'The Cocked Hat'

from P. J. D. Gething

J. E. D. Williams, in a recent note to Forum[1], gives a simple argument to prove the result that the probability of being inside a cocked hat formed by three position lines is 25 percent. No doubt he is right that this result is well known to navigators, in spite of suggestions to the contrary by the BBC Open University. It is certainly known to many involved in radio direction-finding operations, where bearing lines take the place of position lines.

The essence of J. E. D. Williams' argument, which he says he has never seen in print, was given forty years ago by Daniels[2]. Daniels considered polygons of more than three position lines as well as triangles; his general result was that the largest closed polygon formed by n lines is a confidence region of probability P_n (expressed as a fraction) given by

$$P_n = 1 - \frac{n}{2^{(n-1)}}$$

Cocked hats may have their uses, but in this age of computers I cannot help commenting that they are 'old hat'. For DF fixes it has been the standard practice for many years to calculate and report a 90 percent probability region of minimum area, or a rectangular approximation to it, for every fix.

REFERENCES

1 Williams, J. E. D. (1991). The cocked hat. This *Journal* **44**, 269.

2 Daniels, H. E. (1951). The theory of position finding. *Journal of the Royal Statistical Society*, **13B**, 186.

from D. William Swift

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1. INTRODUCTION. I was delighted to read the paper by J. E. D. Williams[1] in the May 1991 issue of the *Journal of Navigation* because it addressed an area which has concerned me for some time. In particular I believe that it is essential that anyone trying to determine their position from bearings should have a good understanding of the level of uncertainty resulting; not to have such an understanding may be very dangerous. I especially liked the simple approach to the question of determining the probability of being within the 'cocked hat', since not all navigators (of small pleasure yachts for example) are technically well qualified, and it is important for all and not only the technically expert to have the understanding. I should note that this is now somewhat academic to many people because they will obtain their fixes from a box of electronics operating from satellites. What I have to say does not concern them, whether they be professional navigators or wealthy amateurs, but I am sure that there must still be a great many pleasure yachtsmen who rely on a hand-bearing compass. Probably few of them read the *Journal of Navigation* but their instructors in the navigational arts possibly do, and so it is to them that I address this brief contribution.

2. GENERAL POPULATION KNOWLEDGE. My concerns with the Williams paper are that I do not believe that all amateur navigators are aware that their probability of being within the cocked hat is as low as 25 percent (at maximum), or that because of this, the cocked hat is an unsatisfactory estimate of possible position error. In fact, the probability is generally even less than 25 percent and actually reduces to zero if the three bearing lines all pass through the same point. I certainly believe that all professionals know this, and with a little experience the amateur will learn it; but about a decade ago, I got into discussion about the subject at my yacht club and was alarmed to discover that the majority of those present – even the experienced ones – were convinced that the cocked hat was a good measure of the maximum error likely in their fixes. Several were even convinced that the true position was certain to be within the cocked hat and that the only problem was to decide where. Some felt that the centre of the triangle was the best bet and others that the corner nearest to the nearest hazard was better because safer. This last point is valid, but the consensus opinion was that this was completely safe and even over-cautious, which is most certainly *not* valid. I was even more alarmed to learn that they had been taught this by a qualified navigation instructor. I hope that this was just a one-off 'bad egg' but nonetheless it made me worry. A full mathematical treatment of the problem is complex (e.g. Daniels, 1951[2]) and inaccessible to most non-statisticians, but fortunately it is not necessary.

3. IMPROVED PRACTICE. So how can one obtain a better measure of position uncertainty? Fortunately, one has more information available about a bearing than merely that it divides the sea surface into two zones of equal probability, and in particular one knows something about the distribution of probability about the bearing itself. It is not necessary to concern oneself with the nature of this distribution (probably Gaussian) but merely to have some idea of the magnitude. When any bearing is taken it should always have associated with it a standard deviation or a probable error. Since most amateur sailors will not have come across these ideas and will not be interested in learning any mathematics, it is only important that they have some concept of a distribution which has a measure of spread – uncertainty – associated with it. I have called it 'probable accuracy' because I think that that is psychologically more acceptable, but it is not intended to have any precise relationship to standard deviation or probable error. It will vary over a wide range depending upon the instrument used, the sea state, the weather, the person taking the bearing, and the state of that person (well or ill, alert or tired, sober or drunk, etc.), so it is impossible to give 'typical' figures, but anyone taking bearings needs to have an approximate value in mind – and to remember that it is variable. (This will come with experience but in the absence of such experience it may be wise to carry out a few experiments under different conditions.) Then, in principle, extra lines can be drawn on the chart as the bearings $\pm$ the probable accuracy as shown in Fig. 1 (where the magnitude of the error angle is exaggerated for clarity). It is not important in practice whether the lines are intended to have 90, 80, 70 or 60 percent probability of including the true bearings, since they are only estimates anyhow. They do not need to be drawn carefully, or even drawn at all once the idea is clear, as long as it is possible to get a 'feel' for the area common to all three bearing 'spreads'; but this should not be interpreted too literally as the common polygon. Rather, it is just an area of roughly that size and shape. It should never be forgotten that statistical random errors are under consideration and that the figures in use are estimates and not exact laboratory values and so precision would be spurious. If the bearing accuracy lines are estimated to be such that there is a 90 percent chance of being within them, say, then there is a high chance of being within the common area. Notice that most of the problems which occur are automatically dealt with; bearing mark distance

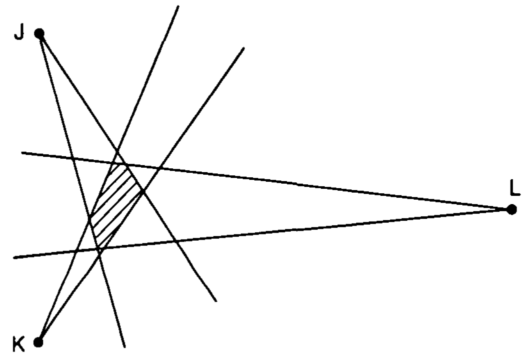

Fig. 1

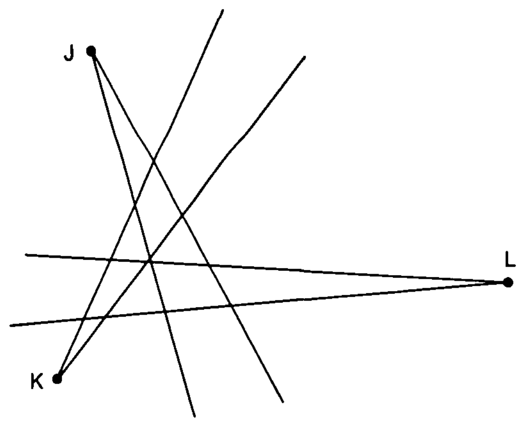

Fig. 2

is appropriately taken into account, for example. If the bearing accuracies have been correctly estimated, the bearing triangles will overlap significantly, as in Fig. 1. If they do not, as in Fig. 2 for example, then either one (or more) of the bearings was in error – i.e. a 'careless' error and not merely a random error – or else the accuracy with which bearings can be measured has been overestimated. As long as the latter is correct, this situation clearly identifies careless errors, but the real point is that the system becomes self-checking. Ideally, the probable accuracy lines should be plotted on the chart along with the bearings, but this would add work and is not really necessary. As long as a sensible probable accuracy angle is sketched in, or even (with practice) just imagined, it will give a much better impression of positional uncertainty and that must be beneficial for safety.

4. HAND-BEARING COMPASS. In the previous section I commented that the bearing accuracy depends upon the instrument used, the sea state, the weather, and the state of the bearing-taker (among other things). It is perhaps worth making a few comments about the first of these; specifically, that is, about the hand-bearing compass. Ideally, optically, such an instrument should satisfy three criteria, namely:

(i) It should present to the user an image of the distant scene with superimposed upon it or immediately against it an image of a compass card which is

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(ii) optically conjugate with the distant scene – i.e. at the same distance so that it is in focus at the same time – and which is

(iii) appropriately magnified to appear in one-to-one relationship with the scene.

Instruments which satisfy these criteria, or at least some of them, are available nowadays, but many of the older instruments do not and are therefore rather unsatisfactory. A hand-bearing compass where one must line up a fiducial line with a distant mark and then read a compass card, all of them at different optical distances from the eye, is particularly difficult and slow to use, and is therefore much less accurate. This is especially true for those of us over fifty with presbyopia; that is, an inability to focus our eyes between near and distant images. Bifocal or progressive spectacles are a great help, but less useful with a hand-bearing compass because it is necessary to change viewing direction to vary the optical power and that takes time, during which the boat and/or the bearing-taker may be lurching from one position to another. Even for a young navigator with good eyesight it takes time to divert the view from one direction to another, and even more time to refocus the eyes from one distance to another; hence the criteria for hand-bearing compasses mentioned above are important for all users and not only for one group. Bear this in mind if you are in the market for such a compass.

5. CONCLUSIONS. The cocked hat is not a satisfactory measure of the uncertainty of a positional fix obtained from bearings, despite the fact that it is sometimes regarded as such by amateur navigators. A much better measure is obtained with very little extra trouble if sensible angular uncertainties are plotted or just sketched in when the bearings are plotted.

REFERENCES

[1] Williams, J. E. D. (1991). The cocked hat. This *Journal*, **44**, 269.

[2] Daniels, H. E. (1951). The theory of position finding. *Journal of the Royal Statistical Society*, Series B, **13**, 186–207.

KEY WORDS

1. Errors. 2. Statistics.

The Author Replies

I am grateful to P. J. D. Gething for introducing me to the paper by Dr Daniels, which I should have read. It was noted (by J. B. Parker) in this *Journal* shortly after it appeared (see Volume **5**, 249) and several times subsequently. The formula was quoted (by M. G. Pearson) in this *Journal* **36**, 80. As the 25 percent probability of being within the cocked hat has been mentioned many times in the *Journal*, and at least once in *Navigation News* (to say nothing of the *Admiralty Manual of Navigation*), navigators should know of it. The opposite was true 50 years ago. The History of Air Navigation Study Group seeks material on the beginnings of the scientific approach to position estimation during the second world war (to which Daniels contributed). Any help would be welcomed.

I must disappoint Lincoln Lee (see forum section of previous issue of the *Journal*, **44**, 433). My argument and the more general elegant result of Daniels both depend on the Principle of Indifference. If all we know is either A or B and not both, A and B each have a probability of 0·5. It is a statement of ignorance and my ignorance precludes me from answering him except to say that I made no assumptions about the shape of the triangle. If we know something about error distribution, the probabilities are different. If the