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Versal deformations in spaces of polynomials of fixed weight

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Abstract. This work was largely inspired by a paper of Shustin, in which he proves that for a plane curve of given degree n whose singularities are not too complicated the singularities are versally unfolded by embedding the curve in the space of all curves of degree n; however, our methods are very different.

The main result gives fairly explicit lower bounds on the sum of the Tjurina numbers at the singularities of a deformation of a weighted-homogeneous hypersurface, when the deformation is the fibre over an unstable point of an appropriate unfolding. The result is sufficiently flexible to cover a variety of applications, some of which we describe. In particular, we will deduce a generalisation of Shustin's result.

Properties of discriminant matrices of unfoldings of weighted-homogeneous functions are crucial to the arguments; the parts of the theory needed are described.

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0. Introduction

It is a well known principle of singularity theory, established by Thom [9], that a generic map is transverse to any submanifold of its target, or more generally of any jet space. There are several versions of the proof; an essential point is that we are contemplating deformations in the (infinite-dimensional) space of all maps.

There are also in the literature several more delicate transversality theorems in which the given map is only permitted to vary in a finite dimensional space; here of course there must be some restrictions on the manifolds to which generic transversality can be established. Some such results follow from Bertini's Theorem in algebraic geometry; others may be found, for example, in [1]. We shall prove further results of this kind.

Our work was largely inspired by a paper of Shustin [8], in which he proves that for a plane curve of given degree n whose singularities are not too complicated the singularities are versally unfolded by embedding the curve in the space of all curves of degree n; however, our methods are very different.

Our main result gives fairly explicit lower bounds on the sum of the Tjurina numbers at the singularities of a deformation of a weighted-homogeneous hypersurface, when the deformation is the fibre over an unstable point of an appropriate unfolding. The result is sufficiently flexible to cover a variety of applications, some of which we describe. In particular, we will deduce a generalisation of Shustin's result.

Properties of discriminant matrices of unfoldings of weighted-homogeneous functions will be crucial to our arguments. We discuss the parts of the theory that we need in the first section. The main result is proved in the second section, and applications given in the third and fourth.

1. Theory of the discriminant

Let $f_0\colon \mathbb{C}^r \to \mathbb{C}$ have an isolated singular point at the origin: for the purposes of this paper, we also suppose f_0 weighted homogeneous. As is well known, it follows that the ideal $Jf_0 := \langle \partial f_0/\partial x_1, \dots, \partial f_0/\partial x_r\rangle$ has finite codimension in the ring $\mathcal{O}_x$ of convergent power series in the x variables. Choose a basis $\phi_0 = 1, \phi_1, \dots, \phi_{\mu-1}$ of the quotient vector space $\mathcal{O}_x/Jf_0$: it is traditional, and usually convenient, to choose the ϕ_i to be monomials, and we arrange them in some order of non-decreasing weight. Now define the unfolding $F(\mathbf{x},\mathbf{u}) = (y,\mathbf{v}) = (f(\mathbf{x},\mathbf{u}),\mathbf{u})$, where $\mathbf{x} = (x_1,\dots,x_r)$, $\mathbf{u} = (u_1,\dots,u_{\mu-1})$, $\mathbf{v} = (v_1,\dots,v_{\mu-1})$, and $f(\mathbf{x},\mathbf{u}) = f_0(\mathbf{x}) + \Sigma_{i=1}^{\mu-1} u_i\phi_i(\mathbf{x})$.

This is the usual construction of a versal unfolding of f_0: we recall that the resulting map F has a stable germ at the origin, and using the $\mathbb{C}^*$-action arising from the homogeneity (where u_i and v_i are assigned weight wt(f_0)-wt(ϕ_i)), it follows that all germs of F are stable: F is *locally stable* in the sense of [2].

Now write $\mathcal{O}_{x,u}$ for the ring of convergent power series in the x and u variables, and $J_xf := \langle \partial f/\partial x_1, \dots, \partial f/\partial x_r\rangle$ for the indicated ideal in it. It follows from Nakayama's Lemma that the ϕ_i form a free basis of the quotient $\mathcal{O}_u$-module $\mathcal{O}_{x,u}/J_xf$. Hence there exist uniquely determined elements $a_{i,j} \in \mathcal{O}_u$ such that

$$f\phi_i \equiv \sum_{j=0}^{\mu-1} a_{i,j}(\mathbf{u})\phi_j \quad \text{mod } J_xf\ (0 \leqslant i < \mu). \tag{1}$$

Then the transpose of $A = (a_{i,j}(\mathbf{v}) - y\,\delta_{i,j})$ is a *discriminant matrix* in the sense of [2], p. 421. Its importance here is largely in its relation to vector fields. Replacing the congruence (1) by an equation gives, say

$$f(\mathbf{x},\mathbf{u})\phi_i = \sum_{j=0}^{\mu-1} a_{i,j}(\mathbf{u})\phi_j + \sum_{k=1}^{r} b_{i,k}(\mathbf{x},\mathbf{u})\frac{\partial f}{\partial x_k}. \tag{2}$$

Then the vector field ξ_i defined on the source of F by

$$\xi_i = \sum_{k=1}^{r} b_{i,k}(\mathbf{x},\mathbf{u})\frac{\partial}{\partial x_k} + \sum_{j=1}^{\mu-1}(a_{i,j}(\mathbf{u}) - f(\mathbf{x},\mathbf{u})\delta_{i,j})\frac{\partial}{\partial u_j}$$

lifts the vector field η_i defined on the target by

$$\eta_i = \sum_{j=0}^{\mu-1}(a_{i,j}(\mathbf{v}) - y\delta_{i,j})\frac{\partial}{\partial v_j},$$

where $\partial/\partial v_0$ denotes $-\partial/\partial y$.

We next observe the

LEMMA 1.1. *The linear relation $\Sigma c_j(a_{j,k} - y\delta_{j,k}) = 0$ between the rows of the discriminant matrix holds at a point p of the target of F if and only if the function $\psi = \Sigma c_j\phi_j$ satisfies the condition $\psi f_p \in J(f_p)$, where f_p is the restriction of f to the fibre of F over p.*

Proof. Let $p = (y,\mathbf{v})$. We have the relation $\Sigma c_i(a_{i,j}(v) - y\delta_{i,j}) = 0$ for each j. Add c_i times the ith Equation (2), and subtract the sum of ϕ_j times the jth relation. The terms in $a_{i,j}$ then all cancel. Write $f_p(x) = f(x,\mathbf{v}) - y$ and (as above) $\psi = \Sigma c_j\phi_j$. Then the expression may be written as

$$\psi(x)f_p(x) = \Sigma_{i,k}c_i b_{i,k}\frac{\partial f_p}{\partial x_k}.$$

This establishes the forward implication: for the converse, if any relation of the form $\psi(x)f_p(x) = \Sigma\alpha_k\partial f_p/\partial x_k$ holds, then reversing the argument shows that at the point $p = (y,\mathbf{v})$ in question, $\Sigma c_i(a_{i,j}(v) - y\delta_{i,j})\phi_j$ lies in Jf_p, whence the conclusion. $\square$

We define two points $\mathbf{y}$ and $\mathbf{y}'$ in the target of a stable map $F\colon N \to P$ to be *equivalent* if the germs of F at the sets $\Sigma(F,\mathbf{y})$ and $\Sigma(F,\mathbf{y}')$ are $\mathcal{K}$- and hence $\mathcal{A}$-equivalent (here $\Sigma(F,\mathbf{y})$ denotes $\Sigma F \cap F^{-1}(\mathbf{y})$). The equivalence classes form a partition of P; here we shall refer to the parts as *leaves*.

The following results are well known: a convenient reference for them (in more generality than we require here) is Looijenga's book [5], Chapter 6.

- A vector field on P is liftable if and only if it is tangent to the discriminant ΔF.
- The leaf containing **y** is smooth there, with tangent space given by the values at **y** of the liftable vector fields.
- The codimension of this leaf equals the sum τ_y of the Tjurina numbers of the singularities of F at the points of $\Sigma(F,\mathbf{y})$.
- The η_i form a free basis of the $\mathcal{O}_{y,v}$-module of vector fields tangent to ΔF.

The *instability locus* of a map f is the set of points y in the target such that the germ of f at $\Sigma(f, y)$ is *not* a stable germ. Our objective is, under suitable conditions, to describe this set. More precisely, we consider the partial unfoldings

$$F^k(\mathbf{x}, u_1, \dots, u_{\mu-1-k}) = (f_u^k(\mathbf{x}), u_1, \dots, u_{\mu-1-k}),$$

where

$$f_u^k(\mathbf{x}) = f_0(\mathbf{x}) + \sum_1^{\mu-1-k} u_i \phi_i(\mathbf{x}).$$

Thus $F = F^0$ is the above versal unfolding of f_0, and F^k is obtained by omitting the last k unfolding monomials.

By Mather's criterion [6] for stable map-germs, the instability locus of $F^k: N \to P$ is the support of the *instability module* $\theta(K^k)/tF^k(\theta_N) + \omega F^k(\theta_P)$, i.e. of the cokernel of $\overline{\omega F^k}$. We begin by analysing the module $M(F^k) = \theta(F^k)/tF^k(\theta_N)$: in fact, we start with $M(F)$.

We have an isomorphism of $\mathcal{O}_{x,u}/J_x f$ onto $M(F)$, since $tF(\partial/\partial u_i)$ has one coordinate 1 and all the rest, save the first, 0. In turn, we have identified $\mathcal{O}_{x,u}/J_x f$ with the free $\mathcal{O}_v$-module on the classes of the ϕ_i, and hence with the cokernel of the map of free $\mathcal{O}_{y,v}$-modules defined by the discriminant matrix. Since F^k is the restriction of F where the source variables u_i and target variables v_i are set equal to 0 for $i > \mu - 1 - k$, we obtain $M(F^k)$ from $M(F)$ by factoring out by these v_i.

We now study the ωF^k: again we begin with ωF. This is a map of $\mathcal{O}_{y,v}$-modules; it suffices to identify the generators. The source θ_P of the map is generated by $\partial/\partial y$ and the $\partial/\partial v_i$; which are mapped into θF by composing with F. When we project to $M(F) \cong \mathcal{O}_{x,u}/J_x f$, the first generator maps to 1 and, since $tF(\partial/\partial u_i) = \phi_i \partial/\partial y + \partial/\partial v_i$, the ith generator maps to $-\phi_i$ for $i \geqslant 1$. These images are thus up to sign the elements we chose as $\mathcal{O}_u$-generators of this module.

For ωF^k with $k > 0$, as well as setting the variables $v_i = 0$ for $i > \mu - 1 - k$, the corresponding generators are not available. Thus $\theta(F^k)/tF^k(\theta_N) + \omega F^k(\theta_P)$ is obtained from the cokernel of $A^T: \mathcal{O}_{y,v}^\mu \to \mathcal{O}_{y,v}^\mu$ by setting $v_i = 0$ for $i > \mu - k$ and factoring out the generators corresponding to ϕ_i for $i \leqslant \mu - 1 - k$: thus (after the substitution) we have the cokernel of the map defined by the last k rows of the discriminant matrix A^T, so by the last k columns of A. The stage is thus set for application of the theory of the discriminant matrix.

The next result we require concerns the symmetry of this matrix. The following result is a special case of a result of Mond and Pellikaan [7]; for a proof in the present context see [2], (10.5.30).

THEOREM 1.2. *Let $f_0: (\mathbb{C}^s, 0) \to (\mathbb{C}, 0)$ be a weighted homogeneous germ of finite singularity type. Then any homogeneous unfolding of f_0 admits a symmetric homogeneous discriminant matrix.*

We have seen above that the instability locus of F^k coincides with the locus where the rank of the submatrix consisting of a certain set of k rows of the discriminant matrix drops below k. In the case when the matrix is symmetric, we can replace this by a submatrix consisting of k of the columns. To use this, we need to be able to identify the columns in question, so now assume that the remaining unfolding monomials have strictly lower weight than those omitted.

PROPOSITION 1.3 ([2], Theorem 10.5.32.). *If F^k is the unfolding obtained by omitting the k unfolding monomials of highest weight, the instability module for F^k is the submodule of $\mathcal{O}^k$ spanned by the k columns of lowest weight of any discriminant matrix for F.*

For the k rows of lowest weight of the discriminant matrix must correspond by symmetry to the k columns of lowest weight.

COROLLARY 1.4. *For f as above, a point $p = (y, \mathbf{v})$ lies in the instability locus of F^k if and only if there exists a linear combination $\psi(x)$ of the k unfolding monomials of least weight such that $\psi f_p \in J(f_p)$.*

This follows at once from the Proposition on recalling Lemma 1.1.

In what follows, we will not need to study the full instability locus of F^k, but only the *positive* instability locus, which is the intersection of the instability locus of F^k with the subspace of positive weight of its target (obtained by setting $v_i = 0$ whenever $\mathrm{wt}(v_i) \leqslant 0$).

2. A bound on Tjurina numbers of unstable maps

Let f_0 be a weighted homogeneous function of finite singularity type; write $H(f_0)$ for its Hessian determinant. Let c be a positive integer, and define

$$k(c) = \#\{i > 0 \mid \mathrm{wt}(\phi_i) > \mathrm{wt}(H(f_0)) - c\},$$

by duality, $k(c) = \#\{i \geqslant 0 \mid \mathrm{wt}(\phi_i) < c\}$.

We consider the corresponding maps F and $F^{k(c)}$, as above. Define also

$$\nu_c(f_0) = \max\left\{\dim_{\mathbb{C}} \frac{\mathcal{O}_r}{J(f_0) + \mathcal{O}_r g}\right\},$$

where the supremum is taken over homogeneous polynomials g not contained in $J(f_0)$ and of weight less than c. The theory we present is independent of the value of c, but usually it will be small: we will mainly be interested in cases where $c \leqslant \mathrm{wt}\, H(f_0) - \mathrm{wt}\,(f_0)$, so that the unfolding monomials omitted have weight at least as great as f_0.

THEOREM 2.1. *The positive instability locus of $F^{k(c)}$ presents only multi-germs of Tjurina number $\geqslant \mu(f_0) - \nu_c(f_0)$.*

We may observe that when $c = 1$ this is essentially equivalent to the algebraic part of Wirthmüller's Theorem [12]. For clearly $\nu_1 = 0$, and the positive instability locus of $F^{k(1)}$ thus presents only multi-germs of Tjurina number $\geqslant \mu(f_0)$. But the only (multi-)germ of Tjurina number $(\geqslant)\,\mu(f_0)$ is presented at 0, and so the positive instability locus is $\{0\}$.

We shall require a number of preliminaries before commencing the proof proper. We begin by re-stating a consequence of the theory of the discriminant. For a point p of the target of F, write f_p for the corresponding map, considered globally, and τ_p for the sum of the Tjurina numbers at the singular points of $f_p^{-1}(0)$.

LEMMA 2.2. *The following numbers are equal:*

(i) *the difference* $\tau_0 - \tau_p$*;*
(ii) *the dimension of the leaf through* p*;*
(iii) *the dimension of the span of the vectors* η_i *at* p*;*
(iv) *the dimension of the quotient* U/U_p*, where* U *is the vector space spanned by the* ϕ_i*, and* U_p *the subspace of those* ψ *such that* $\psi.f_p \in J(f_p)$.

Proof. The equality of (i) to (iii) follows from the general properties of the discriminant cited (from [5]) in Section 1; equality with (iv) now follows from Lemma 1.1. □

Since f_0 is weighted homogeneous, $\tau_0 = \tau(f_0) = \mu(f_0)$ and $f_0 \in J(f_0)$.

We will, from now on, suppose p to be contained in the subspace of positive weight of F, so that f_p differs from f_0 by terms of weight less than wt(f_0).

We will work in the polynomial ring $\mathbb{C}[\mathbf{x}]$. Define K_p to be the ideal in it generated by the ψ such that $\psi.f_p \in J(f_p)$.

LEMMA 2.3. *We have* $\tau_0 - \tau_p = \dim_{\mathbb{C}} \mathbb{C}[\mathbf{x}]/K_p$.

Proof. Since $U_p = U \cap K_p$, the conclusion follows from the equality $\mathbb{C}[\mathbf{x}] = U + K_p$. In fact we assert that $\mathbb{C}[\mathbf{x}] = U + J(f_p)$. For this holds when $p = 0$, by definition of U. The generators $\partial f_p/\partial x_k$ are obtained from the weighted homogeneous polynomials $\partial f_0/\partial x_k$ by adding terms of lower order. Thus for any $g \in \mathbb{C}[\mathbf{x}]$, if we write $g = \Sigma a_i\phi_i + \Sigma b_k \partial f_0/\partial x_k$, then the difference $g - \Sigma a_i\phi_i + \Sigma b_k\partial f_p/\partial x_k$ has lower degree than g. The result thus follows by induction. □

We may suppose the weights w_k of the coordinates x_k to be positive integers. Introduce an additional parameter t, of weight 1. We can then define a homogeneous function F_p of $\mathbf{x}, t$: if f_0 has weight d, set $F_p(x_1, \ldots, x_r, t) = t^d f_p(t^{-w_1}x_1, \ldots, t^{-w_r}x_r)$. Write J_xF_p for the ideal in $\mathbb{C}[\mathbf{x}, t]$ generated by the $\partial F_p/\partial x_k$ and

$$\tilde{I}_p = \{\psi \in \mathbb{C}[\mathbf{x}, t] \mid \psi.F_p \in J_xF_p\}.$$

Substituting the value 1 for t, or, equivalently, factoring out the ideal $\langle t-1\rangle$, projects this to an ideal I_p in $\mathbb{C}[\mathbf{x}]$.

LEMMA 2.4. *We have* $I_p = K_p$.

Proof. The inclusion $I_p \subset K_p$ is immediate.

Now suppose that $\psi \in K_v$, so that we can write $\psi.f_p = \Sigma_k g_k \partial f_p/\partial x_k$. As above, we introduce t to turn this into a homogeneous equation, say

$$\tilde{\psi}.F_p = \sum_k \tilde{g}_k \partial F_p/\partial x_k.$$

Thus $\tilde{\psi} \in \tilde{I}_p$, and now substituting $t = 1$ we deduce $\psi \in I_p$. □

Substituting the value 0 for t, or, equivalently, factoring out the ideal $\langle t \rangle$, projects $\tilde{I}_p$ to an ideal L_p in $\mathbb{C}[\mathbf{x}]$.

LEMMA 2.5. *We have* $\dim \mathbb{C}[\mathbf{x}]/L_p \geqslant \dim \mathbb{C}[\mathbf{x}]/I_p$.

Proof. Choose homogeneous functions $\{\alpha_1, \ldots, \alpha_m\}$ in $\mathbb{C}[\mathbf{x}]$ whose classes in $\mathbb{C}[\mathbf{x}]/L_p$ form a basis. We will show that the α_r span $\mathbb{C}[\mathbf{x}, t]/I$ as $\mathbb{C}[t]$-module. It follows, substituting $t = 1$, that they also span $\mathbb{C}[\mathbf{x}]/I_p$, which proves the result.

For any element $g(\mathbf{x}, t) \in \mathbb{C}[\mathbf{x}, t]$, we can find elements $\lambda_r \in \mathbb{C}[\mathbf{x}]$ such that $(g(\mathbf{x}, 0) - \Sigma_r \lambda_r \alpha_r) = \beta' \in L_p$. Choose $\beta \in I$ which projects onto β'. Then $g(\mathbf{x}, t) - \Sigma_r \lambda_r \alpha_r - \beta$ is divisible by t, so is of form tg', where g' has lower degree than g. The result thus follows by induction on degree. □

We are now ready for the

Proof of Theorem 2.1. It follows from the above lemmas that

$$\tau_0 - \tau_p = \dim U/U_p = \dim \mathbb{C}[\mathbf{x}]/K_p = \dim \mathbb{C}[\mathbf{x}]/I_p \leqslant \dim \mathbb{C}[\mathbf{x}]/L_p.$$

Since $\tilde{I}_p$ contains $J_x(F)$, it certainly follows that $J(f_0) \subset L_p$. It follows that $\mathbb{C}[\mathbf{x}]/L_p$ is supported at the origin, and has the same dimension as $\mathcal{O}_x/L_p\mathcal{O}_x$. We will show that $L_p - J(f_0)$ contains a homogeneous polynomial g of weight less than c; if so, then L_p contains $J(f_0) + \mathbb{C}[\mathbf{x}].g$, so $\dim \mathbb{C}[\mathbf{x}]/L_p \leqslant \dim \mathcal{O}_x/\langle J(f_0), g\rangle$; but this is $\leqslant \nu_c(f_0)$, completing the proof.

We have not yet used the condition that p belongs to the instability locus. By Corollary 1.4, it follows from this that there is a non-trivial linear combination ψ of the k unfolding monomials of least weight – and so of weight less than c – with $\psi \in K_p$. Making this homogeneous using t gives a homogeneous $\tilde{\psi} \in \tilde{I}_p$, of weight less than c, and not contained in $\langle t, J_x(F_p)\rangle$. Now setting $t = 0$ gives a non-zero homogeneous polynomial $g \in L_p - (f_0)$, and of weight less than c. This concludes the proof. □

3. Applications to hypersurfaces

To apply Theorem 2.1, we need to estimate the numbers $\nu_c(f_0)$. In the case of homogeneous functions, such an estimate is in some cases obtained from

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PROPOSITION 3.1. *Let* $\mathbf{f} = (f_1, \ldots, f_n)$ *be a regular sequence of homogeneous polynomials of degree* k *in* $\mathbb{C}[\mathbf{x}]$ *(with* $\mathbf{x} = (x_1, \ldots, x_n)$*). Let* h *be a nonzero homogeneous polynomial of degree* r*. Then*

$$\dim \mathbb{C}[\mathbf{x}]/\langle f_1, \ldots, f_n, h\rangle \leqslant rk^{n-1}.$$

Proof. We can regard $\mathbf{f}$ as a map $\mathbb{C}^n \to \mathbb{C}^n$, which is finite since $(f_1, \ldots, f_n)$ is a regular sequence. The variety $V \subset \mathbb{C}^n$ given by $h = 0$ has dimension $n-1$, hence also $f(V)$ has dimension $n-1$. Choose a point $P \notin f(V)$: then since everything is homogeneous, the line $\mathbb{C}P$ meets $f(V)$ only in the origin.

Making a non-singular linear transformation of $\mathbf{f}$ (which does not affect our hypothesis), we may suppose this line given by $f_2 = \cdots = f_n = 0$. Then h is a nonzero-divisor modulo $\langle f_2, \ldots, f_n\rangle$, so $\{f_2, \cdots, f_n, h\}$ is a regular sequence. Hence $\dim \mathbb{C}[\mathbf{x}]/\langle f_2, \ldots, f_n, h\rangle = rk^{n-1}$, and the result follows. □

COROLLARY 3.2. *If* $f_0\colon \mathbb{C}^n \to \mathbb{C}$ *is homogeneous of degree* d *and of finite singularity type, then* $\nu_c(f_0) \leqslant (c-1)(d-1)^{n-1}$.

This follows from the Proposition on setting $f_i = \partial f_0/\partial x_i$ and taking h as any homogeneous polynomial of degree $< c$.

Of course, the estimate is non-trivial only if $c < d$.

We now consider a hypersurface Γ of degree d in projective space $P^n(\mathbb{C})$. We suppose that Γ has isolated singularities, so that we can choose a hyperplane, which we may take as $x_0 = 0$, transverse to Γ. If the equation of Γ is $\gamma(\mathbf{x}) = 0$, we define $f_0(x_1, \ldots, x_n) = \gamma(0, x_1, \ldots, x_n)$: then f_0 is homogeneous of degree d, and has an isolated singularity at the origin (with $\mu = (d-1)^n$). Set $g(x_1, \ldots, x_n) = \gamma(1, x_1, \ldots, x_n)$.

The function g occurs as a fibre in the unfolding F of f_0 obtained by taking all monomials of degree $\leqslant d$ as unfolding monomials. Since the Hessian $H(f_0)$ has degree $nd - 2n$, this is an unfolding, trivial by [2], (10.2.1), of the unfolding $F^{k(c)}$, where $c = nd - 2n - d$. Thus g is equivalent to f_p, for some p in (the positive subspace of) the target of $F^{k(c)}$. Theorem 2.1 gives a condition for p to belong to the instability locus of $F^{k(c)}$. Since an unfolding is stable if and only if it is versal, this coincides with the condition that the singularities of g are not simultaneously versally unfolded by $F^{k(c)}$. As Γ has no singular points on $x_0 = 0$, this is in turn equivalent to the condition that the singularities of Γ are not simultaneously versally unfolded by the family of all hypersurfaces of degree d in $P^n(\mathbb{C})$.

A similar argument applies if we consider the family of hypersurfaces whose intersections with $x_0 = 0$ coincide with that of Γ. Here we must take $c = nd - 2n - d + 1$.

A direct application of Theorem 2.1, using the estimate of Proposition 3.1 for ν_c, now gives

LEMMA 3.3. *Let Γ be a hypersurface of degree d in $P^n(\mathbb{C})$ with isolated singularities only.*

(i) *Suppose $1/n + 1/d > \frac{1}{2}$; set $\delta = (d-1)^{n-1}(2n + 2d - nd)$. If $\tau_{\text{tot}}(\Gamma) < \delta$, then the family of hypersurfaces of degree d induces simultaneous versal deformations of all the singularities of Γ.*

(ii) *Suppose $1/n + 1/d > \frac{1}{2}(1 + 1/nd)$; set $\delta' = (d-1)^{n-1}(2n + 2d - nd - 1)$. Let H be any hypersurface transverse to Γ. If $\tau_{\text{tot}}(\Gamma) < \delta'$, then the family of hypersurfaces of degree d with the same restriction to H as Γ induces simultaneous versal deformations of all the singularities of Γ.*

The results given are, in fact, best possible. This, together with similar results in cases where the hypotheses above on $1/n + 1/d$ do not hold, will be discussed elsewhere.

If $n = 2$, the hypotheses are satisfied and we obtain:

COROLLARY 3.4. *Let Γ be a reduced (projective) plane curve of degree d.*

(i) *If the sum of the Tjurina numbers of all the singularities of Γ is $< 4(d-1)$, then the family of curves of degree d induces a simultaneous versal unfolding of all the singularities of Γ.*

(ii) *Let L be any line transverse to Γ. If the sum of the Tjurina numbers of all the singularities of Γ is $< 3(d-1)$, then the family of curves of degree d with the same restriction to L as Γ induces a simultaneous versal unfolding of all the singularities of Γ.* □

Here (i) is the result of Shustin [8] which inspired this paper. A different generalisation of Shustin's result has been obtained by Greuel and Lossen [4]. Case (ii) in the case $d = 5$ was cited in [11].

When $n > 2$, Lemma 3.3(i) applies, apart from the trivial cases $d = 2$ and $d = 3$, $n = 3$ (where all hypersurfaces with only isolated singularities are versally unfolded by the family), in four further cases; these are summarised in the following table.

n	3	3	4	5
d	4	5	3	3
δ	18	16	16	16.

These will be considered further elsewhere.

When $n > 2$, Lemma 3.3(ii) applies, apart from the trivial cases $d = 2$ (where all hypersurfaces are versally unfolded by the family), in three further cases; these are summarised in the following table.

n	3	3	4
d	3	4	3
δ	8	9	8.

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The case $n = 3$, $d = 3$, where '$c = 1$', follows from Wirthmüller's Theorem [12], as discussed earlier; the other two cases will be considered further elsewhere.

4. Further applications

We now consider examples with distinct weights. Begin with $f_0 = y^3 + x^b$, of weight $3b$ if wt $x = 3$ and wt $y = b$; we have $\mu(f_0) = 2b - 2$, and the singularity has modality m where $m = [b/3] - 1$ $(3m + 3 \leqslant b \leqslant 3m + 5)$. For any value of $c \leqslant m$, we have $k = c$. The critical value for $\nu_c(f)$ is obtained by taking $g = x^{c-1}$ (the only monomial of this weight), yielding $2(c - 2)$. Thus any combination of singularities with total Tjurina number less than $2(b - c + 1)$ is versally unfolded by F^c.

It is shown in [3] that this result too is best possible. Indeed consider deformations of the form $y^3 + \prod_{i=1}^{c-1}(x - \xi_i)^{b_i}$ with $\Sigma b_i = b$: these have Tjurina numbers $2(b - c + 1)$ and, as is shown in [3], fail to be versally unfolded by the deformation in question.

In fact, in [3] we give a complete determination of the unstable deformations of the weighted homogeneous singularities in the E, Z and Q series. To save repetition, let us agree that ϕ can stand for any of $y^3 + x^b$ (here we set $w = b/3$), $y^3 + uyx^{2w} + vx^{3w}$ with $4u^3 + 27v^2 \neq 0$, and $y^3 + yx^a$ (here we set $w = a/2$). For the E series we set $f_0 = \phi$ and consider the unfolding F^k, where we require $k \leqslant w - 1$ so that no monomial of degree at most that of f_0 is omitted in the unfolding. Since there is only one monomial in the relevant degrees, $c = k$ and

$$\nu_c(f_0) = \dim \mathcal{O}_{x,y}/(J(f_0) + \mathcal{O}_{x,y}.x^{k-1}) = \dim \mathcal{O}_{x,y}/\langle y^2, x^{k-1}\rangle = 2k - 2.$$

The instability locus of F^k is the union of the images of the following, where $s \leqslant k$ and the ξ_i are all distinct in each case

$$\begin{aligned}
&y^3 + \prod_1^s (x - \xi_i)^{b_i} && \textstyle\sum b_i = 3w && \textstyle\sum b_i\xi_i = 0,\\
&y^3 + uy\prod_1^s (x - \xi_i)^{2c_i} + v\prod_1^s (x - \xi_i)^{3c_i} && \textstyle\sum c_i = w && \textstyle\sum c_i\xi_i = 0,\\
&y^3 + y\prod_1^s (x - \xi_i)^{2a_i} && \textstyle\sum a_i = 2w && \textstyle\sum a_i\xi_i = 0.
\end{aligned}$$

In each case, the Milnor numbers (which equal the Tjurina numbers, since the singularities are weighted homogeneous) of the singularities at the singular points $(\xi_i, 0)$ add up to $6w - 2s$; since $\tau_0 = 6w - 2$, we have $\tau_0 - \tau_v = 2s - 2 \leqslant 2k - 2$. Thus all the deformations with $s = k$ realise the bound obtained above.

For the Z series, matters are a little more complicated. We take $f_0 = x\phi$, with ϕ as above, and consider the unfoldings F^k with $k \leqslant w$. Again we have $c = k$ and $\nu_c(f_0) = \dim \mathcal{O}_{x,y}/\langle y^3, xy^2, x^{k-1}\rangle = 2k - 1$.

The unstable deformations are, with $s \leqslant k-1$ in each case

$$
\begin{array}{ll}
xy^3 + \prod_{*}^{s}(x-\xi_i)^{b_i} & \sum b_i = 3w+1, \\
x(y^3 + uy\prod_{1}^{s}(x-\xi_i)^{2c_i} + v\prod_{1}^{s}(x-\xi_i)^{3c_i}) & \sum c_i = w, \\
xy^3 + y\prod_{*}^{s}(x-\xi_i)^{2a_i} & \sum a_i = 2w+1,
\end{array}
$$

where ξ_i is nonzero for $i \geqslant 1$, and either $* = 1$ or $* = 0$ and $\xi_0 = 0$. Here we find that the sum of the Milnor numbers is $6w+2-2s$ for the first deformation with $* = 1$, and $6w+3-3s$ in all the other cases, so that $\tau_0 - \tau_v = 2s+1$ or $2s$ correspondingly. Thus our bound is achieved by the first deformation, with $* = 1$ and $s = k-1$: this is available only from $xy^3 + x^b$: i.e. from the strata $Z^*_{(-1)}, Z^*_{(1)}$ and the part of Z^*_0 with $u = 0$.

For the Q series we take $f_0 = xz^2 + \phi$ with ϕ as above, and consider the unfoldings F^k with $k \leqslant w$. We again find only one monomial x^c in the relevant degrees c, $c = k$ and $\nu_c(f_0) = \dim \mathcal{O}_{x,y,z}/\langle y^2, z^2, xz, x^{k-1}\rangle = 2k$.

The unstable deformations are, with $s \leqslant k-1$ in each case

$$
\begin{array}{lll}
(1) & xz^2 + y^3 + \prod_{*}^{s}(x-\xi_i)^{b_i} & \sum b_i = 3w. \\
(2) & xz^2 + y^3 + uy\prod_{1}^{s}(x-\xi_i)^{2c_i} + v\prod_{1}^{s}(x-\xi_i)^{3c_i} & \sum c_i = w. \\
(3) & xz^2 + y^3 + y\prod_{*}^{s}(x-\xi_i)^{2a_i} & \sum a_i = 2w. \\
(4) & x^{-1}\{(xz+C)^2 + xy^3 + \prod_{1}^{s}(x-\xi_i)^{b_i}\} & \sum b_i = 3w+1,
\end{array}
$$

where (as before) ξ_i is nonzero for $i \geqslant 1$, and either $* = 1$ or $* = 0$ and $\xi_0 = 0$, and in the final case, C is such that the division is possible. Here we find that the sum of the Milnor numbers is $6w - 2s$ for (1) and (3) with $* = 1$ and for (2), and $6w + 2 - 3s$ in all the other cases, so that $\tau_0 - \tau_v = 2s + 2$ or $2s$ correspondingly. Thus in each case there are some deformations with $s = k-1$ producing equality.

Finally we consider other weighted homogeneous germs of low modality, and the unfolding omitting all unfolding monomials of weight at least that of f.

For parabolic (simple elliptic) and exceptional unimodal (triangle) singularities, we take $c = 1$. Here, as discussed earlier, the instability locus consists only of the origin.

For quadrilateral and exceptional bimodal singularities, there are two unfolding monomials, and two candidates for g, one of which is 1 (when $c = 1$, with conclusion as above) and the other is the variable – say x – of least weight. The calculation of ν then turns out to depend only on the series, and we exhibit it in tabular form.

Series	Equation	$J(f) + \langle x \rangle$	ν
E	$y^3 + ya(x) + b(x)$	$\langle y^2, x \rangle$	2
Z	$xy^3 + ya(x) + b(x)$	$\langle x, y^3 \rangle$	3
W	$y^4 + y^2a(x) + yb(x) + c(x)$	$\langle x, y^3 \rangle$	3
Q	$xz^2 + y^3 + ya(x) + b(x)$	$\langle x, y^2, z^2 \rangle$	4
S	$xz^2 + y^2z + za(x) + yb(x) + c(x)$	$\langle x, y^2, yz, z^2 \rangle$	3
U	$y^3 + z^3 + yza(x) + yb(x) + zc(x) + d(x)$	$\langle x, y^2, z^2 \rangle$	4

We have already seen that for most bimodals in the E, Z and Q series these estimates are best possible. This is true for all bimodals in the remaining cases: the deformations for the W series appear in [2], Chapter 11 and we have obtained the others subsequently.

These calculations of ν are valid for all singularities in the series in question, provided that only the case $k = 2$ is under consideration.

It is similarly easy to compute ν for all weighted homogeneous trimodals (these were enumerated in [10]), and again the estimate is best possible in most cases. We defer the details to a subsequent paper.

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