POLYNOMIAL APPROXIMATIONS ON A POLYDISC[1]

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1. Introduction and Results

Throughout this paper, we will use the terminologies and notations as in [4]. Thus, U^N denotes the open unit polydisc in the space $\mathbb{C}^N$ of N complex variables, T^N the distinguished boundary of U^N and

$$V^N = \{(z_1, \cdots, z_N) \in \mathbb{C}^N : |z_j| > 1 \text{ for } j = 1, \cdots, N\}.$$

We say that $\boldsymbol{n} = (n_1, \cdots, n_N)$ tends to infinity if $n_j \to \infty$ for each $j = 1, \cdots, N$. A polynomial P of N complex variables $(z_1, \cdots, z_N)$ is said to be of *order* $\boldsymbol{n} = (n_1, \cdots, n_N)$ if for each j, $1 \leqq j \leqq N$, $(\partial^k / \partial z_j^k) P(z_1, \cdots, z_N)$ is not identically zero for $k = n_j$ but is the zero function for each $k > n_j$. Let P be a polynomial in $\mathbb{C}^N$. If the only zeros of P in $\bar{U}^N \cup \bar{V}^N$ lie on T^N, then P will be called a T^N-polynomial. Hence, for $N = 1$, $T = T^1$, a T-polynomial is a polynomial such that all its zeros lie on the unit circle T. In the case of one complex variable, different kinds of T-polynomial approximation theorems were obtained in [1, 2, and 3]. In this note, we establish these theorems for any $N \geqq 1$.

THEOREM 1. *If f is holomorphic and does not vanish in U^N, there exist T^N-polynomials $Q_{\boldsymbol{m}}$ which converge to f uniformly on every compact subset of U^N.*

THEOREM 2. *If $f \in H^p = H^p(U^N)$, where $1 \leqq p \leqq \infty$, and does not vanish in U^N, there exist T^N-polynomials $Q_{\boldsymbol{m}}$ which converge to f uniformly on every compact subset of U^N and satisfy $\| Q_{\boldsymbol{m}} \|_p \leqq 2 \|f\|_p$ for all $\boldsymbol{m}$.*

Here, uniform convergence on compact subsets of U^N cannot be replaced by convergence in H^p. For $p = \infty$, it is clear, and for $1 \leqq p < \infty$, it is proved for $N = 1$ in [2].

Let $\mathscr{H}^p = \mathscr{H}^p(U^N)$ $(1 \leqq p < \infty)$ be the class of all holomorphic functions f in U^N such that

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$$\|f\|_p = \left\{1/\pi^N \int_{U^N} |f|^p\right\}^{1/p} < \infty.$$

It is clear that each $\mathscr{H}^p$ with the norm $\|\ \|_p$ is a Banach space. For the spaces $\mathscr{H}^p$, we have a stronger result.

THEOREM 3. *If* $f \in \mathscr{H}^p (1 \leqq p < \infty)$ *and does not vanish in* U^N, *there exist* T^N*-polynomials* Q_m *such that*

$$\|Q_m - f\|_p \to 0.$$

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2. Proofs of the above theorems

For $z = (z_1, \cdots, z_N)$ where $z_j \neq 0$, $j = 1, \cdots, N$, we use the notation $1/z = (1/z_1, \cdots, 1/z_N)$. Let P be a polynomial in $\mathbb{C}^N$ with no zero in $\bar{U}^N$ and let $M(z) = z_1^{n_1}, \cdots, z_N^{n_1}$ be a monomial of sufficiently large order so that

$$Q(z) = P(z) + M(z)\tilde{P}(1/z) \tag{1}$$

is a polynomial. Here, $\tilde{P}$ is the polynomial whose coefficients are the complex conjugates of the coefficients of P [cf. 4]. Then $\overline{P(w)} = \tilde{P}(1/w)$ for all $w \in T^N$. Hence,

$$|M(z)\tilde{P}(1/z)/P(z)| = 1 \tag{2}$$

for each z on T^N. Since P has no zero in $\bar{U}^N$, by the maximum principle, we conclude from (1) and (2) that Q has no zero in $\bar{U}^N$, except possibly on T^N. Now, since $M(z)\tilde{M}(1/z) = 1$, we have

$$M(z)\tilde{Q}(1/z) = Q(z). \tag{3}$$

Hence, $\tilde{Q}(1/z)$ does not vanish in $\bar{U}^N$, except possibly on T^N. That is, $\tilde{Q}(z)$, and hence $Q(z)$, has no zero in $\bar{U}^N$, except possibly on T^N. Therefore, Q is a T^N-polynomial.

Now, let f be holomorphic in U^N and $f(z) \neq 0$ for all z in U^N. Then for each r, $0 < r < 1$, the function f_r defined by $f_r(z) = f(rz)$, where $rz = (rz_1, \cdots, rz_N)$, is holomorphic and does not vanish in $(1/r)U^N$, and can then be uniformly approximated on $\bar{U}^N$ by polynomials which do not vanish in $\bar{U}^N$. But $f_r \to f$ uniformly on each compact subset of U^N as $r \uparrow 1$. Hence, f can be approximated uniformly on each compact subset of U^N by polynomials P_n which do not vanish on $\bar{U}^N$. Let

$$Q_{m,n}(z) = P_n(z) + M_m(z)\tilde{P}_n(1/z) \tag{4}$$

where M_m are monomials of sufficiently large order m. By (2) and the maximum principle, we see that

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(5) $$|M_m(z)\tilde{P}_n(1/z)| \leqq |P_n(z)|$$

in $\bar{U}^N$ for all sufficiently large m. Since $P_n \to f$ uniformly on compact subsets of U^N, $M_m(z)\tilde{P}_n(1/z) \to 0$ on compact subsets of U^N as n and suitable $m = m(n)$ tend to infinity. That is, a sequence of T^N-polynomials can be chosen from $Q_{m,n}$ to approximate f uniformly on every compact subset of U^N. This proves the first theorem. If, in addition, f is in H^p $(1 \leqq p \leqq \infty)$, we can choose the P_n so that $\|P_n\|_p \leqq \|f\|_p$ for all n. Hence, using (4) and (5), we have $\|Q_{m,n}\|_p \leqq 2\|f\|_p$ for all n and all sufficiently large m, proving Theorem 2. Now, suppose that $f \in \mathscr{H}^p$ $(1 \leqq p < \infty)$ and does not vanish in U^N. We can choose the P_n, which do not vanish in $\bar{U}^N$, such that $▮P_n - f▮_p \to 0$. For each r, $0 < r < 1$, let $K_r = r\bar{U}_N$ and let D_r be the complement of K_r with respect to U^N. Since $f \in \mathscr{H}^p$ and the ($2N$-dimensional) Lebesgue measure of D_r tends to zero as $r \uparrow 1$, we have

$$\lim_{r \uparrow 1} \int_{D_r} |f|^p = 0.$$

Hence, for each $\varepsilon > 0$, we can choose $1 - r > 0$ so small that

$$\int_{D_r} |P_n|^p < \varepsilon$$

for all large n. Now, for all sufficiently large m, we obtain, using (5),

$$▮M_m(z)\tilde{P}_n(1/z)▮_p \leqq \max_{K_r} |M_m(z)\tilde{P}_n(1/z)| + \left\{\frac{1}{\pi^N} \int_{D_r} |P_n|^p\right\}^{1/p}.$$

Again, since $P_n \to f$ uniformly on K_r, the $m = m(n)$ can be chosen such that $▮M_m(z)\tilde{P}_n(1/z)▮_p \to 0$ as n and m tend to infinity. Hence, a sequence of T_N-polynomials Q_m can be chosen from the $Q_{m,n}$ such that $▮Q_m - f▮_p \to 0$. This completes the proof of the third theorem.

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