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Remarks on recent advances concerning boundary effects and the vanishing viscosity limit of the Navier–Stokes equations

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Abstract

This contribution covers the topic of my talk at the 2016-17 Warwick-EPSRC Symposium: "PDEs and their applications". As such it contains some already classical material and some new observations. The main purpose is to compare several avatars of the Kato criterion for the convergence of a Navier–Stokes solution, to a regular solution of the Euler equations, with numerical or physical issues like the presence (or absence) of anomalous energy dissipation, the Kolmogorov $\frac{1}{3}$ law or the Onsager $C^{0,\frac{1}{3}}$ conjecture. Comparison with results obtained after September 2016 and an extended list of references have also been added.

1.1 Introduction and uniform estimates.

In this contribution I will describe the main topics of my talk at the 2016-17 Warwick-EPSRC Symposium: *PDEs in Fluid Mechanics* in September 2016. Most of these issues are the results of a long term collaboration with Edriss Titi. I will also comment on some more recent (after September 2016) results (also collaboration with Edriss Titi and several other coworkers). In the same way I am going to include (mostly with no details) some recent results of other researchers and an extended list of references whenever they contribute to the understanding of the problems. Eventually one of the guidelines is the comparison between the use of weak convergence and the use of a statistical theory of turbulence. Hence the paper is organized as follows. After introducing some basic and well-known estimates, the zero viscosity limit of solutions of the Navier–

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Stokes equations is considered with no-slip boundary condition but in the presence, for the same initial data, of a Lipschitz solution of the Euler equations. This leads to an extension of Kato's theorem and to the introduction of several (equivalent) criteria for convergence to a smooth solution and for the absence of anomalous energy dissipation. Comparison of these criteria with physical observations or classical ansatz are made. In particular emphasis is given to the issue of the anomalous energy dissipation which leads to the comparison with the Kolmogorov $\frac{1}{3}$ law in the statistical theory of turbulence. Then this leads to the issue of the Onsager $C^{0,\frac{1}{3}}$ conjecture.

As a starting point consider solutions of the Euler equations and of the Navier–Stokes equations in a space-time domain

$$\Omega \times [0,T] \subset \mathbb{R}^d \times \mathbb{R}_t^+, \quad d = 2, 3.$$

We assume that the boundary $\partial\Omega$ is a C^1 manifold with $\vec{n}(x)$ denoting the outward normal at any point x in $\partial\Omega$. Then we introduce the function

$$d(x) = d(x, \partial\Omega) = \inf_{y \in \partial\Omega} |x - y| \geq 0$$

and the set

$$\mathcal{U}_\eta = \{x \in \Omega \,, d(x) < \eta\},$$

which have the following classical geometrical properties.

Proposition 1.1 *For $0 < \eta < \eta_0$ small enough $d(x)_{|\mathcal{U}_\eta} \in C^1(\overline{\mathcal{U}_\eta})$ and for any $x \in \mathcal{U}_\eta$ there exists a unique point $\sigma(x) \in \partial\Omega$ such that $d(x) = |x - \sigma(x)|$. Moreover for every $x \in \mathcal{U}_\eta$ we have*

$$x = \sigma(x) - d(x)\vec{n}(\sigma(x)) \quad \textit{and} \quad \nabla_x d(x) = -\vec{n}(\sigma(x)). \tag{1.1}$$

To focus on the boundary effects, first, we consider a smooth (Lipschitz) solution $u(x,t)$ of the incompressible Euler equations with the impermeability condition:

$$\begin{aligned} &\nabla \cdot u = 0 \quad \text{and} \quad \partial_t u + u \cdot \nabla u + \nabla p = 0 \text{ in } \ \Omega \times [0,T] \\ &\text{and} \quad\ u \cdot \vec{n} = 0 \quad \text{ on } \partial\Omega \times [0,T]\,. \end{aligned} \tag{1.2}$$

The value of such solution for $t = 0$ is denoted by $u_0(x) = u(x,0)$. For the same initial data $u_\nu(x,0) = u_0(x)$ and for any $\nu > 0$ one considers a family $u_\nu(x,t)$ of Leray–Hopf solutions of the Navier–Stokes equations

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with the no-slip boundary condition:

$$\partial_t u_\nu + u_\nu \cdot \nabla u_\nu - \nu \Delta u_\nu + \nabla p_\nu = 0 \quad \text{and} \quad \nabla \cdot u_\nu = 0 \text{ in } \Omega \times [0,T]$$
$$\text{with } \; u_\nu = 0 \,\text{on}\, \partial\Omega \times [0,T]. \tag{1.3}$$

For Lipschitz solutions of the Euler equations we have the obvious energy balance relation

$$\int_\Omega \frac{|u(x,t)|^2}{2}\,\mathrm{d}x = \int_\Omega \frac{|u(x,0)|^2}{2}\,\mathrm{d}x, \quad \text{for all } t \in [0,T], \tag{1.4}$$

while for any Leray–Hopf solution of the Navier–Stokes equations we obtain

$$\int_\Omega \frac{|u_\nu(x,t)|^2}{2}\,\mathrm{d}x + \nu \int_0^t \int_\Omega |\nabla u_\nu(x,s)|^2\,\mathrm{d}x\,\mathrm{d}s \le \int_\Omega \frac{|u_\nu(x,0)|^2}{2}\,\mathrm{d}x, \tag{1.5}$$

for all $t \in [0,T]$.

It is well known that in dimension two the solution u_ν is smooth, unique and (1.5) is actually an equality instead of an inequality. The issue of the regularity of the solutions of (1.3) plays no role in the present contribution which focuses on the zero viscosity limit. It turns out there are no other estimates uniformly valid for all positive ν, and in particular for ν going to zero, other that the one that follows from (1.3). It implies the existence of (may be not unique) limits, in the weak star $L^\infty(0,T;L^2(\Omega))$ topology, of subsequence of solutions u_ν of (1.3). Any such limit is denoted by $\overline{u_\nu}$, and the main question is whether or not we have

$$\overline{u_\nu} = u \;\text{ in } \quad \Omega \times [0,T].$$

As shown in Kato (1972) and Constantin (2005), in the absence of physical boundaries (torus or the whole space) u_ν converges to u.

In the presence of physical boundaries, this is much more subtle. The obvious difficulty comes from the fact that when $\nu \to 0$ only the impermeability boundary condition remains while (here τ denotes the tangential component at the boundary) the relation $(u_\nu)_\tau = 0$ does not persist. Therefore the solution has to become singular near the boundary. It creates a shear flow near the boundary, in solutions of (1.3), which generates vorticity that may propagate inside the domain by the advection term and by the effect of the pressure. This turns out to be the most natural way to generate turbulence (even homogeneous turbulence far from the boundary).

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For any Lipschitz vector field w we denote by $S(w)$ its symmetric stress tensor

$$S(w) = \frac{\nabla w + (\nabla w)^{\perp}}{2}.$$

Denote by $(\cdot,\cdot)$ the $L^2(\Omega)$ scalar product. Then since both u_ν and u are divergence-free Lipschitz vector fields and since u is tangent to the boundary of Ω we obtain, by integration by parts, the classical formula

$$(u_\nu \cdot \nabla u_\nu - u \cdot \nabla u, u_\nu - u) = (u_\nu - u, S(u)(u_\nu - u)). \tag{1.6}$$

From (1.2) and (1.3) we also have

$$\partial_t (u_\nu - u) + u_\nu \cdot \nabla u_\nu - u \cdot \nabla u - \nu \Delta u_\nu + \nabla p_\nu - \nabla p = 0. \tag{1.7}$$

Taking the $L^2(\Omega)$ inner product of (1.7) with $(u_\nu - u)$ and observing that on the boundary Ω we have $u_\nu = 0$ and $u \cdot \vec{n} = 0$, thanks to (1.6), we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}|u_\nu - u|^2_{L^2(\Omega)} + \nu \int_\Omega |\nabla u_\nu|^2 \,\mathrm{d}x \\ \le |(u_\nu - u, S(u)(u_\nu - u))| + \nu \int_\Omega (\nabla u_\nu : \nabla u)\,\mathrm{d}x \\ - \nu \int_{\partial\Omega} (\partial_{\vec{n}} u_\nu)_\tau \cdot u \,\mathrm{d}\sigma. \end{aligned} \tag{1.8}$$

The analysis of the term

$$-\nu \int_{\partial\Omega} (\partial_{\vec{n}} u_\nu)_\tau \cdot u \,\mathrm{d}\sigma,$$

which appears in the right-hand side of (1.8) is, in this section and in the next one, the cornerstone of this contribution. We observe that $(\partial_{\vec{n}} u_\nu)_\tau$ is the tangential component of the stress at the boundary. It creates a shear flow near the boundary and generates vorticity. In order to see this more clearly notice that since $(u_\nu)_\tau = 0$ on the boundary of Ω we obtain the following equality

$$-(\partial_{\vec{n}} u_\nu)_\tau \cdot u = (\nabla \wedge u_\nu) \cdot (\vec{n} \wedge u). \tag{1.9}$$

Therefore all the considerations concerning the left hand-side of (1.9) do have their counterpart on the right-hand side, i.e. in terms of the trace of the vorticity of u_ν on $\partial\Omega$.

Moreover, from (1.8) it follows the very easy, but essential result.

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Proposition 1.2 *Let u be a Lipschitz solution of the Euler equations* (1.2) *and u_ν the solutions of the Navier–Stokes equations* (1.3) *with initial data $u_\nu(x,0) = u(x,0) = u_0(x)$. Then under the hypothesis*

$$\begin{aligned} \limsup_{\nu\to 0} \int_0^T -\nu \int_{\partial\Omega} (\partial_{\vec{n}} \cdot u_\nu)_\tau u \, \mathrm{d}\sigma \, \mathrm{d}t \\ = \limsup_{\nu\to 0} \nu \int_0^T \int_{\partial\Omega} -((\partial_{\vec{n}} u_\nu)_\tau \cdot u_\tau)_- \, \mathrm{d}\sigma \, \mathrm{d}t \leq 0 \end{aligned} \tag{1.10}$$

any weak limit $\overline{u_\nu}$ coincides with u in $\Omega \times [0,T]$.

In the proposition and throughout the paper we use $(X)_- = \inf(0, X)$.

Proof From (1.8), using the Cauchy–Schwarz and Young inequalities we deduce that

$$\begin{aligned} |u_\nu - u|^2_{L^2(\Omega)}(t) + \nu \int_0^t \int_\Omega |\nabla u_\nu(x,s)|^2 \, \mathrm{d}x \, \mathrm{d}s \\ \leq \nu \int_0^t \int_\Omega |\nabla u|^2 \, \mathrm{d}x \, \mathrm{d}s \\ + 2|S(u)|_{L^\infty(\Omega\times[0,T])} \int_0^t |(u_\nu - u)(s)|^2_{L^2(\Omega)} \, \mathrm{d}s \\ + 2 \int_0^t -\nu \int_{\partial\Omega} (\partial_{\vec{n}} u_\nu)_\tau \cdot u \, \mathrm{d}\sigma \, \mathrm{d}s. \end{aligned} \tag{1.11}$$

Then, under the hypothesis (1.10), we have

$$\begin{aligned} \limsup_{\nu\to 0} |(u_\nu - u)(t)|^2_{L^2(\Omega)} \\ \leq |S(u)|_{L^\infty(\Omega\times[0,T])} \int_0^t \limsup_{\nu\to 0} |(u_\nu - u)(s)|^2_{L^2(\Omega)} \, \mathrm{d}s, \end{aligned} \tag{1.12}$$

which implies, by Gronwall's inequality, that

$$\limsup_{\nu\to 0} |(u_\nu - u)(t)|^2_{L^2(\Omega)} = 0, \quad \text{for all } t \in [0,T],$$

and consequently, the relation

$$|\overline{u_\nu - u}|^2_{L^2(\Omega)}(t) \leq \limsup_{\nu\to 0} |(u_\nu - u)(t)|^2_{L^2(\Omega)} \tag{1.13}$$

implies $\overline{u_\nu} = u$ in $\Omega \times [0,T]$. □

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1.2 Kato criterion for convergence to the regular solution.

In a remarkable paper Kato (1984) related the convergence to the smooth solution of the Euler equations to the absence of anomalous energy dissipation in a boundary layer of size ν. At present it turns out that this criterion (this is the object of the Theorem 1.3 below) has several equivalent forms (see Theorem 4.1 in Bardos & Titi (2013) and Constantin et al (2018) for more references). Some of these equivalent forms (in particular the above hypothesis (1.10)) have natural physical interpretations.

Theorem 1.3 *Assume the existence of a Lipschitz solution $u(x,t)$ of the incompressible Euler equations in $\Omega\times]0,T[$. Let $u_\nu(x,t)$ be a Leray–Hopf weak solution of the Navier–Stokes equations (1.3) with no slip boundary condition, that coincides with u at the time $t=0$. Define the region*

$$\mathcal{U}_\nu = \Omega \cap \{d(x,\partial\Omega) < \nu\}.$$

Then the following facts are equivalent:

$$\lim_{\nu\to 0} \nu \int_0^T \int_{\partial\Omega} ((\partial_{\vec{n}} u_\nu)_\tau \cdot u_\tau)_- \, \mathrm{d}\sigma \, \mathrm{d}t = 0, \tag{1.14a}$$

$$u_\nu(t) \to u(t) \ \textit{in } L^2(\Omega) \ \textit{uniformly in } t \in [0,T], \tag{1.14b}$$

$$u_\nu(t) \to u(t) \ \textit{weakly in } L^2(\Omega) \ \textit{for each } t \in [0,T], \tag{1.14c}$$

$$\lim_{\nu\to 0} \nu \int_0^T \int_\Omega |\nabla u_\nu(x,t)|^2 \, \mathrm{d}x \, \mathrm{d}t = 0, \tag{1.14d}$$

$$\lim_{\nu\to 0} \nu \int_0^T \int_{\mathcal{U}_\nu} |\nabla u_\nu(x,t)|^2 \, \mathrm{d}x \, \mathrm{d}t = 0, \tag{1.14e}$$

$$\lim_{\nu\to 0} \frac{1}{\nu} \int_0^T \int_{\mathcal{U}_\nu} |(u_\nu(x,t))_\tau|^2 \, \mathrm{d}x \, \mathrm{d}t = 0, \ \textit{and} \tag{1.14f}$$

$$\lim_{\nu\to 0} \nu \int_0^T \int_{\partial\Omega} (\frac{\partial u_\nu}{\partial \vec{n}}(\sigma,t))_\tau \cdot w(\sigma,t) \, \mathrm{d}\sigma \, \mathrm{d}t = 0 \quad \textit{for all } w(x,t) \in Lip(\partial\Omega \times [0,T]) \ \textit{tangent to } \partial\Omega. \tag{1.14g}$$

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Proof The proof is an updated version (cf. Bardos & Titi, 2013) of the basic result of Kato (1984). First observe that (1.14a) is (with $w = u$) a direct consequence of (1.14g).

The fact that (1.14a) implies (1.14b) was observed in the previous section, while (1.14c) clearly follows from (1.14b).

From (1.14c), for any $0 < t < T$, we deduce

$$\begin{aligned}\lim_{\nu\to 0} 2\nu \int_0^t \int_\Omega |\nabla u_\nu(x,s)|^2 \, \mathrm{d}x \, \mathrm{d}s \\ &\le \int_\Omega |u(x,0)|^2 \, \mathrm{d}x - \liminf_{\nu\to 0} \int_\Omega |u_\nu(x,t)|^2 \, \mathrm{d}x \\ &\le \int_\Omega |u(x,0)|^2 \, \mathrm{d}x - \int_\Omega |u(x,t)|^2 \, \mathrm{d}x \le 0,\end{aligned} \tag{1.15}$$

which gives (1.14d) from which (1.14e) easily follows, as $\mathcal{U}_\nu \subset \Omega$.

Since $u_\nu = 0$ on $\partial\Omega\times]0,T[$ the estimate (1.14f) is deduced from (1.14e) using the Poincaré inequality.

The only non trivial statement is the fact that (1.14f) implies (1.14g) and its proof is inspired by the construction of Kato (1984). We introduce a cut-off function

$$\Theta \in C^\infty(\mathbb{R}), \text{ with } \Theta(0) = 1 \quad \text{and} \quad \Theta(s) = 0 \text{ for } s > 1. \tag{1.16}$$

Then, with $\nu < \eta_0$, use Proposition 1.1 to extend w to a Lipschitz, divergence-free, tangent to the boundary vector field $\hat{w}_\nu$ according to the formula:

$$\begin{aligned}&\hat{w}_\nu(x,t) = 0, \text{ for } \quad x \notin \mathcal{U}_\nu, \\ &\hat{w}_\nu(x,t) = \nabla \wedge (\vec{n}(\sigma) \wedge w(\sigma,t) d(x,\partial\Omega)\Theta(\frac{d(x,\partial\Omega)}{\nu})), \\ &\qquad\qquad \text{for } x = \sigma(x) - d(x,\partial\Omega)\vec{n}(\sigma(x)) \in \mathcal{U}_\nu.\end{aligned} \tag{1.17}$$

Multiplication of the Navier–Stokes equation satisfied by u_ν and integrating by part gives

$$\begin{aligned}&\nu \int_{\partial\Omega} (\frac{\partial u_\nu}{\partial \vec{n}}(\sigma,t))_\tau w(\sigma,t) \, \mathrm{d}\sigma \\ &\quad = \nu(\nabla u_\nu, \nabla \hat{w}_\nu)_{L^2(\Omega)} - (u_\nu \otimes u_\nu, \nabla \hat{w}_\nu)_{L^2(\Omega)} + (\partial_t u_\nu, \hat{w}_\nu)_{L^2(\Omega)} \,.\end{aligned} \tag{1.18}$$

To show that the right-hand side of (1.18) goes to 0 with ν observe that, the only non trivial terms to consider are those that contain the highest power of ν^{-1}.

We have the following estimates, where C denotes any constant which

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depends on the geometry and on the Jacobian of the transformation defined on $\mathcal{U}_\nu$ by the relation $x = \sigma(x) - d(x, \partial\Omega)\vec{n}(\sigma(x))$.

$$
\begin{aligned}
\nu\Big|\int_0^T \int_{\mathcal{U}_\nu} |(\nabla u_\nu, \nabla \hat{w}_\nu)|\, \mathrm{d}x\, \mathrm{d}t\Big| \\
&= -\nu \Big|\int_0^T \int_{\mathcal{U}_\nu} u_\nu : \Delta \hat{w}_\nu \, \mathrm{d}x\, \mathrm{d}t\Big| \\
&\le \nu C \int_0^T \int_0^\nu \int_{\sigma\in\partial\Omega} |(u_\nu)_\tau(\sigma, s)||w(\sigma)| \frac{s}{\nu^3} |\Theta'''(\frac{s}{\nu})|\, \mathrm{d}s\, \mathrm{d}\sigma \mathrm{d}t \\
&\quad + o(\nu)
\end{aligned}
\tag{1.19}
$$

and

$$
\begin{aligned}
&\Big|\int_0^T \int_{\mathcal{U}_\nu} (u_\nu \otimes u_\nu, \nabla \hat{w}_\nu)_{L^2(\Omega)}\, \mathrm{d}t\Big| \\
&\le \Big|\int_0^T \int_{\mathcal{U}^\nu} ((u_\nu)_\tau (u_\nu)_n \partial_n(\hat{w}_\tau)\, \mathrm{d}x\, \mathrm{d}t\Big| + o(\nu) \\
&\le C \int_0^T \int_0^\nu \int_{\sigma\in\partial\Omega} |(u_\nu)_\tau(\sigma, s)||(u_\nu)_n(\sigma, s)||w(\sigma, t)| \frac{s}{\nu^2} \Theta''(\frac{s}{\nu})\, \mathrm{d}s\, \mathrm{d}\sigma\, \mathrm{d}t \\
&\qquad + o\,(\nu).
\end{aligned}
\tag{1.20}
$$

Therefore using Cauchy–Schwarz we obtain from (1.19)

$$
\begin{aligned}
\Big|\nu \int_0^T \int_{\mathcal{U}_\nu} (\nabla u_\nu, \nabla \hat{w}_\nu)\, \mathrm{d}x\, \mathrm{d}t\Big| \\
&\le C \frac{1}{\nu^2} \Big(\int_0^T \int_0^\nu \int_{\sigma\in\partial\Omega} |(u_\nu)_\tau(\sigma, s)|^2 \, \mathrm{d}s\, \mathrm{d}\sigma\, \mathrm{d}t\Big)^{\frac12} \\
&\qquad \times \Big(\int_0^T \int_{\partial\Omega} \int_0^\nu s^2 \, \mathrm{d}s\, \mathrm{d}\sigma\, \mathrm{d}t\Big)^{\frac12} \\
&\le C \Big(\frac{1}{\nu} \int_0^T \int_0^\nu \int_{\sigma\in\partial\Omega} |(u_\nu)_\tau(\sigma, s)|^2 \, \mathrm{d}s\, \mathrm{d}\sigma\, \mathrm{d}t\Big)^{\frac12}
\end{aligned}
\tag{1.21}
$$

and similarly for (1.20) we have

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$$\begin{aligned}
&\left|\int_0^T \int_{\mathcal{U}_\nu} |(u_\nu \otimes u_\nu, \nabla \hat{w}_\nu)| \,\mathrm{d}x\,\mathrm{d}t\right| \\
&\leq \left|\int_0^T \int_{\mathcal{U}^\nu} ((u_\nu)_\tau (u_\nu)_n \partial_n(\hat{w}_\tau) \,\mathrm{d}x\,\mathrm{d}t\right| + o(\nu) \\
&\leq C \int_0^T \int_0^\nu \int_{\sigma\in\partial\Omega} |(u_\nu)_\tau(\sigma,s)||(u_\nu)_n(\sigma,s)||w(\sigma,t)| \frac{s}{\nu^2} \Theta''(\frac{s}{\nu}) \,\mathrm{d}s\,\mathrm{d}\sigma\,\mathrm{d}t \\
&\qquad + o(\nu) \\
&\leq \frac{C}{\nu} \int_0^T \int_0^\nu \int_{\sigma\in\partial\Omega} |(u_\nu)_\tau(\sigma,s)||(u_\nu)_n(\sigma,s)||w(\sigma,t)| \,\mathrm{d}s\,\mathrm{d}\sigma\,\mathrm{d}t + o(\nu) \\
&\leq C \left(\frac{1}{\nu} \int_0^T \int_0^\nu \int_{\sigma\in\partial\Omega} |(u_\nu)_\tau(\sigma,s)|^2 \,\mathrm{d}s\,\mathrm{d}\sigma\mathrm{d}t \right)^{\frac{1}{2}} \\
&\qquad\qquad \times \left(\frac{1}{\nu} \int_0^T \int_0^\nu \int_{\sigma\in\partial\Omega} |(u_\nu)_n(\sigma,s)|^2 \,\mathrm{d}s\,\mathrm{d}\sigma\,\mathrm{d}t \right)^{\frac{1}{2}} + o(\nu).
\end{aligned} \tag{1.22}$$

Moreover, since $u_\nu = 0$ on $\partial\Omega$, with the Poincaré inequality, we have

$$\begin{aligned}
&\int_0^T \int_0^\nu \int_{\partial\Omega} |(u_\nu)_n(\sigma,s,t)|^2 \,\mathrm{d}s\,\mathrm{d}\sigma\,\mathrm{d}t \\
&\qquad \leq \nu^2 \int_0^T \int_0^\nu \int_{\partial\Omega} |(u_\nu)_n|^2 \,\mathrm{d}s\,\mathrm{d}\sigma\,\mathrm{d}t \leq C\|u_0(x)\|^2_{L^2(\Omega)}.
\end{aligned} \tag{1.23}$$

Therefore the last term of both (1.21) and (1.22) is uniformly bounded by

$$C\frac{1}{\nu} \int_0^T \int_{\mathcal{U}_\nu} |(u_\nu(x,t))_\tau|^2 \,\mathrm{d}x\,\mathrm{d}t + o(\nu)$$

and this shows that (1.14f) implies (1.14g), completing the proof. □

1.3 Mathematical and physical interpretation of Theorem 1.3

1.3.1 Recirculation

Since $u_\nu = 0$ on $\partial\Omega$ and u is tangent to the boundary, the fact that

$$(\frac{\partial u_\nu}{\partial \vec{n}}(\sigma,t))_\tau u_\tau = ((\nabla \wedge u_\nu) \wedge \vec{n}) \cdot u < 0$$

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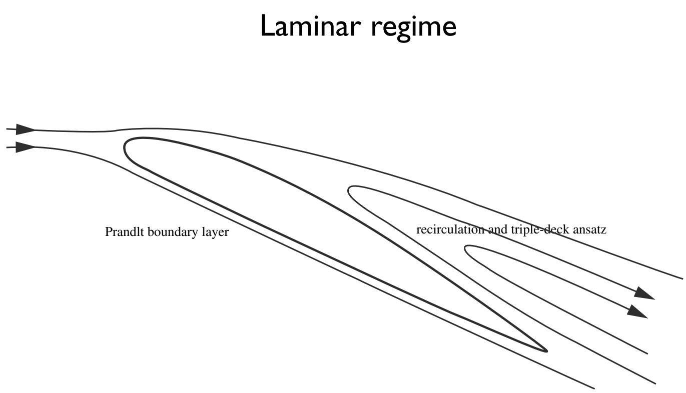

Figure 1.1 Laminar flow with recirculation around an airfoil.

for ν small enough, indicates that somewhere near the boundary the viscous flows u_ν go in the opposite direction to the base flow u that solves the Euler equations, or equivalently that this flow exhibits some backward vorticity. This configuration is known as "recirculation" and does not prevent the fluid from remaining laminar or from having an asymptotic behavior given by the Euler equations, as long as such recirculation is not too big. And this moderate recirculation, shown in Figure 1.1, corresponds to the hypothesis (1.10).

1.3.2 The Prandtl equations and the Stewartson triple-deck ansatz.

As already observed, in the zero-viscosity limit, the boundary condition $(u_\nu)_\tau = 0$ may not persist; hence some type of singularity has to appear near the boundary. However, for linear parabolic problems of the form

$$\partial_t u_\nu - \nu\Delta u_\nu = 0, \qquad u_\nu(x,0) = u_0(x), \qquad u_\nu(x,t)_{|\partial\Omega} = 0 \tag{1.24}$$

and also, for the linearised Navier–Stokes equations (cf. Ding & Jiang, 2018), the solution converges strongly away from the boundary and near the boundary, in a layer $B_{\sqrt{\nu}} = \{x \in \Omega, d(x,\partial\Omega) < \sqrt{\nu}\}$ of size $\sqrt{\nu}$. It can be described in the laminar regime by a "parabolic scaling", that is,

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by a smooth function of the form

$$x = x_\tau - d(x,\partial\Omega)\vec{n}(x_\tau) \mapsto U\left(\frac{d(x,\partial\Omega)}{\sqrt{\nu}}, x_\tau\right).$$

In fact, Prandtl (1904) proposed to represent the solution of the Navier–Stokes equation according to the formula

$$u_\nu(x,t) \simeq U_\tau\left(\frac{d(x,\partial\Omega)}{\sqrt{\nu}}, x_\tau, t\right) + \sqrt{\nu}U_{\vec{n}}\left(\frac{d(x,\partial\Omega)}{\sqrt{\nu}}, x_\tau, t\right), \qquad (1.25)$$

where the indices τ and $\vec{n}$ refer to tangent and normal components of the fluid velocity $U(x,t)$. To describe the behaviour near the boundary one requires that

$$\begin{gathered} U_\tau(0,x_\tau,t) + \sqrt{\nu}U_{\vec{n}}(0,x_\tau,t) = 0 \quad \text{and} \\ \lim_{y\to\infty} U_\tau\left(\frac{y}{\sqrt{\nu}}, x_\tau, t\right) + \sqrt{\nu}U_{\vec{n}}\left(\frac{y}{\sqrt{\nu}}, x_\tau, t\right) = u_\tau(x_\tau,t)\,. \end{gathered} \qquad (1.26)$$

Much later, Stewartson (1974) proposed an ansatz that would incorporate a certain amount of recirculation. To do so he considered three layers of fluid near the boundary and hence called this ansatz “the triple deck”.

1. In the Upper Deck $\{x \in \Omega \big| \sqrt{\nu} < d(x,\partial\Omega)\}$ the solution is described by the Euler flow.

2 In the Lower Deck $\{x \in \Omega \big| 0 < d(x,\partial\Omega) < \nu^{\frac{5}{8}}\}$ the solution is described by the above Prandtl boundary layer ansatz.

3 In Middle Deck $\{x \in \Omega \big| \nu^{\frac{5}{8}} < d(x,\partial\Omega) < \sqrt{\nu}\}$, which connects the two above layers the following scaling is proposed:

$$u_\nu(x,t) \simeq (\nu^{\frac{1}{8}}U_\tau(\frac{d(x,\partial\Omega)}{\nu^{\frac{5}{8}}}, \frac{x_\tau}{\nu^{\frac{3}{8}}}, t), \nu^{\frac{3}{8}}U_{\vec{n}}(\frac{d(x,\partial\Omega)}{\nu^{\frac{5}{8}}}, \frac{x_\tau}{\nu^{\frac{3}{8}}}, t)). \qquad (1.27)$$

Observe that the Prandtl and the triple Deck ansatzs share in common the following property. If they have, for $0 < t < T$, a smooth solution and if this solution gives an accurate description of the genuine behaviour of the solution u_ν of the Navier–Stokes equation then u_ν satisfies any (and of course all) the criteria of Theorem 1.3.

However, to the best of our knowledge there are no mathematical results concerning the validity of the triple deck ansatz.

Additionally, results concerning the Prandtl equations have been established near a flat boundary. The problem in general is ill posed. It is well posed for analytic initial data (see Asano, 1991 and Sammartino & Caflisch, 1998) or for Gevrey initial data (see Gerard-Varet & Masmoudi, 2015). However even with such data the solution may blow up in

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a finite time, see E & Enquist (1997). Eventually one can construct examples where both the Navier–Stokes and Prandtl equations have, with the same initial data, a smooth solution for a finite time. But in this special case the Prandtl equations do not provide a correct approximation of Navier–Stokes equations as $\nu \to 0$.

Since a boundary layer of size ν is much smaller (as $\nu \to 0$) than a boundary layer of size $\sqrt{\nu}$ it follows from the above considerations that if the criteria (1.14e) is not satisfied asymptotic ansatz like the Prandtl equation or the triple deck cannot be uniformly valid for the description of the zero viscosity limit.

1.3.3 Von Karman turbulent Layer

From the above considerations on boundary layers one deduces that turbulence generation in a region of size ν around the boundary is the basic cause for the non convergence to the smooth solution of the Euler equations. Despite the absence of any type of proof I think that it may be useful to compare this issue with the empirical rule for turbulence at the boundary. Since convergence in a strong norm is not expected, a turbulent boundary layer for $\overline{u_\nu}$ should be present, in general, around some part of the boundary.

To the best of my knowledge, the only practical thing available is a description based on experiment, numerical analysis and dimensional analysis: the Prandtl–Von Karman turbulent layer (1932). It provides an ansatz for the tangential component of the velocity $u_\tau(x_{\vec{n}}, x_\tau, t)$ in the layer

$$B_{\text{turbulent}} = \{x \in \Omega, d(x, \Omega) < \nu\} \cap \mathcal{W},$$

with $\mathcal{W}$ denoting a neighborhood of some part of the boundary.

On $\partial\Omega \cap \overline{\mathcal{W}}$ the quantity

$$u^* = \sqrt{\nu \partial_{\vec{n}} u_\tau}, \tag{1.28}$$

which has the dimension of a velocity, is assumed to be of the order of one.

Then in $B_{\text{turbulent}}$ we have

$$u_\tau(x_{\vec{n}}, x_\tau) = u^* U_\tau(s)\,, \quad s = u^* \frac{x_{\vec{n}}}{\nu}, \tag{1.29}$$

with $U_\tau(s)$ an intrinsic function of the quantity s. With phenomenological arguments this function is almost linear for $0 < s < 20$ and given

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by a Prandtl–Von Karman wall law of the form

$$U_\tau(s) = \kappa \log s + \beta \quad \text{for} \quad 20 < s < 100. \tag{1.30}$$

However, either with (1.28), which implies that

$$\nu \partial_{\vec{n}} (u_\tau)_{|\partial\Omega} \geq \alpha > 0$$

or with (1.29), which implies that

$$\nu \int_{\{x\in\Omega \backslash d(x,\partial\Omega)<\nu^{\frac{1}{2}}\}} |\nabla u_\nu(x,,t)|^2 \,\mathrm{d}x \geq \varepsilon > 0 \tag{1.31}$$

we observe that the existence of such a boundary layer is consistent with the fact that $\overline{u_\nu}$ does not converge to the smooth solution u. This is necessary for the appearance of a turbulent wake.

1.3.4 Energy limit and d'Alembert paradox.

Eventually the avatar

$$\lim_{\nu\to 0} \frac{1}{\nu} \int_0^T \int_{\{x\in\Omega | d(x,\partial\Omega)<\nu\}} |(u_\nu(x,t))_\tau|^2 \,\mathrm{d}x \,\mathrm{d}t = 0 \tag{1.32}$$

of the Kato criteria represents the asymptotic (for $\nu \to 0$) amount energy exerted on the boundary of the vessel (or on the obstacle, such as an airfoil for instance) by the flow. This may contribute to the explanation of the d'Alembert paradox.

In a modern setting such a paradox would concern a fluid defined in a domain $\Omega \subset \mathbb{R}^3$ which is the complement of a bounded obstacle K with a divergence-free velocity u constant at infinity.

$$\lim_{|x|\to\infty} u(x) = (u_\infty, 0, 0). \tag{1.33}$$

If the evolution of the fluid is described by the Euler equations, with initial data $u_0(x)$, satisfying condition (1.33) and $\nabla \wedge u_0 = 0$, then the vector field u would remain irrotational and satisfy the boundary condition

$$u(x,t)\cdot \vec{n} = 0 \quad \text{on} \quad \partial\Omega.$$

Therefore u would be a potential flow $u = \nabla\varphi$. And from the relation

$$(u\cdot\nabla u)_i = \sum_{1\leq j\leq d} u_j \partial_{x_j} u_i = \sum_{1\leq j\leq d} \partial_{x_j}\varphi \partial_{x_j}\partial_{x_i}\varphi = \frac{1}{2}\partial_{x_i}|\nabla\varphi|^2 \tag{1.34}$$

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we deduce the equation

$$u \cdot \nabla u - \nabla(\frac{|u|^2}{2}) = 0, \tag{1.35}$$

which implies that $u(x,t)$ is a stationary solution $u(x,t) = U(x)$ satisfying:

$$\nabla \cdot U = \Delta\varphi = 0 \quad \text{in} \quad \Omega\,. \tag{1.36}$$

The force exerted on the obstacle $K = \mathbb{R}^3 \backslash \partial\Omega$ can be computed using Green's formula and we have (see Marchioro & Pulvirenti, 1994 page 54):

$$\begin{aligned} F &= \int_{\partial\Omega} p\vec{n}\,\mathrm{d}\sigma = \int_{\partial\Omega} (p\vec{n} + (\vec{n}\cdot U)U)\,\mathrm{d}\sigma \\ &= \int_{\Omega\cap\{|x|<R\}} (\nabla p + U\cdot\nabla U)\,\mathrm{d}x - \int_{|x|=R} (\frac{\vec{x}}{|x|}p + (\frac{\vec{x}}{|x|}\cdot U)U)\,\mathrm{d}\sigma \\ &= -\lim_{R\to\infty} \int_{|x|=R} (\frac{\vec{x}}{|x|}p + (\frac{\vec{x}}{|x|}\cdot U)U)\,\mathrm{d}\sigma = 0, \end{aligned} \tag{1.37}$$

where we have taken R large enough so that K is contained in $|x| < R$. This fact leads to the following conclusion.

If such stationary flows are described by the Euler equations birds and planes cannot glide! To resolve the paradox one observes that the flow around the obstacle cannot be a regular solution of the Euler equation but on the other hand may well be the zero-viscosity limit of a flow described by the Navier–Stokes equations with no slip boundary condition. In such case the non-zero limit of the right-hand side of (1.32) would represent the amount of energy needed to sustain the flight. This also indicates that the case where the Kato criterion is satisfied seems to be the exception rather than the general rule.

1.4 Kato's criterion, anomalous energy dissipation, and turbulence

Statistical theory of turbulence is based on a notion of an average $\langle . \rangle$. It may denote the ensemble average, average of different quantities of the same fluid at some point, time average, spatial average, etc. It involves in particular two objects:

- the mean energy dissipation denoted by $\varepsilon(\nu) = \nu\langle|\nabla u_\nu|^2\rangle$, and

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- the mean flow velocity increment denoted by $\langle |u_\nu(x+r) - u_\nu(x)|\rangle$.

With the idea that the notion of weak convergence is a deterministic counterpart of the notion of average, as used in the statistical theory of turbulence, among other things to justify the Prandtl–Von Karman law mentioned above it is interesting to compare what is presently known for the relation between anomalous energy dissipation and loss of regularity.

Under the universality assumption about homogeneous isotropic turbulence, these quantities are related by the Kolmogorov 1/3 law:

$$\left\langle \frac{|u_\nu(x+r) - u_\nu(x)|}{|r|^{\frac{1}{3}}} \right\rangle \simeq \varepsilon^{\frac{1}{3}} = \left(\nu \langle |\nabla u_\nu|^2 \rangle \right)^{\frac{1}{3}}. \tag{1.38}$$

One of the starting observations, object of the Kato criterion, is the fact that if a weak limit $\overline{u_\nu}$ of a sequence of solutions of the Navier–Stokes equation is of constant energy, that is,

$$\forall t \in [0,T] \quad \int_\Omega |\overline{u_\nu}(x,t)|^2 \, \mathrm{d}x = \int_\Omega |\overline{u_\nu}(0,t)|^2 \, \mathrm{d}x, \tag{1.39}$$

then there is no anomalous energy dissipation, i.e.

$$\lim_{\nu \to 0} \int_0^T \int_\Omega |\nabla u_\nu(x,t)|^2 \, \mathrm{d}x \, \mathrm{d}t = 0. \tag{1.40}$$

In particular, in passing to the limit, some regularity implies the absence of anomalous energy dissipation. On the other hand, to prove that (1.40) implies the convergence to a smooth solution of the Euler equations Kato uses, in an essential way, an extra hypothesis: the existence of such a smooth solution. To the best of our knowledge the Kato criterion provides the only deterministic (without involving any statistical theory of turbulence) configuration where there is a complete equivalence between regularity and absence of anomalous energy dissipation. In particular, with simple examples (cf. Bardos & Titi, 2010), one can show the existence of solutions $u_\nu(x,t)$ of Navier–Stokes which converge weakly in $L^\infty(0,T;L2(\Omega))$ to a non regular solution $\overline{u_\nu}$ of the Euler equation that conserves energy and hence without anomalous energy dissipation.

Then, from the Kolmogorov relation (1.38), one may infer that solutions of the Euler equations which belong to $L^3((0,T);C^{0,\alpha}(\Omega))$ with $\alpha > \frac{1}{3}$ are of constant total energy. This is the so called Onsager conjecture. Mathematical proofs of this conjecture were given by Eyink (1994) and by Constantin, E, & Titi (1994).

The pertinence of such a threshold was confirmed by the recent conclusion by Buckmaster et al. (2018a) of the program initiated by C. De Lellis

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and L. Székelyhidi about convex integration and wild solutions. In Buckmaster et al. (2018a) the following fact is proved: given any $\beta < 1/3$, a time interval $[0,T]$ and any smooth energy profile $e : [0,T] \mapsto (0,\infty)$, there exists a weak solution $v \in C^\beta([0,T]\times\mathbb{T}^3)$ of the three-dimensional Euler equations with

$$e(t) = \int_{\mathbb{T}^3} |v(x,t)|^2 \, \mathrm{d}x \, .$$

Extension of some of these results to domains with boundary have also been recently obtained. The conservation of energy for solutions in $L^3((0,T); W^{\alpha,3}(\Omega)$ with $\alpha > \frac{1}{3}$ has been proven by Robinson, Rodrigo, & Skipper (2018) when $\Omega = \mathbb{T}^2 \times \mathbb{R}^+$ using a symmetry argument. The case of a general domain has been treated by Bardos & Titi (2018) and by Bardos, Titi, & Wiedemann (2018) where first one proves the validity of the local energy conservation:

$$\partial_t \frac{|u|^2}{2} + \nabla_x \cdot \left(\left(\frac{|u|^2}{2} + p \right) u \right) = 0 \quad \text{in } \mathcal{D}'(0,T)\times\Omega) \tag{1.41}$$

under a "local" $C^{0,\alpha}$ (with $\alpha > \frac{1}{3}$) hypothesis, using an approach inspired by a paper of Duchon & Robert (2000) and then extending the property up to the boundary.

In contrast to what is done in Robinson et al. (2018) for the symmetric case, some weak hypotheses on the pressure are requested. This can be related to the following facts:

1 In the absence of boundary ($\Omega = \mathbb{R}^3$ or $\Omega = \mathbb{T}^3$) the presence of a smooth solution u means that any admissible weak solution $\widetilde{u}$ that is a solution in the sense of distributions and that satisfies the relation

$$\forall 0 < t < T \ \int_\Omega \frac{|\widetilde{u}(x,t)|^2}{2} \, \mathrm{d}x \le \int_\Omega \frac{|u(x,0)|^2}{2} \, \mathrm{d}x \quad \text{with } \widetilde{u}(x,0) = u(x,0) \tag{1.42}$$

coincides with $u(x,t)$ for $t \in (0,T)$ as proven in De Lellis & Székelyhidi (2010).

2 However in the presence of boundary effects the above hypothesis are not enough to show the coincidence of $\widetilde{u}$ and u and some extra hypothesis on $\widetilde{u}$ and the corresponding pressure $\widetilde{p}$ are compulsory (cf. Bardos, Székelyhidi, & Wieidemann, 2014). With these "natural hypothesis" one obtains (cf. Theorem 5.1. in Bardos et al., 2014) sufficient conditions for the convergence to a smooth solution of the Euler equation and for the absence of anomalous energy dissipation. It should be emphasized that the above conditions are compatible with the appearance

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of a parabolic boundary layer in the tangent component of the velocity in the neighborhood of the boundary, as described by the Prandtl ansatz.

As a final remark, observe that the above results concerning the energy conservation and the zero viscosity limit can be extended (under some convenient hypothesis and basically before the formation of shocks) to the solutions of the "compressible" Navier–Stokes and Euler equations, see for instance Carrillo, Feireisl, & Gwiazda (2017), Feireisl et al. (2107), and Bardos & Toan (2016).

References

Asano, A. (1991) Zero-viscosity limit of the incompressible Navier–Stokes equations: *Conference at the IVth Workshop on Mathematical Aspects of Fluid and Plasma dynamics*, Kyoto.

Bardos, C., Székelyhidi, L., Jr., & Wieidemann, E. (2014) On the absence of uniqueness for the Euler equations: the effect of the boundary. (Russian) *Uspekhi Mat. Nauk* **69**, no. 2(416), 3–22; translation in *Russian Math. Surveys* **69**, no. 2, 189–207.

Bardos, C. & Titi, E. (2010) Loss of smoothness and energy conserving rough weak solutions for the 3d Euler equations. *Discrete Contin. Dyn. Syst. Ser. S* **3** no. 2, 185–197.

Bardos, C. & Titi, E. (2013) Mathematics and turbulence: where do we stand? *Journal of Turbulence* **14**, no. 3, 42–76.

Bardos, C. & Titi, E. (2018) Onsager's Conjecture for the Incompressible Euler Equations in Bounded Domains. *Arch. Rational Mech. Anal.*, **228**, no. 1, 191–207.

Bardos C., Titi, E., & Wiedemann, E. (2012) The vanishing viscosity as a selection principle for the Euler equations: the case of 3D shear flow.*C. R. Math. Acad. Sci. Paris* **350**, no. 15-16, 757–760.

Bardos C., Titi, E., & Wiedemann, E. (2018) Onsager's Conjecture with Physical Boundaries and an Application to the Viscosity Limit. *Submitted*, arXiv:1803.04939.

Bardos, C. & Toan, N. (2016) Remarks on the inviscid limit for the compressible flows *Recent Advances in Partial Differential Equations and Applications* Contemporary Mathematics **666** (Edited by Radulescu, D., Sequeira, A., & Solonnikov, V.A.) 66–76.

Buckmaster, T., De Lellis. C, & Székelyhidi Jr, L. (2016) Dissipative Euler flows with Onsager-critical spatial regularity. *Comm. Pure and Applied Math.* **69**, no. 9, 1613–1670.

Buckmaster, T., De Lellis, C., Székelyhidi Jr, L., & Vicol, V. (2018) Onssger's conjecture for admissible weak solutions. Preprint arXiv:1701.08678.

https://doi.org/10.1017/9781108610575.002 Published online by Cambridge University Press

Carrillo, J., Feireisl, E. Gwiazda, P., & Świerczewska-Gwiazda, A. (2017) A Weak solutions for Euler systems with non-local interactions. *J. Lond. Math. Soc* (2) **95**, no. 3, 705–724.

Cheskidov, A., Constantin, P., Friedlander, S., & Shvydkoy, R. (2008) Energy conservation and Onsager's conjecture for the Euler equations. *Nonlinearity* **21**, no. 6, 1233–1252.

Constantin, P. (2005) Euler equations, Navier–Stokes equations and turbulenc, in *Mathematical Foundation of Turbulent Viscous Flows*. Lectures given at the C.I.M.E. Summer School, Martina Franca, Italy. Editors M. Cannone & T. Miyakawa, Springer Lecture Notes in Mathematics **1871**, 1–43.

Constantin, P., E, W., & Titi, E. (1994) Onsager's Conjecture on the Energy Conservation for Solutions of Euler's equation. *Comm. Math. Phys.* **165**, 207–209.

Constantin, P., Elgindi, T., Ignatova, M., & Vicol, V. (2018) Remarks on the inviscid limit for the Navier–Stokes equations for uniformly bounded velocity fields. Preprint arXiv:1512.05674.

Constantin, P., Kukavica, I., & Vicol, V. (2015) On the inviscid limit of the Navier–Stokes equations. *Proc. Amer. Math. Soc.* **143**, no. 7, 3075–3090.

Ding, Y. & Jiang, N. (2018) Zero viscosity and thermal diffusivity limit of the linearized compressible Navier–Stokes-Fourier equations in the half plane. Preprint arXiv:1402.1390.

De Lellis, C. & Székelyhidi, Jr., L. (2009) The Euler equations as a differential inclusion. *Ann. of Math. (2)* **170**, no. 3, 1417–1436.

De Lellis, C., & Székelyhidi, Jr., L. (2010) On admissibility criteria for weak solutions of the Euler equations. *Arch. Ration. Mech. Anal.* **195** no. 1, 225–260.

Duchon, J. & Robert, R. (2000) Inertial energy dissipation for weak solutions of incompressible Euler and Navier–Stokes equations. *Nonlinearity* **13**, 249–255.

E, W. & Engquist, B. (1997) Blowup of solutions of the unsteady Prandtl's equation. *Comm. Pure Appl. Math.* **50** , no. 12, 1287–1293.

Eyink, G.L. (1994) Energy dissipation without viscosity in ideal hydrodynamics, I. Fourier analysis and local energy transfer. *Phys. D* **78**, 222–240.

Feireisl, E., Gwiazda, P., Świerczewska-Gwiazda, A., & Wiedemann, E. (2017) Regularity and energy conservation for the compressible Euler equations, *Arch. Ration. Mech. Anal.* **223**, 1375–139

Frisch, U. (1995) *Turbulence*. Cambridge University Press, Cambridge.

Gerard-Varet, D. & Masmoudi, N. (2015) Well-posedness for the Prandtl system without analyticity or monotonicity. *Ann. Sci. Ec. Norm. Supér.* **(4)** 48, no. 6, 1273–1325

Kato, T, (1972) Nonstationary flows of viscous and ideal fluids in $\mathbb{R}^3$. *J. Functional Analysis* **9**, 296–305.

Kato, T. (1984) Remarks on zero viscosity limit for nonstationary Navier–Stokes flows. with boundary, in *Seminar on Nonlinear Partial Differential Equations*, (Berkeley, Calif., 1983), Math. Sci. Res. Inst. Publl., **2**, Springer, New York, 85–98.

https://doi.org/10.1017/9781108610575.002 Published online by Cambridge University Press

Kukavica, I. & Vicol, V. (2013) On the local existence of analytic solutions to the Prandtl boundary layer equations. *Communications in Mathematical Sciences* **11**, no. 1, 269–292.

Landau, L.D. & Lifshitz, E.M. (1959) *Fluid Mechanics*. Pergamon Press, New York.

Marchioro, C. & Pulvirenti, M. (1994) *Mathematical Theory of Incompressible Nonviscous Fluids*. Applied Mathematical Sciences **96**, Springer–Verlag, New York.

Mohammadi, B., Pironneau , O. (1994) *Analysis of the K-Epsilon Turbulence Model*. Wiley–Masson, Paris.

Onsager, L. (1949) Statistical Hydrodynamics. *Nuovo Cimento* **6** 2, 279–287.

Prandtl, L. (1904) Uber flussigkeits-bewegung bei sehr kleiner reibung, in Actes du 3me Congrès international des Mathematiciens, Heidelberg. Teubner, Leipzig, 484–491.

Robinson, J.C., Rodrigo, J.L., & Skipper J.W.D. (2018) Energy conservation in the 3D Euler equations on $\mathrm{T}^2 \times \mathbb{R}^+$, in *Partial Differential Equations in Fluid Mechanics*. Eds. C.L. Fefferman, J.C. Robinson, & J.L. Rodrigo. Cambridge University Press.

Sammartino, M. & Caflisch, R. (1998) Zero viscosity limit for analytic solutions, of the Navier–Stokes equation on a half-space. I. Existence for Euler and Prandtl equations. *Comm. Math. Phys.* **192**, no. 2, 433–461.

Scheffer, V. (1993) An inviscid flow with compact support in space-time. *J. of Geom. Anal.* **3** , 343–401.

Shnirelman, A. (1997) On the nonuniqueness of weak solutions of the Euler equations. *Comm. Pure Appl. Math.* **50**, 1261–1286.

Stewartson, K. (1974) Multistructured boundary layer on flat plates and related bodies. *Advances Appl. Mech.* **14**, no. 1, 45–239.

https://doi.org/10.1017/9781108610575.002 Published online by Cambridge University Press