ALMOST-DEDEKIND RINGS

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Throughout we assume all rings are commutative with identity. We denote the lattice of ideals of a ring R by $L(R)$, and we denote by $L(R)^*$ the subposet $L(R) - R$.

A classical result of commutative ring theory is the characterization of a Dedekind domain as an integral domain R in which every element of $L(R)^*$ is a product of prime ideals (see Mori **[5]** for a history). This result has been generalized in a number of ways. In particular, rings which are not necessarily domains but which otherwise satisfy the hypotheses (i.e. general ZPI-rings) have been widely studied (see, for example, Gilmer **[3]**), as have rings in which only the principal ideals are assumed to satisfy the hypothesis (i.e. π-rings).

General ZPI-rings and π-rings can both be thought of as "almost Dedekind". In both cases, one gets a representation as the finite direct product of integral domains of the same type (Dedekind domains in the first case, π-domains in the second case) and quotients of discrete (rank one) valuation rings (i.e. special principal ideal rings—or SPIRS as they have come to be called).

Note that ZPI-rings are rings in which every ideal in $L(R)^*$ satisfies the "product of prime ideals" condition, whereas only the principal ideals of a π-ring are assumed to satisfy this condition. This naturally raises consideration of rings in which every ideal of $L(R)^*$ generated by n elements is a product of prime ideals. Any UFD is a π-ring; so a π-ring need not be a general ZPI-ring. In this regard, Levitz **[4, 5]** has obtained the very interesting result that π-rings are the single exception. If every doubly generated ideal in $L(R)^*$ is the product of prime ideals, then every ideal in $L(R)^*$ is.

Butts and Gilmer **[3]** have characterized ZPI-rings in a somewhat different manner. They have shown that ZPI-rings are characterized by the property that every ideal in $L(R)^*$ is a finite intersection of powers of prime ideals.

In this paper, we obtain the analogue of Levitz's theorem for the Butts-Gilmer characterization of general ZPI-rings. That is, we show that, once again, two elements suffice: if R is a ring in which every double generated ideal in $L(R)^*$ is the intersection of powers of prime ideals, then every ideal in $L(R)^*$ is.

For convenience, we will say that a ring R satisfies "Property D" if every doubly generated ideal in $L(R)^*$ is the intersection of powers of prime ideals.

We begin with a simple but useful observation.

LEMMA 1. *Let (R, M) be a quasi-local ring satisfying Property D. If $x, y \in M$ then there are only a finite number of primes minimal over (x, y).*

Proof. (x, y) is the finite intersection of powers of prime ideals, say $(x, y) = \bigcap_{i=1}^{n} P_i^{e_i}$. Then any prime minimal over (x, y) is one of the primes P_i.

We also note the following.

LEMMA 2. *If R satisfies Property D and if S is a multiplicatively closed subset of R, then R_S satisfies Property D.*

Proof. $(a/s_1, b/s)R_S = (a, b)R_S$.

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The following appears in a different form in [**1**]. It reduces the problem for quasi-local rings to two cases.

LEMMA 3. *Let (R, M) be a quasi-local ring satisfying Property D. Then either M is principal or $M = M^2$.*

Proof. Assume $M \neq M^2$. Choose $x \in M - M^2$ and let y be any element of M. Then (x, y^2) is the intersection of prime powers, say $(x, y^2) = \bigcap_{i=1}^{s} P_i^{e_i}$. As $x \notin M^2$, it follows that $(x, y^2) = \bigcap_{i=1}^{s} P_i$. But $y^2 \in P_i$ implies $y \in P_i$; so also $(x, y) = \bigcap_{i=1}^{s} P_i = (x, y^2)$. By Nakayama's lemma, it follows that $y \in (x)$. By the choice of y, it follows that $M = (x)$.

The following addresses the quasi-local case with M principal.

LEMMA 4. *Let (R, M) be a quasi-local ring satisfying Property D. If M is a principal ideal of R, then every nonzero ideal of R is a power of M.*

Proof. If M is a minimal prime of R, then M is the only prime ideal of R. It follows that every nonzero principal ideal of R is a power of M, and hence that every nonzero ideal of R is a power of M.

Hence, assume $M = (x)$ is not a minimal prime ideal of R. If P is any prime ideal of R, then $P = P \cap (x) = (P:x)x = PM$; so $P \subseteq \bigcap_{i=1}^{\infty} M^n$. Set $P_0 = \bigcap_{i=1}^{\infty} M^n$. Note that $M^n = M^{n+1}$ would imply $M^n = 0$ (by Nakayama's lemma), which would, in turn, contradict the assumption that M is not minimal. Hence $M^n \neq M^{n+1}$ for all n. Then $x \in M^r - M^{r+1}$ and $y \in M^s - M^{s+1}$ imply $(x) = M^r$ and $(y) = M^s$, whence $(xy) = (x)(y) = M^{r+s}$. It follows that $xy \notin M^{r+s+1}$, and hence that that P_0 is prime. Note that P_0 contains all prime ideals of R different from M.

If P_0 is a minimal prime ideal of R, then either $P_0 = 0$, in which case R has exactly two prime ideals and both are principal, or else $P_0 \neq 0$ but $P_0^n = 0$ for some $n \geq 2$. In this case, choose $z \in P_0 - P_0^2$. Then (z) is of the form P_0^i, so $(z) = P_0$. Once again R has exactly two prime ideals and both are principal.

Hence if P_0 is a minimal prime of R, then (R, M) is a local Noetherian domain with M a principal ideal. If $A \in L(R)^*$ with $A \subseteq M^n$ and $A \not\subseteq M^{n+1}$, then $A = A \cap M^n = (A:M^n)M^n$ with $(A:M^n) \not\subseteq M$; so $(A:M^n) = R$ and $A = M^n$.

Now, assume P_0 is not a minimal prime. By Lemma 1, the number of minimal primes is finite. Choose $z \in P_0$ outside of all minimal primes of R. Let Q be a prime minimal over (z). Then zR_Q is a power of QR_Q and $zR_Q \neq z^2R_Q$, so $QR_Q \neq Q^2R_Q$. Choose $y \in Q - (Q^2)_Q$. Then (z, y) is the intersection of powers of distinct primes, say $(z, y) = \bigcap_{i=1}^{t} P_i^{e_i}$, with (say) $Q = P_1$. Then $e_1 = 1$ and $(z, y)R_Q = QR_Q = (z^2, y)R_Q$. It follows that $QR_Q = yR_Q$.

We now show that the powers of Q are primary. Hence, assume $rs \in Q^n$ and $s \notin Q$. Then $r \in Q$ and, as above, (r, y^n) is an intersection of powers of distinct primes, $(r, y^n) = \bigcap_{i=1}^{t} P_i^{e_i}$, with (say) $Q = P_1$. Then $(r, y^n)R_Q = Q^{e_1}$. Since also $(rs, y^n)R_Q =$

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$(r, y^n)R_Q = Q^{e_1}R_Q$, it follows that $e_1 = n$, whence $(r, y^n) \subseteq Q^n = P_1^{e_1}$. Hence $r \in Q^n$, and Q^n is Q-primary.

The ideal (y) also is an intersection of powers of distinct primes, say $(y) = \bigcap_{i=1}^{t} Q_i^{e_i}$. Here, it can be assumed that $Q_i \subseteq P_0$, $i = 1, \ldots, t$. For $i = 1, \ldots, t$, let E_i be a prime ideal which is both minimal over (y) and contained in Q_i.

By the above, the powers of E_i are primary for E_i. From $0 = (0 : M)M \subseteq E_i^{e_i}$ and $M \not\subseteq E_i$, it follows that $(0 : M) \subseteq E_i^{e_i}$ for all $i = 1, \ldots, t$. Then $(0 : M) \subseteq Q_i^{e_i}$, $i = 1, \ldots, t$, as well. From this and the assumption that M is principal, we have $MQ^i = Q^i$, $i = 1, \ldots, t$ and $M(y) = M\bigcap_{i=1}^{t} Q_i^{e_1} = \bigcap_{i=1}^{t} MQ_i^{e_i} = \bigcap_{i=1}^{t} Q_i^{e_i} = (y)$. By Nakayama's lemma, $y = 0$, which contradicts the choice of y and completes the proof.

The following addresses the quasi-local case with $M = M^2$.

LEMMA 5. *Let (R, M) be a quasi-local ring satisfying Property D in which $M = M^2$. Then every ideal of R is a power of M.*

Proof. Assume M is not a minimal prime of R. Let Q_i, $i = 1, \ldots, t$, be the minimal primes of R. Choose $x \in M - \bigcup_{i=1}^{t} Q_i$, and let P_i, $i = 1, \ldots, r$, be the primes minimal over (x). If $M \in \{P_i\}$, $i = 1, \ldots, t$, then $t = 1$ and (x) is a power of M. It follows that $(x) = M$, and hence that $M = 0$. Hence, $M \neq P_i$, $i = 1, \ldots, r$. Note that any rank one prime containing x is minimal over (x).

Choose $y \in M - \bigcup_{i=1}^{r} P_i$, so (x, y) is not contained in any rank zero prime or any rank one prime. Let Q be minimal over (x, y). By passing to R_Q, we may assume that Q is maximal and that (x, y) is a power of Q. Since $(x, y) \neq (x, y)^2$, it follows that $QR_Q \neq Q^2R_Q$, and hence (by Lemma 3) that QR_Q is principal in R_Q. By the previous lemma, Q has rank at most one, which contradicts the choice of Q.

Hence M is minimal over 0, and necessarily is the only prime ideal of R. It follows that every principal ideal of R is a power of M, and hence that every ideal of R is a power of M.

We now globalize.

THEOREM 6. *Let R be a ring satisfying Property D. Then R is the finite direct product of special principal ideal rings, fields and (one-dimensional) Dedekind domains.*

Proof. Let $0 = \bigcap_{i=1}^{n} P_i^{e_i}$ be a representation of 0 as an intersection of powers of distinct primes. In this case, it can be assumed that each prime P_i is minimal over 0. By the previous lemmas, each maximal ideal of R contains exactly one minimal ideal of R, so the primes $P_1, \ldots, P_n$ are comaximal. It follows that $R = R_1 \times \cdots \times R_n$, where each R_i is isomorphic to $R/P_i^{e_i}$. Note that each factor R_i inherits Property D from R. By passage to one of the factors, we can assume that R has a unique minimal prime P.

If P is maximal, then R is local, and R is either a field or a special principal ideal ring. Hence, assume P is not maximal. By the previous lemma, it follows for each

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maximal ideal M that $M \neq M^2$ and that R_M is a DVR. Hence R is an integral domain and every nonzero prime ideal of R is maximal. It follows from Property D that every doubly generated ideal of R is the product of prime ideals.

The result now follows from Levitz's theorem.

THEOREM 7. *Let R be a ring. Then R is a general ZPI-ring if, and only if, every doubly generated ideal in $L(R)^*$ is the finite intersection of prime powers.*

Proof. This is immediate from Theorem 6.

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