Waring's expression for a symmetric function in terms of sums of like powers.

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1. In his *Miscellanea Analytica* of the year 1762, Waring stated the identities

$$\Sigma\alpha^p\beta^q = s_p s_q - s_{p+q},$$
$$\Sigma\alpha^p\beta^q\gamma^r = s_p s_q s_r + 2s_{p+q+r} - s_p s_{q+r} - s_q s_{r+p} - s_r s_{p+q},$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

where s_p stands for the sum of the p^{th} powers of n quantities $\alpha, \beta, \gamma, \dots$. If the identities be written in the uncontracted form

$$\Sigma\alpha^p\beta^q = (\alpha^p + \beta^p + \dots)(\alpha^q + \beta^q + \dots) - (\alpha^p\alpha^q + \beta^p\beta^q + \dots)$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

it is readily seen that they hold not merely for certain powers of certain quantities, but for any quantities whatever ; and that, indeed, for the elements of the array

$$\begin{array}{llll} \alpha^p, & \beta^p, & \gamma^p, & \dots \\ \alpha^q, & \beta^q, & \gamma^q, & \dots \\ \dots & \dots & \dots & \dots \end{array}$$

we may legitimately substitute in order the elements of the array

$$\begin{array}{llll} a_1, & a_2, & a_3, & \dots \\ b_1, & b_2, & b_3, & \dots \\ \dots & \dots & \dots & \dots \end{array}$$

thus obtaining the identities

$$\Sigma a_1 b_2 = \Sigma a_1 \Sigma b_1 - \Sigma a_1 b_1,$$
$$\Sigma a_1 b_2 c_3 = \Sigma a_1 \Sigma b_1 \Sigma c_1 + 2\Sigma a_1 b_1 c_1 - \Sigma a_1 \Sigma b_1 c_1 - \Sigma b_1 \Sigma c_1 a_1 - \Sigma c_1 \Sigma a_1 b_1.$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

These latter were given by Binet in 1812, and are usually, but with little real justification, spoken of as "Binet's Identities."

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2. Among the first to give grounds for the validity of the identities and to try to specify the law of formation of the right-hand members was Meier Hirsch in his *Sammlung von Aufgaben* . . . of 1809 (see especially pp. 33–41). Hirsch's mode of proof, which makes any one case dependent on the case immediately preceding, continues to be used at the present day.

3. A fresh step was attempted to be taken in 1857 by Bellavitis (see *Sposizione elem.* §91) when dealing with Binet's form of the identities. Noticing the resemblance between the first two identities and the identities

$$\begin{vmatrix} a & b \\ b & c \end{vmatrix} = ac - b^2,$$

$$\begin{vmatrix} a & f & e \\ f & b & d \\ e & d & c \end{vmatrix} = abc + 2def - ad^2 - be^2 - cf^2,$$

he hastily concluded that the resemblance was general, and directed that, in order to obtain any other of the identities, the reader should first find the development of the corresponding axisymmetric determinant, and then perform certain substitutions. An examination of the very next case would have shown him his error.

4. A similar observation was independently made about the same time by Faà di Bruno, who, when he came in 1876 to publish his *Théorie des Formes Binaires*, went into details regarding it (pp. 10-11). In his case it was Waring's form which was under consideration, and his assertion was that the symmetric function $\Sigma\alpha^p\beta^q\gamma^r\delta^t\ldots$ "peut se mettre sous la forme

$$\begin{vmatrix} s_p & s_{(p)} & s_{(p)} & s_{(p)} & \cdots \\ s_{(q)} & s_q & s_{(q)} & s_{(q)} & \cdots \\ s_{(r)} & s_{(r)} & s_r & s_{(r)} & \cdots \\ s_{(t)} & s_{(t)} & s_{(t)} & s_t & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

en admettant qu'après avoir affectué les opérations on change les produits symboliques $s_{(p)}s_{(q)}s_{(r)}$ en $s_{p+q+r+}\ldots$." Again, however, the

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generalisation was too hastily made. Bruno, it is true, seemed to obtain therefrom the proper expression for $\Sigma\alpha^p\beta^q\gamma^r\delta^t$ by changing the determinant of the 4^{th} order into

$$s_p\begin{vmatrix} s_q & s_{(q)} & s_{(q)} \\ s_{(r)} & s_r & s_{(r)} \\ s_{(t)} & s_{(t)} & s_t \end{vmatrix} - s_{(q)}\begin{vmatrix} s_{(p)} & s_{(p)} & s_{(p)} \\ s_{(r)} & s_r & s_{(r)} \\ s_{(t)} & s_{(t)} & s_t \end{vmatrix} - s_{(r)}\begin{vmatrix} s_{(p)} & s_{(p)} & s_{(p)} \\ s_{(q} & s_q & s_{(q)} \\ s_{(t)} & s_{(t)} & s_t \end{vmatrix} - s_{(t)}\begin{vmatrix} s_{(p)} & s_{(p)} & s_{(p)} \\ s_{(q)} & s_q & s_{(q)} \\ s_{(r)} & s_{(r)} & s_r \end{vmatrix},$$

and expanding in the ordinary way. His development, however, of the three similar determinants here of the 3rd order is quite unjustifiable, the last, for example, being put equal to

$$s_{(p)}s_qs_r - s_{(p)}s_{q+r} - s_qs_{(p)+r} - s_rs_{q+(p)} + 2s_{(p)+q+r},$$

where among other anomalies we notice that the negative term $s_{(p)}s_{(q)}s_{(r)}$ is made $s_{(p)}s_{q+r}$, and the positive term $s_{(p)}s_{(q)}s_{(r)}$ is made $s_{(p)+q+r}$.

By any one familiar with the determinant

$$\begin{vmatrix} x & a & a & a & \ldots \\ b & y & b & b & \ldots \\ c & c & z & c & \ldots \\ d & d & d & u & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

difficulties might have been anticipated, for the development of it is known to be for the 4th order

$$xyzu - \Sigma xyzcd + 2\Sigma xbcd - 3abcd,$$

for the 5th order

$$xyzuv - \Sigma xyzde + 2\Sigma xycde - 3\Sigma xbcde + 4abcde,$$

and so on,—forms which are most unlikely to give rise to Waring's expressions. Besides, these forms are reached after certain terms have disappeared by mutual cancellation, the original number $n\,!$ being thereby changed to $(n-2)2^{n-1}+2$; whereas Waring's expressions contain $n\,!$ terms in every case.

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5. It is nevertheless true that a real analogy with determinants exists, and that it can be utilised by means of the form of determinant specified by Bruno. All that is wanted is to institute the condition that *the law of quasi-multiplication, exemplified by*

$$s_p . s_q = s_{p+q} \quad \textit{and} \quad s_p . s_q . s_r = s_{p+q+r},$$

shall not be applied unless the place-indices of the pair of elements $s_{(p)}, s_{(q)}$ *or of the triad* $s_{(p)}, s_{(q)}, s_{(r)}$ *form a cyclic group.* For example, in the case of the determinant of the 4th order, the term $s_{(p)}s_{(q)}s_{(r)}s_{(t)}$ when originating in the places

$$(1, 2), \ (2, 1), \ (3, 4), \ (4, 3) \text{ is replaced by } s_{p+q} . s_{r+t},$$

and when originating in the places

$$(1, 3), \ (2, 4), \ (3, 2), \ (41) \text{ is replaced by } s_{p+q+r+t}.$$

It will be remembered that the separation of the double-indexed elements of a term of a determinant into cyclical groups, for example, the separation of

$$a_{06}a_{12}a_{23}a_{35}a_{44}a_{51}a_{60} \text{ into } a_{06}a_{60} . a_{12}a_{23}a_{35}a_{51} . a_{44} ;$$

or, what is the same thing, the resolution of a substitution into cyclic substitutions, for example, the resolution of

$$\binom{0123456}{6235410} \text{ into } \binom{06}{60} \cdot \binom{1235}{2351} \cdot \binom{4}{4}$$

was introduced by Cauchy for the purpose of deciding on the sign of any term of a determinant: and that the permutations of the integers 1, 2, 3, ..., n thereby became classifiable into sets according to the form of the result reached by the said separation or resolution. Thus, the 120 permutations of 1, 2, 3, 4, 5 became classifiable into

1	like	12345	with	cyclic	groups	1, 2, 3, 4, 5	and sign	$(-1)^{5-5}$
10	„	12354	„	„	„	1, 2, 3, 45	„	$(-1)^{5-4}$
20	„	12453	„	„	„	1, 2, 345	„	$(-1)^{5-3}$
15	„	13254	„	„	„	1, 23, 45	„	$(-1)^{5-3}$
30	„	13452	„	„	„	1, 2345	„	$(-1)^{5-2}$
20	„	21453	„	„	„	12, 345	„	$(-1)^{5-2}$
24	„	23451	„	„	„	12345	„	$(-1)^{5-1}$

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In accordance with this Waring's expression for $\Sigma\alpha^p\beta^q\gamma^r\delta^t\epsilon^u$ would consist of

1 term like $s_p s_q s_r s_t s_u$
10 terms like $s_p s_q s_r s_{t+u}$
20 terms like $s_p s_q s_{r+t+u}$
15 terms like $s_p s_{q+r} s_{t+u}$
30 terms like $s_p s_{q+r+t+u}$
20 terms like $s_{p+q} s_{r+t+u}$
24 terms like $s_{p+q+r+t+u}$;

and since the numbers of *different* terms of the said types are respectively 1, 10, 10, 15, 5, 10, 1 the expression itself would be

$$s_p s_q s_r s_t s_u - \Sigma s_p s_q s_r s_{t+u} + 2\Sigma s_p s_q s_{r+t+u} + \Sigma s_p s_{q+r} s_{t+u}$$
$$- 6\Sigma s_p s_{q+r+t+u} - 2\Sigma s_{p+q} s_{r+t+u} + 24 s_{p+q+r+t+u}$$

as we know otherwise that it really is.