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ON DEFINABILITY OF NONMEASURABLE SETS

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In [**1**], Solovay constructed a model of ZFC in which every set of reals in $OD(\mathbf{R})$ is Lebesgue measurable. Here we construct a model in which every equivalence class of sets of reals modulo null sets that is in $OD(\mathbf{R})$ consists of Lebesgue measurable sets. This result immediately implies Solovay's, since the equivalence class of any set of reals in $OD(\mathbf{R})$ is itself $OD(\mathbf{R})$. As a consequence, one cannot provably explicitly define a non-measurable set modulo null sets within ZFC. We do not know whether this holds in the model Solovay uses (where an inaccessible cardinal is collapsed to ω_1). Instead, our model is a generic extension of his model.

In the model we construct, a somewhat stronger statement holds: every set in $OD(\mathbf{R})$ of sets of reals which has $< 2^c$ inequivalent elements modulo null sets, consists entirely of Lebesgue measurable sets of reals. Also just as in Solovay's work, everything works just as well for category. We therefore present a general result which encompasses these extensions.

Let I be a family of sets of reals closed under finite unions. Two sets of reals are called *I-equivalent* if their symmetric difference is included in an element of I.

THEOREM. *Let M be a countable transitive model of $ZF + DC + V = L(\mathbf{R})$, and let α be any ordinal in M. Then there is a generic extension N of M such that a) the reals in N are the same as the reals in M b) choice, $c = \omega_1$ and $\alpha < 2^c$ hold in N c) let $I \in N$, $I \subseteq M$, be a family of sets of reals closed under finite unions, and let K be a family of sets of reals which has $< 2^c$ I-inequivalent elements, where K is in the $OD(\mathbf{R})$ of N. Then every element of K is I-equivalent to some set of reals in M.*

We begin the proof by considering, for each cardinal κ, the notion of forcing $\mathscr{P}_\kappa$ whose conditions consist of countable partial functions from κ into 2, under inclusion.

LEMMA 1. *It can be proved in $ZF + (\exists x)(V = L[x])$ that there are arbitrarily large cardinals κ with* $\mathrm{cf}(\kappa) > \omega_1$ *such that there is no set of pairwise incompatible conditions in $\mathscr{P}_\kappa$ of power κ.*

Proof. Let M be a countable model of $ZF + (\exists x)(V = L[x])$. Then M has a generic extension M^* which satisfies $V = L[a]$ for some $a \subset \omega$, and hence choice and GCH. Therefore in M^*, if $\mathrm{cf}(\kappa) > \omega_1$ and κ is a cardinal,

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then there is no set of pairwise incompatible conditions in $\mathscr{P}_\kappa$ of power κ. It is obvious that these properties of κ hold also in M. So we have established the conclusion of the theorem in all countable models of $ZF + (\exists x)(V = L[x])$, and so we are done by the downward Skolem-Lowenheim theorem.

We now fix M to be a countable transitive model of $ZF + V = L[\mathbf{R}] + DC$, and let κ be as in Lemma 1. We force over M with $\mathscr{P}_\kappa$. We view the generic object as $\mathbf{f} : \kappa \rightarrow 2$. We write $N = M[\mathbf{f}]$.

LEMMA 2. *In N, choice and CH hold. In addition, $\mathbf{R}^M = \mathbf{R}^N$, $\omega_1{}^M = \omega_1{}^N$, and κ remains a cardinal of cofinality $> \omega_1$.*

Proof. $\mathbf{R}^M = \mathbf{R}^N$ follows from DC in M. Hence $\omega_1{}^M = \omega_1{}^N$. Obviously, every $g : \omega \rightarrow 2$ in M is of the form $(\lambda n)(\mathbf{f}(\lambda + n))$, for some ordinal $\lambda \leqq \omega_1{}^M$. Hence $N = L[\mathbf{f}]$, and so N satisfies choice and CH. Also $\operatorname{cf}(\kappa)$ remains greater than ω_1 in N since $\operatorname{cf}(\kappa) > \omega_1$ in M and there is no set of pairwise incompatible conditions of power κ in M.

LEMMA 3. *In N, $2^{\omega_1} = 2^c = \kappa$.*

Proof. In N, for each ordinal $\alpha < \kappa$, consider the function

$$(\lambda\beta < \omega_1)(\mathbf{f}(\alpha \cdot \omega_1 + \beta)).$$

These functions must all be different, and hence in N, $2^{\omega_1} \geqq \kappa$. On the other hand, since $\operatorname{cf}(\kappa) > \omega_1$, we see that every subset of ω_1 is in $M[\mathbf{f} \restriction \lambda]$, for some ordinal $\lambda < \kappa$. Now each $M[\mathbf{f} \restriction \lambda]$ can have at most $\max(\omega_2, \operatorname{card}(\lambda))$ subsets of ω_1. Hence $2^{\omega_1} \leqq \kappa$.

LEMMA 4. *For every forcing term t there is an ordinal $\lambda < \kappa$ such that for every condition p and real number x in M, $p \Vdash x \in t$ if and only if $p \restriction \lambda \Vdash x \in t$.*

Proof. In N, we can construct an $A \subset \kappa$ of power ω_1 such that for conditions p with $\operatorname{Dom}(p) \subset A$ and reals x, if some extension of p forces $x \notin t$ then some extension of p with domain $\subset A$ forces $x \notin t$. It is clear that $p \Vdash x \in t$ if and only if $p \restriction A \Vdash x \in t$. Hence choose $\lambda < \kappa$ to be such that $A \subset \lambda$.

For ordinals λ, $\alpha < \kappa$ and conditions p, we let $p_{\lambda,\alpha}$ be the condition given by

$$p_{\lambda,\alpha}((\lambda \cdot \alpha) + \beta) \simeq p(\beta), \text{ for } \beta < \lambda;$$

$$p_{\lambda,\alpha}(\beta) \simeq p((\lambda \cdot \alpha) + \beta), \text{ for } \beta < \lambda;$$

$$p_{\lambda,\alpha}(\gamma) \simeq p(\gamma) \text{ for } \gamma \notin [0, \lambda) \cup [\lambda \cdot \alpha, \lambda \cdot (\alpha + 1)).$$

LEMMA 5. *For every forcing term t, and ordinals λ, $\alpha < \kappa$, there is a term $t_{\lambda,\alpha}$ such that for all forcing statements $\psi(t)$ and conditions p, we have*

$p \Vdash \psi(t)$ *if and only if* $p_{\lambda,\alpha} \Vdash \psi(t_{\lambda,\alpha})$. (*Here it is understood that* $\psi(x)$ *does not mention the generic object* $\mathbf{f}$.)

Proof. The mapping which sends each condition p to $p_{\lambda,\alpha}$ is an automorphism of the conditions, which induces an automorphism of the forcing terms and forcing statements in the standard way.

LEMMA 6. *Let* t *and* λ *be as in Lemma* 4, *and let* $t_{\lambda,\alpha}$ *for* $\alpha < \kappa$ *be as in Lemma* 5. *Then for any condition* p *and real* x *in* M, $p \Vdash x \in t_{\lambda,\alpha}$ *if and only if* $p \restriction [\lambda \cdot \alpha, \lambda \cdot (\alpha + 1)) \Vdash x \in t_{\lambda,\alpha}$.

Proof. We have $p_{\lambda,\alpha} \Vdash x \in t_{\lambda,\alpha}$ if and only if

$$p \Vdash x \in t$$

if and only if

$$p \restriction [0, \lambda) \Vdash x \in t$$

if and only if

$$p \restriction [0, \lambda)_{\lambda,\alpha} \Vdash x \in t_{\lambda,\alpha}$$

if and only if

$$p_{\lambda,\alpha} \restriction [\lambda \cdot \alpha, \lambda \cdot (\alpha + 1)) \Vdash x \in t_{\lambda,\alpha}.$$

Since any condition is of the form $p_{\lambda,\alpha}$, we are done.

Towards the proof of the Theorem, we now let $I \in N$, $I \subset M$ be a family of sets of reals closed under finite unions, and let K be a family of sets of reals which has $< 2^c$ I-inequivalent elements, where K is in the $OD(\mathbf{R})$ of N. Let t be any forcing term such that $N \models t \in K$. We will assume that $N \models$ "t is not I-equivalent to any set of reals in M", and obtain a contradiction.

Let p be any condition such that $p \subset \mathbf{f}$, $p \Vdash t \in K$, and p forces that t is not I-equivalent to any set of reals in M. Let $\lambda < \kappa$ be such that $\mathrm{Dom}(p) \subset \lambda$ and for every condition p^* and real number x in M, $p^* \Vdash x \in t$ if and only if $p^* \restriction \lambda \Vdash x \in t$.

LEMMA 7. *In* N, $\{\alpha < \kappa : p_{\lambda,\alpha} \subset \mathbf{f}\}$ *has power* κ.

Proof. It is enough to show that for each $\beta < \kappa$ there is an $\alpha \in [\omega_1 \cdot \beta, \omega_1 \cdot (\beta + 1))$ such that $p_{\lambda,\alpha} \subset \mathbf{f}$. This is obvious by the genericity of $\mathbf{f}$.

LEMMA 8. *There are ordinals* $0 < \alpha < \beta < \kappa$ *and a condition* $q \subset \mathbf{f}$ *such that* $p_{\lambda,\alpha} \cup p_{\lambda,\beta} \subset q$, *and for some* $A \in I$,

$$q \Vdash (\forall x)(x \in \mathbf{R} - A \rightarrow (x \in t_{\lambda,\alpha} \leftrightarrow x \in t_{\lambda,\beta})).$$

Proof. By Lemma 7, choose $0 < \alpha < \beta < \kappa$ such that $p_{\lambda,\alpha} \cup p_{\lambda,\beta} \subset \mathbf{f}$

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and such that in N, $t_{\lambda,\alpha}$ and $t_{\lambda,\beta}$ are I-equivalent. Let $A \in I$ be such that

$$N \models (\forall x)(x \in \mathbf{R} - A \rightarrow (x \in t_{\lambda,\alpha} \leftrightarrow x \in t_{\lambda,\beta})).$$

Let $r \subset \mathbf{f}$ force

$$(\forall x)(x \in \mathbf{R} - A \rightarrow (x \in t_{\lambda,\alpha} \leftrightarrow x \in t_{\lambda,\beta})).$$

Take $q = r \cup p_{\lambda,\alpha} \cup p_{\lambda,\beta}$.

We now fix the ordinals α, β, the set A, and the condition q of Lemma 8.

LEMMA 9. *For all reals $x \notin A$, $q \Vdash x \in t_{\lambda,\alpha}$ or $q \Vdash x \notin t_{\lambda,\alpha}$.*

Proof. Suppose $q \subset q_1$, $q \subset q_2$, $q_1 \Vdash x \in t_{\lambda,\alpha}$, and $q_2 \Vdash x \notin t_{\lambda,\alpha}$. We can assume without loss of generality that

$$q_1 \restriction [\lambda \cdot \beta, \lambda \cdot (\beta + 1)) = q_2 \restriction [\lambda \cdot \beta, \lambda \cdot (\beta + 1)) = q \restriction [\lambda \cdot \beta, \lambda \cdot (\beta + 1)),$$

by Lemma 6. We now have $q_1 \Vdash x \in t_{\lambda,\beta}$, $q_2 \Vdash x \notin t_{\lambda,\alpha}$. But this contradicts Lemma 6.

LEMMA 10. *q forces that $t_{\lambda,\alpha}$ is I-equivalent to an element of M.*

Proof. It is clear that q forces that t is I-equivalent to

$$\{x \in \mathbf{R} : q \Vdash x \in t_{\lambda,\alpha}\}.$$

LEMMA 11. *$q_{\lambda,\alpha}$ forces that t is I-equivalent to an element of M.*

Proof. This follows from Lemmas 5 and 10.

We now have our desired contradiction, since $p \Vdash$ "t is not I-equivalent to any element of M", and $p \subset q_{\lambda,\alpha}$. This concludes the proof of the Theorem.

COROLLARY. *Assume that there is a model of ZFC in which there exists an inaccessible cardinal. Then there is a model of ZFC in which every equivalence class of sets of reals modulo null sets (meager sets) that is in $OD(\mathbf{R})$, consists of Lebesgue measurable sets (sets with the property of Baire). More generally, this is true of any family of sets of reals which has $< 2^c$ equivalence classes represented.*

Proof. Immediate from the Theorem since from [**1**], there is a model of $ZF + DC + V = L(\mathbf{R})$ in which all sets of reals are Lebesgue measurable (have the property of Baire), and if a set of reals is measurable (has the Baire property), in M, it remains so in N.

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