# Appendix E
# Symmetry properties of matrix elements

In this appendix we derive symmetry properties of matrix elements of the electromagnetic multipole operators that follow from hermiticity of the current and time-reversal invariance of the strong and electromagnetic interactions [Pr65, Wa84].[1] The electromagnetic current is an observable and an hermitian operator

$$\begin{aligned} \hat{\mathbf{J}}(\mathbf{x})^\dagger &= \hat{\mathbf{J}}(\mathbf{x}) \\ \hat{\rho}(\mathbf{x})^\dagger &= \hat{\rho}(\mathbf{x}) \end{aligned} \tag{E.1}$$

The properties of the spherical and vector spherical harmonics under complex conjugation follow by inspection

$$\begin{aligned} Y_{JM}^\star &= (-1)^M \, Y_{J,-M} \\ \mathscr{Y}_{JJ1}^{M\star} &= (-1)^{1+M} \, \mathscr{Y}_{JJ1}^{-M} \end{aligned} \tag{E.2}$$

The adjoints of the multipole operators then follow from their definition

$$\begin{aligned} \hat{\mathscr{T}}_{JM_J}(\kappa)^\dagger &= (-1)^{M_J+\eta} \, \hat{\mathscr{T}}_{J,-M_J}(\kappa) \\ \eta &\equiv 1 \qquad ; \text{ current multipoles} \\ &\equiv 0 \qquad ; \text{ charge multipoles} \end{aligned} \tag{E.3}$$

It is useful to include isospin in the analysis. Define spherical components of $\boldsymbol{\tau}$

$$\begin{aligned} \tau_{\pm 1} &= \mp \frac{1}{\sqrt{2}}(\tau_1 \pm i\tau_2) \\ \tau_0 &= \tau_3 \end{aligned} \tag{E.4}$$

[1] Selection rules from parity invariance of these interactions are discussed in the text.

Now isolate the isospin dependence of a multipole operator in a factor

$$\begin{aligned} I_{TM_T} &\equiv \frac{1}{2} \qquad\qquad ; \; T=0 \\ &\equiv \frac{1}{2}\tau_{1,M_T} \qquad ; \; T=1 \end{aligned} \tag{E.5}$$

It follows that the multipole adjoints further satisfy

$$\hat{\mathscr{T}}^{\dagger}_{TM_T} \;=\; (-1)^{M_T}\hat{\mathscr{T}}_{T,-M_T} \tag{E.6}$$

A combination of these results gives the full adjoints of the multipole operators

$$\hat{\mathscr{T}}_{JM_J;TM_T}(\kappa)^{\dagger} = (-1)^{M_T+M_J+\eta}\,\hat{\mathscr{T}}_{J,-M_J;T,-M_T}(\kappa) \tag{E.7}$$

We shall now derive from this the following relation on a general reduced matrix element of a multipole operator

$$\langle J_f T_f \,⁞⁞\, \hat{\mathscr{T}}_{J,T}(\kappa) \,⁞⁞\, J_i T_i\rangle^{\star} \;=\; (-1)^{J_f-J_i+T_f-T_i+\eta}\langle J_i T_i \,⁞⁞\, \hat{\mathscr{T}}_{J,T}(\kappa) \,⁞⁞\, J_f T_f\rangle \tag{E.8}$$

Here the symbol ⁞⁞ indicates a reduced matrix element with respect to both angular momentum and isospin. The proof of this relation follows from the Wigner–Eckart theorem [Ed74]

$$\begin{aligned} \langle J_f M_f T_f \bar{M}_f|\hat{\mathscr{T}}_{JM_J;TM_T}|J_i M_i T_i \bar{M}_i\rangle &= (-1)^{J_f-M_f}\begin{pmatrix} J_f & J & J_i \\ -M_f & M_J & M_i \end{pmatrix} \\ &\quad \times[J \rightleftharpoons T] \times \langle J_f T_f \,⁞⁞\, \hat{\mathscr{T}}_{J,T} \,⁞⁞\, J_i T_i\rangle \end{aligned} \tag{E.9}$$

Now take the complex conjugate of this relation and use the definition of the adjoint $\langle f|\hat{\mathscr{T}}|i\rangle^{\star} = \langle i|\hat{\mathscr{T}}^{\dagger}|f\rangle$

$$\begin{aligned} &(-1)^{J_f-M_f}\begin{pmatrix} J_f & J & J_i \\ -M_f & M_J & M_i \end{pmatrix} \times [J \rightleftharpoons T] \times \langle J_f T_f \,⁞⁞\, \hat{\mathscr{T}}_{J,T} \,⁞⁞\, J_i T_i\rangle^{\star} \\ &= (-1)^{M_J+M_T+\eta}(-1)^{J_i-M_i}\begin{pmatrix} J_i & J & J_f \\ -M_i & -M_J & M_f \end{pmatrix} \\ &\times[J \rightleftharpoons T] \times \langle J_i T_i \,⁞⁞\, \hat{\mathscr{T}}_{J,T} \,⁞⁞\, J_f T_f\rangle \end{aligned} \tag{E.10}$$

Here the Wigner–Eckart theorem has been used once more on the last matrix element. Now use the properties of the 3-j symbols [Ed74] to rewrite the right hand side

$$\begin{aligned} \text{r.h.s} \;&=\; (-1)^{J_f-J_i+T_f-T_i+\eta}(-1)^{J_f-M_f}\begin{pmatrix} J_f & J & J_i \\ -M_f & M_J & M_i \end{pmatrix} \\ &\quad \times[J \rightleftharpoons T] \times \langle J_i T_i \,⁞⁞\, \hat{\mathscr{T}}_{J,T} \,⁞⁞\, J_f T_f\rangle \end{aligned} \tag{E.11}$$

Equation (E.8) has now been established.

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Let us now investigate the restrictions imposed by time-reversal invariance. Recall that the time-reversal operator is *anti-unitary* and satisfies

$$\begin{aligned} \hat{T}\, i\, \hat{T}^{-1} &= -i \\ \langle f|\hat{T}^{-1}|i\rangle &= \langle Tf|i\rangle^{\star} \end{aligned} \tag{E.12}$$

The properties of the electromagnetic current under time reversal follow from classical correspondence

$$\begin{aligned} \hat{T}\, \hat{\mathbf{J}}(\mathbf{x})\, \hat{T}^{-1} &= -\hat{\mathbf{J}}(\mathbf{x}) \\ \hat{T}\, \hat{\rho}(\mathbf{x})\, \hat{T}^{-1} &= \hat{\rho}(\mathbf{x}) \end{aligned} \tag{E.13}$$

Thus the multipole operators satisfy

$$\hat{T}\, \hat{\mathscr{T}}_{JM_J,TM_T}\, \hat{T}^{-1} = (-1)^{M_J}\, \hat{\mathscr{T}}_{J,-M_J;TM_T} \tag{E.14}$$

Note that the current only involves $M_T = 0$ and hence time reversal does not affect the isospin here. Our states are defined to transform according to [2]

$$\hat{T}\, |JM_J; TM_T\rangle = (-1)^{J+M_J}|J, -M_J; TM_T\rangle \tag{E.15}$$

Time-reversal invariance then says

$$\begin{aligned} &\langle J_f M_f T_f \bar{M}_f|\hat{\mathscr{T}}_{JM_J;TM_T}|J_i M_i T_i \bar{M}_i\rangle \\ &\quad= \langle J_f M_f T_f \bar{M}_f|\hat{T}^{-1}\hat{T}\, \hat{\mathscr{T}}_{JM_J;TM_T}\, \hat{T}^{-1}\hat{T}|J_i M_i T_i \bar{M}_i\rangle \\ &\quad= (-1)^{J_i+M_i}(-1)^{J_f+M_f}(-1)^{M_J} \\ &\qquad \times\langle J_f, -M_f T_f \bar{M}_f|\hat{\mathscr{T}}_{J,-M_J;TM_T}|J_i, -M_i T_i \bar{M}_i\rangle^{\star} \end{aligned} \tag{E.16}$$

Now use the Wigner–Eckart theorem on both sides and the properties of the 3-j symbols [Ed74]

$$\begin{aligned} &(-1)^{J_f-M_f}\begin{pmatrix} J_f & J & J_i \\ -M_f & M_J & M_i \end{pmatrix} \times [J \rightleftharpoons T]_{(1)} \times \langle J_f T_f \vdots\vdots\, \hat{\mathscr{T}}_{J,T}\, \vdots\vdots\, J_i T_i\rangle \\ &\quad= (-1)^{J_i+M_i}(-1)^{J_f+M_f}(-1)^{M_J}(-1)^{J_f+M_f}\begin{pmatrix} J_f & J & J_i \\ M_f & -M_J & -M_i \end{pmatrix} \\ &\qquad \times[J \rightleftharpoons T]_{(1)} \times \langle J_f T_f \vdots\vdots\, \hat{\mathscr{T}}_{J,T}\, \vdots\vdots\, J_i T_i\rangle^{\star} \end{aligned} \tag{E.17}$$

Since the isospin factors are identical, this relation implies

$$\langle J_f T_f \vdots\vdots\, \hat{\mathscr{T}}_{J,T}\, \vdots\vdots\, J_i T_i\rangle^{\star} = (-1)^{J}\langle J_f T_f \vdots\vdots\, \hat{\mathscr{T}}_{J,T}\, \vdots\vdots\, J_i T_i\rangle \tag{E.18}$$

[2] Note that this involves a *phase convention.*

Table E.1. Selection rules for multipole operators from parity and time reversal in elastic scattering; this quantity must be +1.

| | $\hat{M}_{JM}(\kappa)$ | $\hat{T}^{\mathrm{el}}_{JM}(\kappa)$ | $\hat{T}^{\mathrm{mag}}_{JM}(\kappa)$ |
|---|---|---|---|
| Parity | $(-1)^J$ | $(-1)^J$ | $(-1)^{J+1}$ |
| Time Reversal | $(-1)^J$ | $(-1)^{J+1}$ | $(-1)^{J+1}$ |

A combination of Eq. (E.8) and Eq. (E.18) then leads to

$$\langle J_f T_f \,\vdots\vdots\, \hat{\mathscr{T}}_{J,T} \,\vdots\vdots\, J_i T_i \rangle = (-1)^{J+\eta+J_f-J_i+T_f-T_i} \langle J_i T_i \,\vdots\vdots\, \hat{\mathscr{T}}_{J,T} \,\vdots\vdots\, J_f T_f \rangle \quad \text{(E.19)}$$

This is the basic result of this appendix. It follows from the hermiticity of the current, time-reversal invariance of the strong and electromagnetic interactions, *and* a phase convention on the states. This relation allows one to turn around the matrix elements. If the initial and final states are identical, as is the case in elastic electron scattering, this relation leads to a *selection rule*. It states that

$$(-1)^{J+\eta} = 1 \qquad ; \text{ elastic scattering} \quad \text{(E.20)}$$

Thus $J + \eta$ must be an even integer in elastic scattering. Hence only the even charge multipoles and odd current multipoles can contribute to elastic scattering. The selection rules for the various multipoles from both parity and time reversal in the case of elastic scattering are shown in Table E.1. For the charge and transverse magnetic multipoles, time-reversal and parity invariance lead to identical selection rules, that is, only charge multipoles with even $J$ and transverse magnetic multipoles with odd $J$ contribute to elastic electron scattering. For the transverse electric multipoles, parity implies $J$ must be even while time reversal implies $J$ must be odd. Hence invariance under *both* parity and time-reversal invariance implies there are no transverse electric multipoles in elastic electron scattering

$$\langle i | \hat{T}^{\mathrm{el}}_{JM}(\kappa) | i \rangle = 0 \qquad ; \text{ parity and time reversal} \quad \text{(E.21)}$$