THE COMPLETION OF A LATTICE ORDERED GROUP

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(Received 8 May 1967)

1. Introduction

A lattice ordered group ('l-group') is called *complete* if each set of elements that is bounded above has a least upper bound (and dually). A complete l-group is archimedean and hence abelian, and each archimedean l-group has a completion in the sense of the following theorem.

THEOREM 1.1. *If G is an archimedean l-group, then there exists a complete l-group $G^\wedge$ with the following properties*

(1) *G is an l-subgroup of $G^\wedge$, and*

(2) *if $h \in G^\wedge$, then $h = \bigvee\{g \in G | g \leqq h\}$.*

Moreover, if H is any complete l-group with these properties, then there exists a unique l-isomorphism σ of $G^\wedge$ onto H such that $g\sigma = g$ for all $g \in G$.

Note that for each $h \in G^\wedge$, $-h = \bigvee\{g \in G | g \leqq -h\}$ so that

(2′) $h = \bigwedge\{g \in G | g \geqq h\}$

also holds.

The complete l-group $G^\wedge$ described in this theorem is the Dedekind-MacNeille 'completion by cuts' of G (see [4] or [12] for details of its construction). Although Theorem 1.1 gives a more abstract characterisation of $G^\wedge$ it does not enable one to compute the completion of an archimedean l-group.

In Theorem 2.4 we give an abstract characterisation of $G^\wedge$ which makes no mention of infinite suprema or infima, and using this result we can precisely describe the completion of an archimedean l-group with a basis. As a further application of Theorem 2.4 we show that the completion of an l-group of real valued functions is an l-group of real valued functions over the same domain (Theorem 3.2) and give a reasonably decent description of this completion (Theorem 3.3).

We show (Example VI) that if G is an l-subgroup of a complete l-group H then H need not contain a completion of G. In fact the completion of the free abelian l-group G of rank $\geqq 2$ is a complete vector lattice, but G is a

[1] This research was supported by a grant from the National Science Foundation.

https://doi.org/10.1017/S1446788700005760 Published online by Cambridge University Press

subdirect sum of integers. We give necessary and sufficient conditions on a subdirect sum of integers in order that its completion should be a subdirect sum of integers. The concepts of singular elements and minimal primes are crucial to the examination of a subdirect sum of integers. In Section 4 we develop the theory of singular elements for an arbitrary l-group G. For example the subgroup of G that is generated by the set S of singular elements is an abelian l-ideal of G and is a subdirect sum of integers. If each strictly positive element of G exceeds a singular element, then G is a subdirect sum of totally ordered groups ('o-groups') if and only if S is in the center of G. Any value of a singular element is a minimal prime. In Section 5 we derive methods for finding minimal primes.

In the final section we give an example of a complete l-group that is a subdirect sum of discrete o-groups, but not a subdirect sum of integers. Thus we have a counter example to a theorem of Iwasawa [13] that asserts that a complete l-group is the cardinal sum of a vector lattice and a subdirect sum of integers.

Throughout this paper let Z denote the group of integers under the natural order and let R denote the corresponding additive group of reals. If A and B are non-empty subsets of a set, then $A||B$ means $A \nsubseteq B$ and $B \nsubseteq A$ and $A \setminus B = \{a \in A | a \notin B\}$. If $\{G_\lambda : \lambda \in \Lambda\}$ is a set of l-groups, then $\Pi G_\lambda (\Sigma G_\lambda)$ will denote the large (small) cardinal sum of the G_λ.

2. The completion of an l-group

Throughout this section G denotes an l-group. Let X be a subset of G then the polar of X (in G) is the set

$$X' = \{g \in G | |g| \wedge |x| = 0 \text{ for all } x \in X\};$$

X' is a convex l-subgroup of G. Every cardinal summand of G is a polar and conversely, by a theorem of Riesz, in a complete l-group each polar is a cardinal summand. A convex l-subgroup C of G is *closed* (in G) if, whenever $\{c_\lambda | \lambda \in \Lambda\} \subseteqq C$ and $g = \bigvee_G \{c_\lambda | \lambda \in \Lambda\}$ exists, $g \in C$. Each polar in G is closed. (For proofs of these results, see [4] or [12]).

Let G be a subgroup of an l-group H. Then G is *dense* in H if, for each $0 < h \in H$, there exists $g \in G$ such that $0 < g \leqq h$. From Theorem 1.1, any archimedean l-group G is dense in its completion.

LEMMA 2.1. *Let G be a convex l-subgroup of an l-group H. Then the following conditions on G are equivalent*

(1) *G is dense in H;*
(2) *$G' = 0$;*
(3) *$G'' = H$.*

Proof. Clearly (1) implies (2) and (2) and (3) are equivalent. Let $0 < h \in H$ and suppose that (3) holds. Then there exists $0 < a \in G$ such that $g = a \wedge h > 0$; by the convexity of G, $g \in G$. Hence G is dense in H.

Corollary. *Let K be a convex l-subgroup of an l-group G. Then K is dense in K''.*

Proof. K is a convex l-subgroup of K'' and no strictly positive element of K'' is orthogonal from K. Hence by (2) K is dense in K''.

Lemma 2.2. (Bernau) (1) *If G is a dense l-subgroup of an l-group H then all joins and intersections in G agree with those in H.*

(2) *If S is a dense subgroup of an archimedean l-group G and $0 < g \in G$ then $g = \bigvee \{x \in S | 0 < x \leqq g\}$.*

(For a proof of (1) see [3], page 116, for a proof of (2), see [2], page 604).

Corollary. *A closed l-ideal K of an archimedean l-group G is a polar.*

Proof. By the corollary to Lemma 2.1, K is dense in K''. Hence, if $0 < g \in K''$, $g = \bigvee_{K''}\{k \in K | 0 < k \leqq g\}$. But, since K'' is an l-ideal of G, $\bigvee_{K''}\{k \in K | 0 < k \leqq g\} = \bigvee_G\{k \in K | 0 < k \leqq g\}$. Hence, since K is closed, $g \in K$. Thus $K = K''$.

A. Bigard has informed us that he has also proved this corollary and Johnson and Kist [16] have obtained the result under the additional assumption that G is a vector lattice.

Lemma 2.3. *Let G be a dense l-subgroup of a complete l-group H.*

(1) *If K is an l-subgroup of H that contains G and is complete then K is an l-ideal of H.*

(2) *The l-ideal K of H generated by G is the completion of G; it is the unique l-subgroup of H that is a completion of G.*

(3) *If J is an l-ideal of G then the completion of J is the l-ideal of H that is generated by J. Thus, if J is complete, it is an l-ideal of H.*

Proof. (1) Let $0 < h < k \in K$; then, by Lemma 2.2,

$$h = \bigvee \{g_i \in G | 0 < g_i \leqq h\}$$

Since $\{g_i | 0 < g_i \leqq h\}$ is bounded above by k and K is dense and complete, it follows from Lemma 2.2(1) that $h \in K$.

(2) $K = \{k \in H | g_1 \leqq k \leqq g_2$ for some $g_1, g_2 \in G\}$ and since K is an l-ideal of H it is complete. If $k \in K \diagdown G$ then $k > g$ for some $g \in G$ and so $k-g = \bigvee \{g_i \in G | g_i \leqq k-g\}$ thus $k = \bigvee \{g_i+g | g_i+g \leqq k\}$. Hence, by Theorem 1.1, K is the completion of G.

Suppose that L is an l-subgroup of H that contains G and is complete.

https://doi.org/10.1017/S1446788700005760 Published online by Cambridge University Press

By (1), L is an l-ideal of H and hence $L \supseteqq K$. If $0 < x \in L \setminus K$ then no element of G exceeds x and thus, by the remarks after Theorem 1.1, L is not the completion of G.

(3) Let J^* denote the polar of J in H. Then, by Riesz's theorem, $H = J^* \oplus J^{**}$. By the corollary to Lemma 2.1, J is dense in J^{**} which, since it is an l-ideal of H, is complete. Thus the completion of J is the l-ideal of J^{**} generated by J; but this is the l-ideal of H generated by J.

From Lemma 2.3, we have the following abstract characterisation of the completion of an archimedean l-group.

THEOREM 2.4. *Let G be an l-subgroup of a complete l-group H. Then the following are equivalent*

(a) *H is the completion of G;*

(b) *If $0 < h \in H$ then $0 < g_1 \leqq h \leqq g_2$ for some $g_1, g_2 \in G$;*

(c) *G is dense in H and no proper l-subgroup of H contains G and is complete.*

PROOF. That (a) implies (b) is an immediate consequence of Theorem 1.1. If (b) holds then G is clearly dense in H and no proper l-ideal of H contains G. Thus, from Lemma 2.3 (1), (c) holds. From Lemma 2.3, it is immediate that (c) implies (a).

COROLLARY. *If $\{G_\lambda : \lambda \in \Lambda\}$ is a set of l-groups, then $(\Pi G_\lambda)\hat{} = \Pi \hat{G}_\lambda$ and $(\Sigma G_\lambda)\hat{} = \Sigma \hat{G}_\lambda$.*

A strictly positive element s of G is called *basic* if the set

$$\{x \in G \mid 0 \leqq x \leqq s\}$$

is totally ordered. A *basis* for G is a maximal (pairwise) disjoint subset $\{s_\lambda \mid \lambda \in \Lambda\}$ of G where, in addition, each s is basic. In [8] it is shown that an archimedean l-group has a basis if and only if there exists an l-isomorphism σ such that $\Sigma R_\lambda \subseteqq G\sigma \subseteqq \Pi R_\lambda$, where the R_λ are subgroups of R. As a further application of Lemma 2.3, we have the following result.

THEOREM 2.5. *Let G be an archimedean l-group with a basis. Then there exists an l-isomorphism σ of G such that $\Sigma R_\lambda \subseteqq G\sigma \subseteqq \Pi R_\lambda$. For each $\lambda \in \Lambda$ let*

$$T_\lambda = \begin{cases} R_\lambda \text{ if } R_\lambda \text{ is cyclic} \\ R \text{ otherwise.} \end{cases}$$

Then ΠR_λ is an l-subgroup of ΠT_λ and the completion of G is the l-ideal of ΠT_λ generated by $G\sigma$.

PROOF. An l-subgroup of R is either cyclic or dense in R ([12] page 45). Hence ΠR_λ is dense in ΠT_λ. But $G\sigma$ is clearly dense in ΠR_λ and thus $G\sigma$

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is dense in ΠT_λ. Since ΠT_λ is the cardinal product of complete O-groups it is itself complete and the result is now immediate from Lemma 2.3.

COROLLARY. *H is a complete l-group with a basis if and only if it is (isomorphic to) an l-ideal of ΠR_λ where $R_\lambda = R$ or Z for each $\lambda \in \Lambda$.*

As an illustration of Theorem 2.5, the completion of all convergent real sequences is all bounded sequences; the latter is also the completion of all eventually constant real sequences.

For an l-group G the radical $R(G)$ is defined as follows. For each non-zero $g \in G$ let L_g be the join of all l-ideals of G which do not contain g. Then

$$R(G) = \bigcap \{L_g \mid 0 \neq g \in G\}.$$

If G is archimedean and B is the set of all basic elements, then ([11] Lemma 5.4)

$$R(G) = \bigcap \{b' \mid b \in B\} = \bigcap \text{ all maximal polars of } G.$$

Thus $R(G)^+ = \{g \in G^+ \mid g$ does not exceed a basic element$\}$; in particular $R(G) = 0$ if and only if G has a basis.

For the remainder of this section we *assume that G is archimedean*. An l-ideal M of G is a *value* of $g \in G$ if it is maximal with respect to not containing g. A strictly positive element $g \in G$ has a unique value M if and only if it is basic and, if this is the case, $M = g'$ and $G = g' \oplus g''$; [9].

A. Bigard has also proved the following proposition.

PROPOSITION 2.6. *$G/R(G)$ is archimedean and $R(G/R(G)) = 0$. Thus Theorem 2.5 describes the completion of $G/R(G)$.*

PROOF. $R(G)$ is a polar and hence $G/R(G)$ is archimedean ([11] Lemma 3.3). If X is a strictly positive element of $G/R(G)$ then $X = R(G)+x$ where $0 < x \in G \setminus R(G)$ and hence $x > b$ for some basic element b. Since $b \notin R(G)$ the value b' of b contains $R(G)$ and clearly $b'/R(G)$ is the only value of $R(G)+b$. Thus $R(G)+b$ is basic and, since $X \geqq R(G)+b$, it follows that $X \notin R(G/R(G))$.

PROPOSITION 2.7. *Let G be a dense l-subgroup of an archimedean l-group H. Then*

(i) $R(G) = R(H) \cap G$;
(ii) *$R(G)$ is dense in $R(H)$;*
(iii) *$R(G) = 0$ if and only if $R(H) = 0$;*
(iv) *$R(G) = G$ if and only if $R(H) = H$.*

PROOF. (i) If s is basic in H then $s \geqq g > 0$ for some $g \in G$ and clearly g is basic in G. Let $x \in G$ and suppose $x \wedge s = h > 0$ for some basic $s \in H$. Then, for any basic $g \in G$ such that $s \geqq g > 0$, $g \wedge h > 0$. Hence $g \wedge x > 0$.

Thus $x \in R(G)$ implies $x \in R(H)$. Conversely if g is basic in G then g is basic in H. Hence $R(H) \cap G \subseteqq R(G)$.

(ii) Let $0 < h \in R(H)$ then $h \geqq g > 0$ for some $g \in G$. Since h exceeds no basic element neither does g. Thus $g \in R(G)$.

(iii) If $R(H) = 0$ then, by (i), $R(G) = 0$; if $R(G) = 0$ then by (ii) $R(H) = 0$.

(iv) If $R(H) = H$ then, by (i), $R(G) = G$; if $R(G) = G$ then no element of G and hence no element of H is basic, thus $R(H) = H$.

In particular, if H is the completion of G, the conclusions of the proposition hold. Unfortunately $R(H)$ need not be the completion of $R(G)$; see Example IV and the following proposition.

PROPOSITION 2.8. Let G be an archimedean l-group and let H be the completion of G. Then $R(H)$ is the completion of $R(G)$ if and only if $G = R(G) \oplus R(G)'$.

PROOF. Suppose that $R(H)$ is the completion of $R(G)$ and let $R(H)^*$ denote the polar of $R(H)$ in H; by Riesz's theorem, $H = R(H) \oplus R(H)^*$. Let $0 < g \in G$, then $g = h_1 + h_2$ where $0 \leqq h_1 \in R(H)$, $0 \leqq h_2 \in R(H)^*$. Since $R(H)$ is the completion of $R(G)$, there exists $g_1 \in R(G)$ such that $g_1 > h_1$. By Proposition 2.7 (i), $g_1 \in R(H)$ hence $g_1 \wedge h_2 = 0$ and thus $h_1 = g \wedge g_1 \in G \cap R(H)$ so $h_1 \in R(G)$. This implies $h_2 \in G \cap R(H)^*$ which, since $R(G) \subseteqq R(H)$, requires that h_2 is disjoint from $R(G)$. Hence $g \in R(G) \oplus R(G)'$.

Conversely, if $G = R(G) \oplus R(G)'$, then $H = H_1 \oplus H_2$ where H_1 is the completion of $R(G)$ and H_2 that of $R(G)'$. Then $R(H) = R(H_1) \oplus R(H_2)$ and, by Proposition 2.7, $R(H_1) = H_1$, $R(H_2) = 0$. Thus $R(H) = H_1$.

G is said to be *laterally complete* if each set of pairwise disjoint elements has a least upper bound. If G is laterally complete then $G = R(G) \oplus R(G)'$ where $R(G)'$ is (isomorphic to) a cardinal sum ΠR_λ of subgroups R_λ of R ([11], Theorem 5.5). Hence the completion of G is the direct sum of the completion of $R(G)$ and that of $R(G)'$. By Theorem 2.5, the completion of ΠR_λ is ΠT_λ where $T_\lambda = R_\lambda$ if R_λ is cyclic and $T_\lambda = R$ otherwise.

3. The completion of a subdirect sum of reals

Let G be an l-subgroup of an l-group H and let $\mathscr{G}(\mathscr{H})$ be the lattice of all convex l-subgroups of $G(H)$. For $M \in \mathscr{G}$, $C \in \mathscr{H}$ define

$$M\sigma = \bigcap \{\text{all convex } l\text{-subgroups of } H \text{ that contain } M\}$$
$$C\tau = C \cap G.$$

It is shown in [10] that $M\sigma\tau = M$ and $C\tau\sigma \subseteqq C$. Thus σ is a 1—1 mapping of $\mathscr{G}$ into $\mathscr{H}$ and τ is a mapping of $\mathscr{H}$ onto $\mathscr{G}$. Moreover σ is an l-isomorphism and σ, σ^{-1} preserve infinite joins. The above results have also been proven by S. Wolfenstein. In [11] it is shown further that if G is dense in H then τ induces a $1-1$ mapping of the polars of H onto the polars of G.

$M \in \mathscr{G}$ is *prime* if $a, b \in G^+ \setminus M$ implies $a \wedge b \in G^+ \setminus M$. Clearly if C is prime in H then $C\tau$ is prime in G. In [9] it is shown that for $M \in \mathscr{G}$ the following are equivalent

(1) M is prime,

(2) if $a \wedge b = 0$ then $a \in M$ or $b \in M$,

(3) the set G/M of cosets $M+a$ is totally ordered where, by definition, $M+a \leqq M+b$ if and only if $a \leqq m+b$ for some $m \in M$.

If G is abelian then (3) implies G/M is an O-group.

$M \in \mathscr{G}$ is *regular* if it is maximal without containing some element $g \in G$. In this case M is also called a *value* of g. Each regular subgroup of G is prime.

LEMMA 3.1. *Let G be an l-subgroup of an l-group H and let M be a value of $0 < g \in G$;*

(1) *there exists a value N of g in H such that $N \supseteqq M\sigma$; for such a value $N \cap G = M$;*

(2) *the mapping: $M+x \overset{\rho_N}{\to} N+x$ for $x \in G$ is a $1-1$ order preserving mapping of G/M into H/N; if H is abelian ρ_N is an l-isomorphism;*

(3) *if M is a maximal convex l-subgroup of G and $0 < h \in H$ implies $h < g$ for some $g \in G$ then N is maximal in H.*

PROOF. (1) Since $M = M\sigma\tau = M\sigma \cap G$, $g \notin M\sigma$ and so, by Zorn's Lemma, there exists a value N of g in H such that $N \supseteqq M\sigma$. Now $M \subseteqq N \cap G$ and the latter is a convex l-subgroup of G without g. Therefore $M = N \cap G$.

(2) $M+x \leqq M+y$ implies $x \leqq m+y$ for some $m \in M \subseteqq N$; thus $N+x \leqq N+y$. If $N+x = N+y$ then $x-y \in N \cap G = M$; hence $M+x = M+y$. Therefore ρ_N is $1-1$ and order preserving. If H is abelian then clearly ρ_N is a group homomorphism and hence, since it is order preserving and $1-1$, it is an l-isomorphism.

(3) For any subset X of H let $H(X)$ denote the convex l-subgroup of H generated by X; thus by hypothesis $H(G) = H$. Let $0 < h \notin N$; then, since N is a value of g, $g \in H(N \cup h) \cap G$. But, since M is a maximal convex l-subgroup of G, this implies $G \subseteqq H(N \cup h)$ and hence $H = H(N \cup h)$. Thus N is maximal in H.

REMARK. If G is archimedean and H is the completion of G, the conclusions of Lemma 3.1 apply. Thus, if M is a maximal l-ideal of G, N is a maximal l-ideal of H. Example II shows that there may be an infinite number of maximal l-ideals N of H such that $N \cap G = M$. However, if M

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is closed, then, by the corollary to Lemma 2.2, M is a maximal polar and hence $M = s'$ for some basic element s. Thus $G = s'' \oplus s'$ and $H = D \oplus N$ where D is the completion of the subgroup s'' of R. Here N is the unique maximal l-ideal of H such that $G \cap N = M$.

Now suppose that G is archimedean and let H be the completion of G. Let $\mathscr{M}$ be a collection of regular l-ideals of G such that $\bigcap \{M | M \in \mathscr{M}\} = 0$. Then the mapping $g \to (-, M+g, -)$ is the natural l-isomorphism of G onto a subdirect sum of the cardinal product $\Pi G/M$. For each $M \in \mathscr{M}$ pick $0 < g \in G$ such that M is a value of g in G and a value N of g in H with $N \supseteqq M\sigma$. Then $(\bigcap N) \cap G = \bigcap (N \cap G) = \bigcap M = 0$ and, since G is dense in H, $\bigcap N = 0$.

By Lemma 3.1, the diagram

$$\begin{tikzcd} G \arrow[r, "\eta_M"] \arrow[d, "\gamma"'] & G/M \arrow[d, "\rho_N"] \\ H \arrow[r, "\eta_N"'] & H/N \end{tikzcd}$$

where γ is the inclusion of G into H and η_M, η_N are the natural l-homomorphisms, commutes for each $M \in \mathscr{M}$. Hence the mapping δ:

$$(-, M+g, -)\delta = (-, N+g, -)$$

is an l-isomorphism of $\Pi G/M$ into $\Pi H/N$ such that the diagram below commutes where α, β are the natural l-isomorphisms of G onto a subdirect sum of $\Pi G/M$, H onto a subdirect sum of $\Pi H/N$.

$$\begin{tikzcd} G \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & \Pi G/M \arrow[d, "\delta"] \\ H \arrow[r, "\beta"'] & \Pi H/N \end{tikzcd}$$

Thus making use of the fact that G is a subdirect sum of subgroups of the reals if and only if it contains a family $\mathscr{M}$ of maximal l-ideals M such that $\bigcap M = 0$, we have the following theorem.

THEOREM 3.2. *If G is a subdirect sum of real groups then so is its completion H. In fact, if θ is an l-isomorphism of G into ΠR_λ, where $R_\lambda = R$ for each $\lambda \in \Lambda$, there exists an l-isomorphism φ of H into ΠR_λ such that the diagram*

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$$\begin{tikzcd} G \arrow[r, "\theta"] \arrow[d, "\gamma"'] & \Pi R_\lambda \\ H \arrow[ur, "\varphi"'] & \end{tikzcd}$$

commutes, where γ is the inclusion of G into H.

The next theorem gives a method of finding the completion of an l-group of real valued functions. [2]

THEOREM 3.3. *Let G be an l-subgroup of ΠR_λ where each $R_\lambda = R$, $\lambda \in \Lambda$, and let $L(G)$ denote the l-ideal of ΠR_λ generated by G. Let M be a maximal l-subgroup of $L(G)$ in which G is dense; then M is a completion of G. Further every completion of G in ΠR_λ is obtained in this way.*

PROOF. By Theorem 3.2, M has a completion $M^\wedge$ in ΠR_λ. Let $0 < a \in M^\wedge$; then there exists $m \in M$ such that $m > a$. Since $M \subseteqq L(G)$ and the latter is convex, this implies $a \in L(G)$; thus $M^\wedge \subseteqq L(G)$. But G is dense in M and M is dense in $M^\wedge$ hence G is dense in $M^\wedge$. Thus, from the maximality of M, $M = M^\wedge$.

Let $0 < m \in M$; then since G is dense in M there exists $g_1 \in G$ such that $0 < g_1 \leqq m$ while, since $M \subseteqq L(G)$, there exists $g_2 \in G$ such that $m < g_2$. Hence, by Theorem 2.4, M is the completion of G.

Conversely, suppose that M is a completion of G in ΠR_λ. Then G is dense in M and, by Theorem 2.4(b), $M \subseteqq L(G)$. By Zorn's Lemma there exists a maximal l-subgroup N of $L(G)$ such that $M \subseteqq N$ and G is dense in N. From the first part of the theorem N is a completion of G and, by Theorem 2.4, it is the unique completion of G contained in N. Thus since M is a completion of G we must have $M = N$ so that M is a maximal l-subgroup of $L(G)$ in which G is dense.

There is another way of describing the completion of G in ΠR_λ. Let M be a maximal l-subgroup of ΠR_λ in which G is dense. Then, as above, M is complete and hence, by Lemma 2.3, the l-ideal of M generated by G is the completion of G.

If in the above theorem ΠR_λ is replaced by a complete l-group H such that each l-subgroup of H, in which G is dense, has a completion in H, then the subgroup M obtained in the theorem is the completion of G. However we shall show in Section 6 that the free abelian l-group of rank > 1 is a subdirect sum of integers but its completion is a real vector lattice.

[2] It follows from [21] that, if $G^\wedge$ is an l-subgroup of ΠR_λ where each $R_\lambda \subseteq R$ and $G^\wedge$ is complete, suprema and infima in $G^\wedge$ are the same as those in ΠR_λ if and only $G^\wedge$ has a basis. Hence in the completion of an l-subgroup G of ΠR_λ suprema and infima are pointwise if and only if G has a basis.

Thus an l-subgroup of a complete l-group H need not have a completion in H.

REMARK. If $0 < g \in G$ and G is archimedean, then the l-ideal of G generated by g,

$$G(g) = \{x \in G \mid |x| < ng \text{ for some } n > 0\}$$

is a subdirect sum of reals and hence Theorems 3.2 and 3.3 apply 'locally' to all archimedean l-groups. These theorems can be applied 'globally' in the following way. For each $a \in G^+$ let $G^\wedge(a)$ be the completion of $G(a)$. For each pair $a, b \in G^+$ with $a \leqq b$, $G(a)$ in an l-ideal of $G(b)$ and hence, by Lemma 2.3 (3) the l-ideal of $G^\wedge(b)$ generated by $G(a)$ is the unique completion of $G(a)$ contained in $G^\wedge(b)$. Thus there is a unique isomorphism π_a^b of $G^\wedge(a)$ into $G^\wedge(b)$ such that the diagram

$$\begin{tikzcd} G(a) \arrow[r] \arrow[d] & G^\wedge(a) \arrow[d, "\pi_a^b"] \\ G(b) \arrow[r] & G^\wedge(b) \end{tikzcd}$$

commutes where the un-named mappings are the natural inclusions. Further one can show that $G^\wedge$ is the direct limit of the $G^\wedge(a)$ under the homomorphisms π_a^b.

4. Singular elements in an l-group

An element s in an l-group G is said to be *singular* if $s > 0$ and, for $a \in G$,

$$0 \leqq a < s \text{ implies } a \wedge (s-a) = 0.$$

This concept was introduced by Iwasawa [13] and was used in his proof that an archimedean l-group is commutative.

The following Lemma was proved by Iwasawa for archimedean l-groups.

LEMMA 4.1. *Let G be an l-group.*

(1) *If $s \geqq a > 0$ and s is singular, then a is singular and a and s commute.*

(2) *If s_1 and s_2 are singular and $0 < x_i \in G(s_i)$, $i = 1, 2$, then x_1 and x_2 commute.*

PROOF. (1) Since $a \wedge (s-a) = 0$, $a+(s-a) = (s-a)+a$ and hence $a+s = s+a$. If $0 < b < a$ then $0 \leqq b \wedge (a-b) \leqq b \wedge (s-b) = 0$ and so a is singular.

https://doi.org/10.1017/S1446788700005760 Published online by Cambridge University Press

(2) There exists $n > 0$ such that $x_i < ns_i$, $i = 1, 2$, and hence, by the interpolation property for l-groups,

$x_i = x_{i1} + \cdots + x_{in}$ where $0 \leqq x_{ij} \leqq s_i$, $i = 1, 2$, $j = 1, \cdots, n$. By (1), each x_{ij} is singular hence it suffices to show that singular elements commute. Let t_1, t_2 be singular elements and let $y = t_1 \wedge t_2$. Then

$$\begin{aligned} t_1+t_2 &= (t_1-y)+y+(t_2-y)+y \\ &= (t_1-y)+(t_2-y)+y+y && \text{since } y \wedge (t_2-y) = 0 \\ &= (t_2-y)+(t_1-y)+y+y && \text{since } (t_1-y) \wedge (t_2-y) = 0 \\ &= (t_2-y)+y+(t_1-y)+y && \text{since } (t_1-y) \wedge y = 0 \\ &= t_2+t_1. \end{aligned}$$

LEMMA 4.2. *Let G be an l-group. Then each l-homomorphism onto an l-group H maps a singular element of G onto a singular element or zero. In particular, if s is singular then so is $-g+s+g$ for all $g \in G$.*

PROOF. Let M be an l-ideal of G and let $s \in G \setminus M$ be singular. If

$$M+s > M+a > M$$

then $s > m+a$ for some $m \in M$ and hence

$$s \geqq y = (m+a) \vee 0.$$

Since s is singular $y \wedge (s-y) = 0$ and thus

$$M = M+y \wedge (M+s-M+y).$$

But $M+y = M+a$ so that

$$M = M+a \wedge (M+s-M+a)$$

which shows that $M+s$ is singular in G/M.

Now let π be an l-homomorphism of G onto H with kernel M. Then $H \cong G/M$ and so, since $M+s$ is singular in G/M, $s\pi$ is singular in H.

THEOREM 4.3. *Let G be an l-group and let S be the set of singular elements of G. Then*

(a) *$\bigvee\{G(s) | s \in S\}$ is an abelian l-ideal of G;*
(b) *$\bigvee\{G(s) | s \in S\}$ is the subgroup $[S]$ of G generated by S;*
(c) *each strictly positive element of $[S]$ exceeds a singular element.*

PROOF. (a) By definition, $\bigvee G(s)$ is the convex l-subgroup of G generated by the $G(s)$ but, c.f. [9] or [18], this is just the subgroup of G generated by the $G(s)$. Thus, by Lemma 4.1, $\bigvee G(s)$ is abelian. By Lemma 4.2, each inner automorphism induces a permutation of S and hence $\bigvee G(s)$ is an l-ideal.

(b) Clearly $[S] \subseteqq \bigvee G(s)$. Let $s \in S$; then it follows from the proof of Lemma 4.1 that each positive element of $G(s)$ is a sum of singular elements.

Further each positive element of $\bigvee G(s)$ is a sum of positive elements of the $G(s)$, $s \in S$. Hence each positive element of $\bigvee G(s)$ is contained in $[S]$, and thus $\bigvee G(s) \subseteqq [S]$.

(c) Since each strictly positive element of $\bigvee G(s)$ is a sum of singular elements, this is obvious.

COROLLARY. *For a l-group G, the following are equivalent*

(a) *each strictly positive element exceeds a singular element*;

(b) $S' = 0$;

(c) $S'' = G$.

PROOF. (a) clearly implies (b) which is equivalent to (c). By Lemma 2.1, (c) implies $[S]$ is dense in G. Thus since each strictly positive element of $[S]$ is a sum of singular elements, (a) is satisfied.

LEMMA 4.4. *Let G be an l-group and suppose that $G = S''$. Let α be an l-isomorphism of G onto a subdirect sum of the cardinal product ΠG_i of o-groups G_i, $i \in I$. Let ρ be the projection of $\Pi_I G_i$ onto $\Pi_{I \setminus K} G_i$ where*

$$K = \{k \in I \mid (s\alpha)_k = 0 \ \text{ for all } \ s \in S\}$$

and let $\beta = \alpha\rho$. Then β is an l-isomorphism of G onto a subdirect sum of $\Pi_{I\setminus K} G_i$. Further, if $s \in S$ and $(s\beta)_i \neq 0$ then $(s\beta)_i$ is the least positive element of G_i. Hence G is a subdirect sum of discrete o-groups.

PROOF. Suppose that $g\beta = 0$ where $g > 0$. By the corollary to Theorem 4.3, $g \geqq s$ where $s \in S$ and, for such an s, $g\beta = 0$ implies $s\beta = 0$. This implies $s\alpha = 0$ and so contradicts the fact that α is $1-1$. Hence β is an l-isomorphism.

Let $s \in S$ and suppose that $(s\beta)_i > (a\beta)_i > 0$ for some $a \in G^+$; then $b = s \wedge a > 0$ and $b < s$. Thus, by the singularity of s, $b \wedge (s-b) = 0$. This implies $(b\beta)_i \wedge (s\beta - b\beta)_i = 0$ which is impossible in the o-group G_i. Hence $(s\beta)_i$ is the least positive element of G_i.

COROLLARY 1. *There is an l-isomorphism β of the l-ideal $[S]$ of G onto a subdirect sum of ΠZ_i, where Z_i is the group of integers for each $i \in I$. Further β can be chosen so that, for each $s \in S$, $i \in I$, $(s\beta)_i = 1$ or 0. In particular, $[S]$ is archimedean.*

PROOF. $[S]$ is abelian and, by the corollary to Theorem 4.3, each strictly positive element is over a singular element. By the lemma, there is an l-isomorphism β of $[S]$ onto a subdirect sum of discrete o-groups G_i, $i \in I$, where, for each $i \in I$, $([S]\beta)_i$ is the subgroup generated by the least positive element of G_i. But this group is isomorphic to Z; hence we have the result.

COROLLARY 2. *If G is a subdirect sum of a cardinal product ΠG_λ of* *o-groups G_λ then $0 < s \in G$ is singular if and only if, for each λ, either $s_\lambda = 0$ or s_λ is the least positive element of G_λ.*

LEMMA 4.5. *Let G be an l-group and let $s \in G$ be singular. Then each value M of s in G is a minimal prime. Thus if N is a prime subgroup of G which is not minimal then $[S] \subseteqq N$.*

PROOF. Suppose that $M \supset Q$ where Q is prime and let $0 < h \in M \diagdown Q$. Since Q is prime, $g = s \wedge h \in M \diagdown Q$. Thus $s > g > 0$ and $Q+s > Q+g > Q$ so that $Q+s-g > Q$. Since s is singular $g \wedge (s-g) = 0$. Therefore

$$Q = Q+g \wedge Q+(s-g) = \min\{Q+g, Q+(s-g)\};$$

but each of $Q+g$, $Q+(s-g)$ is strictly greater than Q so we have a contradiction. Thus M is a minimal prime.

If $s \notin N$ then $N \subseteqq M$ for some value M of s. Since M is a minimal prime, this is impossible. Hence $[S] \subseteqq N$.

Example III shows that if $G = S''$ and M is a minimal prime then it need not be the value of a singular element.

Suppose that G is *representable*; this means that there exists a set $\mathscr{M}$ of l-ideals M of G such that G/M is an o-group and $\bigcap M = 0$. Then the mapping: $g \to (-, M+g, -)$ is a representation of G; equivalently, we say that $\mathscr{M}$ is a representation of G. By Lemma 4.5, each representation $\mathscr{M}$ of G must include a value for each singular element $s \in G$; for each $M \in \mathscr{M}$ is prime and since $\bigcap M = 0$ there exists $M \in \mathscr{M}$ such that $s \notin M$.

Byrd [6] has shown that an l-group G is representable if and only if each minimal prime is normal. Thus, if G is representable, $G = S''$ and $\mathscr{M}$ is the set of all values of singular elements, then $\mathscr{M}$ is a representation and any representation can be refined to a subset of $\mathscr{M}$.

THEOREM 4.6. *Let S be the set of all singular elements of an l-group G and let $H = S''$. Then the following are equivalent.*

(1) *H is representable;*
(2) *S is in the center of H.*

PROOF. $[1 \to 2]$. By Lemma 4.4, we may assume that $H \subseteqq \Pi H_i$ where the H_i are discrete o-groups and $s \in S$ if and only if $s_i = 1$ or 0 where 1 is the least strictly positive element of H_i. If $s \in S$ then clearly $(-h+s+h)_i = s_i$ for each $h \in H$, $i \in I$ so that s is in the center of H.

$[2 \to 1]$. It suffices to show that if M is a value of $s \in S$ then $M \lhd H$. Suppose, on the contrary, that $-h+M+h = N \neq M$. Then, by Lemma 4.5, M and N are minimal primes and hence there exists $0 < g \in N \diagdown M$. Then $0 < k = g \wedge s \in N \diagdown M$ and k is singular and hence in the center of H. But then $k = h+k-h \in h+N-h = M$; a contradiction.

The main consideration of the remainder of this section is finding necessary and sufficient conditions under which the completion of an l-group of integer valued functions is again a group of integer valued functions.

LEMMA 4.7. *Let G be a dense l-subgroup of an archimedean l-group H and let s be a singular element of G. Then s is singular in H.*

PROOF. If $0 < h < s$, $h \in H$ then, by Lemma 2.2,

$$h = \bigvee\{g_i \in G | 0 < g_i \leqq h\}$$

and hence

$$s-h = s-\bigvee g_i = s+\bigwedge -g_i = \bigwedge(s-g_i).$$

For each $j \in I$, $g_j \wedge (s-g_j) = 0$ and hence, since $\bigwedge_I(s-g_i) \leqq s-g_j$,

$$g_j \wedge (\bigwedge_I(s-g_i)) = 0.$$

Thus

$$0 = \bigvee_J(g_j \wedge (s-h)) = (s-h) \wedge (\bigvee_J g_j) = (s-h) \wedge h.$$

Hence s is singular in H.

COROLLARY. *If $S'' = G$ and G is a dense l-subgroup of an archimedean l-group H then both G and H are subdirect sums of discrete o-groups.*

PROOF. Since each singular element of G is also singular in H, each strictly positive element of H is over a singular element. Both G, H are archimedean, hence commutative, so we can apply Theorem 4.6. Thus, by Lemma 4.4, G and H are subdirect sums of discrete o-groups.

The next theorem shows that under certain conditions, which are actually necessary (Theorem 4.9), the completion of a subdirect sum of integers can be obtained, as a subdirect sum of integers, using a method analogous to that in Theorem 3.3.

THEOREM 4.8. *Let G be a subdirect sum of reals and suppose that $G = S''$. Then, without loss of generality, we may assume that $G \subseteqq \Pi Z_i$, where $Z_i = Z$ for each $i \in I$, and*

(a) *for each $i \in I$, there exists $s \in S$ such that $s_i \neq 0$,*
(b) *if $s_i \neq 0$ for $s \in S$ then $s_i = 1$.*

Let $L(G)$ be the l-ideal of ΠZ_i generated by G and let M be a maximal l-subgroup of $L(G)$ in which G is dense. Then M is a completion of G and every completion of G in ΠZ_i is obtained in this way.

PROOF. By Lemma 4.4, we may assume that G is embedded in ΠZ_i in the manner described. Let σ denote the natural embedding of ΠZ_i in ΠR_i where each $R_i = R$. Then σ defines an embedding of G in ΠR_i. By Theorem 3.2, $G \subseteqq H \subseteqq \Pi R_i$ for some completion H of G. Let $0 < h \in H$ then h_i is an integer for each $i \in I$. For if not we can pick $s \in S$ with $s_i = 1$

and, without loss of generality we can assume that $0 < h_i < s_i$ and $h < s$. By Lemma 4.7, s is singular in H hence $h \wedge (s-h) = 0$. But, clearly, $h_i \wedge (s_i - h_i) \neq 0$ so we have a contradiction. Thus $H \subseteqq \Pi Z_i$.

The argument used in the proof of Theorem 3.3 now completes the result.

THEOREM 4.9. *Let G be an archimedean l-group. Then the completion H of G is a subdirect sum of integers if and only if*

(a) *G is a subdirect sum of integers and*
(b) *$G = S''$.*

PROOF. By Theorem 4.8, conditions (a), and (b) are sufficient. Conversely, if H is a subdirect sum of integers, then clearly so is G. Hence it suffices to show that $G = S''$. To do this, we need only show that each strictly positive element h of H exceeds a singular element $s \in H$. For then, by denseness, there exists $g \in G$ such that $s \geqq g > 0$. By Lemma 4.1, g is singular in H and thus in G. Hence each strictly positive element of H, and in particular of G, is over a singular element of G.

By way of contradiction, suppose that $0 < h \in H$ does not exceed a singular element. If $h_i \neq 0$ then, since there exists $a \in H$ with $a_i = 1$, we may assume that $h_i = 1$ and hence there does not exist $k \in H$ with $h = 2k$. Because of this, the following lemma gives an immediate contradiction to our assumption that h does not exceed a singular element.

LEMMA 4.10. (Iwasawa). *Let H be a complete l-group. If $0 < h \in H$ does not exceed a singular element then $h = 2k$ for some $k \in H$.*

PROOF. If h does not exceed a singular element then there exists $0 < u < h$ such that $d = u \wedge (h-u) > 0$. Then $h = h-u+u \geqq 2d > 0$. Let $k = \bigvee\{x \in H | 0 < 2x \leqq h\}$; then $h \geqq \bigvee\{2x | 0 < 2x \leqq h\}$ and, from [2], the latter is equal to $2k$. If $h > 2k > 0$ then $h > h-2k > 0$ and so $h-2k$ exceeds no singular element of H. Thus $h-2k > 2d > 0$ for some $0 < d \in H$. This means $h > 2(k+d) > 2k$ whence $k < k+d = \bigvee\{x+d | 0 < 2x \leqq h\}$. But $2(x+d) \leqq h$ for each x such that $0 < 2x \leqq h$. Hence $k+d \leqq k$ which is impossible.

An entirely similar argument proves that a complete l-group with no singular elements is divisible; hence a real vector lattice [15].

COROLLARY 1. (Iwasawa). *If H is a complete l-group with no singular elements, then H is a real vector lattice.*

COROLLARY 2. *Let G be a complete l-group. Then $G = S' \oplus S''$ where S' is a real vector lattice and S'' is a subdirect sum of discrete o-groups.*

PROOF. By Riesz's theorem, each polar is a cardinal summand and hence $G = S' \oplus S''$. By Lemma 4.4, S'' is a subdirect sum of discrete o-groups. Since S' is complete and has no singular elements, it follows from the last corollary that it is a complete vector lattice.

Iwasawa 'proves' that S'' is a subdirect sum of integers that contains the small sum. Example V however shows this to be false.

Let G be an l-subgroup of the cardinal product ΠZ_i of copies of the integers. Then we call an element $0 < a \in G$ *bounded* if there exists an integer M such that $a_i \leqq M$ for all $i \in I$. If $0 < a \in G$ is bounded, then we write

$$||a|| = \min \{M \in Z | a_i \leqq M \text{ for all } i \in I\}.$$

Thus a is bounded if and only if it is bounded regarded as a function from I into Z.

S. J. Bernau has sent the authors a preprint of a paper in which he gives necessary and sufficient conditions on an l-group G in order that there is an l-isomorphism σ of G onto a subdirect sum of ΠZ_i, where $Z_i = Z$ for each $i \in I$, such that, for each $0 < a \in G$, $a\sigma$ is over a bounded element of $G\sigma$. The following proposition shows that, for subdirect sums of integers, the latter boundedness condition is equivalent to $G = S''$.

PROPOSITION 4.11. *Let $G \subseteqq \Pi Z_i$ be a subdirect sum of the cardinal product ΠZ_i of copies of the integers. Then the following are equivalent*

(a) $G = S''$,

(b) *each strictly positive element of G exceeds a bounded element.*

PROOF. By Theorem 4.3, (a) clearly implies (b). Suppose conversely that (b) holds and (a) is false. Let $0 < a \in G$ exceed no singular element and let a be such that $||a||$ is minimal among such elements. Since a exceeds no singular element, there exists $0 < y < a$ such that $d = y \wedge (a-y) > 0$ and $d, y, a-y$ exceed no singular elements. By the minimality of $||a||$, $||d|| = ||y|| = ||a-y||$ and hence, since $d \leqq y, a-y \leqq a$, there exists $i \in I$, such that $d_i = y_i = (a-y)_i = a_i = ||a||$; this is impossible. Hence each element of G exceeds a singular element.

If G is an l-group and $G = [S]$ then, by the first corollary to Lemma 4.4, G is a subdirect sum of integers and, since each positive element of $[S]$ is a sum of singular elements, each element of G is bounded. However, if $G \subseteqq \Pi Z_i$ and each element of G is bounded it does not follow that $G = [S]$; see Example III.

If G is an archimedean l-group and $G = S''$ then, by the corollary to Theorem 4.3, $[S]$ is dense in G. Hence, by Lemma 1.2, each positive element of G is a join of elements of $[S]$. The last theorem of this section shows that the converse is also true. Note however that the group in Example VI satisfies (a) and (b) of the theorem but it is not a subdirect sum of integers.

THEOREM 4.12. *Let G be an l-group. Then the following conditions on G are equivalent.*

(a) *each $0 < g \in G$ is the join of elements from $[S]^+$;*
(b) *G is archimedean and $G = S''$.*

If either condition is satisfied then G is representable as a subdirect sum of discrete o-groups.

PROOF. We have already pointed out that (b) implies (a); hence it remains to show the converse. Let $s \in S$ and let $0 < g \in G$ where $g = \bigvee s_i$ for $s_i \in [S]^+$. Then

$$s+g = s+\bigvee s_i = \bigvee(s+s_i) = \bigvee(s_i+s) = (\bigvee s_i)+s = g+s.$$

Thus S is in the center of G and hence, by Theorem 4.6, we may assume that G is a subdirect sum of ΠG_λ where, for each λ, G_λ is a discrete o-group. Let C_λ be the smallest non-zero convex subgroup of G_λ. Then $C_\lambda = [c_\lambda]$ where c_λ is the least positive element of G.

Suppose that $0 \leqq na < b$ for $n = 1, 2, \cdots$; $b = (-, b_\lambda, -)$ and $a = (-, a_\lambda, -)$. If $a_\lambda \neq 0$, then $b_\lambda > na_\lambda$ for $n = 1, 2, \cdots$ hence $b_\lambda \notin C_\lambda$ so that b_λ exceeds each element of C_λ. But $b = \bigvee x_i$ where each $x_i \in [S]^+$; in particular, each $x_i \in \Pi C_\lambda$. Then $b_\lambda \geqq b_\lambda - a_\lambda \geqq (x_i)_\lambda$ for each $\lambda \in \Lambda$, $i \in I$, thus $b \geqq b-a \geqq x_i$. Consequently, $b \geqq b-a \geqq \bigvee x_i = b$ which implies $a = 0$. Hence G is archimedean.

5. Minimal prime subgroups

We have already made use of the fact that the minimal prime subgroups of an l-group G determine whether or not G is representable. Further we have seen that each value of a singular element is a minimal prime. To establish some of the properties of Example V, we require a method for constructing minimal primes. The material in this section, especially the connection between minimal primes and filters, is quite similar to that developed by other authors (see, for example, [1], [5], [14]). However most of their results are expressed in terms of filters and not in terms of subgroups.

Let G be an l-group and let P be a prime subgroup of G. Then $K = G^+ \setminus P$ satisfies

(*) $$0 < h \wedge k \in K \text{ for all } h, k \in K;$$

a subset of G^+ which is maximal with respect to property (*) is called an *ultrafilter* on G. The following theorem shows that the minimal prime subgroups of G are completely determined by the ultrafilters on G. The theorem was proved by Johnson and Kist [17] for the case in which G is abelian and is also essentially contained in [1].

THEOREM 5.1. *The mapping* $K \to \bigcup\{k' | k \in K\}$ *is a one to one mapping of the set of ultrafilters on* G *onto the set of minimal prime subgroups of* G*; the inverse mapping is* $\bar{K} \to G^+ \setminus \bar{K}$.

PROOF. For an ultrafilter K, let $K\eta$ denote $\bigcup\{k' | k \in K\}$; for a minimal prime subgroup $\bar{K}$, let $\bar{K}\rho$ denote $G^+ \setminus \bar{K}$.

Let K be an ultrafilter and let $h, k \in K$ then, since $0 < h \wedge k$ and $(h \wedge k)' \supseteqq h' \cup k'$, the set $\{k' | k \in K\}$ is a directed set of convex 1-subgroups of G. Hence $K\eta$ is a convex l-subgroup of G. If $a \wedge b = 0$ where $a > 0$, $b > 0$ then either $a \wedge k > 0$ for all $k \in K$ or $a \in K\eta$. In the first case $K \cup \{a \wedge k | k \in K\} \cup \{a\}$ satisfies (*) and hence, by the maximality of K, $a \in K$; thus $b \in K\eta$. Hence $K\eta$ is prime. Since $K\eta$ is prime, $G^+ \setminus K\eta$ satisfies (*) and it clearly contains K. The maximality of K then gives $K = G^+ \setminus K\eta = K\eta\rho$.

Let $\bar{K}$ be a minimal prime subgroup and suppose that $\bar{K}\rho$ is not maximal with respect to (*). Then there exists K maximal with respect to (*) which properly contains $\bar{K}\rho$. By the first part, $K\eta$ is a minimal prime subgroup of G and, since $\bar{K}\rho = G^+ \setminus \bar{K} \subset K = G^+ \setminus K\eta$, $K\eta \subset \bar{K}$. This contradicts the minimality of $\bar{K}$.

Hence η is a mapping of the set of ultrafilters on G into the set of minimal primes of G and ρ is a mapping of the set of minimal primes of G into the set of ultrafilters on G. Further, for any ultrafilter K on G, $K\eta\rho = K$ and, for any minimal prime subgroup $\bar{K}$ of G, $\bar{K}\rho\eta = \bar{K}$. Thus η and ρ are mutually inverse and we have the result.

NOTATION. If, in what follows, K denotes an ultrafilter on a lattice group G then $\bar{K}$ denotes the minimal prime subgroup $\bigcup\{k' | k \in K\}$ of G produced by Theorem 5.1.

COROLLARY 1. *Let* M *be a prime subgroup of an* l*-group* G *and let* K *be the intersection of all minimal prime subgroups contained in* M*. Then* $K = \bigcup\{a' | a \in G^+ \setminus M\}$*. Thus* K *is prime if and only if* M *exceeds a unique minimal prime subgroup.*

PROOF. Clearly

$$\bigcup\{a' | a \in G^+ \setminus M\} \subseteqq K.$$

Suppose

$$0 < b \notin \bigcup\{a' | a \in G^+ \setminus M\}.$$

Then $a \wedge b > 0$ for all $a \in G^+ \setminus M$ and hence

$$\{b\} \bigcup \{a \wedge b | a \in G^+ \setminus M\} \bigcup G^+ \setminus M$$

obeys (*) and thus is contained in a maximal such subset N. By Theorem 5.1, $\bar{N}$ is a minimal prime subgroup of G and, since $G^+ \setminus M \subseteqq G^+ \setminus \bar{N}$, $\bar{N} \subseteqq M$ and $b \notin \bar{N}$. Therefore $b \notin K$.

COROLLARY 2. *For a prime subgroup N of an l-subgroup G, the following are equivalent.*

(1) *N is a minimal prime subgroup;*
(2) $N = \bigcup \{a' | a \in G^+ \setminus N\}$;
(3) *$G^+ \setminus N$ is an ultrafilter on G.*

That (1) implies (2), in the above corollary, has been shown by Byrd and Lloyd.

We now apply Theorem 5.1 to a cardinal product $G = \Pi\{G_\lambda | \lambda \in \Lambda\}$ of 0-groups. Let $\mathscr{S}$ denote the set of all proper subsets of Λ, including the null set.

COROLLARY 3. *Let $\mathscr{P}$ be a subset of $\mathscr{S}$ that is maximal with respect to*

$$A \cup B \in \mathscr{P} \text{ for all } A, B \in \mathscr{P}.$$

Then $M = \{g \in G |$ support of g belongs to $\mathscr{P}\}$ is a minimal prime subgroup of G and each minimal prime subgroup of G is of this form.

PROOF. $\mathscr{P}^* = \{\Lambda \setminus A | A \in \mathscr{P}\}$ is an ultrafilter on Λ. Let

$$K = \{0 < g \in G | \text{ support of } g \in \mathscr{P}^*\}$$

then K is maximal with respect to (*) and hence $M = \bigcup \{k' | k \in K\}$ is a minimal prime.

Note that if A and B are proper subsets of Λ, $A \cup B = \Lambda$ and $A \cap B = \square$, then $A \in \mathscr{P}$ or $B \in \mathscr{P}$. For since $\mathscr{P}^*$ is an ultrafilter, $A \in \mathscr{P}^*$ or $B \in \mathscr{P}^*$.

In the remainder of this section, we use Theorem 5.1 to obtain some properties of prime subgroups which are required for the examples in Section 6. In particular we have an easy method for describing the values of singular elements.

For a strictly positive element g in an l-group G let X be a subset of $\{x \in G | 0 < x \leqq g\}$ that is maximal with respect to

$$(*) \qquad 0 < a \wedge b \in X \text{ for all } a, b \in X.$$

PROPOSITION 5.2. *$N = \bigcup \{x' | x \in X\}$ is a minimal prime subgroup of G which does not contain g. Moreover each minimal prime subgroup of G may be obtained in this way.*

PROOF. Let $Y = \{y \in G | y \geqq x \text{ for some } x \in X\}$; then Y is a subset of G that obeys (*). If $h \in G$ is such that $h \wedge y > 0$ for all $y \in Y$ then, in particular, $(h \wedge g) \wedge x = h \wedge x > 0$ for all $x \in X$. Hence, by the maximality of X, $h \wedge g \in X$ and so $h \in Y$. Thus Y is maximal with respect to (*). Further it is clear that $N = \bigcup \{y' | y \in Y\}$ and hence, by Theorem 5.1, N is a minimal prime subgroup of G which certainly does not contain g.

On the other hand, if Y is maximal with respect to (*) and contains g then $X = Y \cap \{x \in G | 0 < x \leqq g\}$ is a maximal subset of $\{x \in G | 0 < x \leqq g\}$ with respect to (*) and $Y = \{y \in G | y \geqq x \text{ for some } x \in X\}$. Hence, as above, $\bigcup \{y' | y \in Y\} = \bigcup \{x' | x \in X\}$.

COROLLARY 1. *If g is singular, then N is a value of g and each value of g is obtained in this way.*

PROOF. From Lemma 4.5, the values of g are the minimal primes not containing g.

COROLLARY 2 (Lloyd). *If g is basic, then g' is the unique minimal prime not containing g and conversely.*

PROOF. If g is basic then $\{x | 0 < x \leqq g\}$ is totally ordered and hence satisfies (*). Thus $N = \bigcup \{x' | 0 < x \leqq g\} = g'$. If g is not basic then it exceeds a pair of strictly positive disjoint elements so that there exist at least two minimal primes not containing g.

PROPOSITION 5.3. *If, for each $0 < g \in G$, $G = g' \oplus g''$ then, for each proper prime subgroup M, $N = \bigcup \{a' | a \in G^+ \setminus M\}$ is a minimal prime. Hence, in particular, each proper prime subgroup contains a unique minimal prime.*

PROOF. By Theorem 5.1, Corollary 1, it suffices to show that N is prime. Suppose $g \wedge h = 0$ and pick $0 < s \notin M$. By hypothesis,

$$G = (g \wedge s)'' \oplus (g \wedge s)'.$$

Thus $s = u \vee v$ where $u \in (g \wedge s)''$, $v \in (g \wedge s)'$. Since $s \geqq g \wedge s$, $u \geqq g \wedge s$ and hence $u'' \supseteqq (g \wedge s)''$. But, since $u \in (g \wedge s)''$, $u'' \subseteqq (g \wedge s)''$ so that $u'' = (g \wedge s)''$ and $u' = (g \wedge s)'$. Now

$$g \wedge s = g \wedge (u \vee v) = (g \wedge u) \vee (g \wedge v) = (g \wedge u) + (g \wedge v).$$

Thus, since $v \in (g \wedge s)'$, $g \wedge v = 0$. If $v \notin M$ then $g \in v' \subseteqq N$; if $v \in M$ then $u \notin M$ for otherwise $s \in M$, hence $h \in (g \wedge s)' = u' \subseteqq N$. Therefore either g or h belongs to N which is thus prime.

COROLLARY 1 (Banaschewski [1]). *If G is a complete l-group then each proper prime l-ideal exceeds a unique minimal prime.*

COROLLARY 2. *If, for each $0 < g \in G$, $G = g' \oplus g''$ then each pair of prime subgroups A, B with $A || B$ generates G.*

PROOF. Let $A \vee B$ denote the subgroup of G generated by A, B. Then $A \vee B$ is prime and exceeds at least two distinct minimal primes. Thus $G = A \vee B$.

The following is a partial converse of Proposition 5.3. Whether or not the converse of this proposition is true is an open question; it is true for finite valued l-groups.

PROPOSITION 5.4. *If each prime subgroup of G exceeds a unique minimal prime and, for each $0 < g \in G$ there exists $0 < x \in G$ such that $g' = x''$ then $G = g' \oplus g''$ for each $0 < g \in G$.*

PROOF. Consider $0 < g \in G$ and suppose that $0 < y \in G \diagdown g' \oplus g''$. Let $M \supseteqq g' \oplus g''$ be a value of y. Then $N = \bigcup \{a' | a \in G^+ \diagdown M\}$ is a minimal prime. If $g \in N$ then $g \wedge a = 0$ for some $a \notin M$ and hence $a \in g' \subseteqq M$; a contradiction. Thus $g \notin N$ and, since N is prime, $g' \subseteqq N$. By hypothesis, $g' = x''$ and $x \in x'' = g' \subseteqq N$. Thus $x \wedge a = 0$ for some $a \notin M$ which implies $a \in x' = g'' \subseteqq N$; a contradiction.

6. Examples

I. Let G be the l-group of all real sequences that are eventually constant. It follows, from Theorem 2.5, that the completion H of G is the l-group of all bounded sequences. Note that

$$a = (1, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \cdots)$$
$$b = (\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{16}, \cdots)$$

are positive elements of H but no element of G lies between them. Thus the fact that G is dense in its completion does not imply that between any two comparable elements of H there is an element of G.

II. Let G be the l-group of all integral valued sequences that are eventually constant. It follows from Theorem 4.8 that the completion H of G is the l-group of all bounded integral valued sequences. For each positive integer n let

$$G_n = \{g \in G | g_n = 0\}.$$

Then it can easily be shown that the prime l-ideals of G are $G_1, G_2, \cdots$ and $\sum = \sum Z_i$. $\sum$ is an l-ideal in H, but no longer prime.

Let A be a value of $a = (1, 0, 1, 0, \cdots)$ in H that contains $\sum$ and let B be a value of $b = (0, 1, 0, 1, \cdots)$ in H that contains $\sum$. Since $a \wedge b = 0$, $a \in B \diagdown A$ and $b \in A \diagdown B$ so that $A || B$. By Lemma 3.1, A, B are maximal in H and hence are values of $c = (1, 1, 1, \cdots)$. It therefore follows that there are an infinite number on values N of c in H such that $N \cap G = \sum$.

III. Let G be the subgroup of $\prod_1^\infty Z_i$ generated by $\sum_1^\infty Z_i$ and $(2, 2, \cdots)$. Then $\sum Z_i = [S]$ is a minimal prime subgroup of G and is clearly not the value of any singular element.

IV. Example of an archimedean l-group such that $G \neq S' \oplus S''$ and $R(H)$ is not the completion of $R(G)$ where H is the completion of G.

The second example of page **233** of [7] has these properties.

V. Let G be the l-group of all integer valued functions on $[0, 1]$. Then $C = \{g \in G | g \text{ has countable support}\}$ is an l-ideal of G. Let $H = G/C$; then H is an archimedean l-group in which $H = S''$ but H is not a subdirect sum of integers.

1) *H is archimedean.*

PROOF. A positive element $X = C+x = C+(-x_i-)$ in H is in *normal form* if each $x_i \geqq 0$ and, if $x_i > 0$, then there exists an uncountable number of components equal to x_i. Clearly each positive element has a normal form. Let $X = C+(-x_i-)$ and $Y = C+(-y_i-)$ be positive elements of H in normal form and suppose that $X > nY$ for all $n > 0$. Since $Y < X$ we may assume that the support of $y = (-y_i-)$ is contained in the support of $x = (-x_i-)$.

For each $n > 0$ let $\alpha_n = \{i \in [0, 1] | x_i = n\}$. Then, if $y > 0$, there exists n such that $\beta = \alpha_n \cap \text{support } y$ is uncountable. For each $i \in \beta$ define $z_i = 1$, for $i \notin \beta$ define $z_i = 0$. Then $Z = C+z > C$ and $C+y \geqq C+z$ however, clearly, $C+x \not> (n+1)(C+z)$; thus $X \not> (n+1)Y$.

2) *Each strictly positive element exceeds a singular element.*

PROOF. Let $C+h > C$ be in normal form and, for each $i \in [0, 1]$, define $g_i = 1$ if $h_i > 0$ and $g_i = 0$ otherwise. Then $C+g > C$ and $C+h \geqq C+g$. Since g is singular in $\prod Z_i$, it follows, from Lemma 4.2, that $C+g$ is singular in H.

3) *The only maximal l-ideals of $\prod Z_i$ are the values of basic elements. Hence H has no maximal l-ideals and so is not a subdirect sum of integers.*

PROOF. We use the theory of minimal primes developed in Section 5. Let N be a proper prime l-ideal of $\prod Z_i$ which is not the value of a basic element and let M be the unique minimal prime contained in N. Then

$$M = \{g \in \prod Z_i | \text{ support of } g \in \mathscr{P}\}$$

where $\mathscr{P}$ is a family of proper subsets of $[0, 1]$ that is maximal with respect to $A, B \in \mathscr{P}$ implies $A \cup B \in \mathscr{P}$.

Let

$$\begin{aligned} T_1 &= [0, \tfrac{1}{2}), & T_2 &= [\tfrac{1}{2}, 1] \\ T_{11} &= [0, \tfrac{1}{4}), & T_{12} &= [\tfrac{1}{4}, \tfrac{1}{2}), \\ T_{21} &= [\tfrac{1}{2}, \tfrac{3}{4}), & T_{22} &= [\tfrac{3}{4}, 1] \quad \text{etc.} \end{aligned}$$

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Since M is prime, either T_1 or $T_2 \in \mathscr{P}$. If $T_1 \in \mathscr{P}$ then either T_{21} or T_{22} is in $\mathscr{P}$ etc. Thus we get a family $U_1, U_2, \cdots$ of disjoint members of $\mathscr{P}$ such that $[0, 1] = \bigcup \{U_n | 1 \leqq n\}$. Let $0 < g \in \prod Z_i \setminus N$ and define $h \in \prod Z_i$ by $h_i = ng_i$ if $i \in U_n$. Then $(h-ng)_i > 0$ if $i \in \bigcup \{U_m | m > n\}$. Hence $N+h \geqq N+ng > N$ for all $n > 0$. This implies $\prod Z_i/N$ is not archimedean, and hence N is not a maximal l-ideal.

Any maximal l-ideal of H is the image of a maximal l-ideal of $\prod Z_i$. But, since $\sum Z_i \subsetneqq C$, each maximal l-ideal of $\prod Z_i$ is mapped onto H. Hence H has no maximal l-ideals.

4) *The completion of H is a subdirect sum of discrete 0-groups but is not a subdirect sum of integers.*

Proof. This is an immediate consequence of the corollary to Lemma 4.7 and properties (2), (3).

5) *H is not complete or laterally complete.*

Proof. Divide $[0, 1]$ into an uncountable number of subsets α each of which is uncountable. For each α, let $(x_\alpha)_a = 1$ if $a \in \alpha$ and 0 otherwise; let $X_\alpha = C + x_\alpha$.

Suppose that Y is the least upper bound of the X_α. Then $Y = C + y$ where each component of y is either 0 or 1; since $Y \geqq X_\alpha$ for each α, all but a countable number of components of y are 1.

For each α, pick $a \in \alpha$ such that $y_a = 1$ and define

$$\bar{y}_x = \begin{cases} y_x & \text{if } x \neq a \\ 0 & \text{if } x = a \end{cases}$$

and let $\bar{Y} = C + \bar{y}$. Then $\bar{Y} \geqq X_\alpha$ for each α but $Y > \bar{Y}$ since $y_x > \bar{y}_x$ for an uncountable number of x.

Remark. If $G = \prod Z_\lambda$ where $Z_\lambda = Z$ for each $\lambda \in \Lambda$ and Λ is not countable and $C = \{g \in G | g \text{ has countable support}\}$, then it can be shown that $H = G/C$ is archimedean with a completion that is a subdirect sum of discrete 0-groups but not a subdirect sum of integers. It is not known whether property (3) holds for $\prod Z_\lambda$.

Methods similar to those used in establishing property (3) show that the only maximal l-ideals of a cardinal product $\prod \{G_\lambda | \lambda \in \Lambda\}$ of abelian 0-groups G_λ, $|\Lambda| \leqq |R|$ are the subgroups of the form $M_\lambda \oplus \prod \{G_\delta | \delta \neq \lambda\}$, where M_λ is the maximal convex l-subgroup of G_λ. Hence

(1) if no G_λ has a maximal convex l-subgroup then $\prod G_\lambda$ has no maximal convex l-subgroups;

(2) if $G_\lambda \subsetneqq R$ for each $\lambda \in \Lambda$, no factor group of $\prod G_\lambda / \sum G_\lambda$ has any maximal l-ideals.

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VI. *The completion of a free abelian l-group.*

Weinberg [22] has shown that the free abelian l-group F of rank β can be constructed in the following way. Let G be the free abelian group of rank β and form the cardinal product $\prod(G, T)$ of G over all possible total orders T of G. Then F is the l-subgroup of $\prod(G, T)$ generated by the long constants $\langle g \rangle$, $g \in G$. Thus, in particular, each non-zero element of F is of the form $f = \bigvee_I \bigwedge_J \langle g_{ij} \rangle$ where $g_{ij} \in G$ and I, J are finite. In addition, Weinberg [23] has shown that F is a subdirect sum of integers and is thus archimedean. The latter results may also be obtained by adapting the proof given for free vector lattices in [20]. As we shall use this representation to show that the free abelian l-group has no singular elements, we describe it below.

Let G be any torsion free abelian group and let $x_1, \cdots, x_n$ be any finite set of non-zero elements of G. Then it can be shown that G admits a total order T with a maximal convex subgroup C not containing any of $x_1, \cdots, x_n$ such that G/C is an 0-group of rank 1; in particular, if G is free T and C may be chosen so that G/C is cyclic. By Weinberg's result, above, each strictly positive element f of F is of the form $\bigvee_I \bigwedge_J \langle g_{ij} \rangle$, where $g_{ij} \in G$ and I, J are finite. Now G admits a total order T with a convex subgroup C, not containing any non-zero g_{ij}, such that G/C is cyclic. Let ρ_f be the mapping $h \to h_T \to C + h_T$ of F onto $G/C \cong Z$; then clearly ρ_f is an l-homomorphism, further one can show that, in particular, $f\rho_f > 0$. Hence the family $\{\rho_f | 0 < f \in F\}$ separates the points of F so that F is a subdirect union of cyclic groups G/C.

To show that F has no singular elements, it suffices, by Lemma 4.4, Corollary 2, to show that, for each $0 < f \in F$ there exists $0 < h \in F$ such that $f\rho_h > 1$. That is, it suffices to show that there exists a total order T on G with a convex subgroup C such that $C + \bigvee_I \bigwedge_J g_{ij} > C + a$ where $f = \bigvee_I \bigwedge_J \langle g_{ij} \rangle$ and $C + a$ covers C. Now $G = \sum \{Z_\alpha | \alpha \in A$ where $|A| = \beta\}$ and, since there are only a finite number of the g_{ij}, we can write

$$G = \sum_1^m Z_{\alpha_k} \oplus \sum \{Z_\alpha | \alpha \neq \alpha_k, k = 1, 2, \cdots, m\} = X \oplus Y$$

where m is finite and, for each $i \in I$, $j \in J$, $g_{ij}(\alpha) = 0$ if $\alpha \neq \alpha_k$ for some k. If we can find a total order of X with the above property, then a lexicographic extension of Y by X will be the desired total order for G. Thus there is no loss of generality in assuming G is of finite rank m.

By the remarks above, there exists a total order on G with a convex subgroup C such that $C + \bigvee_I \bigwedge_J g_{ij} > C$. Thus, for some i^*, $C + \bigwedge_J g_{i^*j} > C$; denote the g_{i^*j} by $g(1), \cdots, g(k)$ where each $g(j)$ is an m-tuple of integers $(g_1(j), \cdots, g_m(j))$. Any convex subgroup of a totally ordered group is pure and hence, in particular C is a pure subgroup of G. Thus the fact that G/C

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is cyclic implies $G = C \oplus I$ where I is cyclic and G is ordered by $(c, i) \geqq (0, 0)$ if and only if $i > 0$ or $i = 0$, $c \geqq 0$. The transformation from $G = \sum Z_k$ to $C \oplus I$ is effected by a non-singular unimodular integral matrix:

$$(x_1, \cdots, x_m) \begin{bmatrix} a_{11}, & \cdots, & a_{1m} \\ & & \\ a_{m1}, & \cdots, & a_{mm} \end{bmatrix} = (\cdots, x_1 a_{1m} + \cdots + x_m a_{mm}).$$

In particular, $C + (x_1, \cdots, x_m) > C$ if and only if $x_1 a_{1m} + \cdots + x_m a_{mm} > 0$. Hence we have only to show that there exist $b_1, \cdots, b_m$ with

$$(|b_1|, \cdots, |b_m|) = 1$$

such that

$$b_1 g_1(i) + \cdots + b_m g_m(i) > 1 \text{ for } i = 1, 2, \cdots, k.$$

We know there exists

$$a_1 x_1 + \cdots + a_m x_m = 0$$

such that

$$(|a_1|, \cdots, |a_m|) = 1$$

and

$$a_1 g_1(i) + \cdots + a_m g_m(i) \geqq 1 \text{ for } i = 1, 2, \cdots, k.$$

If only one of the a_j, $j = 1, 2, \cdots, m$ is non-zero, say a_1, then $|a_1| = 1$. Let $z = \max \{|g_2(i)| \mid 1 \leqq i \leqq k\}$ and let a be an odd positive integer such that $a > 2z+1$. Then $a_i a g_1(i) + 2g_2(i) > 1$ for $i = 1, 2, \cdots, k$.

If more than one a_j is non-zero, we may suppose $a_1 \neq 0 \neq a_m$. Let $b_j = p^\alpha a_j$, $j \neq m$, $b_m = q^\beta a_m$ where $p > q$ are primes not dividing any of the a_j, $j = 1, 2, \cdots, m$. Then

$$b_1 x_1 + \cdots + b_m x_m = p^\alpha (a_1 x_1 + \cdots + a_m x_m) - a_m (p^\alpha - q^\beta) x_m.$$

Hence, for any $g_1(i), \cdots, g_m(i)$,

$$\begin{aligned} b_1 g_1(i) + \cdots + b_m g_m(i) &= p^\alpha (a_1 g_1(i) + \cdots + a_m g_m(i)) - a_m (p^\alpha - q^\beta) g_m(i) \\ &\geqq p^\alpha - a_m (p^\alpha - q^\beta) g_m(i) \\ &> 1 \end{aligned}$$

if

$$\frac{p^\alpha - 1}{|p^\alpha - q^\beta|} > |a_m| |g_m(i)|.$$

This will be true if

$$\frac{1 - 1/p^\alpha}{|1 - q^\beta / p^\alpha| |a_m|} > \max \{|g_m(i)| | 1 \leqq i \leqq k\}$$

and clearly we can choose such α and β. Since p, q are prime to all the a_j, it is clear that $(|b_1|, \cdots, |b_m|) = 1$. Hence in either case we can find

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$$b_1x_1+\cdots+b_mx_m=0$$

with $(|b_1|, \cdots, |b_m|) = 1$ such that, for $i = 1, 2, \cdots, k$,

$$b_1g_1(i)+\cdots+b_mg_m(i) > 0.$$

Thus, as pointed out above, F has no singular elements.

REMARK I. Let G be the free abelian l-group of rank n and let T be a total order of G with a convex subgroup C such that G/C is cyclic. As above, $G = C \oplus I$ and the l-homomorphism of G onto G/C is equivalent to a mapping $(x_1, \cdots, x_m) \to a_1x_1+\cdots+a_mx_m$, where $(|a_1|, \cdots, |a_m|) = 1$, of G onto Z; let $\varphi(a_1, \cdots, a_m)$ denote this mapping. Then the mapping

$$f = \bigvee_I \bigwedge_J \langle g_{ij} \rangle \to f_T = \bigvee_I \bigwedge_J g_{ij} \to \bigvee_I \bigwedge_J g_{ij}\varphi(a_1, \cdots, a_m)$$

is an l-homomorphism of F onto Z and we have pointed out that such homomorphisms separate the points of F. Hence, if S is the set of all m-tuples of integers $(a_1, \cdots, a_m)$ such that $(|a_1|, \cdots, |a_m|) = 1$ and $\prod\{Z_s | s \in S\}$ is the cardinal product of S copies of the integers, then F is the l-subgroup of $\prod Z_s$ generated by the elements

$$(-, a_1x_1+\cdots+a_mx_m, -).$$

This gives a more constructive description of the free abelian l-group of rank m than those given in [20], [22], [23].

REMARK II. If, in the remark above, we replace Z by R we obtain the free vector lattice V of rank m. If we embed Z_s naturally in R_s for each $s \in S$ then V is the subspace of $\prod R_s$ generated by F. Since F has no singular elements, it follows from Lemma 4.7 and Corollary II to Lemma 4.10 that the completion $F^\wedge$ of F in $\prod R_s$ is a vector lattice and hence contains V. Since F is dense in $F^\wedge$ so clearly is V and we thus have

$$F \subset V \subset F^\wedge = V^\wedge.$$

We end the paper by listing the following open questions.

1. If G is an l-subgroup of a complete vector lattice H does H contain a completion of G?

2. Characterise those l-subgroups of a complete l-group H that have a unique completion in H.

3. Characterise those l-groups which are subdirect sums of copies of Z.

4. If $G = \prod G_\lambda$ is the cardinal product of 0-groups G_λ, are there any maximal l-ideals of G which contain $\sum G_\lambda$?

Added in proof. Alain Bigard has shown that the converse to Proposition 5.3 is false [C.R. Acad. Sc. Paris 266 (1968) 261—262] and Norman Reilly has constructed an example of a cardinal sum $\prod G_\lambda$ of 0-groups for which $\sum G_\lambda$ is contained in a maximal l-ideal [to appear in the Duke Math. J.]. Finally Conrad has obtained an answer to question 3 [to appear in the Proc. Amer. Math. Soc.].

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