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Quantitative laws of large numbers in noncommutative probability*

Junming Cao, Yong Jiao, Sijie Luo† and Dejian Zhou

Abstract. **In this paper, we investigate the quantitative law of large numbers for noncommutative random variables. Firstly, we establish a Baum-Katz theorem for noncommutative successively independent random variables, which resolves an open problem posed by Stoica ([48]). Our approach differs from the classical treatment but relies on the theory of asymmetric maximal inequality for noncommutative martingales. Additionally, we derive a moderate deviation inequality for noncommutative successively independent sequences, and extend this result together with the Baum-Katz theorem to noncommutative martingales. Finally, we conclude the paper by applying our results to derive a noncommutative Marcinkiewicz-Zygmund type strong laws of large numbers theorem, which extends the result of Łuczak ([37]) in some aspects.**

1 Introduction

The convergence rate of the law of large numbers plays an essential role in probability theory, which is closely related to the large deviation theory (see [4, 29, 39, 45, 46]), concentration of measure phenomenon (see [34]), Markov chains (see [5, 32, 41, 44]) and related areas (see [9, 30]). One of the most fundamental results in this area, known as the *Baum-Katz theorem*, which is stated as follows.

Theorem* 1.1** ***(Buam-Katz-Chow) *Let* $\{X, X_n, n \geq 1\}$ *be a sequence of i.i.d. random variables and* $S_n = \sum_{k=1}^n X_k$ *for* $n \geq 1$*. For* $r \geq 1$*,* $p > 0$ *and* $r/p > 1/2$*, the following statements are equivalent:*

(1) $\mathbb{E}|X|^p < \infty$;
(2) $\sum_{n=1}^\infty n^{r-2}\mathbb{P}\left\{|S_n - nb| > \varepsilon n^{r/p}\right\} < \infty,$ *for all* $\varepsilon > 0$;
(3) $\sum_{n=1}^\infty n^{r-2}\mathbb{P}\left\{\max_{1\leq m\leq n}|S_m - mb| > \varepsilon n^{r/p}\right\} < \infty,$ *for all* $\varepsilon > 0$*,*

where $b = 0$ *if* $r/p > 1$ *and* $b = \mathbb{E}[X]$ *if* $1/2 < r/p \leq 1$.

The equivalence between (1) and (2) was proved by Baum and Katz [2], and that between (1) and (3) was by Chow [8]. Since then, a great deal of effort has been devoted to

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†The corresponding author.

generalizing the Baum-Katz theorem to general sequences and sharpening the parameters in the theorem. We refer to [33, 14, 18, 36, 50] for more details. Notably, if there are no independence assumptions on the sequence $\{X_n\}_{n=1}^{\infty}$, condition (3) is strictly stronger than (2). Hence, in the sequel, we will only investigate the equivalence between (1) and (3) in the noncommutative framework.

Motivated by the study of operator algebras and noncommutative analysis, numerous significant probability inequalities in the noncommutative case have been developed in the past decades. One of the most critical developments in this area is the Burkholder-Gundy inequality for noncommutative martingales established by Pisier and Xu [40]. Since then, fundamental martingale inequalities such as the Doob maximal inequality [24] and the Burkholder/Rosenthal inequality [25, 26] have been well developed in the noncommutative setting, leading the noncommutative martingale theory into an attractive area in probability theory. We refer to [6, 3, 16, 20, 21, 42] for more information. In the present paper, we aim to continue the fruitful line of investigating noncommutative deviation inequalities.

In 2009, Stoica [48, Theorem 1] adapted the method of Katz [27, Theorem 2(a)] and provided a noncommutative extension of Theorem 1.1 for successively $i.i.d.$ sequence with $1/2 < r/p \leq 1$. However, it was stated by Stoica that his approach cannot tackle the case for $r/p > 1$ (see [48, p. 321]), leaving it as an open problem. In the present paper, we apply the weak type asymmetric maximal inequality for noncommutative martingales to establish a noncommutative Baum-Katz theorem for $r/p > 1$, settling the problem of Stoica.

To state our results precisely, we begin with recalling basic concepts and notations from noncommutative analysis. By a noncommutative probability space $(\mathcal{M}, \tau)$, we mean that $(\mathcal{M}, \tau)$ is a tracial von Neumann algebra equipped with a normal faithful tracial state τ. Let $L_0(\mathcal{M})$ be the $*$-algebra of all τ-measurable operators with respect to $(\mathcal{M}, \tau)$. The noncommutative counterpart of the *maximal column tail probability* introduced in [28] is stated as follows, which is closely related to the almost uniform convergence in $L_0(\mathcal{M})$. For a sequence $(x_n)_{n\geq 1} \subseteq L_0(\mathcal{M})$ and $t > 0$, define

$$\mathrm{Prob}_{\tau}^{c}\left\{\sup_{n\geq 1}|x_n| > t\right\} \coloneqq \inf_{e\in\mathcal{M}_\pi}\left\{\tau(1-e) : \|x_n e\|_\infty \leq t \text{ for all } n \geq 1\right\},$$

where $\mathcal{M}_\pi$ stands for the lattice of projections in $\mathcal{M}$. Our noncommutative Baum-Katz theorem for successively independent sequence (see the definition in §2) is stated as follows.

***Theorem* 1.2** *Suppose that $\{x_k\}_{k\geq 1}$ is a sequence of successively independent random variables in $L_0(\mathcal{M})$, which have the same distribution of $x \in L_0(\mathcal{M})$. Let $S_n = \sum_{k=1}^{n} x_k$ be the partial sum. For $r \geq 1$ and $p > 0$ with $r/p > 1/2$, let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous increasing function with $\psi(t) = O(t^{r-1+r/p})$ for each $t > 0$ such that $\{\psi(n)/n^{r/p}\}_{n=1}^{\infty}$ forms an increasing sequence and*

$$\sum_{n=1}^{\infty} n^{r-1}\tau\left[e_{(\psi(n),\infty)}(|x|)\right] < \infty. \tag{1.1}$$

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Then we have

$$\sum_{n=1}^{\infty} n^{r-2}\tau\left[e_{(\varepsilon\psi(n),\infty)}(|S_n - nb|)\right] < \infty, \quad \forall\varepsilon > 0, \tag{1.2}$$

and

$$\sum_{n=1}^{\infty} n^{r-2}\mathrm{Prob}_{\tau}^{c}\left\{\sup_{1\le m\le n}|S_m - mb| > \varepsilon\psi(n)\right\} < \infty, \quad \forall\varepsilon > 0, \tag{1.3}$$

where $b = 0$ *if* $r/p > 1$ *and* $b = \tau(x)$ *if* $1/2 < r/p \le 1$.

From [23, Remark 3.4], we see that (1.3) implies (1.2). It is worthwhile to mention that Theorem 1.2 is a noncommutative Baum-Katz theorem under a more general integrability condition, which enables one to deal with ψ with $n^{r/p} \le \psi(n) \le n^{r-1+r/p}$ for sufficiently large $n \ge 1$. In particular, if $\psi(n) = n^{r/p}$, (1.1) goes back to Theorem 1.1 (1). Thanks to the noncommutative Borel-Cantelli lemma, we get the following noncommutative strong law of large numbers by Łuczak [37, Theorem 3.6] by applying Theorem 1.2.

***Corollary* 1.3** *Let* $0 < p < 2$ *and* $\{x_k\}_{k\ge1} \subseteq L_p(\mathcal{M})$ *be a successively independent identically distributed sequence. Then*

$$n^{-1/p}\sum_{k=1}^{n}(x_k - b\mathbf{1}) \xrightarrow{a.u.} 0 \;\; as \;\; n \to \infty,$$

where $b = 0$ *for* $0 < p < 1$ *and* $b = \tau(x_1)$ *for* $1 \le p < 2$.

Note here that, for the case $\psi(n) = (n\log n)^{r/p}$, the Baum-Katz theorem holds only for $r > 1$ in the commutative setting. For the case $r = 1$, the estimate $\mathbb{P}\left\{|S_n| > \varepsilon(n\log n)^{1/p}\right\}$ is called *the moderate deviation*, which was first studied by Davis [11] and later generalized by Rohatgi [43]. Hence, the second main result of the present paper is the noncommutative moderate deviation inequality stated as follows.

***Theorem* 1.4** *Let* $\{x_k\}_{k\ge1} \subseteq L_p(\mathcal{M})$ *be a sequence of successively independent random variables with the same distribution of* $x \in L_p(\mathcal{M})$. *Let* $S_n = \sum_{k=1}^{n} x_k$ *for each* $n \ge 1$. *Then, for any* $\varepsilon > 0$, *we have*

(1) for $p = 2$,

$$\sum_{n=1}^{\infty}\frac{\log n}{n}\tau\left(e_{(\varepsilon(n\log n)^{1/2},\infty)}(|S_n - n\tau(x)|)\right) < \infty,$$

(2) for $1 < p < 2$,

$$\sum_{n=1}^{\infty}\frac{\log n}{n}\mathrm{Prob}_{\tau}^{c}\left\{\sup_{1\le m\le n}|S_m - m\tau(x)| > \varepsilon(n\log n)^{1/p}\right\} < \infty.$$

It is worthwhile to mention that for the case $p = 2$, we apply the noncommutative Fuk-Nagaev inequality in the proof of Theorem 1.4, rather than the symmetrization

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method used in Davis' original approach [11]. Moreover, motivated by the law of the iterated logarithm [12], we adapt the proof of Theorem 1.4 (1) to obtain a corresponding result for $\mathbb{P}\left\{|S_n| > \varepsilon(n\log\log n)^{1/2}\right\}$ in Corollary 3.11.

Remark 1.5 The original proof of the equivalence between (1) and (2) in Theorem 1.1 relies on the Levý inequality. However, the noncommutative version of the Levý inequality is not yet to be well-developed, which is one of the main obstructions in our approach of proving the noncommutative analogue of Theorem 1.1 in full generality.

Motivated by the recent development of noncommutative martingale theory, we also extend the Baum-Katz theorem to the noncommutative martingale framework. In the commutative setting, it has been pointed out by Lesigne and Volný [35] that $\mathbb{P}(|S_n| > \varepsilon n) = O\left(n^{-p/2}\right)$ if $\sup_{k\geq 1}\mathbb{E}|X_k|^p < \infty$ with $p \geq 2$, and the exponent $p/2$ is optimal, even for strictly stationary and ergodic sequences of martingale differences. Thus, for the martingale case, the Baum-Katz theorem does not hold without additional assumptions.

Following Chung [10], we consider a non-decreasing function $f : [0,\infty) \to \mathbb{R}^+$ such that

$$\sum_{n=1}^{\infty}\frac{1}{f\left(2^n\right)} < \infty. \tag{1.4}$$

Specifically, if $f(t) = \left(\log^+|t|\right)^{1+\varepsilon}$, $t \in [0,\infty)$ and $\varepsilon > 0$, it is easy to see that f fulfills (1.4). Hence, the following results contain several important examples in studying the law of large numbers. Suppose that $\{X_j\}_{j=1}^{\infty}$ is a sequence of random variables fulfilling the *uniform moment condition*: $\sup_{j\geq 1}\mathbb{E}[|X_j|^p f(|X_j|)] < \infty$. Chung [10, Corollary 1] derived the Marcinkiewicz-Zygmund strong law of large numbers for independent random variables $\{X_j\}_{j=1}^{\infty}$ satisfying the uniform moment condition. Later, under the same condition, Chung's result was further extended by Stout [49] and Balka et al. [1] in related areas. The following two theorems extend results in [1] to noncommutative martingales.

***Theorem* 1.6 (*Noncommutative martingales with* $1 < p < 2$)** *For* $1 < p < 2$ *and* $r \geq 1$ *or* $r = p = 1$*, let* $\{dx_i\}_{i\geq 1}$ *be a martingale difference sequence satisfying* $\sup_{i\geq 1}\tau\left(|dx_i|^p f\left(|dx_i|\right)\right) < \infty$*, where* f *fulfills* (1.4)*. Then the following holds*

$$\sum_{n=1}^{\infty} n^{r-2}\mathrm{Prob}_\tau^c\left\{\sup_{1\leq k\leq n}\left|\sum_{i=1}^{k} dx_i\right| > \varepsilon n^{r/p}\right\} < \infty, \quad \forall \varepsilon > 0.$$

***Theorem* 1.7 (*Noncommutative martingales with* $p \geq 2$)** *For* $p \geq 2$ *and* $r/p > 1/2$*, let* $q = 2p(r-1)/(2r-p)$ *and* $\{dx_i\}_{i\geq 1}$ *be a martingale difference sequence satisfying* $\sup_{i\geq 1}\tau\left(|dx_i|^q f\left(|dx_i|\right)\right) < \infty$*, where* f *fulfills* (1.4)*. The following holds*

$$\sum_{n=1}^{\infty} n^{r-2}\mathrm{Prob}_\tau^c\left\{\sup_{1\leq k\leq n}\left|\sum_{i=1}^{k} dx_i\right| > \varepsilon n^{r/p}\right\} < \infty, \quad \forall \varepsilon > 0.$$

Remark 1.8 It was pointed out in [10] implicitly that the uniform moment condition mentioned in the previous theorems is necessary even for commutative martingales. More precisely, the series in Theorem 1.6 and Theorem 1.7 diverge whenever (1.4) fails.

A direct consequence of Theorem 1.6 is the following law of large numbers for noncommutative martingales, which extends the result of Łuczak [37, Theorem 3.6] (see also Corollary 1.3) at some points.

***Corollary* 1.9** *Let $1 \le p < 2$ and $\{x_n\}_{n\ge 1}$ in $L_0(\mathcal{M})$ be a martingale difference sequence such that $\sup_{n\ge 1} \tau\left(|x_n|^p f(|x_n|)\right) < \infty$, where f fulfills (1.4). Then*

$$\sum_{k=1}^{n} x_k/n^{1/p} \xrightarrow{a.u.} 0 \ \ as \ \ n \to \infty. \tag{1.5}$$

Recently, it has been shown by Ricard and Hong [17] that the noncommutative L_p martingales need not be convergent uniformly whenever $1 \le p < 2$. Thus, Corollary 1.9 is of independent interest in studying the convergence theorem for noncommutative martingales.

Inspired by the recent work on noncommutative large deviation inequalities of Jiao et al. [19], we obtain following result on moderate deviation, extending [47, Theorem 1] of Stoica to the noncommutative martingales.

***Theorem* 1.10** *Let $p > 4$ and $\{dx_i\}_{i\ge 1}$ be an L_p-bounded noncommutative martingale difference sequence. Then*

$$\sum_{n=1}^{\infty} \frac{\log n}{n} \mathrm{Prob}_{\tau}^{c} \left\{ \sup_{1\le k\le n} \left| \sum_{i=1}^{k} dx_i \right| > \varepsilon (n \log n)^{1/2} \right\} < \infty, \quad \forall \varepsilon > 0.$$

For the case $1 \le p \le 4$, it has been pointed out by Miao, Yang and Stoica in [38, Theorem 2.4] that Theorem 1.10 fails in general. Furthermore, by applying a more specific calculations in the proof of Theorem 1.10, we obtain a slight variant of Theorem 1.10 in Corollary 4.9.

Our paper is organized as follows. In Section 2, we collect necessary materials from noncommutative analysis, such as noncommutative Lebesgue spaces, the noncommutative independence, and noncommutative martingales. In Section 3, we provide detailed proofs of the noncommutative Baum-Katz type theorem and the noncommutative moderate deviation inequality for successively independent sequences. In Section 4, we extend results obtained in the previous section from the successively independent case to the noncommutative martingales case. In Section 5, as an application of our result, we conclude the paper with a noncommutative Marcinkiewicz-Zygmund type strong laws of large numbers.

Throughout the paper, let $(\mathcal{M}, \tau)$ be a fixed noncommutative probability space. We write $A \lesssim_p B$ to mean that $A \le C_p B$ with a positive constant C_p depending only on the parameter p, and we ignore the subscript when the constant is universal.

2 Preliminaries

2.1 Noncommutative L_p-spaces

Let $L_0(\mathcal{M})$ denote the topological $*$-algebra of measurable operators with respect to $(\mathcal{M}, \tau)$. Equipping $L_0(\mathcal{M})$ with the topology induced by the convergence in measure makes $L_0(\mathcal{M})$ a topological vector space, and the elements in $L_0(\mathcal{M})$ are called *(noncommutative) random variables*. In the sequel, all random variables are noncommutative if no confusions arise. Given $0 < p < \infty$, the noncommutative L_p space is defined by

$$L_p(\mathcal{M}) \coloneqq \left\{ x \in L_0(\mathcal{M}) : \tau\left(|x|^p\right)^{\frac{1}{p}} < \infty \right\},$$

equipped with (quasi-) norm $\|x\|_p \coloneqq \tau\left(|x|^p\right)^{\frac{1}{p}}$ for $x \in L_p(\mathcal{M})$. As usual, we let $L_\infty(\mathcal{M}) \coloneqq \mathcal{M}$ and $\|\cdot\|_\infty$ is the operator norm on $\mathcal{M}$.

For $x \in L_0(\mathcal{M})$, its *distribution function* is defined by

$$\lambda_s(x) = \tau\left(e_s^{\perp}(|x|)\right), \quad \forall s > 0,$$

where $e_s^{\perp}(|x|) = e_{(s,\infty)}(|x|)$ is the spectral projection of $|x|$ associated with the interval (s, ∞). For any $s, t > 0$ and $x, y \in L_0(\mathcal{M})$, their distribution functions satisfy that

$$\lambda_{s+t}(|x+y|) \le \lambda_s(|x|) + \lambda_t(|y|). \tag{2.1}$$

For $x \in L_p(\mathcal{M})$ with $0 < p < \infty$, we have the Chebyshev inequality

$$\lambda_s(x) \le \frac{\|x\|_p^p}{s^p}, \quad \forall s > 0. \tag{2.2}$$

The concept of *almost uniform convergence* was introduced by Lance [31], which can be viewed as a noncommutative substitution of the *almost everywhere convergence* in $L_0(\mathcal{M})$.

***Definition* 2.1** A sequence $\{x_n\}_{n\ge1} \subseteq L_0(\mathcal{M})$ is said to *converge almost uniformly* (a.u. in short) to $x \in L_0(\mathcal{M})$, if for any $\varepsilon > 0$ there is a projection $e \in \mathcal{M}$ such that

$$\tau(\mathbf{1} - e) < \varepsilon \quad \text{and} \quad \lim_{n\to\infty} \|(x_n - x)e\|_\infty = 0.$$

Moreover, $\{x_n\}_{n\ge1}$ is said to *converge bilaterally almost uniformly* (b.a.u. in short) to x if for any $\varepsilon > 0$ there is a projection $e \in \mathcal{M}$ such that

$$\tau(\mathbf{1} - e) < \varepsilon \quad \text{and} \quad \lim_{n\to\infty} \|e(x_n - x)e\|_\infty = 0.$$

It is obvious that the a.u.convergence implies the b.a.u. convergence. When $(\mathcal{M}, \tau)$ is commutative, the Egorov theorem entails that the a.u. convergence coincides with the a.e. convergence in $L_0(\mathcal{M})$.

We conclude this subsection by introducing the column weak space $\Lambda_{p,\infty}(\mathcal{M}; \ell_\infty^c)$, which is deeply connected with the probability of $\mathrm{Prob}_\tau^c\left(\sup_{n\ge1}|x_n| > t\right)$. The space $\Lambda_{p,\infty}(\mathcal{M}; \ell_\infty^c)$ is defined as the sequences $(x_n)_{n\ge1}$ satisfying that

$$\left\|(x_n)_{n\ge1}\right\|_{\Lambda_{p,\infty}(\mathcal{M};\ell_\infty^c)} = \sup_{\lambda>0} \inf_{q\in\mathcal{M}_\pi} \left\{ \lambda\tau((\mathbf{1}-q))^{\frac{1}{p}} : \|x_n q\|_\infty \le \lambda \text{ for all } n \ge 1 \right\} < \infty,$$

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where $\mathcal{M}_\pi$ is the lattice of projections in $\mathcal{M}$. One can also define the row space by taking adjoints.

2.2 Noncommutative Orlicz spaces

Let Φ be an Orlicz function on $[0, \infty)$. We say Φ satisfies the $\triangle_2$-condition if $\Phi(2t) \leq C\Phi(t)$ for all $t > 0$ and some $C > 0$. It is not hard to check that Φ satisfies the $\triangle_2$-condition if and only if for any $\alpha > 0$, there is a constant $C_\alpha > 0$ such that $\Phi(\alpha t) \leq C_\alpha \Phi(t)$ for all $t > 0$. For $1 \leq p \leq q < \infty$, we say that Φ is p-convex if the function $t \mapsto \Phi\left(t^{1/p}\right)$ is convex, and is q-concave if the function $t \mapsto \Phi\left(t^{1/q}\right)$ is concave. In addition, Φ satisfies the Δ_2-condition if and only if it is q-concave for some $q < \infty$. For $x \in L_0(\mathcal{M})$, we have (see e.g. [13, Corollary 2.8])

$$\tau(\Phi(|x|)) = \int_0^\infty \Phi(t) \mathrm{d}\tau\left(e_t\right), \tag{2.3}$$

where $\tau(e_t) = \tau(e_{(-\infty,t)}(|x|))$ is the spectral measure. For $(x_i)_{i=0}^n \subset L_0(\mathcal{M}, \tau)$, $\lambda_i \in (0, 1)$ with $\sum_{i=0}^n \lambda_i \leq 1$, we have the following Jensen's inequality (see e.g. [13, Theorem 4.4])

$$\tau\left[\Phi\left(\left|\sum_{i=0}^n \lambda_i x_i\right|\right)\right] \leq \sum_{i=0}^n \lambda_i \tau\left(\Phi\left(|x_i|\right)\right). \tag{2.4}$$

The noncommutative Orlicz space $L_\Phi(\mathcal{M})$ is defined by

$$L_\Phi(\mathcal{M}) = \left\{x \in L_0(\mathcal{M}) : \tau\left[\Phi\left(\frac{|x|}{c}\right)\right] < \infty, \text{ for some } c > 0\right\},$$

equipped with the norm

$$\|x\|_\Phi = \inf\{c > 0 : \tau(\Phi(|x|/c)) \leq 1\}.$$

2.3 Noncommutative martingales

The following materials on noncommutative martingales are standard. Let $\{\mathcal{M}_n\}_{n\geq 0}$ be an increasing sequence of von Neumann subalgebras of $\mathcal{M}$ such that the union of the $\mathcal{M}_n$ is w^*-dense in $\mathcal{M}$. For every $n \geq 0$, let $\mathcal{E}_n$ be the conditional expectation from $\mathcal{M}$ onto $\mathcal{M}_n$ satisfying

(1) $\mathcal{E}_n(xy) = \mathcal{E}_n(x)y, \mathcal{E}_n(yx) = y\mathcal{E}_n(x), n \geq 0$ and $y \in \mathcal{M}_n$;
(2) $\mathcal{E}_n\mathcal{E}_m = \mathcal{E}_n$ for $m \geq n$;
(3) $\tau\left(\mathcal{E}_n(x)\right) = \tau(x), n \geq 0$.

The noncommutative martingale is a sequence $x = \{x_n\}_{n\geq 1}$ in $L_1(\mathcal{M})$ satisfying $\mathcal{E}_n\left(x_{n+1}\right) = x_n$ for all $n \geq 1$. The martingale differences sequence $(dx_k)_{k\geq 1}$ is defined by $dx_1 \coloneqq x_1$ and $dx_k \coloneqq x_k - x_{k-1}$ with $k \geq 1$. Given a martingale x in $L_p(\mathcal{M})$, its L_p norm is given by $\|x\|_p \coloneqq \sup_{n\geq 1} \|x_n\|_p$. If $\|x\|_p < \infty$, x is called an L_p-bounded martingale. More generally, given an Orlicz function Φ, we say that $x = (x_n)_{n\geq 1}$ is an L_Φ-bounded martingale if $\|x\|_\Phi = \sup_{n\geq 0} \|x_n\|_\Phi < \infty$.

Now we introduce an important noncommutative weak type asymmetric maximal inequality taken from [15, Theorem A(ii)].

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***Theorem* 2.1** *For every $n \geq 0$, let $\mathcal{E}_n$ be the conditional expectation defined as above. For $1 \leq p \leq 2$, we have*

$$\left\|(\mathcal{E}_n(x))_{n\geq 1}\right\|_{\Lambda_{p,\infty}(\mathcal{M};\ell_\infty^c)} \leq c_p \left\|\left(\sum_{n\geq 1}|dx_n|^2\right)^{1/2}\right\|_{L_p(\mathcal{M})},$$

where $c_p > 0$ is a constant depending only on p.

2.4 Noncommutative independence

Recall the *noncommutative independences* of Junge and Xu (see [26]) as follows.

***Definition* 2.2** Assume that $\{\mathcal{M}_k\}_{k\geq 0}$ are von Neumann subalgebras in $\mathcal{M}$.

(1) We say that $\{\mathcal{M}_k\}_{k\geq 0}$ are independent with respect to τ, if $\tau(xy) = \tau(x)\tau(y)$ holds for each $x \in \mathcal{M}_k$ and y belongs to the von Neumann algebra generated by $\{\mathcal{M}_j\}_{j\neq k}$.
(2) We say that $\{\mathcal{M}_k\}_{k\geq 0}$ are successively independent with respect to τ, if $vN(\mathcal{M}_0, \dots, \mathcal{M}_{k-1})$ and $vN(\mathcal{M}_k)$ are independent for each $k \geq 1$, where $vN(\mathcal{M}_0 \dots, \mathcal{M}_{k-1})$ denotes the von Neumann subalgebra generated by $\mathcal{M}_0, \dots, \mathcal{M}_{k-1}$.
(3) We say that the random variables sequence $\{x_k\}_{k\geq 0}$ is independent or successively independent, if for each $k \geq 0$, the unital von Neumann subalgebras $\mathcal{M}_k$ generated by x_k are independent or successively independent with respect to τ.

Remark 2.2 Assume that $\{\mathcal{M}_k\}_{k\geq 0}$ are von Neumann subalgebras in $\mathcal{M}$.

(1) Let $\{\mathcal{M}_k\}_{k\geq 0}$ be independent with respect to τ. Then $\{\mathcal{M}_k\}_{k\geq 0}$ are successively independent with respect to τ; see [26, Lemma 1.2].
(2) Let $(\mathcal{M}_k)_{k\geq 0}$ be successively independent with respect to τ. Then for each $k \geq 0$, [26, Remark 1.1] yields that

$$\mathcal{E}_{vN(\mathcal{M}_0,\dots,\mathcal{M}_k)}(x) = \tau(x), \quad x \in \mathcal{M}_j,\ j > k,$$

where $\mathcal{E}_{vN(\mathcal{M}_0,\dots,\mathcal{M}_k)}$ is the conditional expectation from $\mathcal{M}$ to $vN(\mathcal{M}_0, \dots, \mathcal{M}_k)$. Therefore, if $x_k \in L_p(\mathcal{M}_k)$ with $\tau(x_k) = 0$, then $\{x_k\}_{k\geq 0}$ is a martingale difference sequence with respect to the filtration $(vN(\mathcal{M}_0, \dots, \mathcal{M}_k))_{k\geq 0}$.

We say $\{x_n\}_{n\geq 1}$ is identically distributed if the distribution functions $\tau\left(e_s^{\perp}(|x_n|)\right)$ are identical for all $n \in \mathbb{N}$.

We conclude the preliminaries section with the noncommutative Rosenthal inequality established by Junge and Xu [26, Theorem 2.1], which is one of the key ingredients in our proof of Theorem 1.2.

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***Theorem* 2.3 (*Junge-Xu*)** *Let $2 \leq p < \infty$, $\{x_k\}_{k\geq 1} \in L_p(\mathcal{M})$ with $\tau(x_k) = 0$. If $\{x_k\}_{k\geq 1}$ is successively independent, then*

$$\frac{C}{p^2}\left\|\sum_{k\geq 1} x_k\right\|_p \leq \max\left\{\left(\sum_{k\geq 1}\|x_k\|_p^p\right)^{\frac{1}{p}}, \left(\sum_{k\geq 1}\|x_k\|_2^2\right)^{\frac{1}{2}}\right\} \leq 2\left\|\sum_{k\geq 1} x_k\right\|_p,$$

where C is an absolute constant.

3 Noncommutative Baum-Katz theorem and moderate deviation inequality

3.1 Noncommutative Baum-Katz theorem

This subsection is devoted to proving Theorem 1.2. We now begin with a sequence of elementary lemmas on the estimate of the maximal tail probabilities. The first lemma is the weak type asymmetric maximal inequality associated with a convex function.

***Lemma* 3.1** *Let Ψ be a 2-convex, p-concave Orlicz function with $2 \leq p < \infty$ and $x = (x_n)_{n\geq 1}$ be an L_Ψ-bounded martingale. Then for any $\lambda > 0$,*

$$\Psi(\lambda)\mathrm{Prob}_\tau^c\left\{\sup_{n\geq 1}|x_n| > \lambda\right\} \leq c_p\tau\left(\Psi\left(|x|\right)\right),$$

where c_p is a constant depending only on p.

Proof By [21, Lemma 5.4], it follows that for a 1-convex, r-concave Orlicz function Φ with $1 \leq r < \infty$, there exists a projection q such that $\sup_n \|qx_nq\| \leq t$ for any $t > 0$, and

$$\Phi(t)\tau(1-q) \leq c_r\tau(\Phi(|x|)). \tag{3.1}$$

On the other hand, we have

$$\begin{aligned}
&\sup_{\lambda>0}\inf_e\left\{\Psi(\lambda)\tau(1-e) : \sup_{n\geq 1}\|x_ne\|_\infty \leq \lambda\right\}\\
=&\sup_{\lambda>0}\inf_e\left\{\Psi(\lambda^{1/2})\tau(1-e) : \sup_{n\geq 1}\left\|e|x_n|^2e\right\|_\infty \leq \lambda\right\}\\
\leq&\sup_{\lambda>0}\inf_e\left\{\Psi(\lambda^{1/2})\tau(1-e) : \sup_{n\geq 1}\left\|e\mathcal{E}_n(|x|^2)e\right\|_\infty \leq \lambda\right\},
\end{aligned}$$

where we use $\mathcal{E}_n(x)^*\mathcal{E}_n(x) \leq \mathcal{E}_n(x^*x)$ in the inequality ($\mathcal{E}_n$ is the conditional expectation from $\mathcal{M}$ to $\mathcal{M}_n$). By (3.1), we deduce that for any $\lambda > 0$,

$$\Psi(\lambda)\mathrm{Prob}_\tau^c\left\{\sup_{n\geq 1}|x_n| > \lambda\right\} \leq c_p\tau\left(\Psi\left(\left(\mathcal{E}_n(|x|^2)\right)^{1/2}\right)\right) \leq c_p\tau\left(\Psi\left(|x|\right)\right),$$

which completes the proof.

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***Lemma* 3.2** *Let $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ be sequences of random variables. Then, for any $t>0$, we have*

$$\mathrm{Prob}^c_\tau\left\{\sup_{n\geq 1}|x_n+y_n|>t\right\}\leq \mathrm{Prob}^c_\tau\left\{\sup_{n\geq 1}|x_n|>t/2\right\}+\mathrm{Prob}^c_\tau\left\{\sup_{n\geq 1}|y_n|>t/2\right\}. \quad (3.2)$$

Proof For projections p_1 and p_2 satisfying $\|x_np_1\|\leq t/2$ and $\|y_np_2\|\leq t/2$ for all $n\geq 1$, we set $p=p_1\wedge p_2$. Then the triangle inequality yields that

$$\|(x_n+y_n)p\|_\infty\leq \|x_np\|_\infty+\|y_np\|_\infty=\|x_np_1p\|_\infty+\|y_np_2p\|_\infty\leq t.$$

Hence, we have

$$\mathrm{Prob}^c_\tau\left\{\sup_{n\geq 1}|x_n+y_n|>t\right\}\leq \tau(1-p_1)+\tau(1-p_2),$$

and the desired estimate follows by taking the infimum on the right hand side of the inequality.

***Lemma* 3.3** *Let $(x_n)_{n\geq 1}$ be a sequence of random variables. Assume that there exists a projection q such that $x_nq=x_n$ holds for each $n\geq 1$. Then we have*

$$\mathrm{Prob}^c_\tau\left\{\sup_{n\geq 1}|x_n|>t\right\}\leq \tau(q)\quad \textit{for } t>0. \quad (3.3)$$

Proof It suffices to observe that $\|x_n(1-q)\|_\infty=0<t$.

The following four technical lemmas regarding the estimate of the distribution for noncommutative random variables are essential in our proof of Theorem 1.2.

***Lemma* 3.4** *Let $r\geq 1$, $p>0$ and $1/2<r/p\leq 1$. Let $x\in L_0(\mathcal{M})$ and $\psi:\mathbb{R}^+\to\mathbb{R}^+$ be a continuous increasing function with $\psi(t)=O(t^{r-1+r/p})$ for each $t>0$ such that $\{\psi(n)/n^{r/p}\}_{n=1}^\infty$ forms an increasing sequence and (1.1) holds. Then*

$$n\tau\left(xe_{(\psi(n),\infty)}(|x|)\right)/\psi(n)\to 0,\quad n\to\infty.$$

Proof Recall that the generalized inverse ψ^{-1} is defined by

$$\psi^{-1}(s)=\inf\{t>0:\psi(t)>s\},\quad \forall s>0,$$

with $\mathbb{1}_{(n,\infty)}\left(\psi^{-1}(\cdot)\right)=\mathbb{1}_{(\psi(n),\infty)}(\cdot)$ almost everywhere. Thus, by (1.1),

$$\left\|\psi^{-1}(|x|)\right\|_r^r\lesssim\sum_{n=1}^\infty n^{r-1}\tau\left(e_{(n,\infty)}\left(\psi^{-1}(|x|)\right)\right)<\infty.$$

By the functional calculus of $|x|$, we have

$$\left|\tau\left(xe_{(\psi(n),\infty)}(|x|)\right)\right|\leq\tau\left(\left|xe_{(\psi(n),\infty)}(|x|)\right|\right)\leq\tau\left(|x|e_{(\psi(n),\infty)}(|x|)\right).$$

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Since $\{\psi(n)/n^{r/p}\}_{n\geq 1}$ is increasing,

$$n\left|\tau\left(xe_{(\psi(n),\infty)}(|x|)\right)\right|/\psi(n)\lesssim n^{1-r/p}\tau\left(|x|e_{(n,\infty)}\left(\psi^{-1}\left(|x|\right)\right)\right)$$
$$<\tau\left(\left(\psi^{-1}(|x|)\right)^{1-r/p}|x|e_{(n,\infty)}\left(\psi^{-1}\left(|x|\right)\right)\right).$$

We claim that $\left(\psi^{-1}(t)\right)^{1-r/p}t\lesssim\left(\psi^{-1}(t)\right)^{r}$ for each $t>0$. Indeed, this is equivalent to that $\psi\left(t^{p/(rp+r-p)}\right)\lesssim t$, i.e., $\psi(t)=O\left(t^{r-1+r/p}\right)$. Hence, by the fact $\left\|\psi^{-1}(|x|)\right\|_r<\infty$, it follows that

$$n\left|\tau\left(xe_{(\psi(n),\infty)}(|x|)\right)\right|/\psi(n)\lesssim\tau\left(\left(\psi^{-1}(|x|)\right)^{r}e_{(n,\infty)}\left(\psi^{-1}\left(|x|\right)\right)\right)\to 0,\quad n\to\infty.$$

***Lemma* 3.5** *Let $r\geq 1$, $p>0$ and $r/p>1$. Let $x\in L_0(\mathcal{M})$ and $\psi:\mathbb{R}^+\to\mathbb{R}^+$ be a function such that $\{\psi(n)/n^{r/p}\}_{n=1}^{\infty}$ forms an increasing sequence and (1.1) holds. Then*

$$n\tau\left(xe_{(0,\psi(n)]}(|x|)\right)/\psi(n)\to 0,\quad n\to\infty.$$

Proof By the functional calculus of $|x|$, for any $1\leq N\leq n$, we have

$$\left|\tau\left(xe_{(0,\psi(n)]}(|x|)\right)\right|\leq\tau\left(\left|xe_{(0,\psi(n)]}(|x|)\right|\right)\leq\tau\left(|x|e_{(0,\psi(n)]}(|x|)\right)$$
$$=\tau\left(|x|e_{(0,\psi(N)]}(|x|)\right)+\sum_{i=N+1}^{n}\tau\left(|x|e_{(\psi(i-1),\psi(i)]}(|x|)\right).$$

It is clear that $\{n/\psi(n)\}_{n\geq 1}$ is decreasing. Therefore,

$$\psi(n)^{-1}n\tau\left(|x|e_{(0,\psi(n)]}(|x|)\right)\lesssim n^{1-r/p}\psi(N)+\psi(n)^{-1}n\sum_{i=N+1}^{n}\psi(i)\tau\left(e_{(\psi(i-1),\psi(i)]}(|x|)\right)$$
$$\leq n^{1-r/p}\psi(N)+\sum_{i=N+1}^{n}i\tau\left(e_{(\psi(i-1),\psi(i)]}(|x|)\right).$$

By (1.1), i.e., $\sum_{i=2}^{\infty}i\tau\left(e_{(\psi(i-1),\psi(i)]}(|x|)\right)=\sum_{i=1}^{\infty}\tau(\left(e_{(\psi(i),\infty)}(|x|)\right)<\infty$, the desired convergence follows from taking $n\to\infty$ and $N\to\infty$.

***Lemma* 3.6** *For $r\geq 1$ and $p\geq 2$, let $x\in L_0(\mathcal{M})$ and $\psi:\mathbb{R}^+\to\mathbb{R}^+$ be a function such that $\{\psi(n)/n^{r/p}\}_{n=1}^{\infty}$ forms an increasing sequence and (1.1) holds. Then*

$$n\tau\left(|x|^2e_{(0,n^{r/p}]}(|x|)\right)/\psi(n)^2\leq C_r n^{1-2r/p}.$$

Proof Put $\psi(0)=0$ and we have

$$\tau\left(|x|^2e_{(0,\psi(n)]}(|x|)\right)=\sum_{k=1}^{n}\tau\left(|x|^2e_{(\psi(k-1),\psi(k)]}(|x|)\right)$$
$$\leq\sum_{k=1}^{n}\psi(k)^2\tau\left(e_{(\psi(k-1),\psi(k)]}(|x|)\right).$$

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For the positive increasing sequence $\{\psi(n)n^{-r/p}\}_{n\geq 1}$, it follows that

$$\psi(k)/\psi(n) \leq (k/n)^{r/p}$$

for any $1 \leq k \leq n$. Therefore,

$$n\tau\left(|x|^2 e_{(0,\psi(n)]}(|x|)\right)/\psi(n)^2 \leq n^{1-2r/p}\sum_{k=1}^{n} k^{2r/p}\tau\left(e_{(\psi(k-1),\psi(k)]}(|x|)\right).$$

It follows from $p \geq 2$ that

$$\begin{aligned}\sum_{k=1}^{n} k^{2r/p}\tau\left(e_{(\psi(k-1),\psi(k)]}(|x|)\right) &\leq \sum_{k=1}^{n} k^{r}\tau\left(e_{(\psi(k-1),\psi(k))}(|x|)\right)\\ &= \sum_{k=0}^{\infty}\left[(k+1)^r - k^r\right]\tau\left(e_{(\psi(k),\infty)}(|x|)\right)\\ &\leq 1 + r2^{r-1}\sum_{k=1}^{\infty} k^{r-1}\tau\left(e_{(\psi(k),\infty)}(|x|)\right) \coloneqq C_r < \infty,\end{aligned}$$

which yields the desired result.

***Lemma* 3.7** *For $r \geq 1$, $p > 0$, and $s > p$, let $x \in L_0(\mathcal{M})$ and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a function such that $\{\psi(n)/n^{r/p}\}_{n=1}^{\infty}$ forms an increasing sequence and (1.1) holds. Then*

$$\sum_{n=1}^{\infty} n^{r-1}\tau\left(|x|^s e_{(0,\psi(n)]}(|x|)\right)/\psi(n)^s < \infty.$$

Proof Observe that

$$\tau\left(|x|^s e_{(0,\psi(n)]}|x|\right) = \sum_{k=1}^{n}\tau\left(|x|^s e_{(\psi(k-1),\psi(k)]}(|x|)\right) \leq \sum_{k=1}^{n}\psi(k)^s\tau\left(e_{(\psi(k-1),\psi(k)]}(|x|)\right).$$

Hence,

$$\begin{aligned}\sum_{n=1}^{\infty} n^{r-1}\tau\left(|x|^s e_{(0,\psi(n)]}|x|\right)/\psi(n)^s &= \sum_{k=1}^{\infty}\psi(k)^s\tau\left(e_{(\psi(k-1),\psi(k)]}(|x|)\right)\sum_{n=k}^{\infty} n^{r-1}/\psi(n)^s\\ &\leq \sum_{k=1}^{\infty} k^{sr/p}\tau\left(e_{(\psi(k-1),\psi(k)]}(|x|)\right)\sum_{n=k}^{\infty} n^{r-1-sr/p}\\ &\lesssim_{r,p,s} \sum_{k=1}^{\infty} k^{r}\tau\left(e_{(\psi(k-1),\psi(k)]}(|x|)\right)\\ &\lesssim_{r,p,s} 1 + r2^{r-1}\sum_{k=1}^{\infty} k^{r-1}\tau\left(e_{(\psi(k),\infty)}(|x|)\right) < \infty,\end{aligned}$$

where the first inequality is due to the fact $\psi(k)/\psi(n) \leq (k/n)^{r/p}$ for any $n \geq k$.

Now we are ready to prove Theorem 1.2.

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Proof of Theorem 1.2 It suffices to show that (1.1) implies (1.3). By Lemma 3.4 and 3.5, (1.3) is equivalent to

$$\sum_{n=1}^{\infty} n^{r-2}\mathrm{Prob}_{\tau}^{c}\left\{\sup_{1\le m\le n}\left|\sum_{k=1}^{m}\left(x_k-\tau\left(x_k e_{(0,\psi(n)]}|x_k|\right)\right)\right|>\varepsilon\psi(n)\right\}<\infty,\quad \forall\varepsilon>0.$$

Thanks to (3.2), it follows that

$$\mathrm{Prob}_{\tau}^{c}\left\{\sup_{1\le m\le n}\left|\sum_{k=1}^{m}\left(x_k-\tau\left(x_k e_{(0,\psi(n)]}(|x_k|)\right)\right)\right|>\varepsilon\psi(n)\right\}\le A_n+B_n,$$

where

$$A_n\coloneqq \mathrm{Prob}_{\tau}^{c}\left\{\sup_{1\le m\le n}\left|\sum_{k=1}^{m}\left(x_k e_{(\psi(n),\infty)}(|x_k|)\right)\right|>(\varepsilon/2)\psi(n)\right\},$$

and

$$B_n\coloneqq \mathrm{Prob}_{\tau}^{c}\left\{\sup_{1\le m\le n}\left|\sum_{k=1}^{m}\left(x_k e_{(0,\psi(n)]}|x_k|-\tau\left(x_k e_{(0,\psi(n)]}|x_k|\right)\right)\right|>(\varepsilon/2)\psi(n)\right\}.$$

Applying Lemma 3.3, we get

$$A_n\le\tau\left(\bigvee_{k=1}^{n}e_{(\psi(n),\infty)}(|x_k|)\right)\le\sum_{k=1}^{n}\tau\left(e_{(\psi(n),\infty)}(|x_k|)\right).$$

From (1.1), it is immediate that $\sum_{n=1}^{\infty}n^{r-2}A_n<\infty$. It now remains to show that $\sum_{n=1}^{\infty}n^{r-2}B_n<\infty$. For any $s\ge 2$, it follows from Remark 2.2 (2) and Lemma 3.1 with $\Psi(t)=t^s$ for all $t\ge 0$ that

$$B_n\lesssim_{s,\varepsilon}\psi(n)^{-s}\left\|\sum_{k=1}^{n}\left(x_k e_{(0,\psi(n)]}|x_k|-\tau\left(x_k e_{(0,\psi(n)]}|x_k|\right)\right)\right\|_s^s.$$

Note here that $\{x_k e_{(0,\psi(n)]}|x_k|-\tau\left(x_k e_{(0,\psi(n)]}|x_k|\right)\}_{k\ge1}$ is a mean zero sequence, then it follows from Theorem 2.3 that

$$\begin{aligned}&\left\|\sum_{k=1}^{n}\left(x_k e_{(0,\psi(n)]}|x_k|-\tau\left(x_k e_{(0,\psi(n)]}|x_k|\right)\right)\right\|_s^s\\ &\lesssim_{s,\varepsilon}\left(\sum_{k=1}^{n}\left\|x_k e_{(0,\psi(n)]}(|x_k|)\right\|_s^s+\left(\sum_{k=1}^{n}\left\|x_k e_{(0,\psi(n)]}|x_k|\right\|_2^2\right)^{s/2}\right)\\ &\lesssim_{s,\varepsilon}n\tau\left(|x|^s e_{(0,\psi(n)]}(|x_k|)\right)+\left(n\tau\left(|x|^2 e_{(0,\psi(n)]}(|x_k|)\right)\right)^{s/2}.\end{aligned}$$

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Therefore, the following inequality holds

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-2} B_n \\
&\lesssim \sum_{n=1}^{\infty} n^{r-1}\psi(n)^{-s}\tau\left(|x|^s e_{(0,\psi(n)]}(|x_k|)\right) + \sum_{n=1}^{\infty} n^{r-2}\psi(n)^{-s}\left(n\tau\left(|x|^2 e_{(0,\psi(n)]}(|x_k|)\right)\right)^{s/2} \\
&:= I_1 + I_2.
\end{aligned}$$

For $p \geq 2$, let $s > (p - p/r)/(1 - p/2r)$ (i.e., $r - 2 - sr/p + s/2 < -1$). Then it follows from Lemma 3.6 that

$$\begin{aligned}
\psi(n)^{-s}\left(n\tau\left(|x|^2 e_{(0,\psi(n)]}(|x_k|)\right)\right)^{s/2} &= \left(\psi(n)^{-2}\cdot n\tau\left(|x|^2 e_{(0,\psi(n)]}(|x_k|)\right)\right)^{s/2} \\
&\lesssim n^{-sr/p+s/2}.
\end{aligned}$$

Thus $I_2 \lesssim \sum_{n=1}^{\infty} n^{r-2-sr/p+s/2} < \infty$. Note that $s > (p - p/r)/(1 - p/2r) > p$. Then it follows from Lemma 3.7 that $I_1 < \infty$. For $0 < p < 2$, let $s = 2$ and note that $I_1 = I_2$ in this case. Hence, Lemma 3.7 implies that $I_1 = I_2 < \infty$, which completes our proof in full generality.

3.2 Moderate deviation inequality

In this subsection, we provide a proof of Theorem 1.4, and the following two lemmas are needed in our approach. Lemma 3.9 is a noncommutative analogue of [11, Lemma 1].

***Lemma* 3.8** *Let $1 < p < \infty$ and $x \in L_p(\mathcal{M})$. Then*

$$n^{1-1/p}(\log n)^{-1/p}\tau\left(xe_{[(n\log n)^{1/p},\infty)}(|x|)\right) \to 0, \quad n \to \infty.$$

Proof Since $n^{1-1/p}(\log n)^{-1/p} \leq (n\log n)^{1-1/p}$ for $n \geq 3$, we have

$$\begin{aligned}
n^{1-1/p}(\log n)^{-1/p}\left|\tau\left(xe_{[(n\log n)^{1/p},\infty)}(|x|)\right)\right| &\leq (n\log n)^{1-1/p}\tau\left(|x|\, e_{[(n\log n)^{1/p},\infty)}(|x|)\right) \\
&\leq \tau\left(|x|^p\, e_{[(n\log n)^{1/p},\infty)}(|x|)\right).
\end{aligned}$$

The desired result follows from the assumption $x \in L_p(\mathcal{M})$.

***Lemma* 3.9** *Let $1 \leq p < \infty$ and $x \in L_p(\mathcal{M})$. Then*

$$\sum_{n=1}^{\infty} n^{p-1}(\log n)^p \tau\left(e_{(n\log n,\infty)}(|x|)\right) < \infty.$$

Proof It is clear that the function $t \mapsto t\log t$ is an increasing convex function on $[1,\infty)$, and let φ be its inverse function, that is $\varphi(t\log t) = t$ for all $t \geq 1$. For $y = \varphi(|x|)$,

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we have

$$\begin{aligned}\sum_{n=1}^{\infty} n^{p-1}(\log n)^p \tau\left(e_{(n\log n,\infty)}(|x|)\right) &= \sum_{n=1}^{\infty} n^{p-1}(\log n)^p \tau\left(e_{(n,\infty)}(y)\right)\\ &\lesssim \sum_{k=1}^{\infty} \tau\left(e_{(k,k+1]}(y)\right)\sum_{n=1}^{k} n^{p-1}(\log n)^p\\ &\lesssim \sum_{k=1}^{\infty}(k\log k)^p \tau\left(e_{(k,k+1]}(y)\right)\\ &\lesssim \|y\log y\|_p^p = \|\varphi^{-1}(y)\|_p^p = \|x\|_p^p,\end{aligned}$$

which completes the proof.

Now we give the proof of Theorem 1.4 for $1 < p < 2$.

***Proof of Theorem 1.4** (2)* Let $a_n = n\log n$ for $n \geq 1$. By Lemma 3.8, it suffices to prove that

$$\sum_{n=1}^{\infty} \frac{\log n}{n}\mathrm{Prob}_\tau^c\left\{\sup_{1\leq m\leq n}\left|\sum_{k=1}^{m}\left(x_k - \tau\left(x_k e_{(0,a_n^{1/p})}(|x_k|)\right)\right)\right| > \varepsilon a_n^{1/p}\right\} < \infty, \quad \forall\varepsilon > 0.$$

It follows from (3.2) and Lemma 3.3 that

$$\mathrm{Prob}_\tau^c\left\{\sup_{1\leq m\leq n}\left|\sum_{k=1}^{m}\left(x_k - \tau\left(x_k e_{(0,a_n^{1/p}}(|x_k|)\right)\right)\right| > \varepsilon a_n^{1/p}\right\} \leq n\tau\left(e_{[a_n^{1/p},\infty)}(|x_k|)\right)+C_n,$$

where

$$C_n \coloneqq \mathrm{Prob}_\tau^c\left\{\sup_{1\leq m\leq n}\left|\sum_{k=1}^{m}\left(x_k e_{(0,a_n^{1/p})}(|x_k|) - \tau\left(x_k e_{(0,a_n^{1/p})}(|x_k|)\right)\right)\right| > (\varepsilon/2)\, a_n^{1/p}\right\}.$$

From Lemma 3.9, $\|x\|_p < \infty$ implies that $\sum_{n=1}^{\infty}\log n\tau\left(e_{[a_n^{1/p},\infty)}(|x|)\right) < \infty$. It remains to show that $\sum_{n=1}^{\infty}(\log n/n)\, C_n < \infty$. Applying Lemma 3.1 with $\Psi(\cdot) = (\cdot)^2$ and the orthogonality of martingale differences yields that

$$C_n \lesssim (a_n)^{-2/p}\sum_{k=1}^{n}\left\|x_k e_{(0,a_n^{1/p})}(|x_k|)\right\|_2^2 \leq (a_n)^{-2/p} n\left\|x e_{(0,a_n^{1/p})}(|x|)\right\|_2^2.$$

Therefore,

$$\begin{aligned}\sum_{n=2}^{\infty}\frac{\log n}{n}C_n &\lesssim \sum_{n=2}^{\infty} n^{-2/p}(\log n)^{1-2/p}\sum_{i=2}^{n} a_i^{2/p}\tau\left(e_{[a_{i-1},a_i)}(|x|^p)\right)\\ &= \sum_{i=2}^{\infty} a_i^{2/p}\tau\left(e_{[a_{i-1},a_i)}(|x|^p)\right)\sum_{n=i}^{\infty} n^{-2/p}(\log n)^{1-2/p}\\ &\lesssim \sum_{i=2}^{\infty} a_i\tau\left(e_{[a_{i-1},a_i)}(|x|^p)\right) < \infty,\end{aligned}$$

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where the last inequality follows from $\|x\|_p < \infty$. Hence, we complete the proof.

For the case $p = 2$, Davis' symmetrization method heavily relying on the Levý inequality which is not yet well developed in the noncommutative setting. Instead, our approach relies on the following modified noncommutative Fuk-Nagaev inequality.

Lemma 3.10 *Let $\{x_k\}_{k\geq 1} \subseteq L_2(\mathcal{M})$ be a sequence of successively independent self-adjoint random variables, which have the same distribution of $x \in L_2(\mathcal{M})$ such that $\tau(x) = 0$. Let $S_n = \sum_{k=1}^n x_k$ for each $n \geq 1$. Then, for any $t_1, t_2 > 0$,*

$$\begin{aligned}\tau\left(e_{[t_1,\infty)}(|S_n|)\right) &\leq n\tau\left(e_{[t_2,\infty)}(|x|)\right) + 2\left(\frac{en\|x\|_2^2}{n\|x\|_2^2 + t_1t_2}\right)^{t_1/t_2}\\ &\leq n\tau\left(e_{[t_2,\infty)}(|x|)\right) + 2\left(\frac{en\|x\|_2^2}{t_1t_2}\right)^{t_1/t_2}.\end{aligned}$$

Proof By the proof of [7, Theorem 2], it is clear that

$$\begin{aligned}&\tau\left(e_{[t_1,\infty)}(S_n)\right)\\ \leq& n\tau\left(e_{[t_2,\infty)}(x)\right) + \exp\left\{\frac{t_1}{t_2} - \left[\frac{1}{t_2}\left(t_1 - n\tau(xe_{(0,t_2]}(|x|))\right) + \frac{n\|x\|_2^2}{t_2^2}\right]\log\left(\frac{t_1t_2}{n\|x\|_2^2} + 1\right)\right\}.\end{aligned}$$

Since x is a self-adjoint element in $L_2(\mathcal{M})$ and $\tau(x) = 0$, we get that

$$\left|\tau(xe_{(0,t_2]}(|x|))\right| = \left|\tau(xe_{(t_2,\infty)}(|x|))\right| \leq \|x\|_2\tau(e_{(t_2,\infty)}(x))^{1/2} \leq \|x\|_2^2/t_2,$$

where we used the Cauchy-Schwarz inequality in the first inequality and the Chebyshev inequality $\tau(e_{(t_2,\infty)}(|x|)) \leq \|x\|_2^2/t_2^2$ in the last inequality. Therefore, we have

$$\frac{1}{t_2}\left(t_1 - n\tau(xe_{(0,t_2]}(|x|))\right) + \frac{n\|x\|_2^2}{t_2^2} \geq t_1/t_2,$$

and

$$\tau\left(e_{[t_1,\infty)}(S_n)\right) \leq n\tau\left(e_{[t_2,\infty)}(x)\right) + e^{t_1/t_2}\left(\frac{n\|x\|_2^2}{n\|x\|_2^2 + t_1t_2}\right)^{t_1/t_2}.$$

By replacing x with $-x$ and x_k with $-x_k$, the desired result follows.

Now we give the proof of Theorem 1.4 for the case $p = 2$.

***Proof of Theorem** 1.4* (1) For the case $\{x_k\}_{k\geq 1}$ are self-adjoint, we apply Lemma 3.10 with parameters $t_1 = \varepsilon\sqrt{n\log n}$ and $t_2 = \varepsilon\sqrt{n\log n}/3$ to get that

$$\begin{aligned}\sum_{n\geq 2}\frac{\log n}{n}\tau\left(e_{[\varepsilon\sqrt{n\log n},\infty)}(|S_n - n\tau(x)|)\right) &\lesssim \sum_{n\geq 2}\log n\tau\left(e_{[\varepsilon\sqrt{n\log n}/3,\infty)}(|x - \tau(x)|)\right)\\ &+ \sum_{n\geq 2}\frac{1}{n(\log n)^2}.\end{aligned}$$

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Note that the fact $x \in L_2(\mathcal{M})$ implies the convergence of the series. For general random variables, it suffices to split the sequence into the complex combination of two self-adjoint sequences and apply the triangle inequality of distribution function (2.1) to conclude the proof.

It is easy to derive the convergence rate for the law of the iterated logarithm as follows.

***Corollary* 3.11** *Let $\{x_k\}_{k\geq 1} \subseteq L_2(\mathcal{M})$ be a sequence of successively independent self-adjoint random variables, which have the same distribution of $x \in L_2(\mathcal{M})$. Let $S_n = \sum_{k=1}^n x_k$ for each $n \geq 1$. Then for any $\varepsilon > 0$, we have*

$$\sum_{n=3}^{\infty} \frac{1}{n\log n} \tau\left(e_{(\varepsilon(n\log\log n)^{1/2},\infty)}\left(|S_n - n\tau(x)|\right)\right) < \infty.$$

Proof Let $a_n = n\log\log n$ for $n \geq 3$. Applying Lemma 3.10 with $t_1 = \varepsilon\sqrt{a_n}$ and $t_2 = \varepsilon\sqrt{a_n}/2$ yields that

$$\begin{aligned}\sum_{n\geq 3} \frac{1}{n\log n} \tau\left(e_{[\varepsilon\sqrt{a_n},\infty)}(|S_n - n\tau(x)|)\right) &\lesssim \sum_{n\geq 3} \frac{1}{\log n} \tau\left(e_{[\sqrt{a_n}/2,\infty)}(|x-\tau(x)|)\right) \\ &\quad + \sum_{n\geq 3} \frac{1}{n\log n(\log\log n)^2}.\end{aligned}$$

Observe that

$$\begin{aligned}\sum_{n\geq 3} \frac{1}{\log n} \tau\left(e_{[\sqrt{a_n},\infty)}(|x-\tau(x)|)\right) &= \sum_{k\geq 3}\left(\sum_{3\leq n\leq k} \frac{1}{\log n}\right) \tau\left(e_{[a_n,a_{n+1})}(|x-\tau(x)|^2)\right) \\ &\leq \sum_{k\geq 3} a_k \tau\left(e_{[a_n,a_{n+1})}(|x-\tau(x)|^2)\right) \lesssim \|x\|_2^2 .\end{aligned}$$

Hence, we complete the proof.

4 Baum-Katz Theorem and moderate deviation inequality for noncommutative martingales

4.1 Baum-Katz theorem for noncommutative martingales

This subsection is devoted to establishing a Baum-Katz type theorem for noncommutative martingales. We begin with recalling the following technical fact from Balka and Tómács [1].

Fact 4.1 Suppose that f is a positive non-decreasing function defined on $[0,\infty)$ with $\sum_{n=1}^{\infty} \frac{1}{f(2^{cn})} < \infty$ for some $c > 0$. Then the following holds

(1) $\sum_{n=1}^{\infty} 1/(nf(n^c)) < \infty$ for some $c > 0$,
(2) $\sum_{n=1}^{\infty} 1/(nf(\varepsilon n^c)) < \infty$ for all $\varepsilon, c > 0$.

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Remark 4.2 In what follows, we may assume that $\lim_{n\to\infty} f(2^{n+1})/f(2^n) = 1$ if $\sum_{n=1}^{\infty} \frac{1}{f(2^n)} < \infty$. Indeed, if the f satisfies $\sum_{n=1}^{\infty} \frac{1}{f(2^n)} < \infty$, then by [1, Corollary 3.2] there exists a positive non-decreasing function g defined on $[0,\infty)$ such that $\sum_{n=1}^{\infty} \frac{1}{g(2^n)} < \infty$, $\limsup_{t\to\infty} g(t)/f(t) \le 1$ and $\lim_{n\to\infty} g(2^{n+1})/g(2^n) = 1$. It now suffices to replace f by g in the proof.

***Lemma* 4.3** ([1]) *Suppose that f is a positive non-decreasing function defined on $[0,\infty)$ with $\sum_{n=1}^{\infty} \frac{1}{f(2^n)} < \infty$. Then the following holds*

(1) for all $p > 0$ there is a $c_p > 0$ such that the function $h_p(x) = x^{-p} f(c_p x)$ is decreasing for $x \ge 1$,
(2) for all $q \ge 1$ there is a convex increasing function $f_q : [0,\infty) \to \mathbb{R}^+$ and $N_q > 0$ such that f_q is affine on $[0, N_q]$ and $f_q(x) = x^q f(\sqrt{x})$ for $x \ge N_q$.

The following lemma is elementary and we omit its proof.

***Lemma* 4.4** *Let $\{x_n\}_{n\ge1}$ be a sequence of random variables in $L_0(\mathcal{M})$. Let $\phi, \rho : [0,\infty) \to \mathbb{R}^+$be non-decreasing functions such that $\limsup_{t\to\infty} \rho(t)/\phi(t) < \infty$. Then $\sup_{n\ge1} \tau(\phi(|x_n|)) < \infty$ implies that $\sup_{n\ge1} \tau(\rho(|x_n|)) < \infty$.*

Before showing the proof of Theorem 1.6, we need the following weak type asymmetric maximal inequality which can be viewed as a suitable substitute of the weak type $(1,1)$ Doob inequality.

***Lemma* 4.5** *Let $(x_k)_{k\ge1}$ be a noncommutative martingale in $L_1(\mathcal{M})$. Then for any positive integer n, the following holds*

$$\mathrm{Prob}_\tau^c \left\{ \sup_{1\le k\le n} |x_k| > t \right\} \le \frac{C}{t} \sum_{k=1}^{n} \|dx_k\|_1, \quad \forall t > 0,$$

where C is an absolute constant.

Proof Applying Theorem 2.1 with $p = 1$ and Minkowski inequality, it follows that for each $t > 0$ and any positive integer n, there exists a projection e such that $\sup_{1\le k\le n} \|x_k e\|_\infty \le t$ and

$$\begin{aligned} t\tau(1-e) &\le C \left\| \left(\sum_{k=1}^{n} |dx_k|^2 \right)^{1/2} \right\|_1 = C \left\| \sum_{k=1}^{n} |dx_k|^2 \right\|_{1/2}^{1/2} \\ &\le C \sum_{k=1}^{n} \left\| |dx_k|^2 \right\|_{1/2}^{1/2} = C \sum_{k=1}^{n} \|dx_k\|_1 . \end{aligned}$$

We now provide the proof of Theorem 1.6.

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Proof of Theorem 1.6 We assume that $\sup_{i\geq 1}\tau\left(|dx_i|^p f(|dx_i|)\right) = C < \infty$ with $x_k = \sum_{i=1}^k dx_i$ for each $1 \leq k \leq n$ and $n \geq 1$. For each $1 \leq i \leq n$, we define

$$y_{i,n} = dx_i e_{(0,n^{r/p}]}(|dx_i|) - \mathcal{E}_{i-1}\left(dx_i e_{(0,n^{r/p}]}(|dx_i|)\right),$$
$$z_{i,n} = dx_i e_{(n^{r/p},\infty)}(|dx_i|) - \mathcal{E}_{i-1}\left(dx_i e_{(n^{r/p},\infty)}(|dx_i|)\right).$$

Let $A_k^n = \sum_{i=1}^k y_{i,n}$ and $B_k^n = \sum_{i=1}^k z_{i,n}$ for all $1 \leq k \leq n$ and $n \geq 1$. It is easy to see that both $\{A_k^n\}_{1\leq k\leq n}$ and $\{B_k^n\}_{1\leq k\leq n}$ are martingales and $x_k = A_k^n + B_k^n$. By Lemma 3.2, we have

$$\begin{aligned}
&\mathrm{Prob}_\tau^c\left\{\sup_{1\leq k\leq n}|x_k| > \varepsilon n^{r/p}\right\}\\
\leq&\mathrm{Prob}_\tau^c\left\{\sup_{1\leq k\leq n}\left|A_k^n\right| > (\varepsilon/2)n^{r/p}\right\} + \mathrm{Prob}_\tau^c\left\{\sup_{1\leq k\leq n}\left|B_k^n\right| > (\varepsilon/2)n^{r/p}\right\}\\
:=&\mathrm{I}_n + \mathrm{II}_n.
\end{aligned}$$

To estimate x we apply Lemma 4.5 to obtain

$$\mathrm{II}_n \lesssim \left((\varepsilon/2)n^{r/p}\right)^{-1}\sum_{i=1}^n \|z_{i,n}\|_1 \lesssim n^{-r/p}\sum_{i=1}^n \tau\left(|dx_i| e_{(n^{r/p},\infty)}(|dx_i|)\right). \tag{4.1}$$

It follows from the Borel functional calculus for $|dx_i|$ that

$$\begin{aligned}
\tau\left(|dx_i| e_{(n^{r/p},\infty)}(|dx_i|)\right) &\leq \frac{\tau\left(|dx_i|\,|dx_i|^{p-1} f(|dx_i|)\, e_{(n^{r/p},\infty)}|dx_i|\right)}{n^{(p-1)r/p} f\left(n^{r/p}\right)}\\
&= \frac{n^{r/p-r}}{f\left(n^{r/p}\right)}\tau\left(|dx_i|^p f(|dx_i|)\right) \lesssim \frac{n^{r/p-r}}{f\left(n^{r/p}\right)}.
\end{aligned} \tag{4.2}$$

Combining (4.1), (4.2) and Fact 4.1(1) entails that

$$\sum_{n=1}^\infty n^{r-2}\mathrm{II}_n \lesssim \sum_{n=1}^\infty \frac{1}{nf\left(n^{r/p}\right)} < \infty.$$

To estimate I_n, we apply Lemma 3.1 with $\Psi(t) = t^2$ for all $t \geq 0$ to get

$$\begin{aligned}
\mathrm{I}_n &\lesssim \left(\varepsilon^2 n^{2r/p}\right)^{-1}\tau\left(|A_n^n|^2\right) = \frac{1}{\varepsilon^2}n^{-2r/p}\sum_{i=1}^n \tau\left(|y_{i,n}|^2\right)\\
&\lesssim \left(\varepsilon^2 n^{2r/p}\right)^{-1}\sum_{i=1}^n \tau\left(|dx_i|^2 e_{(0,n^{r/p}]}(|dx_i|)\right).
\end{aligned}$$

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Let $q = (2-p)r/p$ and it follows from (2.3) that

$$\begin{aligned}\sum_{n=1}^{\infty} n^{r-2}\mathrm{I}_n &\lesssim \sum_{n=1}^{\infty} n^{-q-2}\sum_{i=1}^{n}\int_0^{\infty} t^2 \mathbb{1}_{(0,n^{r/p}]}(t)\mathrm{d}\tau(e_t)\\ &= \sum_{i=1}^{\infty}\int_0^{\infty} t^2 \sum_{n\geq \max\{i,t^{p/r}\}} n^{-q-2}\,\mathrm{d}\tau(e_t)\\ &\lesssim \sum_{i=1}^{\infty}\int_0^{\infty} t^2\left(\max\left\{i,t^{p/r}\right\}\right)^{-q-1}\,\mathrm{d}\tau(e_t) \coloneqq \sum_{i=1}^{\infty}\left(I_i^1+I_i^2+I_i^3\right),\end{aligned}$$

where

$$I_i^1 = \int_0^1 t^2 i^{-q-1}\,\mathrm{d}\tau(e_t),\ I_i^2 = \int_1^{i^{r/p}} t^2 i^{-q-1}\,\mathrm{d}\tau(e_t),\ I_i^3 = \int_{i^{r/p}}^{\infty} t^{p-p/r}\,\mathrm{d}\tau(e_t).$$

Note here that $\sum_{i=1}^{\infty} I_i^1 \leq \sum_{i=1}^{\infty} i^{-q-1} < \infty$. It follows from Lemma 4.3 (1) that $t^{p-2}f(c_p t)$ is non-increasing for $t \geq 1$ and some $c_p > 0$. Combining that $t^{p/r}f(c_p t)$ is non-decreasing yields that

$$I_i^2 \leq i^{-q-1}\int_1^{i^{r/p}} t^2\frac{t^{p-2}f(c_p t)}{i^{(p-2)r/p}f\left(c_p i^{r/p}\right)}\mathrm{d}\tau(e_t),\ I_i^3 \leq \int_{i^{r/p}}^{\infty} t^{p-p/r}\frac{t^{p/r}f(c_p t)}{if\left(c_p i^{r/p}\right)}\mathrm{d}\tau(e_t).$$

By Remark 4.2 and Lemma 4.4 we have

$$\int_0^{\infty} t^p f(c_p t)\mathrm{d}\tau(e_t) = \tau\left(|dx_i|^p f\left(c_p|dx_i|\right)\right) \lesssim C.$$

Notice that Fact 4.1(2) implies $\sum_{i=1}^{\infty}\left(I_i^2+I_i^3\right) \lesssim \sum_{i=1}^{\infty} C/\left(if\left(c_p i^{r/p}\right)\right) < \infty$. Hence,

$$\sum_{n=1}^{\infty} n^{r-2}\mathrm{I}_n \lesssim \sum_{i=1}^{\infty}\left(I_i^1+I_i^2+I_i^3\right) < \infty.$$

Finally,

$$\sum_{n=1}^{\infty} n^{r-2}\mathrm{Prob}_{\tau}^{c}\left\{\sup_{1\leq k\leq n}|x_k| > \varepsilon n^{r/p}\right\} \leq \sum_{n=1}^{\infty} n^{r-2}\mathrm{I}_n + \sum_{n=1}^{\infty} n^{r-2}\mathrm{II}_n < \infty,$$

which yields the desired conclusion.

To show the case for $p \geq 2$, we recall the following noncommutative Burkholder-Gundy inequality for Orlicz norms [22, Theorem 7.2].

***Theorem* 4.6** *Let Φ be a 2-convex and q-concave Orlicz function for some $2 \leq q < \infty$ and $(x_k)_{k\geq 0}$ be a sequence martingale differences that is bounded in $L_\Phi(\mathcal{M})$. Then*

$$\tau\left(\Phi\left(\left|\sum_{k\geq 0} x_k\right|\right)\right) \approx_\Phi \max\left\{\tau\left(\Phi\left[\left(\sum_{k\geq 0}|x_k|^2\right)^{1/2}\right]\right), \tau\left(\Phi\left[\left(\sum_{k\geq 0}|x_k^*|^2\right)^{1/2}\right]\right)\right\}.$$

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***Lemma* 4.7** *Let f be the function fulfilling (1.4). Then, for any $q \geq 2$, we can modify f to obtain a 2-convex and s-concave function Φ_q for some $2 \leq s < \infty$.*

Proof By Lemma 4.3 (2), there is an increasing convex function $f_{q/2} : [0, \infty) \to (0, \infty)$ and positive numbers N_q, such that $f_{q/2}$ is linear on $[0, N_q]$, and $f_{q/2}(t) = t^{q/2} f(\sqrt{t})$ for all $t \geq N_q$. Define $\Phi_q : [0, \infty) \to [0, \infty)$ by setting $\Phi_q(t) = f_{q/2}(t^2)$. Obviously, Φ_q is increasing and 2-convex. Clearly, $\Phi_q(0) = 0$ and $\Phi_q(t) = ct^2$ for $t \in [0, \sqrt{N_q}]$ with some constant $c > 0$. For $t \geq \sqrt{N}_q$ we have $\Phi_q(t) = t^q f(t)$. It follows from Remark 4.2 that Φ_q satisfies the Δ_2-condition. Hence, there exists an $s \in [2, \infty)$ such that Φ_q is s-concave.

Now we give the proof of Theorem 1.7.

Proof of Theorem 1.7 Assume without loss of generality that $x_k = \sum_{i=1}^k dx_i$ is self-adjoint for each $k \in \mathbb{N}^+$. Since $q = \frac{2p(r-1)}{2r-p} \geq 2$, it follows from Lemma 4.7 and Lemma 4.4 that there exists a 2-convex and s-concave function Φ_q for some $2 \leq s < \infty$ with

$$\sup_{i\geq 1} \tau\left(\Phi_q\left(|dx_i|\right)\right) \lesssim \sup_{i\geq 1} \tau\left(|dx_i|^q f\left(|dx_i|\right)\right) < \infty.$$

Let $K = \sup_{i\geq 1} \tau\left(\Phi_q\left(|dx_i|\right)\right)$ and we apply Lemma 3.1 to the finite martingale $\{x_k/\sqrt{n}\}_{1\leq k\leq n}$ to get that

$$\mathrm{Prob}_\tau^c\left\{\sup_{1\leq k\leq n} |x_k| > \varepsilon n^{r/p}\right\} \lesssim_r \frac{\tau\left[\Phi_q\left(|x_n/\sqrt{n}|\right)\right]}{\Phi_q\left(\varepsilon n^{r/p-1/2}\right)}.$$

It follows from Theorem 4.6 that

$$\tau\left[\Phi_q\left(|x_n|/\sqrt{n}\right)\right] \lesssim \tau\left(\Phi_q\left[\left(\sum_{i=1}^n |dx_i|^2/n\right)^{\frac{1}{2}}\right]\right) = \tau\left(f_{q/2}\left(\sum_{i=1}^n |dx_i|^2/n\right)\right).$$

Applying the Jensen inequality (2.4) to the convex function $f_{q/2}$ yields that

$$\tau\left(f_{q/2}\left(\sum_{i=1}^n |dx_i|^2/n\right)\right) \leq (1/n)\sum_{i=1}^n \tau\left(f_{q/2}\left(|dx_i|^2\right)\right) = (1/n)\sum_{i=1}^n \tau\left(\Phi_q\left(|dx_i|\right)\right) \leq K.$$

For $\varepsilon n^{r/p-1/2} \geq \sqrt{N_q}$, that is, $n \geq \left(\sqrt{N_q}/\varepsilon\right)^{2p/(2r-p)} \coloneqq n_0$, we have

$$\begin{aligned}\mathrm{Prob}_\tau^c\left\{\sup_{1\leq k\leq n} |x_k| > \varepsilon n^{r/p}\right\} &\leq \frac{K}{\Phi_q\left(\varepsilon n^{r/p-1/2}\right)} \\ &= \frac{K}{\varepsilon^q n^{q(r/p-1/2)} f\left(\varepsilon n^{r/p-1/2}\right)} = \frac{Kn^{1-r}}{\varepsilon^q f\left(\varepsilon n^{r/p-1/2}\right)}.\end{aligned}$$

Hence, taking the sum over all $n \in \mathbb{N}^+$ and applying Fact 4.1(2) yield that

$$\sum_{n\geq n_0} n^{r-2}\mathrm{Prob}_\tau^c\left\{\sup_{1\leq k\leq n} |x_k| > \varepsilon n^{r/p}\right\} \lesssim \sum_{n\geq n_0} \frac{K}{\varepsilon^q n f\left(\varepsilon n^{r/p-1/2}\right)} < \infty.$$

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For general martingales, it suffices to apply the hermitian dilation technique to reduce to the self-adjoint martingale case.

4.2 Moderate deviation inequality for martingale differences

Before showing the proof of Theorem 1.10, we need to recall the following deviation inequality for noncommutative martingales taken from [19, Lemma 4.3].

***Lemma* 4.8** *Let $1 < p < \infty$, and $(x_k)_{k\geq 1}$ in $L_p(\mathcal{M})$ be a noncommutative martingale. Suppose that $\sup_k \|dx_k\|_p \leq K$ for some $K > 0$. Then there exists $C_p > 0$ such that*

$$\|x_n\|_p \leq C_p n^{\frac{\max\{2,p\}}{2p}} K, \quad \forall n \geq 1.$$

We now provide the proof of Theorem 1.10 as follows.

Proof of Theorem 1.10 For each $1 \leq i \leq n$ and some $a > 0$, we define

$$\begin{aligned} u_{i,n} &= dx_i e_{(0,(\log n)^a]}(|dx_i|) - \mathcal{E}_{i-1}\left(dx_i e_{(0,(\log n)^a]}(|dx_i|)\right), \\ v_{i,n} &= dx_i e_{((\log n)^a,\infty)}(|dx_i|) - \mathcal{E}_{i-1}\left(dx_i e_{((\log n)^a,\infty)}(|dx_i|)\right). \end{aligned}$$

Let $M_k^n = \sum_{i=1}^k u_{i,n}$ and $N_k^n = \sum_{i=1}^k v_{i,n}$ for all $1 \leq k \leq n$ and $n \geq 1$. It is easy to see that both $\{M_k^n\}_{1\leq k\leq n}$ and $\{N_k^n\}_{1\leq k\leq n}$ are martingales. Denoted by $x_k = \sum_{i=1}^k dx_i$ for $1 \leq k \leq n$, it follows that $x_k = M_k^n + N_k^n$. By Lemma 3.2, we have

$$\begin{aligned} \mathrm{Prob}_\tau^c \left\{ \sup_{1\leq k\leq n} |x_k| > \varepsilon(n\log n)^{1/2} \right\} &\leq \mathrm{Prob}_\tau^c \left\{ \sup_{1\leq k\leq n} \left|M_k^n\right| > (\varepsilon/2)(n\log n)^{1/2} \right\} \\ &\quad + \mathrm{Prob}_\tau^c \left\{ \sup_{1\leq k\leq n} \left|N_k^n\right| > (\varepsilon/2)(n\log n)^{1/2} \right\} \\ &\coloneqq I_n^* + II_n^*. \end{aligned}$$

Applying Lemma 3.1 with $\Psi(t) = t^2$ for $t > 0$, it follows from the orthogonality of martingale differences that

$$II_n^* \lesssim (n\log n)^{-1} \sum_{i=1}^n \left\|v_{i,n}\right\|_2^2 \lesssim (n\log n)^{-1} \sum_{i=1}^n \tau\left(|dx_i|^2 e_{((\log n)^a,\infty)}(|dx_i|)\right).$$

By the Hölder inequality and Chebyshev inequality (2.2),

$$\tau\left(|dx_i|^2 e_{((\log n)^a,\infty)}(|dx_i|)\right) \leq \left\||dx_i|^2\right\|_{p/2} \left\|e_{((\log n)^a,\infty)}(|dx_i|)\right\|_{p/(p-2)} \lesssim (\log n)^{(2-p)a}.$$

Hence, we have $II_n^* \lesssim (\log n)^{(2-p)a-1}$. For some $q > 2$, applying Lemma 4.8 to $\left(M_k^n\right)_{1\leq k\leq n}$ with $K = (\log n)^a$, it follows that

$$\left\|M_n^n\right\|_q \lesssim n^{1/2}(\log n)^a.$$

By Lemma 3.1, we obtain that

$$I_n^* \lesssim (n\log n)^{-q/2} \left\|M_n^n\right\|_q^q \lesssim (\log n)^{aq-q/2}.$$

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Therefore,

$$\sum_{n=2}^{\infty} \frac{\log n}{n} \mathrm{Prob}_{\tau}^{c} \left\{ \sup_{1\le k\le n} |x_k| > \varepsilon (n \log n)^{1/2} \right\} \lesssim \sum_{n=2}^{\infty} \frac{(\log n)^{aq-q/2+1}}{n} + \sum_{n=2}^{\infty} \frac{(\log n)^{(2-p)a}}{n}.$$

The series converges for the range $p > 4$, $q > \frac{4(p-2)}{p-4}$ and $\frac{1}{p-2} < a < \frac{q-4}{2q}$.

Inspired by Zeng's result [51], we study the convergence rate of the iterated logarithm for noncommutative martingales.

***Corollary* 4.9** *Let $p > 2$ and $\{dx_i\}_{i\ge1}$ be an L_p-bounded noncommutative martingale difference sequence. Then for any $\varepsilon > 0$, we have*

$$\sum_{n=3}^{\infty} \frac{1}{n \log n} \mathrm{Prob}_{\tau}^{c} \left\{ \sup_{1\le k\le n} \left| \sum_{i=1}^{k} dx_i \right| > \varepsilon (n \log \log n)^{1/2} \right\} < \infty.$$

Proof By the same arguments used in the proof of Theorem 1.10, for some $a > 0$ and $q > 2$, we have

$$\sum_{n=3}^{\infty} \frac{1}{n \log n} \mathrm{Prob}_{\tau}^{c} \left\{ \sup_{1\le k\le n} \left| \sum_{i=1}^{k} dx_i \right| > \varepsilon (n \log \log n)^{1/2} \right\}$$
$$\lesssim \sum_{n=3}^{\infty} \frac{(\log \log n)^{aq-q/2}}{n \log n} + \sum_{n=3}^{\infty} \frac{(\log \log n)^{(2-p)a-1}}{n \log n}.$$

The series converges by taking $0 < a < (q-2)/2q$.

It is worthwhile to mention that for $1 \le p \le 2$, there exists an L_p-bounded martingale difference sequence such that the series in Corollary 4.9 diverges (see [38, Theorem 2.4]).

5 Marcinkiewicz-Zygmund strong law of large numbers

In 1947, Chung [10] proved the following extension of the Marcinkiewicz-Zygmund strong law of large numbers for independent random variables.

***Theorem* 5.1** *Let $1 \le p < 2$ and $\{X_n\}_{n\ge1}$ be a sequence of mean-zero independent random variables such that $\sup_{n\ge1} \mathbb{E}\left(|X_n|^p f(|X_n|)\right) < \infty$, where f fulfills (1.4). Then $\lim_{n\to\infty} \sum_{k=1}^{n} X_k / n^{1/p} = 0$ almost surely.*

Stout later extended Theorem 5.1 to the cases of martingale differences (see [49, Theorem 3.3.9, Corollary 3.3.5]). In this subsection, we further extend Stout's results to the noncommutative framework. Recall the a.u convergence of sequences from Definition 2.1, and the following lemma is a noncommutative analogue of the Borel-Cantelli lemma enabling one to derive an a.u. convergence result from convergence of specific series of tail probabilities.

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***Lemma 5.2** (**Noncommutative Borel-Cantelli lemma** [28]) Suppose $\{I_n\}_{n=1}^{\infty}$ is a sequence of integers such that $\cup_{n=1}^{\infty} I_n = \{k \in \mathbb{N} : k \geq n_0\}$ for some $n_0 \in \mathbb{N}$. Let (z_n) be a sequence of random variables. If for any $\delta > 0$,*

$$\sum_{n \geq n_0} \mathrm{Prob}_\tau^c \left\{ \sup_{m \in I_n} |z_m| > \delta \right\} < \infty,$$

then

$$z_n \xrightarrow{a.u.} 0 \ \ as \ \ n \to \infty.$$

It is worthwhile to mention that a slight modification in the proof of Lemma 5.2 ensures the validity of the conclusion if the following series

$$\sum_{n \geq n_0} \mathrm{Prob}_\tau^c \left\{ \sup_{m \in I_n} (|z_m/(\sup I_n)^a|) > \delta \right\}$$

converges for some $a > 0$.

Proof of Corollary 1.9 Let $r = 1$ in Theorem 1.6 and note that the series

$$\sum_{n=2}^{\infty} \frac{1}{n \log n} = \infty.$$

It follows that there exists a positive constant C such that

$$\mathrm{Prob}_\tau^c \left\{ \sup_{1 \leq m \leq n} |S_m| > \varepsilon n^{1/p} \right\} \leq \frac{C}{\log n}, \quad \text{for all } n \in \mathbb{N}.$$

Therefore, we obtain $\sum_{n=1}^{\infty} \mathrm{Prob}_\tau^c \left\{ \sup_{1 \leq m \leq 2^{2^n}} |S_m| > \varepsilon 2^{2^n/p} \right\} < \infty$. By Lemma 5.2, $\sum_{k=1}^{n} x_k / n^{1/p}$ converges to 0 almost uniformly.

Remark 5.3 The proof of Corollary 1.3 is analogous to the proof of Corollary 1.9, and we left the details for readers. In the commutative martingale setting, Chung [10] pointed out that the uniform boundedness condition in our results is the best possible, i.e., (1.5) diverges whenever (1.4) diverges. In addition, if we take $f(t) = \left(\log^+ |t|\right)^{1+\epsilon}$ for $t \geq 0$ and $\varepsilon > 0$ in Corollary 1.9, we will see that the logarithmic factor compensates for weakening the assumption concerning independence and identical distributions compared to the classical Marcinkiewicz-Zygmund law.

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School of Mathematics and Statistics, HNP-LAMA, Central South University, Changsha, China
e-mail: caojunming@csu.edu.cn, jiaoyong@csu.edu.cn, sijieluo@csu.edu.cn, zhoudejian@csu.edu.cn.