Bigg|_{\substack{r=r_0 \\ z=z_0}} \neq 0.$$

5.1.1. **Analysis of the symmetry T_1.** We will analyze the integrals that define the functions F_1 and/or K_1, assuming that the vector field associated with system (39) possesses the reversible symmetry T_1.

Lemma 5.1. *Assume that the vector field (2) has the T_1 symmetry. Then, we have the following:*

(1) $\overline{F}_1(r,z) \equiv 0$,
(2) $\overline{K}_1(r,z) \equiv 0$.

Proof. Under the symmetry T_1, we have the following symmetry properties with respect to the angular variable θ,

$$\tilde{f}_1(r,\theta,z) = \tilde{f}_1(r,\pi-\theta,z), \tilde{g}_1(r,\theta,z) = -\tilde{g}_1(r,\pi-\theta,z), \tilde{h}_1(r,\theta,z) = -\tilde{h}_1(r,\pi-\theta,z). \tag{46}$$

Then, it is verified that, $F_1(r,\pi-\theta,z) = -F_1(r,\theta,z)$, $G_1(r,\pi-\theta,z) = G_1(r,\theta,z)$ and $K_1(r,\pi-\theta,z) = -K_1(r,\theta,z)$. Let $u = \pi - \theta$ be a coordinates change, then

$$\int_0^{\pi} F_1(r,\theta,z)d\theta = -\int_{\pi}^{0} F_1(r,\pi-u,z)du = \int_0^{\pi} F_1(r,\pi-u,z)du = -\int_0^{\pi} F_1(r,u,z)du.$$

Returning to the variable θ we have that, $\int_0^{\pi} F_1(r,\theta,z)d\theta = 0$. Similarly, making the change of coordinates $u = 3\pi - \theta$, we can prove that $\int_{\pi}^{2\pi} F_1(r,\theta,z)d\theta = 0$. Analogously to the previous process, we prove that $\int_0^{\pi} K_1(r,\theta,z)d\theta = 0$ and $\int_{\pi}^{2\pi} K_1(r,\theta,z)d\theta = 0$. □

Remark 3. *From the previous lemma, it is clear that the first-order Averaging theory cannot be applied if the vector field in (2) or (39) possesses the reversible symmetry T_1. In particular, the first-order Averaging method fails to yield periodic solutions in our Example 2.1, where f_1, g_1, and h_1 are as given in (15).*

Motivated by the results in [20], we must to examine whether any conclusions about the existence of periodic solutions can be obtained via the second-order averaging theory. For this, it is necessary to calculate the solutions of the system $\tilde{F}_2 = 0$ and $\tilde{K}_2 = 0$ where

$$\begin{aligned} \tilde{F}_2(r,z) &= \int_0^{2\pi} \left[F_2(r,\theta,z) - F_1(r,\theta,z)G_1(r,\theta,z)\right] d\theta, \\ \tilde{K}_2(r,z) &= \int_0^{2\pi} \left[K_2(r,\theta,z) - K_1(r,\theta,z)G_1(r,\theta,z)\right] d\theta. \end{aligned} \tag{47}$$

By the symmetry T_1, we have the following symmetry properties with respect to the angular variable θ,

$$\begin{aligned} \tilde{f}_2(r,\theta,z) =& \tilde{f}_2(r,\pi-\theta,z), \\ \tilde{g}_2(r,\theta,z) =& -\tilde{g}_2(r,\pi-\theta,z), \\ \tilde{h}_2(r,\theta,z) =& -\tilde{h}_2(r,\pi-\theta,z). \end{aligned} \tag{48}$$

Then it is verified the following symmetry properties,

$$\begin{aligned} F_2(r,\pi-\theta,z) &= -F_2(r,\theta,z), \\ G_2(r,\pi-\theta,z) &= G_2(r,\theta,z), \\ K_2(r,\pi-\theta,z) &= -K_2(r,\theta,z). \end{aligned} \tag{49}$$

We present the following result,

Lemma 5.2. *Assume that the vector field (2) or (39) has the T_1 symmetry. Then, we have the following:*

(1) $\tilde{F}_2(r,z) = 0$,
(2) $\tilde{K}_2(r,z) = 0$.

Proof. For item (1), we write

$$\int_0^{2\pi} \left[F_2(r,\theta,z) - F_1(r,\theta,z)G_1(r,\theta,z)\right] d\theta = \int_0^{2\pi} F_2(r,\theta,z)d\theta - \int_0^{2\pi} F_1(r,\theta,z)G_1(r,\theta,z)d\theta.$$

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Since $F_2(r,\theta,z) = -F_2(r,\pi-\theta,z)$, the integral $\displaystyle\int_0^{2\pi} F_2(r,\theta,z) = 0$, the proof is similar to item (1) of Lemma 5.1. Now, we will focus on demonstrating

$$\int_0^{2\pi} F_1(r,\theta,z)G_1(r,\theta,z)d\theta = 0.$$

For this, rewrite the integral as

$$\int_0^{2\pi} F_1(r,\theta,z)G_1(r,\theta,z)d\theta = \int_0^{\pi} F_1(r,\theta,z)G_1(r,\theta,z)d\theta + \int_{\pi}^{2\pi} F_1(r,\theta,z)G_1(r,\theta,z)d\theta.$$

If we take the change of coordinates $u = \pi - \theta$ and by the symmetry properties (49), we have that

$$\begin{aligned} \int_0^{\pi} F_1(r,\theta,z)G_1(r,\theta,z)d\theta &= -\int_{\pi}^{0} F_1(r,\pi-u,z)G_1(r,\pi-u,z)du, \\ &= -\int_0^{\pi} F_1(r,u,z)G_1(r,u,z)du. \end{aligned} \tag{50}$$

So, we obtain that $\displaystyle\int_0^{\pi} F_1(r,\theta,z)G_1(r,\theta,z)d\theta = 0$. On the other hand, if we take the change of coordinates $u = 3\pi-\theta$, we get similarly that, $\displaystyle\int_{\pi}^{2\pi} F_1(r,\theta,z)G_1(r,\theta,z)d\theta = 0$. Analogously, by performing the same procedure, we prove the result for the item (2). □

Remark 4. *Consequently, the second-order averaging method yields no information regarding the existence of periodic solutions. Recursively, following the results in [20], the higher-order averaging theory also fails to provide information on the existence of periodic solutions.*

5.1.2. **Analysis of the symmetry T_2.** Now we proceed to analyze the symmetry T_2. In this case we have the following result.

Lemma 5.3. *Assume that the vector field (39) has the T_2 symmetry. Then, we have the following:*

(1) $\overline{F}_1(r,z) = 0$,
(2) $\overline{K}_1(r,z) = 0$.

Proof. We know that $F_1(r,\theta,z) = \tilde{f}_1(r,\theta,z)\cos\theta + \tilde{g}_1(r,\theta,z)\sin\theta$ and $K_1(r,\theta,z) = \tilde{h}_1(r,\theta,z)$. By the reversible symmetry T_2, the functions f_1 and h_1 are odd in the variable y, while g_1 is even. Consequently, $\tilde{f}_1$ and $\tilde{h}_1$ are odd functions of θ, and $\tilde{g}_1$ is even. It follows that both F_1 and K_1 are odd in θ, and therefore their integrals over a symmetric interval vanish. □

Remark 5. *Analogously to the previous case we cannot apply the Averaging theory of first order if the vector field (2) or (39) possess the reversible symmetry T_2. In particular, we cannot obtain periodic solutions by the Averaging method of first order in our Example 2.2 with f_1, g_1 and h_1 as in (17).*

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5.1.3. **Analysis of the symmetry T_3.** To compare our results with our method and the theory of Averaging, we will concentrate on studying polynomial vector field as in (39). The existence of the reversible implies that f_1, g_1 are odd in z and h_1 is even in z, this implies that

$$f_1(x,y,z) = z\tilde{f}_1(x,y,z^2),\ g_1(x,y,z) = z\tilde{g}_1(x,y,z^2), \text{ and } h_1(x,y,z) = \tilde{h}_1(x,y,z^2),$$

for some polynomials $\tilde{f}_1, \tilde{g}_1$ and $\tilde{h}_1$. Then,

$$\overline{F}_1(r,z) = z\int_0^{2\pi}\left[\tilde{f}_1(r,\theta,z^2)\cos\theta + \tilde{g}_1(r,\theta,z^2)\sin\theta\right]d\theta.$$

Remark 6. *From the previous calculus we observe that $z_0 = 0$ can be part of an equilibrium point of the Averaging criteria of first order and r_0 must be computed by the equation $\overline{K}_1(r_0,0) = 0$. On the other hand, in our main result for the reversible symmetry T_3, that is Theorem 1.2, we have that the continued periodic solutions bifurcate from the circular solution on the plane $z_0 = 0$ and r_0 is computed by the equation $\int_0^{\pi}\tilde{h}_1(r_0,\theta+\theta_0,0)ds = 0$. Therefore, we deduce that in both methods $z_0 = 0$ could generate periodic solutions*

Remark 7. *We consider the Example 3.1 in order to compare the applicability of the averaging method of the first order. We note that*

$$\begin{aligned}\overline{F}_1(r,z) &= \pi r z(a_{101}+b_{011}),\\ \overline{K}_1(r,z) &= \frac{1}{4}\pi\left(8c_{002}z^2 + 4c_{200}r^2 + c_{320}r^4\right).\end{aligned}$$

From this system, we obtain a unique equilibrium point $\left(r_0 = 2\sqrt{-\frac{c_{200}}{c_{320}}}, z_0 = 0\right)$ under the condition $c_{200}c_{320} < 0$. It is verified that the determinant of its monodromy matrix is $-\frac{8\pi^2 c_{200}^2(a_{101}+b_{011})}{c_{320}}$. If we impose that $(a_{101}+b_{011})c_{200} \neq 0$, then by the Averaging theorem of first order, we obtain a unique family of periodic solution of period close to 2π which are close to the plane $z = 0$.
On the other hand, using our method r_0 must satisfy the equation

$$\int_0^{\pi}\tilde{h}_1(r_0,s+\theta_0,0)ds = \frac{1}{24}r^2\left[3\pi\left(4c_{200}+c_{320}r^2\right) + 16c_{210}r\cos^3\theta_0\right] = 0.$$

From where given conditions on the parameters, we obtain 1 or 2 positive roots.

We point out that by Theorem 1.2, we obtain a family of initial conditions parameterized by ε, given by $(r(\varepsilon),\theta(\varepsilon),z(\varepsilon)) = (r_0+\delta r(\varepsilon),\theta_0,0)$, where θ_0 is initially arbitrary. The application of the Averaging method yields a different family of initial conditions of the form $(\tilde{r}(\varepsilon),\tilde{\theta}(\varepsilon),\tilde{z}(\varepsilon)) = (r_0+\mathcal{O}(\varepsilon^3),0,\mathcal{O}(\varepsilon^3))$. It is important to note that these two families of initial conditions are not the same. In the Example 3.1 by the method presented in this work and the Averaging theory, the bifurcation of periodic solutions from the same point $(r_0, z_0 = 0)$. Note theses families of initial conditions are not the same. In the Figure 2 we show how these two different families of initial conditions generated by both methods.

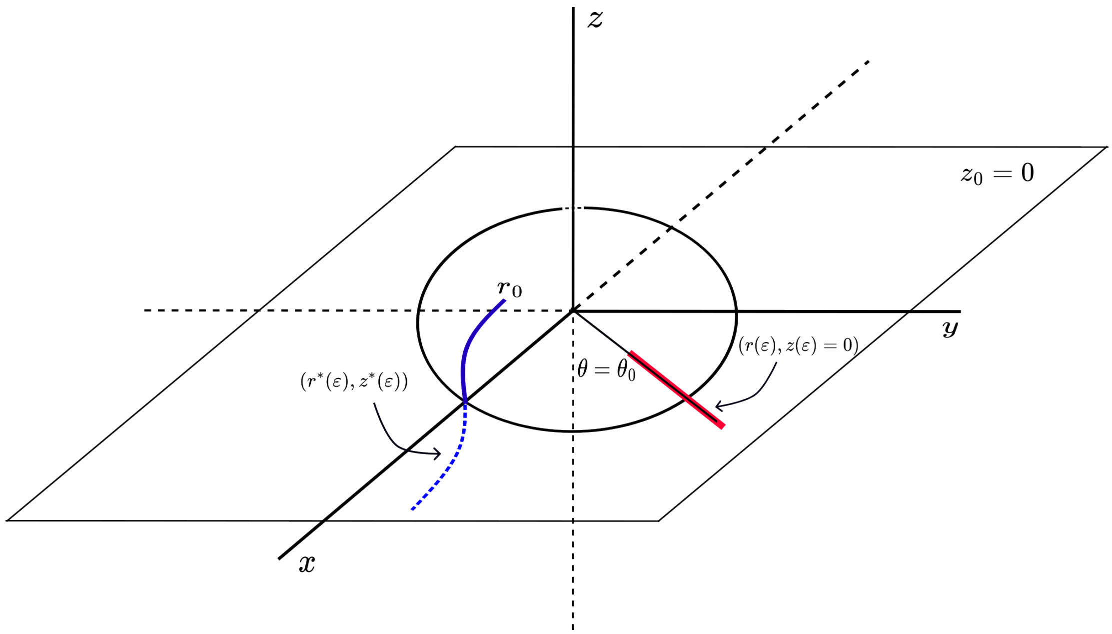

FIGURE 2. Families of initial conditions $(r(\varepsilon), z(\varepsilon) = 0)$ and $(r^*(\varepsilon), z^*(\varepsilon))$ parameterized by ε, obtained by Theorem 1.2 and the Averaging theorem of first order respectively, when the field (2) is under the reversible symmetry T_3. Initially $\theta = \theta_0$ is an arbitrary angle.

5.1.4. **Analysis of the symmetry T_5.** For the purpose of analyzing the symmetry T_5, we will assume that the functions f_j, g_j, h_j are polynomials. So, we have that they can by written as

$$\begin{aligned} f_1(x,y,z) &= f_{11}(x^2,y,z^2), \\ g_1(x,y,z) &= x\ g_{11}(x^2,y,z^2) + z\ g_{12}(x^2,y,z^2), \\ h_1(x,y,z) &= h_{11}(x^2,y,z^2), \end{aligned}$$

For some polynomials f_1, g_1 and h_1. Now, we begin by presenting the following result.

Lemma 5.4. *It is verified the following:*

(1) $\int_0^{2\pi} \cos^{2\ell+1}\theta \sin^m \theta d\theta = 0,$

(2) $\int_0^{2\pi} \cos^{2\ell}\theta \sin^m \theta d\theta = 0$ *if m is odd.*

(3) $\int_0^{2\pi} \cos^{2\ell}\theta \sin^m \theta d\theta = 2\int_0^{\pi} \cos^{2\ell}\theta \sin^m \theta d\theta$ *if m is even.*

(4) $\int_0^{2\pi} \cos^{2\ell+1}\theta d\theta = 0.$

(5) $\int_0^{2\pi} \cos^{2\ell}\theta d\theta = 2\pi\frac{(2l)!}{2^{2l}(l!)^2}.$

for $\ell, m \in \mathbb{N}_0$.

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Lemma 5.5. *Assume that the vector field (2) or (39) has the reversible symmetry T_5. Then,*

(1) $\int_0^{2\pi} F_1(r,\theta,z)d\theta = rz\int_0^{2\pi}\cos\theta\ \tilde{g}_{12}(r^2\cos^2\theta, r\sin\theta, z^2)d\theta,$

(2) $\int_0^{2\pi} K_1(r,\theta,z)d\theta = 2\int_0^{\pi}\tilde{h}_{11}(r^2\cos^2\theta, r\sin\theta, z^2)d\theta.$

Proof. In cylindrical coordinates the functions f_1, g_1 and h_1 with the symmetry T_5 has the form

$$\begin{aligned}
\tilde{f}_1(r,\theta,z) &= \tilde{f}_{11}(r^2\cos^2\theta, r\sin\theta, z^2),\\
\tilde{g}_1(r,\theta,z) &= r\cos\theta\ \tilde{g}_{11}(r^2\cos^2\theta, r\sin\theta, z^2) + z\ \tilde{g}_{12}(r^2\cos^2\theta, r\sin\theta, z^2),\\
\tilde{h}_1(r,\theta,z) &= \tilde{h}_{11}(r^2\cos^2\theta, r\sin\theta, z^2).
\end{aligned}$$

Then, by equations (44) we have that

$$\begin{aligned}
F_1(r,\theta,z) &= \tilde{f}_{11}(r^2\cos^2\theta, r\sin\theta, z^2)\cos\theta + \left[r\cos\theta\ \tilde{g}_{11}(r^2\cos^2\theta, r\sin\theta, z^2) + z\ \tilde{g}_{12}(r^2\cos^2\theta, r\sin\theta, z^2)\right]\sin\theta,\\
K_1(r,\theta,z) &= \tilde{h}_{11}(r^2\cos^2\theta, r\sin\theta, z^2).
\end{aligned}$$

Now, we analyze the following integrals that are part of the first equation in (44)

$$\begin{aligned}
I_1 &= \int_0^{2\pi}\tilde{f}_{11}(r^2\cos^2\theta, r\sin\theta, z^2)\cos\theta d\theta,\\
I_2 &= r\int_0^{2\pi}\cos\theta\sin\theta\ \tilde{g}_{11}(r^2\cos^2\theta, r\sin\theta, z^2)d\theta,\\
I_3 &= rz\int_0^{2\pi}\sin\theta\ \tilde{g}_{12}(r^2\cos^2\theta, r\sin\theta, z^2)d\theta.
\end{aligned}$$

It is verified that $I_1 = 0, I_2 = 0$ by virtue of the previous Lemma 5.4. Since the integrand is polynomial, it results in a combination of sine and cosine functions of that form

$$\cos^{2l+1}\theta\sin^m\theta, sin^{2l}\cos\theta,$$

and using the items of the previous Lemma 5.4, we obtain the result. ☐

Remark 8. *In general, it is not true that $\overline{F}_1 \equiv 0$ and $\overline{K}_1 \equiv 0$. In fact, we obtain*

$$\begin{aligned}
\overline{F}_1(r,z) &= \pi b_{011} rz\\
\overline{K}_1(r,z) &= \frac{1}{8}\pi r^2\left[\left(8z^2(c_{022}+c_{202}) + 2c_{220}r^2 + c_{240}r^4\right)\right].
\end{aligned}$$

Moreover, we obtain a unique zero given by $\left(r_0 = \sqrt{2}\sqrt{-\frac{c_{220}}{c_{240}}}, z_0 = 0\right)$ under the condition $c_{220}c_{240} < 0$. It is verified that the determinant of its monodromy matrix is $\frac{2\pi^2 b_{011}c_{220}^3}{c_{240}^2}$. If we impose that $b_{011}c_{220}c_{240} \neq 0$, we obtain a family of periodic solutions with period close to 2π by the Averaging method.

Now, as we see in Theorem 1.3 with $j = 1$, we obtain a family of initial conditions parameterized by ε as follows $(r(\varepsilon), \theta(\varepsilon), z(\varepsilon)) = (r_0 + \delta r(\varepsilon), \pi/2, 0)$ (with the same value of r_0). However, by the use of the Averaging method, we obtain a family of initial conditions

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given by $(\tilde{r}(\varepsilon), \tilde{\theta}(\varepsilon), \tilde{z}(\varepsilon)) = (r_0 + \mathcal{O}(\varepsilon^3), 0, 0 + \mathcal{O}(\varepsilon^3))$. *Furthermore, we obtain, via our method and the Averaging theory, the bifurcation of periodic solutions from the same point* $(r_0, z_0 = 0)$. *However, theses families of initial conditions are not the same as shown in the figure 3.*

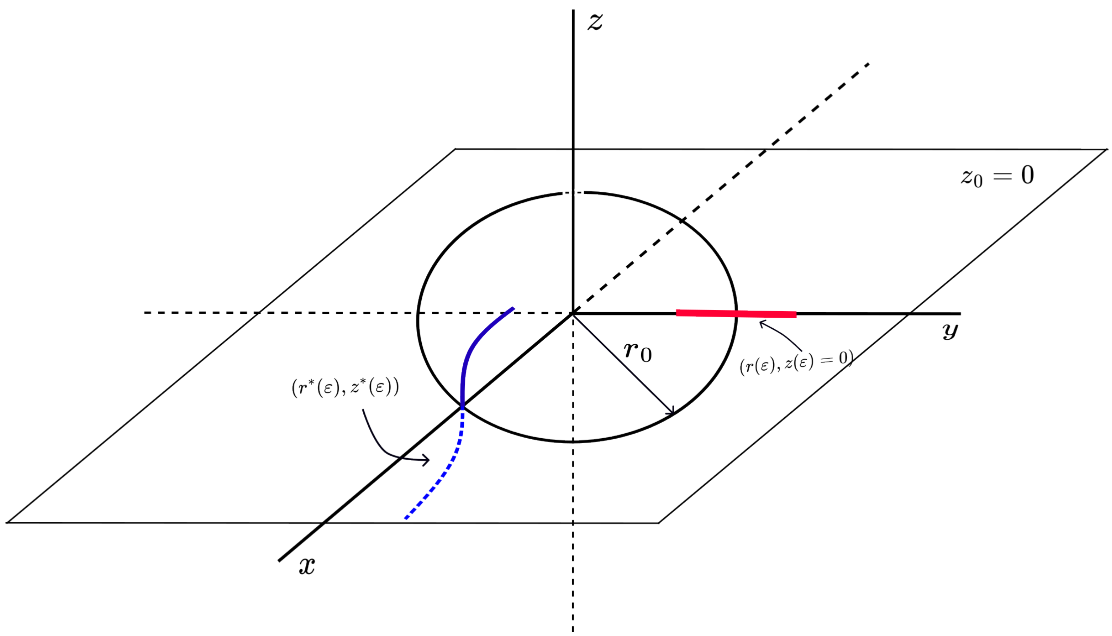

FIGURE 3. Families of initial conditions $(r(\varepsilon), z(\varepsilon) = 0)$ and $(r^*(\varepsilon), z^*(\varepsilon))$ parameterized by ε, obtained by Theorem 1.3 with $j = 1$ (this is the T_5-symmetry) and the Averaging theorem respectively, when the field (2) is under the reversible symmetry T_5 and $\theta_0 = \pi/2$.

5.1.5. **Analysis of the symmetry T_6.** For the purpose of analyzing the symmetry T_6, we will again assume that the functions f_1, g_1, h_1 are polynomials. Then, by virtue of the symmetry we have that

$$\begin{aligned} f_1(x,y,z) &= y\ f_{11}(x,y^2,z^2) + z\ f_{12}(x,y^2,z^2),\\ g_1(x,y,z) &= g_{11}(x,y^2,z^2),\\ h_1(x,y,z) &= h_{11}(x,y^2,z^2). \end{aligned}$$

for some polynomials f_1, g_1 and h_1. In cylindrical coordinates the functions take the form,

$$\begin{aligned} \tilde{f}_1(r,\theta,z) &= r\sin\theta\ \tilde{f}_{11}(r\cos\theta, r^2\sin^2\theta, z^2) + z\ \tilde{f}_{12}(r\cos\theta, r^2\sin^2\theta, z^2),\\ \tilde{g}_1(r,\theta,z) &= \tilde{g}_{11}(r\cos\theta, r^2\sin^2\theta, z^2),\\ \tilde{h}_1(r,\theta,z) &= \tilde{h}_{11}(r\cos\theta, r^2\sin^2\theta, z^2). \end{aligned}$$

Lemma 5.6. *Assume that the vector field* (2) *and* (39) *has the reversible symmetry* T_6. *Then, we have the following:*

$$\int_0^{2\pi} F_1(r,\theta,z)d\theta = z\int_0^{2\pi} \tilde{f}_{12}(r\cos\theta, r^2\sin^2\theta, z^2)\cos\theta d\theta,$$

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$$\int_0^{2\pi} \tilde{h}_{11}(r\cos\theta, r\sin^2\theta, z^2)d\theta = 2\int_0^{\pi} \tilde{h}_{11}(r\cos\theta, r\sin^2\theta, z^2)d\theta.$$

Proof. By the equations (44), the integral of F_1 takes the form

$$\int_0^{2\pi} \Big[\big(r\sin\theta\ \tilde{f}_{11}(r\cos\theta, r^2\sin^2\theta, z^2) + z\ \tilde{f}_{12}(r\cos\theta, r^2\sin^2\theta, z^2)\big)\cos\theta + \tilde{g}_{11}(r\cos\theta, r^2\sin^2\theta, z^2)\sin\theta\Big]\, d\theta.$$

and we write it as the sum of the following three integrals

$$\begin{aligned} I_1 &= \int_0^{2\pi} r\sin\theta \tilde{f}_{11}(r\cos\theta, r^2\sin^2\theta, z^2)\cos\theta d\theta,\\ I_2 &= \int_0^{2\pi} z\ \tilde{f}_{12}(r\cos\theta, r^2\sin^2\theta, z^2)\cos\theta d\theta,\\ I_3 &= \int_0^{2\pi} \sin\theta\ \tilde{g}_1(r\cos\theta, r^2\sin^2\theta, z^2)d\theta. \end{aligned}$$

It is verified that $I_1 = 0$ and $I_3 = 0$ due to the parity of the integrand, since they are odd in the variable θ. For the integral I_2 we cannot say anything because the integrand is even in θ. Now we rewrite the integral

$$\int_0^{2\pi} \tilde{h}_1(r\cos\theta, r\sin^2\theta, z^2) = \int_0^{\pi} \tilde{h}_1(r\cos\theta, r^2\sin^2\theta, z^2)d\theta + \int_{\pi}^{2\pi} \tilde{h}_1(r\cos\theta, r^2\sin^2\theta, z^2)d\theta.$$

Let $u = 2\pi - \theta$ be a change of coordinates and observe that

$$\begin{aligned} \int_{\pi}^{2\pi} \tilde{h}_1(r\cos\theta, r^2\sin^2\theta, z^2)d\theta &= -\int_{\pi}^{0} \tilde{h}_1(r\cos(2\pi-u), r^2\sin^2(2\pi-u), z^2)du,\\ &= \int_0^{\pi} \tilde{h}_1(r\cos u, r^2\sin^2 u, 0)du. \end{aligned}$$

This concludes the proof. □

Remark 9. *By item* (1) *of the previous Lemma 5.6, it is possible to apply the averaging method and obtain information about the existence of periodic solutions when the vector field* (2) *or* (39) *admits the reversible symmetry* T_6*. Our objective now is to present the differences between both methods, highlighting the advantages and disadvantages of each. For this purpose, we will present three examples that we consider important to illustrate the relevance of these differences depending on the symmetry.*

Example 5.1. *Consider the system* (2) *with the following functions that admit symmetry* T_6

$$\begin{aligned} f_1(x_1,x_2,x_3) &= a_{210}x_1^2x_2 + a_{030}x_2^3 + a_{201}x_1^2x_3,\\ g_1(x_1,x_2,x_3) &= b_{120}x_1x_2^2 + b_{102}x_1x_3^2 + b_{022}x_2^2x_3^2,\\ h_1(x_1,x_2,x_3) &= c_{220}x_1^2x_2^2 + c_{020}x_2^2 + c_{002}x_3^2, \end{aligned} \tag{51}$$

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where a_{ij}, b_{ij} and c_{ij} are arbitrary real numbers, and $R_1 = R_2 = R_3 = 0$. The previous functions in cylindrical coordinates takes the form

$$\begin{aligned}\tilde{f}_1(r,\theta,z) =& a_{210}r^3\cos^2\theta\sin\theta + a_{030}r^3\sin^3\theta + a_{201}zr^2\cos^2\theta,\\ \tilde{g}_1(r,\theta,z) =& b_{120}r^3\cos\theta\sin^2\theta + b_{102}rz^2\cos\theta + b_{022}r^2z^2\sin^2\theta,\\ \tilde{h}_1(r,\theta,z) =& c_{220}r^4\cos^2\theta\sin^2\theta + c_{020}r^2\sin^2\theta + c_{002}z^2.\end{aligned}$$

We verified that $\overline{F}_1(r,z) \equiv 0$, thus, we cannot apply the Averaging method of first order.

Next, we present other important example for the analysis.

Example 5.2. *Let*

$$\begin{aligned}f_1(x_1,x_2,x_3) =& a_{210}x_1^2x_2 + a_{030}x_2^3 + a_{201}x_1^2x_3 + a_{301}x^3z,\\ g_1(x_1,x_2,x_3) =& b_{120}x_1x_2^2 + b_{102}x_1x_3^2 + b_{022}x_2^2x_3^2,\\ h_1(x_1,x_2,x_3) =& c_{220}x_1^2x_2^2 + c_{020}x_2^2 + c_{002}x_3^2,\end{aligned} \tag{52}$$

be polynomials that admits the reversible symmetry T_6 where a_{ij}, b_{ij} and c_{ij} are arbitrary real numbers, and $R_1 = R_2 = R_3 = 0$. The previous functions in cylindrical coordinates takes the form

$$\begin{aligned}\tilde{f}_1(r,\theta,z) =& a_{210}r^3\cos^2\theta\sin\theta + a_{030}r^3\sin^3\theta + a_{201}zr^2\cos^2\theta + r^3z\cos^3\theta,\\ \tilde{g}_1(r,\theta,z) =& b_{120}r^3\cos\theta\sin^2\theta + b_{102}rz^2\cos\theta + b_{022}r^2z^2\sin^2\theta,\\ \tilde{h}_1(r,\theta,z) =& c_{220}r^4\cos^2\theta\sin^2\theta + c_{020}r^2\sin^2\theta + c_{002}z^2.\end{aligned}$$

We obtain that

$$\begin{aligned}\overline{F}_1(r,z) &= \frac{3a_{301}\pi}{4}r^3z,\\ \overline{K}_1(r,z) &= \frac{1}{4}\pi\left(8c_{002}z^2 + 4c_{020}r^2 + c_{220}r^4\right).\end{aligned}$$

This system has a zero with radius $r_0 = 2\sqrt{\frac{c_{020}}{c_{220}}}$ and $z = 0$, when the condition $c_{202}c_{220} < 0$ it is satisfied.

Remark 10. *By the Theorem (1.3) with $j = 2$, the point $\left(r_0 = 2\sqrt{\frac{c_{020}}{c_{220}}}, z_0 = 0\right)$ is exactly the point $(r_0, 0)$ from which the T_6-reversible symmetric periodic solution close to the $z = 0$-plane bifurcate. However the curve of initial conditions that generated by the Averaging method is not the same curve of initial condition generated by our method, similiar to happened with the symmetry T_5, so in this case, the curve of initial condition is taken in $z_0 = 0$ and $\theta_0 = 0$ and it is on the x-axis as shown in the Figure 4*

Example 5.3. *Consider the following polynomials*

$$\begin{aligned}f_1(x_1,x_2,x_3) =& a_{210}x_1^2x_2 + a_{030}x_2^3 + z\left(a_{300}x^3 + a_{500}x^5\right),\\ g_1(x_1,x_2,x_3) =& b_{120}x_1x_2^2 + b_{102}x_1x_3^2 + b_{022}x_2^2x_3^2,\\ h_1(x_1,x_2,x_3) =& c_{220}x_1^2x_2^2 + c_{020}x_2^2 + c_{002}x_3^2,\end{aligned} \tag{53}$$

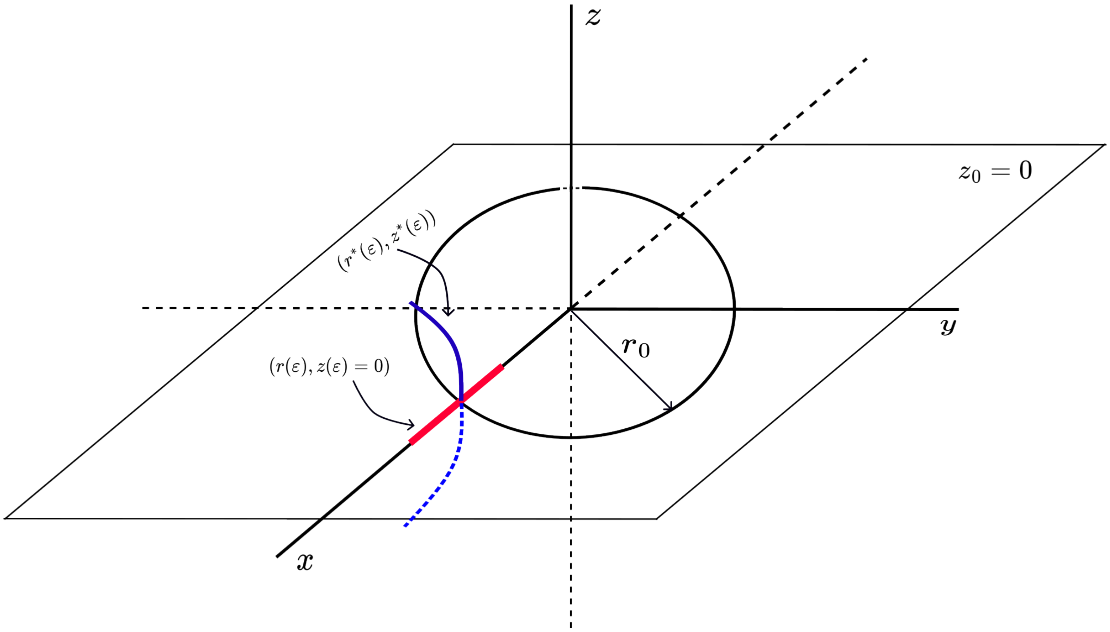

FIGURE 4. Families of initial conditions $(r(\varepsilon), z(\varepsilon) = 0)$ and $(r^*(\varepsilon), z^*(\varepsilon))$ parameterized by ε, obtained by Theorem 1.3 with $j = 2$ (this is, the T_6-symmetry) and the Averaging theorem respectively, when the field (2) has the reversible symmetry T_6 and $\theta_0 = 0$.

that admit the T_6-reversible symmetry, a_{ij}, b_{ij} and c_{ij} are arbitrary real numbers, and $R_1 = R_2 = R_3 = 0$. The previous functions in cylindrical coordinates takes the form

$$\begin{aligned}
\tilde{f}_1(r,\theta,z) =& a_{210}r^3\cos^2\theta\sin\theta + a_{030}r^3\sin^3\theta + z\left(a_{201}r^2\cos^2\theta + r^3\cos^3\theta\right),\\
\tilde{g}_1(r,\theta,z) =& b_{120}r^3\cos\theta\sin^2\theta + b_{102}rz^2\cos\theta + b_{022}r^2z^2\sin^2\theta,\\
\tilde{h}_1(r,\theta,z) =& c_{220}r^4\cos^2\theta\sin^2\theta + c_{020}r^2\sin^2\theta + c_{002}z^2.
\end{aligned}$$

We have that

$$\begin{aligned}
\overline{F}_1(r,z) &= \frac{3a_{301}\pi}{4}r^3 z,\\
\overline{K}_1(r,z) &= \frac{1}{4}\pi\left(8c_{002}z^2 + 4c_{020}r^2 + c_{220}r^4\right).
\end{aligned}$$

Then, solving the equations $\overline{F}_1(r,z) = 0$ and $\overline{K}_1(r,z) = 0$, we obtain three no-degenerate isolated zeros

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$$r_0 = 2\sqrt{\frac{c_{020}}{c_{220}}}, \quad z_0 = 0,$$
$$r_0 = \sqrt{\frac{6}{5}}\sqrt{\frac{a_{300}}{a_{500}}}, \quad z_0 = \frac{3}{10}\frac{\sqrt{10\ a_{300}\ a_{500}\ c_{020} - 3a_{300}^2\ c_{220}}}{a_{500}\ \sqrt{c_{002}}},$$
$$r_0 = \sqrt{\frac{6}{5}}\sqrt{\frac{a_{300}}{a_{500}}}, \quad z_0 = -\frac{3}{10}\frac{\sqrt{10a_{300}\ a_{500}\ c_{020} - 3a_{300}^2\ c_{220}}}{a_{500}\ \sqrt{c_{002}}}.$$

Thus, by the Averaging theorem of first order from each point, there are periodic solutions close to each plane z_0 respectively. Now, using our Theorem 1.3, we obtain the existence of reversible symmetric periodic solutions close to circle with radius $r_0 = 2\sqrt{\frac{c_{020}}{c_{220}}}$ on the plane $z_0 = 0$ under the condition that $c_{202}c_{220} > 0$.

Note that in this work, our analysis does not cover the case of double symmetry, because the vector field of system (2) does not form a group under the composition of its reversible symmetries, unlike what occurs in Hamiltonian systems (see, for example, [14]).

In order to get an approximation of the reversible symmetric periodic solutions, we propose an approximation of the solution for the system (2):

$$\begin{aligned} x(t,\xi,\varepsilon) &= x_0(t) + \varepsilon^\alpha x_1(t) + \mathcal{O}(\varepsilon^{\alpha+1}) \\ y(t,\xi,\varepsilon) &= y_0(t) + \varepsilon^\alpha y_1(t) + \mathcal{O}(\varepsilon^{\alpha+1}) \\ z(t,\xi,\varepsilon) &= z_0(t) + \varepsilon^\alpha z_1(t) + \mathcal{O}(\varepsilon^{\alpha+1}) \end{aligned} \tag{54}$$

where $\xi = (x_0, y_0, z_0)$ is the initial condition. By the change of cylindrical variables, we have that

$$\begin{aligned} x(t) &= r(t)\cos\theta(t) \\ y(t) &= r(t)\sin\theta(t) \\ z(t) &= z(t) \end{aligned} \tag{55}$$

where $(r(t), \theta(t), z(t))$ is as in (7). Thus, the approximation of the solution of the complete system is given by

$$\begin{aligned} x(t) &= (r_0(t) + \varepsilon^\alpha r_1(t) + \mathcal{O}(\varepsilon^\alpha))\cos(\theta_0(t) + \varepsilon^\alpha\theta_1(t) + \mathcal{O}(\varepsilon^{\alpha+1})), \\ y(t) &= (r_0(t) + \varepsilon^\alpha r_1(t) + \mathcal{O}(\varepsilon^\alpha))\sin(\theta_0(t) + \varepsilon^\alpha\theta_1(t) + \mathcal{O}(\varepsilon^{\alpha+1})), \\ z(t) &= z_0(t) + \varepsilon^\alpha z_1(t) + \mathcal{O}(\varepsilon^{\alpha+1}) \end{aligned} \tag{56}$$

when,

$$\begin{array}{llll} x_0(t) & = r_0(t)\cos\theta_0(t), & x_1(t) & = r_1(t)\cos\theta_1(t), \\ y_0(t) & = r_0(t)\sin\theta_0(t), & y_1(t) & = r_1(t)\sin\theta_1(t), \\ z_0(t) & = z_0(t), & z_1(t) & = z_1(t). \end{array} \tag{57}$$

In our approach, it is possible to compute a second approximation of the solution but additional information is required.

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