ANZIAM J. 42(2001), 494–503

BILATERAL ESTIMATES OF THE CRITICAL MACH NUMBER FOR SOME CLASSES OF CARRYING WING PROFILES

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(Received 5 May 1994; revised 14 November 2000)

Abstract

A problem of estimation of the critical Mach number for a class of carrying wing profiles with a fixed theoretical angle of attack is considered. The Chaplygin gas model is used to calculate the velocity field of the flow. The original problem is reduced to a special minimax problem. A solution is constructed for an extended class of flows including multivalent ones, hence M^* is estimated from above. For a fixed interval $[0, \beta_0]$, $\beta_0 \cong 3\pi/8$, an estimate of M^* is given from below.

1. Introduction

An important problem in the theory of gas flow around a body with given shape is to determine the range of Mach number M_∞ of the free stream in order that the flow be subsonic everywhere. The upper bound M^* of the range is called the critical Mach number and serves as a parameter by which aerodynamical characteristics of transonic wing profiles are evaluated.

The critical Mach number is a functional of the profile shape. Estimating M^* for various classes of profiles and determining configurations for which the maximum values of M^* are attained is not a simple problem. This problem was solved for some classes of symmetric profiles with zero lift in [4, 6, 8, 9] (see also [3]) by Gilbarg and Shiffman, and Loewner in 1954, by Brutyan and Lyapunov in 1981 and by Kraĭko in 1987. Moreover, in 1992 Aul'chenko [1] used a method of numerical design for some carrying profiles with increased critical Mach number. An analytical method to estimate M^* under isoperimetrical constraints was proposed in [2]. Thus, estimates of M^* for carrying profiles are of actual importance.

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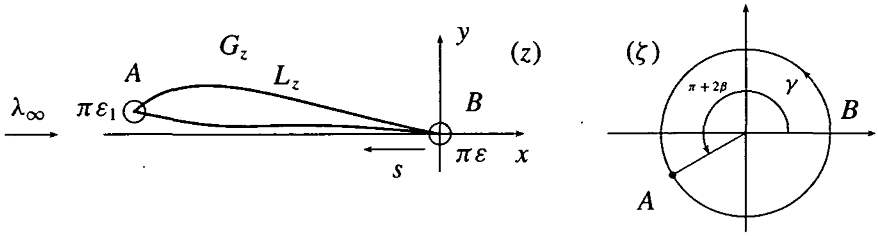

FIGURE 1. Formulation of the problem for an isolated airfoil.

2. Statement of the problem and description of principal results

We consider the isentropic potential flow of an ideal gas with free stream velocity λ_∞ and theoretical angle of attack β. All velocities are referred to the critical velocity v_*.

In this flow we consider an airfoil with closed boundary L_z. This contour is assumed to be smooth except for the sharp edges A and B (see Figure 1). The edges are the leading and trailing critical points of the flow to provide a finite maximum of velocity. The exterior angles at the points A and B are equal to $\pi\varepsilon_1$ and $\pi\varepsilon$ respectively, $1 \leq \varepsilon, \varepsilon_1 \leq 2$. We choose B to be the origin of coordinates. By fixing $\beta \in [0, \pi/2]$ we obtain a class of airfoils.

PROBLEM. For the described class of airfoils it is required to determine a value of the free stream velocity $\lambda^*(\beta)$ such that

(a) for $\lambda_\infty < \lambda^*(\beta)$ there exist airfoils with subsonic velocities;
(b) for $\lambda_\infty > \lambda^*(\beta)$ there exist no airfoils with subsonic velocities.

Obtaining an exact solution of the problem for an isentropic flow is difficult. To simplify the problem we use a model of subsonic gas flow developed by Chaplygin (see for example [3]). We make use of Chaplygin's gas model to guarantee a satisfactory approximation in the whole subsonic region. By Stepanov [10] we find that the relative error of the dependence of the density ρ on the Mach number is larger than the error of an approximate dependence ρ on the relative velocity λ. Therefore, we use Chaplygin's approximation only to determine the velocity λ, and then we calculate M

$$M = \sqrt{\frac{2}{\kappa+1}\lambda} : \sqrt{1 - \frac{\kappa-1}{\kappa+1}\lambda^2}, \tag{1}$$

where κ is the isentropic exponent. In this case $\rho(\lambda) = (1+4c^2\lambda^2)^{-1/2}$, where c^2 is the positive constant chosen on the condition that in Chaplygin's gas model the adiabatic curve has the best linear approximation. In particular, in [10] it was proposed that $c^2 = 0.296$. The choice $c^2 = 0$ corresponds to incompressible fluid ($\rho = \text{const}$).

Combining Chaplygin's gas model with formula (1) means that maximization of M^* is equivalent to maximization of λ_∞ on the condition that $\lambda \le 1$ everywhere in the flow (see [2, 5]).

It is well known [5] that in Chaplygin's model of gas flow the flow region is an image of the region $\{\zeta : |\zeta| > 1\}$ under a quasiconformal mapping by a function $z = z(\zeta)$, which satisfies Beltrami's equation

$$z_{\bar\zeta} + \mu(\zeta) z_\zeta = 0,$$

where

$$\mu(\zeta) = c^2 \overline{h(\zeta)} \exp\left[\overline{\chi(\zeta)} + \chi(\zeta)\right] / h(\zeta),$$

the transition to the z-plane being realized by the formula

$$dz = u_0\{h(\zeta)\exp[-\chi(\zeta)]\,d\zeta - c^2 \overline{h(\zeta)\exp[\chi(\zeta)]\,d\zeta}\}. \tag{2}$$

Here

$$h(\zeta) = \exp(-i\beta)\left[1 - e^{2i\beta}/\zeta^2 + (e^{2i\beta} - 1)/\zeta\right],$$

$$\chi(\zeta) = -\frac{1}{2\pi}\int_0^{2\pi} S(\gamma)\frac{e^{i\gamma} + \zeta}{e^{i\gamma} - \zeta}\,d\gamma, \qquad |\zeta| > 1, \tag{3}$$

and $S(\gamma)$ is an integrable function. Set

$$A_0(c) = \ln\left\{\left[(1 + 4c^2)^{1/2} - 1\right]/(2c^2)\right\}.$$

The constant u_0 in (2) sets a linear scale and does not influence the solution of the problem. It can be uniquely defined, for example, by giving the perimeter L of the profile contour. We set

$$\lambda_\infty = v_\infty/v_*, \qquad \lambda^*_\infty(\beta) = v^*_\infty(\beta)/v_*,$$
$$\lambda^{(k)}_\infty(\beta) = \exp(T_k)/(1 - c^2\exp(2T_k)), \quad k = 1, 2, \tag{4}$$

where T_k is the root of the equation

$$T - A_0(c) + k\sin\beta\frac{1 - c^2\exp(2T)}{1 + c^2\exp(2T)} = 0, \quad k = 1, 2. \tag{5}$$

Our principal result is a proof of the inequalities

$$\lambda^{(2)}_\infty(\beta) \le \lambda^*_\infty(\beta) \le \lambda^{(1)}_\infty(\beta), \tag{6}$$

the right-hand side inequality being proved for each $\beta \in [0, \pi/2]$, and the one on the left-hand side being proved for each $\beta \in [0, \beta_0]$, $\beta_0 \cong 3\pi/8$. The left-hand side inequality is verified by examples which solve a certain auxiliary problem. Equality in the right-hand side case is attained only for multivalent profiles. Thus for real profiles we have $\lambda^*_\infty(\beta) < \lambda^{(1)}_\infty(\beta)$. It means that if $\lambda_\infty \geq \lambda^{(1)}_\infty(\beta)$, then for each profile for which gas flow is in accordance with Chaplygin's model with angle of attack β, there exist parts of the profile contour on which $\lambda > 1$.

From (1) and (6) it follows that both upper and lower estimates for the critical Mach number obey

$$M^{(2)}(\beta) \leq M^*(\beta) \leq M^{(2)}(\beta), \tag{7}$$

where for $k = 1, 2$, $M^{(k)}(\beta)$ is determined by the equality

$$M^{(k)}(\beta) = \sqrt{\frac{2\exp(T_k)[1 - c^2\exp(2T_k)]}{(\kappa+1)[1 - c^2\exp(2T_k)]^2 - (\kappa-1)\exp(2T_k)}}. \tag{8}$$

Notice that we may state another corollary of (6) for the case of an ideal incompressible fluid. Taking $c = 0$ we obtain $A_0(0) = 0$ and $T_k = -k\sin\beta$. We denote by $v^*_{\max}$ the minimum of all $v_{\max}$ over the class of profiles with given v_∞ and β. Then from (6) it follows that

$$\sin\beta \leq \ln(v^*_{\max}/v_\infty) \leq 2\sin\beta. \tag{9}$$

In particular, for any profile in the path of the flow of an ideal incompressible fluid with angle of attack β, we have

$$v_{\max}/v_\infty \geq \exp(\sin\beta). \tag{10}$$

Thus to prove that the estimates (7), (9) and (10) hold it suffices to prove (6).

3. An outline of the proof of (6) and comments on the figures

By (2) and (3), for each 2π-periodic function $S(\gamma) \in L_1[0, 2\pi]$ we have some profile (which may be multivalent) if $S(\gamma)$ satisfies the condition [5]

$$\int_0^{2\pi} S(\gamma)e^{i\gamma}\,d\gamma = 2\pi i e^{i\beta} A(T, \beta), \tag{11}$$

where

$$A(T, \beta) = \sin\beta(1 - c^2\exp(2T))/(1 + c^2\exp(2T)).$$

The condition (11) provides the closeness of the profile contour. The condition $\lambda = v/v_* \leq 1$ is equivalent to the inequality

$$S(\gamma) \leq A_0(c), \quad 0 \leq \gamma \leq 2\pi. \tag{12}$$

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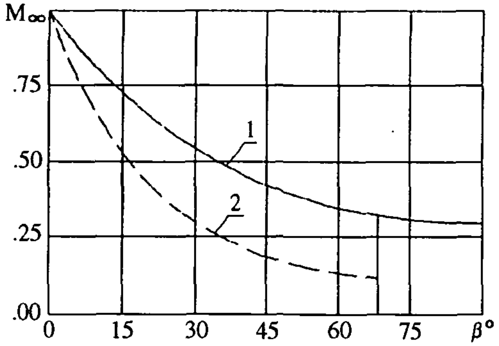

FIGURE 2. Dependences $M = M^{(j)}(\beta)$ for $\kappa = 1.4$.

Next we have $\lambda_\infty = \exp(T)/(1 - c^2 \exp(2T))$, where

$$T = \frac{1}{2\pi} \int_0^{2\pi} S(\gamma)\, d\gamma. \tag{13}$$

Hence maximization of λ_∞ is equivalent to maximization of the functional T for 2π-periodic functions $S(\gamma) \in L_1[0, 2\pi]$ under the restrictions (11), (12).

Using subordinate functions and Lindelöf's principle (see, for example, [7]), we can show that for a fixed T

$$\inf \sup_{|\zeta|>1} \Re[\chi(\zeta) - \chi(\infty)] = A(T, \beta), \tag{14}$$

where infimum is taken for all functions $S(\gamma) \in L_1[0, 2\pi]$ which satisfy (11) and (12) ($\Re$ denotes real part). Since

$$S(\gamma) = T + \Re[\chi(e^{i\gamma}) - \chi(\infty)],$$

the restriction (12) and the equality (14) imply the inequality

$$T - A_0(c) + A(T, \beta) \le 0. \tag{15}$$

As the left-hand part of (15) is monotonic with respect to T, the maximum value of T, satisfying (15), is obtained as the solution of (5) for $k = 1$ (taking into account that (5) has a unique root for any $\beta \in [0, \pi/2]$). Thus $T_1 = \max T$. Hence, by virtue of the given relation between λ_∞ and T, the right-hand inequality in (6) follows. The formula (8) determines $M^{(1)}(\beta)$ by T and the right-hand inequality in (7). In the limit case for $\beta = 0$ we have $T_1 = A_0(c)$, $M^{(1)}(0) = 1$, so we have a symmetric flow around a plate. In the general case the graph of $M^{(1)} = M^{(1)}(\beta)$ for $\kappa = 1.4$ may be seen, labelled as line 1, in Figure 2. Notice that $M^{(1)}(\pi/2) = 0.298$.

To estimate λ_∞^* and M^* from below, that is, to prove that the left-hand inequalities of (6) and (7) hold, it suffices to take these values for some flow which satisfies

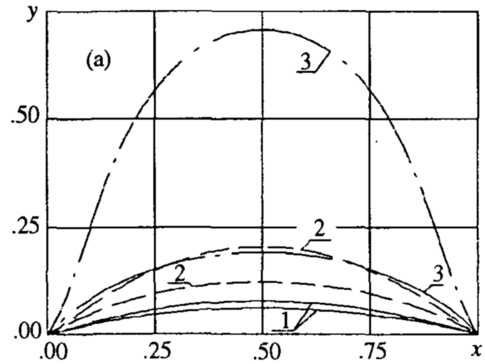

FIGURE 3. Contours of univalent profiles for $\beta = 10°, 20°, 40°$.

the condition $\lambda \leq 1$. We shall construct such a flow as a solution of a certain new variational problem. We define a characteristic of the deviation of the flow from the non-perturbed flow as

$$B[\lambda_\infty, \beta, \lambda, \theta] = \sup\left[\ln^2(\Lambda/\Lambda_\infty) + (\theta_\infty - \theta)^2\right]^{1/2} = \sup\left|\ln\left[\Lambda e^{-i\theta}/(\Lambda_\infty e^{-i\theta_\infty})\right]\right|.$$

Here $\lambda e^{i\theta}$ is a relative velocity vector of Chaplygin's gas flow,

$$\Lambda = \left[(1 + 4c^2\lambda^2)^{1/2} - 1\right]/(2c^2\lambda)$$

is a generalized modulus of the relative velocity and the supremum is taken over all points of the flow region around a single profile with the same angle of attack $\beta \in [0, \pi/2]$. We wish to minimize

$$B[\lambda_\infty, \beta, \lambda, \theta] \tag{16}$$

for given λ_∞ and β. We denote this minimum by $B[\lambda_\infty, \beta]$. By (2) and (3), $\chi(\zeta) = \ln(\Lambda e^{-i\theta})$, and by Lindelöf's principle we have

$$B[\lambda_\infty, \beta] = \min \sup_{|\zeta|>1} |\chi(\zeta) - \chi(\infty)| = 2A(T, \beta),$$

the minimum being attained for the function $\chi(\zeta) = a(T, \beta)/\zeta + \chi(\infty)$, where $a(T, \beta) = 2ie^{i\beta}A(T, \beta)$, $T = \chi(\infty)$ and $\theta_\infty = 0$. Therefore by (2) we can restore a profile which is the minimum of (16). For this profile the condition $\max \lambda = 1$, that is, the condition $\max S(\gamma) = A_0(c)$, implies (5) for $k = 2$, where $T_2 < T_1$. Then we can determine $M^{(2)}(\beta)$ by (8).

The graph of $M^{(2)}(\beta)$ for $\kappa = 1.4$ is shown by the line labelled 2 in Figure 2, where $M^{(2)}(\beta_0) = 0.11$, $\beta_0 \cong 3\pi/8$. The values obtained for $\lambda_\infty^{(2)}(\beta)$ and $M^{(2)}(\beta)$ give

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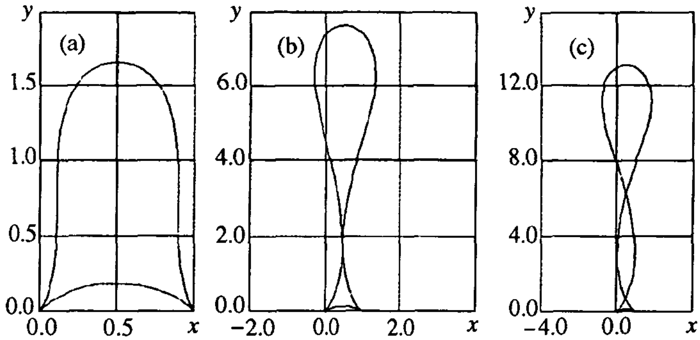

FIGURE 4. Contours of univalent profiles for (a) $\beta = 60°$; (b) $\beta = \beta_0 \cong 3\pi/8$; and contour of non-univalent profile for (c) $\beta = 72°$.

the left-hand side estimates in (6) and (7) if and only if the corresponding profiles are univalent. The calculations show that the functions $\chi(\zeta) = a(T,\beta)/\zeta + T$ for $0 \le T \le T_2$ correspond to univalent flow regions only for $0 \le \beta \le \beta_0 < \pi/2$. The corresponding contours for $T = T_2(\beta)$ and for several values of β are presented in Figures 3 and 5. The contours in Figure 3 correspond to $\beta = 2\pi/9$ (40°) (line 3), $\beta = \pi/9$ (20°) (line 2) and $\beta = \pi/18$ (10°) (line 1), respectively. As β decreases, the contours tend to a plate. If β increases, then at first two points of inflexion appear (for $\beta = \pi/3$ the shape of the contour is shown in Figure 4 (a)), next there appears the self-intersection point on the upper surface of the profile (in Figure 4 (b) $\beta = \beta_0$), and further the flow region becomes multivalent (the contour in Figure 4 (c) corresponds to $\beta = 72°$). In addition the lower surface of the profile becomes straight and again tends to a plate. The authors are not aware of the value of $M^{(2)}(\beta)$ for $\beta > \beta_0$.

4. Generalization to the case of a straight uniserial profiles cascade

Let a cascade of airfoils be disposed along the ordinates axis with a given step t, $t > 0$ (Figure 5). We denote the flow velocities at infinity in front of the cascade and behind the cascade as $\lambda_1 \exp(i\theta_1)$ and $\lambda_2 \exp(i\theta_2)$, respectively. Without loss of generality we suppose $\theta_1 = 0$. From the continuity equation and the condition of flow potentiality it follows that

$$\lambda_2 = \lambda_1\sqrt{1+4d^2}, \qquad \theta_2 = \arcsin\frac{2d\rho(\lambda_1)}{\sqrt{1+4d^2}},$$

where $d = \Gamma/[2t\lambda\rho_1(\lambda_1)]$, $\rho(\lambda_1) = (1+4c^2\lambda^2)^{-1/2}$ and Γ is the velocity circulation.

It is known that the flow domain around the cascade is an image of the infinite Riemann surface R_ζ, and also that the projection of the bounds of R_ζ coincides with

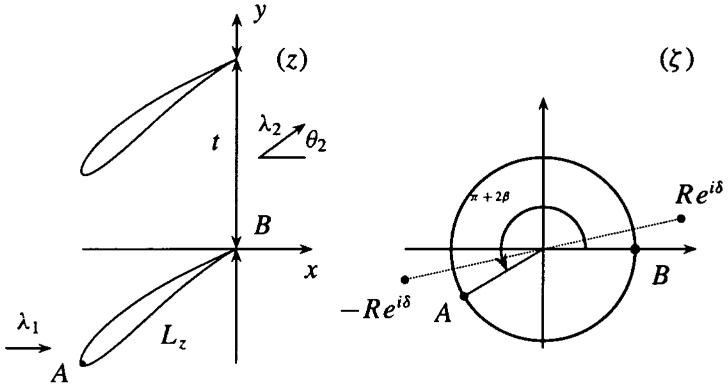

FIGURE 5. Formulation of the problem for an airfoil cascade.

the unit circle. The sheets of R_ζ are the exterior of the unit circle with two branching points $\zeta = \pm R\exp(i\delta)$. The point $\zeta = 1$ corresponds to the airfoils' trailing edges.

For Chaplygin's gas model the flow domain is obtained from (2), namely

$$h(\zeta) = \exp(-i\beta)(\zeta - \zeta_1)(\zeta - 1)[(\zeta^2/R^2 - 1)(\zeta^2 - R^{-2})],$$
$$\zeta_1 = \exp(i(\pi + 2\beta)), \quad u_0 = \lambda_1\rho(\lambda_1)t(R^2 + 1)/[\pi R^3 \cos(\beta + \delta)].$$

The parameters β and δ depend on R and d:

$$\beta = \arcsin \frac{d(R^2 - R^{-2})}{2\sqrt{(R + R^{-1})^2 + d^2(R - R^{-2})}},$$
$$\delta = \beta + \arctan \frac{d(R - R^{-1})}{R + R^{-1}}.$$

So a cascade of the considered class is determined by the function $S(\gamma)$ and two parameters R and d.

The analogue of the problem (11)–(13) is variational:

$$T \equiv \frac{1}{2\pi}\int_0^{2\pi} \Phi(\gamma)S(\gamma)\,d\gamma \to \max,$$
$$\int_0^{2\pi} \Phi(\gamma)S(\gamma)e^{i\gamma}\,d\gamma = \pi e^{-i\delta}(A_1 - iA_2), \quad S(\gamma) \le A_0(c),$$

where

$$\Phi(\gamma) = (R^4 - 1)/[(R^2 + 1)^2 - 4R^2\cos^2\gamma],$$
$$A_1(T) = \frac{(R^2 + 1)}{2R}\ln\frac{H(\gamma) + \sqrt{1 + 4d^2}}{1 + \sqrt{H^2(T) - 4d^2}},$$

$$A_2(T) = -\frac{(R^2-1)}{2R}\arcsin\frac{2d}{H(T)}, \quad H(T) = \sqrt{(1+4d^2)(1+4c^2g^2(T)},$$

and where the monotone function

$$\lambda_1 = g(T) \equiv \frac{\exp T[(1+\sqrt{1+4d^2}c^2\exp 2T)(\sqrt{1+4d^2}+c^2\exp 2T]^{1/2}}{\sqrt{1+4d^2}(1-c^2\exp 4T)}$$

connects λ_1 and T.

The following statements are proved:

(1) A maximal possible value T is the greatest of the roots of the equation

$$\sqrt{A^2(T)+A^2(T)}/2 + T - A_0 = 0; \tag{17}$$

(2) Uniqueness of the root of (17) is provided by the inequality $R \leq 1 + 2/(d_* - 1)$, where

$$d_* = (1+4c^2)\sqrt{1+4d^2}\Big/\left[\sqrt{(1+4c^2+4d^2)(1+4c^2)} - 8c^2d\right].$$

As $R \to \infty$ ($t \to \infty$) we have

$$\lim_{R\to\infty} d = 0, \quad \lim_{R\to\infty}\Phi(\gamma) = 1, \quad \lim_{R\to\infty}(A_1+A_2) = ie^{i\beta}A(T,\beta).$$

Consequently in the limit case we obtain the problem (11)–(13).

Acknowledgements

We wish to express gratitude to G. Yu. Stepanov for his valuable advice and to D. A. Fokin for performing the numerical calculations.

The research described in this publication was made possible in part by Grants 99-01-00173 and 99-01-00366 from the Russian Foundation of Fundamental Research.

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