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HIGHER GENUS GROMOV–WITTEN THEORY OF $\mathsf{Hilb}^n(\mathbb{C}^2)$ AND CohFTs ASSOCIATED TO LOCAL CURVES

RAHUL PANDHARIPANDE[1] and HSIAN-HUA TSENG[2]

[1] Department of Mathematics, ETH Zürich, Rämistrasse 101, 8092 Zürich, Switzerland;
email: rahul@math.ethz.ch

[2] Department of Mathematics, Ohio State University, 100 Math Tower, 231 West 18th Ave.,
Columbus, OH 43210, USA;
email: hhtseng@math.ohio-state.edu

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Dedicated to A. Givental on the occasion of his 60th birthday

Abstract

We study the higher genus equivariant Gromov–Witten theory of the Hilbert scheme of n points of $\mathbb{C}^2$. Since the equivariant quantum cohomology, computed by Okounkov and Pandharipande [*Invent. Math.* **179** (2010), 523–557], is semisimple, the higher genus theory is determined by an R-matrix via the Givental–Teleman classification of Cohomological Field Theories (CohFTs). We uniquely specify the required R-matrix by explicit data in degree 0. As a consequence, we lift the basic triangle of equivalences relating the equivariant quantum cohomology of the Hilbert scheme $\mathsf{Hilb}^n(\mathbb{C}^2)$ and the Gromov–Witten/Donaldson–Thomas correspondence for 3-fold theories of local curves to a triangle of equivalences in all higher genera. The proof uses the analytic continuation of the fundamental solution of the QDE of the Hilbert scheme of points determined by Okounkov and Pandharipande [*Transform. Groups* **15** (2010), 965–982]. The GW/DT edge of the triangle in higher genus concerns new CohFTs defined by varying the 3-fold local curve in the moduli space of stable curves. The equivariant orbifold Gromov–Witten theory of the symmetric product $\mathsf{Sym}^n(\mathbb{C}^2)$ is also shown to be equivalent to the theories of the triangle in all genera. The result establishes a complete case of the crepant resolution conjecture [Bryan and Graber, *Algebraic Geometry–Seattle 2005, Part 1*, Proceedings of Symposia in Pure Mathematics, 80 (American Mathematical Society, Providence, RI, 2009), 23–42; Coates *et al.*, *Geom. Topol.* **13** (2009), 2675–2744; Coates & Ruan, *Ann. Inst. Fourier (Grenoble)* **63** (2013), 431–478].

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0. Introduction

0.1. Quantum cohomology. The *Hilbert scheme* $\mathsf{Hilb}^n(\mathbb{C}^2)$ of n points in the plane $\mathbb{C}^2$ parameterizes ideals $\mathcal{I} \subset \mathbb{C}[x, y]$ of colength n,

$$\dim_{\mathbb{C}} \mathbb{C}[x, y]/\mathcal{I} = n.$$

The Hilbert scheme $\mathsf{Hilb}^n(\mathbb{C}^2)$ is a nonsingular, irreducible, quasiprojective variety of dimension $2n$, see [14, 28] for an introduction. An open dense set of $\mathsf{Hilb}^n(\mathbb{C}^2)$ parameterizes ideals associated to configurations of n distinct points.

The symmetries of $\mathbb{C}^2$ lift to the Hilbert scheme. The algebraic torus

$$\mathsf{T} = (\mathbb{C}^*)^2$$

acts diagonally on $\mathbb{C}^2$ by scaling coordinates,

$$(z_1, z_2) \cdot (x, y) = (z_1 x, z_2 y).$$

The induced T-action on $\mathsf{Hilb}^n(\mathbb{C}^2)$ will play basic role here.

The Hilbert scheme carries a tautological rank-n vector bundle,

$$\mathcal{O}/\mathcal{I} \to \mathsf{Hilb}^n(\mathbb{C}^2), \tag{0.1}$$

with fiber $\mathbb{C}[x, y]/\mathcal{I}$ over $[\mathcal{I}] \in \mathsf{Hilb}^n(\mathbb{C}^2)$, see [22]. The T-action on $\mathsf{Hilb}^n(\mathbb{C}^2)$ lifts canonically to the tautological bundle (0.1). Let

$$D = c_1(\mathcal{O}/\mathcal{I}) \in H_T^2(\mathsf{Hilb}^n(\mathbb{C}^2), \mathbb{Q})$$

be the T-equivariant first Chern class.

The T-equivariant quantum cohomology of $\mathsf{Hilb}^n(\mathbb{C}^2)$ has been determined in [29]. The *matrix elements* of the T-equivariant quantum product count rational curves meeting three given subvarieties of $\mathsf{Hilb}^n(\mathbb{C}^2)$. (The count is virtual.) The (nonnegative) *degree* of an effective curve class

$$\beta \in H_2(\mathsf{Hilb}^n(\mathbb{C}^2), \mathbb{Z})$$

is defined by pairing with D,

$$d = \int_\beta D.$$

(The $\beta = 0$ is considered here effective.) Curves of degree d are counted with weight q^d, where q is the quantum parameter. The ordinary multiplication in T-equivariant cohomology is recovered by setting $q = 0$.

Let M_D denote the operator of T-equivariant quantum multiplication by the divisor D. A central result of [29] is an explicit formula for M_D an as operator on Fock space.

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0.2. Fock space formalism. We review the Fock space description of the T-equivariant cohomology of the Hilbert scheme of points of $\mathbb{C}^2$ following the notation of [29, Section 2.1], see also [16, 28].

By definition, the *Fock space* $\mathcal{F}$ is freely generated over $\mathbb{Q}$ by commuting creation operators α_{-k}, $k \in \mathbb{Z}_{>0}$, acting on the vacuum vector $v_\emptyset$. The annihilation operators α_k, $k \in \mathbb{Z}_{>0}$, kill the vacuum

$$\alpha_k \cdot v_\emptyset = 0, \quad k > 0,$$

and satisfy the commutation relations

$$[\alpha_k, \alpha_l] = k\delta_{k+l}.$$

A natural basis of $\mathcal{F}$ is given by the vectors

$$|\mu\rangle = \frac{1}{\mathfrak{z}(\mu)} \prod_i \alpha_{-\mu_i} v_\emptyset \tag{0.2}$$

indexed by partitions μ. Here,

$$\mathfrak{z}(\mu) = |\mathrm{Aut}(\mu)| \prod_i \mu_i$$

is the usual normalization factor. Let the length $\ell(\mu)$ denote the number of parts of the partition μ.

The *Nakajima basis* defines a canonical isomorphism,

$$\mathcal{F} \otimes_{\mathbb{Q}} \mathbb{Q}[t_1, t_2] \stackrel{\sim}{=} \bigoplus_{n \geqslant 0} H^*_{\mathsf{T}}(\mathsf{Hilb}^n(\mathbb{C}^2), \mathbb{Q}). \tag{0.3}$$

The Nakajima basis element corresponding to $|\mu\rangle$ is

$$\frac{1}{\Pi_i \mu_i} [V_\mu]$$

where $[V_\mu]$ is (the cohomological dual of) the class of the subvariety of $\mathsf{Hilb}^{|\mu|}(\mathbb{C}^2)$ with generic element given by a union of schemes of lengths

$$\mu_1, \ldots, \mu_{\ell(\mu)}$$

supported at $\ell(\mu)$ distinct points of $\mathbb{C}^2$. (The points and parts of μ are considered here to be unordered.) The vacuum vector $v_\emptyset$ corresponds to the unit in

$$1 \in H^*_{\mathsf{T}}(\mathsf{Hilb}^0(\mathbb{C}^2), \mathbb{Q}).$$

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The variables t_1 and t_2 are the equivariant parameters corresponding to the weights of the T-action on the tangent space $\mathrm{Tan}_0(\mathbb{C}^2)$ at the origin of $\mathbb{C}^2$.

The subspace of $\mathcal{F} \otimes_{\mathbb{Q}} \mathbb{Q}[t_1, t_2]$ corresponding to $H^*_{\mathsf{T}}(\mathsf{Hilb}^n(\mathbb{C}^2), \mathbb{Q})$ is spanned by the vectors (0.2) with $|\mu| = n$. The subspace can also be described as the n-eigenspace of the *energy operator*:

$$|\cdot| = \sum_{k>0} \alpha_{-k}\,\alpha_k.$$

The vector $|1^n\rangle$ corresponds to the unit

$$\mathsf{1} \in H^*_{\mathsf{T}}(\mathsf{Hilb}^n(\mathbb{C}^2), \mathbb{Q}).$$

A straightforward calculation shows

$$D = -|2, 1^{n-2}\rangle. \tag{0.4}$$

The standard inner product on the T-equivariant cohomology of $\mathsf{Hilb}^n(\mathbb{C}^2)$ induces the following *nonstandard* inner product on Fock space after an extension of scalars:

$$\langle \mu | \nu \rangle = \frac{(-1)^{|\mu|-\ell(\mu)}}{(t_1 t_2)^{\ell(\mu)}} \frac{\delta_{\mu\nu}}{\mathfrak{z}(\mu)}. \tag{0.5}$$

With respect to the inner product,

$$(\alpha_k)^* = (-1)^{k-1}(t_1 t_2)^{\mathrm{sgn}(k)}\,\alpha_{-k}. \tag{0.6}$$

0.3. Quantum multiplication by D. The formula of [29] for the operator M_D of quantum multiplication by D is:

$$\mathsf{M}_D(q, t_1, t_2) = (t_1 + t_2) \sum_{k>0} \frac{k}{2} \frac{(-q)^k + 1}{(-q)^k - 1}\,\alpha_{-k}\,\alpha_k - \frac{t_1 + t_2}{2} \frac{(-q) + 1}{(-q) - 1} |\cdot| + \frac{1}{2} \sum_{k,l>0} [t_1 t_2\,\alpha_{k+l}\,\alpha_{-k}\,\alpha_{-l} - \alpha_{-k-l}\,\alpha_k\,\alpha_l].$$

The q-dependence of M_D occurs only in the first two terms (which acts diagonally in the basis (0.2)). The two parts of the last sum are known respectively as the splitting and joining terms.

Let μ^1 and μ^2 be partitions of n. The T-equivariant Gromov–Witten invariants of $\mathsf{Hilb}^n(\mathbb{C}^2)$ in genus 0 with 3 cohomology insertions given (in the Nakajima basis) by μ^1, D, and μ^2 are determined by M_D:

$$\sum_{d=0}^{\infty} \langle \mu^1, D, \mu^2 \rangle_{0,d}^{\mathsf{Hilb}^n(\mathbb{C}^2)} q^d = \langle \mu^1 \mid \mathsf{M}_D | \mu^2 \rangle.$$

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Equivalently, denoting the 2-cycle $(2, 1^n)$ by (2), we have

$$\sum_{d=0}^{\infty} \langle \mu^1, (2), \mu^2 \rangle_{0,d}^{\mathsf{Hilb}^n(\mathbb{C}^2)} q^d = \langle \mu^1 | -\mathsf{M}_D | \mu^2 \rangle. \tag{0.7}$$

Let $\mu^1, \ldots, \mu^r \in \text{Part}(n)$. The T-equivariant Gromov–Witten series in genus g,

$$\langle \mu^1, \mu^2, \ldots, \mu^r \rangle_g^{\mathsf{Hilb}^n(\mathbb{C}^2)} = \sum_{d=0}^{\infty} \langle \mu^1, \mu^2, \ldots, \mu^r \rangle_{g,d}^{\mathsf{Hilb}^n(\mathbb{C}^2)} q^d \in \mathbb{Q}[[q]],$$

is a sum over the degree d with variable q. The T-equivariant Gromov–Witten series in genus 0,

$$\langle \mu^1, \mu^2, \ldots, \mu^r \rangle_0^{\mathsf{Hilb}^n(\mathbb{C}^2)} = \sum_{d=0}^{\infty} \langle \mu^1, \mu^2, \ldots, \mu^r \rangle_{0,d}^{\mathsf{Hilb}^n(\mathbb{C}^2)} q^d,$$

can be calculated from the special 3-point invariants (0.7), see [29, Section 4.2].

0.4. Higher genus. Our first result here is a determination of the T-equivariant Gromov–Witten theory of $\mathsf{Hilb}(\mathbb{C}^2, d)$ in *all* higher genera g. We use the Givental–Teleman classification of semisimple Cohomological Field Theories (CohFTs). The Frobenius structure determined by the T-equivariant genus 0 theory of $\mathsf{Hilb}(\mathbb{C}^2, d)$ is semisimple, but *not* conformal. Therefore, the R-matrix is *not* determined by the T-equivariant genus 0 theory alone. Fortunately, together with the divisor equation, an evaluation of the T-equivariant higher genus theory in degree 0 is enough to uniquely determine the R-matrix.

Let $\text{Part}(n)$ be the set of partitions of n corresponding to the T-fixed points of $\mathsf{Hilb}^n(\mathbb{C}^2)$. For each $\eta \in \text{Part}(n)$, let $\text{Tan}_\eta(\mathsf{Hilb}^n(\mathbb{C}^2))$ be the T-representation on the tangent space at the T-fixed point corresponding to η. Let

$$\mathbb{E}_g \to \overline{\mathcal{M}}_g$$

be the Hodge bundle of differential forms over the moduli space of stable curves of genus g.

THEOREM 1. *The R-matrix of the T-equivariant Gromov–Witten theory of $\mathsf{Hilb}^n(\mathbb{C}^2)$ is uniquely determined from the T-equivariant genus 0 theory by the divisor equation and the degree 0 invariants*

$$\langle \mu \rangle_{1,0}^{\mathsf{Hilb}^n(\mathbb{C}^2)} = \sum_{\eta \in \text{Part}(n)} \mu|_\eta \int_{\overline{\mathcal{M}}_{1,1}} \frac{e(\mathbb{E}_1^* \otimes \text{Tan}_\eta(\mathsf{Hilb}^n(\mathbb{C}^2)))}{e(\text{Tan}_\eta(\mathsf{Hilb}^n(\mathbb{C}^2)))},$$

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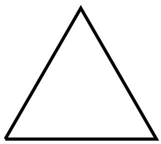
$$\langle\,\rangle^{\mathsf{Hilb}^n(\mathbb{C}^2)}_{g\geqslant 2,0} = \sum_{\eta\in\mathrm{Part}(n)} \int_{\overline{\mathcal{M}}_g} \frac{e(\mathbb{E}_g^*\otimes \mathrm{Tan}_\eta(\mathsf{Hilb}^n(\mathbb{C}^2)))}{e(\mathrm{Tan}_\eta(\mathsf{Hilb}^n(\mathbb{C}^2)))}.$$

The insertion μ in the genus 1 invariant is in the Nakajima basis, and $\mu|_\eta$ denotes the restriction to the T-fixed point corresponding to η—which can be calculated from the Jack polynomial J^η. The integral of the Euler class e in the formula may be explicitly expressed in terms of Hodge integrals and the tangent weights of the T-representation $\mathrm{Tan}_\eta(\mathsf{Hilb}^n(\mathbb{C}^2))$.

Apart from $\langle\mu\rangle^{\mathsf{Hilb}^n(\mathbb{C}^2)}_{1,0}$ and $\langle\,\rangle^{\mathsf{Hilb}^n(\mathbb{C}^2)}_{g\geqslant 2,0}$, all other degree 0 invariants of $\mathsf{Hilb}^n(\mathbb{C}^2)$ in positive genus vanish. Theorem 1 can be equivalently stated in the following form: *the* R*-matrix of the* T*-equivariant Gromov–Witten theory of* $\mathsf{Hilb}^n(\mathbb{C}^2)$ *is uniquely determined from the* T*-equivariant genus 0 theory by the divisor equation and the degree 0 invariants in positive genus.*

0.5. Lifting the triangle of correspondences. The calculation of the T-equivariant quantum cohomology of $\mathsf{Hilb}^n(\mathbb{C}^2)$ is a basic step in the proof [2, 29, 31] of the following triangle of equivalences:

Quantum cohomology
of $\mathsf{Hilb}^n(\mathbb{C}^2)$



Gromov–Witten theory of $\mathbb{C}^2\times\mathbb{P}^1$ | Donaldson–Thomas theory of $\mathbb{C}^2\times\mathbb{P}^1$

Our second result is a lifting of the above triangle to all higher genera. The top vertex is replaced by the T*-equivariant Gromov–Witten theory of* $\mathsf{Hilb}^n(\mathbb{C}^2)$ *in genus* g *with* r *insertions*. The bottom vertices of the triangle are *new* theories which are constructed here.

Let $\overline{\mathcal{M}}_{g,r}$ be the moduli space of Deligne–Mumford stable curves of genus g with r markings. (We always assume g and r satisfy the *stability* condition $2g-2+r>0$.) Let

$$\mathcal{C}\rightarrow\overline{\mathcal{M}}_{g,r}$$

be the universal curve with sections

$$\mathsf{p}_1,\ldots,\mathsf{p}_r:\overline{\mathcal{M}}_{g,r}\rightarrow\mathcal{C}$$

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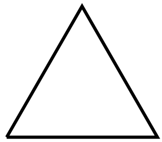
associated to the markings. Let

$$\pi : \mathbb{C}^2 \times \mathcal{C} \to \overline{\mathcal{M}}_{g,r}$$

be the universal *local curve* over $\overline{\mathcal{M}}_{g,r}$. The torus T acts on the $\mathbb{C}^2$ factor. The Gromov–Witten and Donaldson–Thomas theories of the morphism π are defined by the π-relative T-equivariant virtual class of the universal π-relative moduli spaces of stable maps and stable pairs. (While the triangle of equivalences originally included the Donaldson–Thomas theory of ideal sheaves, the theory of stable pairs [36] is much better behaved, see [26, Section 5] for a discussion valid for $\mathbb{C}^2 \times \mathbb{P}^1$. We use the theory of stable pairs here.)

THEOREM 2. *For all genera $g \geqslant 0$, there is a triangle of equivalences of T-equivariant theories:*

Gromov–Witten theory of $\mathsf{Hilb}^n(\mathbb{C}^2)$
in genus g with r insertions



Gromov–Witten theory of $\pi : \mathbb{C}^2 \times \mathcal{C} \to \overline{\mathcal{M}}_{g,r}$

Donaldson–Thomas theory of $\pi : \mathbb{C}^2 \times \mathcal{C} \to \overline{\mathcal{M}}_{g,r}$

The triangle of Theorem 2 may be viewed from different perspectives. First, all three vertices define CohFTs. Theorem 2 may be stated as simply an isomorphism of the three CohFTs. A second point of view of the bottom side of the triangle of Theorem 2 is as a GW/DT correspondence in *families of 3-folds* as the complex structure of the local curve varies. While a general GW/DT correspondence for families of 3-folds can be naturally formulated, there are very few interesting cases studied. (The equivariant GW/DT correspondence is a special case of the GW/DT correspondence for families (and is well studied).)

For a pointed curve of fixed complex structure $(C, p_1, \ldots, p_r)$, the triangle of Theorem 2 is a basic result of the papers [2, 29, 31] since C can be degenerated to a curve with all irreducible components of genus 0.

0.6. Crepant resolution. The Hilbert scheme of points of $\mathbb{C}^2$ is well known to be a crepant resolution of the symmetric product,

$$\epsilon : \mathsf{Hilb}^n(\mathbb{C}^2) \to \mathsf{Sym}^n(\mathbb{C}^2) = (\mathbb{C}^2)^n / S_n.$$

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Viewed as an *orbifold*, the symmetric product $\mathsf{Sym}^n(\mathbb{C}^2)$ has a T-equivariant Gromov–Witten theory with insertions indexed by partitions of n. The T-equivariant Gromov–Witten generating series,

$$\langle \mu^1, \mu^2, \ldots, \mu^r \rangle_g^{\mathsf{Sym}^n(\mathbb{C}^2)} = \sum_{b=0}^{\infty} \langle \mu^1, \mu^2, \ldots, \mu^r \rangle_{g,b}^{\mathsf{Sym}^n(\mathbb{C}^2)} u^b \in \mathbb{Q}[[u]],$$

is a sum over the number of free ramification points b with variable u. (A full discussion of the definition appears in Section 3.2.)

The Frobenius structure determined by the T-equivariant genus 0 theory of $\mathsf{Sym}^n(\mathbb{C}^2)$ is semisimple, but *not* conformal. Again, the R-matrix is *not* determined by the T-equivariant genus 0 theory alone. The determination of the R-matrix of $\mathsf{Sym}^n(\mathbb{C}^2)$ is given by the following result parallel to Theorem 1.

THEOREM 3. *The* R*-matrix of the* T*-equivariant Gromov–Witten theory of* $\mathsf{Sym}^n(\mathbb{C}^2)$ *is uniquely determined from the* T*-equivariant genus 0 theory by the divisor equation and all the unramified invariants,*

$$\langle \mu^1, \ldots, \mu^r \rangle_{g,0}^{\mathsf{Sym}^n(\mathbb{C}^2)},$$

in positive genus.

For each $\eta \in \text{Part}(n)$, let $\mathcal{H}_g^\eta$ be the moduli space étale covers with $\ell(\eta)$ connected components of degrees

$$\eta_1, \ldots, \eta_{\ell(\eta)}$$

of a nonsingular genus $g \geqslant 2$ curve. (For the definition of $\overline{\mathcal{H}}_g^\eta$, we consider the parts of η to be *ordered* and in bijection with the components of the cover.) The degree of the map

$$\mathcal{H}_g^\eta \rightarrow \mathcal{M}_g$$

is an unramified Hurwitz number. Let

$$\overline{\mathcal{H}}_g^\eta \rightarrow \overline{\mathcal{M}}_g$$

be the compactification by admissible covers.

Since the genus of the component of the cover corresponding to the part η_i is $\eta_i(g-1)+1$, there is a Hodge bundle

$$\mathbb{E}^*_{\eta_i(g-1)+1} \rightarrow \overline{\mathcal{H}}_g^\eta$$

obtained via the corresponding component.

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The simplest unramified invariants required in Theorem 3 are

$$\langle\,\rangle_{g\geqslant 2,0}^{\mathsf{Sym}^n(\mathbb{C}^2)} = \sum_{\eta\in\mathrm{Part}(n)} \frac{1}{|\mathrm{Aut}(\eta)|}\int_{\overline{\mathcal{H}}_g^\eta}\prod_{i=1}^{\ell(\eta)}\frac{e(\mathbb{E}^*_{\eta_i(g-1)+1}\otimes\mathrm{Tan}_0(\mathbb{C}^2))}{e(\mathrm{Tan}_0(\mathbb{C}^2))}.$$

However, there are many further unramified invariants in positive genus obtained by including insertions. Unlike the case of $\mathsf{Hilb}^n(\mathbb{C}^2)$, the unramified invariants with insertions for $\mathsf{Sym}^n(\mathbb{C}^2)$ do not all vanish.

In genus 0, the equivalence of the T-equivariant Gromov–Witten theories of $\mathsf{Hilb}^n(\mathbb{C}^2)$ and the orbifold $\mathsf{Sym}^n(\mathbb{C}^2)$ was proven in [1]. (The prefactor $(-i)^{\sum_{i=1}^r\ell(\mu^i)-|\mu^i|}$ was treated incorrectly in [1] because of an arithmetical error. The prefactor here is correct.) Our fourth result is a proof of the equivalence for all genera.

THEOREM 4. *For all genera $g\geqslant 0$ and $\mu^1,\mu^2,\ldots,\mu^r\in\mathrm{Part}(n)$, we have*

$$\langle\mu^1,\mu^2,\ldots,\mu^r\rangle_g^{\mathsf{Hilb}^n(\mathbb{C}^2)} = (-i)^{\sum_{i=1}^r\ell(\mu^i)-|\mu^i|}\langle\mu^1,\mu^2,\ldots,\mu^r\rangle_g^{\mathsf{Sym}^n(\mathbb{C}^2)}$$

after the variable change $-q=e^{iu}$.

The variable change of Theorem 4 is well defined by the following result (which was previously proven in genus 0 in [29]).

THEOREM 5. *For all genera $g\geqslant 0$ and $\mu^1,\mu^2,\ldots,\mu^r\in\mathrm{Part}(n)$, the series*

$$\langle\mu^1,\mu^2,\ldots,\mu^r\rangle_g^{\mathsf{Hilb}^n(\mathbb{C}^2)}\in\mathbb{Q}(t_1,t_2)[[q]]$$

is the Taylor expansion in q of a rational function in $\mathbb{Q}(t_1,t_1,q)$. (As always, g and r are required to be in the stable range $2g-2+r>0$.)

Theorem 4 establishes a complete case of the crepant resolution conjecture [1, 5, 6].

Calculations in closed form in higher genus are not easily obtained. The first nontrivial example occurs in genus 1 for the Hilbert scheme of 2 points:

$$\langle(2)\rangle_1^{\mathsf{Hilb}^2(\mathbb{C}^2)} = -\frac{1}{24}\frac{(t_1+t_2)^2}{t_1t_2}\cdot\frac{1+q}{1-q}. \qquad (0.8)$$

While the formula is simple, our virtual localization [15] calculation of the integral is rather long. Since (0.8) captures all degrees, the full graph sum must be controlled—we use the valuation at (t_1+t_2) as in [29].

The above calculation (0.8) for the Hilbert scheme of 2 points yields new Hodge integral calculations for the bielliptic locus $\overline{\mathcal{H}}_1((2)^{2n})$, the moduli space of double covers of elliptic curves with $2n$ ordered branch points,

$$\overline{\mathcal{H}}_1((2)^{2n}) \to \overline{\mathcal{M}}_{1,2n}.$$

Since the domain curves parameterized by $\overline{\mathcal{H}}_1((2)^{2n})$ have genus $n+1$, there is a Hodge bundle

$$\mathbb{E}^*_{n+1} \to \overline{\mathcal{H}}_1((2)^{2n}).$$

By a direct application of Theorem 4,

$$\sum_{n=1}^{\infty} \frac{u^{2n-1}}{(2n-1)!} \int_{\overline{\mathcal{H}}_1((2)^{2n})} \lambda_{n+1}\lambda_{n-1} = \frac{i}{24} \cdot \frac{1-e^{iu}}{1+e^{iu}}$$
$$= \frac{1}{48}u + \frac{1}{576}u^3 + \frac{1}{5760}u^5 + \cdots .$$

(We follow the standard convention $\lambda_i = c_i(\mathbb{E}_{n+1})$.) The u and u^3 coefficients can be checked geometrically using the bielliptic calculations of [8]. The u^5 coefficient has been checked in [39, Remark 5.14].

The corresponding series for higher n,

$$\langle (2, 1^{n-2}) \rangle_1^{\mathsf{Hilb}^n(\mathbb{C}^2)} \in \mathbb{Q}(t_1, t_2, q),$$

very likely has a simple closed formula. We return to these questions in a future paper.

0.7. Plan of proof. Theorems 2 and 4 are proven together by studying the R-matrices of all four theories. The R-matrix of the CohFT associated to the local Donaldson–Thomas theories of curves is easily proven to coincide with the R-matrix of the T-equivariant Gromov–Witten theory of $\mathsf{Hilb}^n(\mathbb{C}^2)$ determined in Theorem 1. Similarly, the R-matrices of the CohFTs associated to $\mathsf{Sym}^n(\mathbb{C}^2)$ and the local Gromov–Witten theories of curves are straightforward to match (with determination by Theorem 3). The above results require a detailed study of the divisor equation in the four cases.

The main step of the joint proof of Theorems 2 and 4 is to match the R-matrices of Theorems 1 and 3. The matter is nontrivial since the former is a function of q and the latter is a function of u. The matching after

$$-q = e^{iu}$$

requires an analytic continuation. Our method here is to express the R-matrices in terms of the solution of the QDE associated to $\mathsf{Hilb}^n(\mathbb{C}^2)$. Fortunately, the

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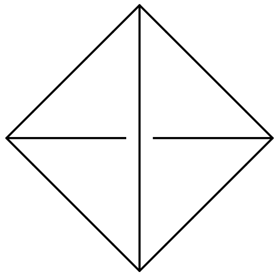
analytic continuation of the solution of the QDE proven in [30] is exactly what is needed, after a careful study of asymptotic expansions, to match the two R-matrices.

The addition of $\mathsf{Sym}^n(\mathbb{C}^2)$ via Theorem 4 to the triangle of Theorem 2 yields a tetrahedron of equivalences of T-equivariant theories (as first formulated in genus 0 in [1]).

Gromov–Witten theory of $\mathsf{Hilb}^n(\mathbb{C}^2)$
in genus g with r insertions



Gromov–Witten theory of
$\pi : \mathbb{C}^2 \times \mathcal{C} \to \overline{\mathcal{M}}_{g,r}$

Donaldson–Thomas theory of
$\pi : \mathbb{C}^2 \times \mathcal{C} \to \overline{\mathcal{M}}_{g,r}$

Orbifold Gromov–Witten theory of $\mathsf{Sym}^n(\mathbb{C}^2)$
in genus g with r insertions

Before the development of the orbifold Gromov–Witten theory of $\mathsf{Sym}^n(\mathbb{C}^2)$, a theory of Hurwitz–Hodge integrals was proposed by Cavalieri [3]. While the orbifold Gromov–Witten theory is formulated in terms of principal S_n-bundles over curves, Cavalieri's theory is formulated in terms of the associated Hurwitz covers of curves. In fact, the virtual class of the orbifold theory of $\mathsf{Sym}^n(\mathbb{C}^2)$ exactly coincides with the Hodge integrand proposed by Cavalieri, so the two theories are *equal*. The orbifold vertex of the above tetrahedron may therefore also be viewed via Cavalieri's definition. (Our discussion concerns Cavalieri's level $(0, 0)$ theory extended naturally over the moduli space of genus g curves. In genus 0, Cavalieri [3] noted the equivalence of his theory to the Gromov–Witten vertex. He also defined theories of other levels (a, b) which do not exactly agree with the corresponding (a, b)-theories of the Gromov–Witten vertex (even in genus 0). We do not explore here the interesting geometry of the (a, b)-level structure over the moduli space of genus g curves.)

1. Cohomological field theory

1.1. Definitions. The notion of a cohomological field theory (CohFT) was introduced in [20], see also [24]. We follow closely here the treatment of [35, Section 0.5], and we refer the reader also to the survey [33].

Let k be an algebraically closed field of characteristic 0. Let A be a commutative k-algebra. Let V be a free A-module of finite rank, let

$$\eta : \mathsf{V} \otimes \mathsf{V} \to \mathsf{A}$$

be an even, symmetric, nondegenerate pairing, and let $\mathbf{1} \in V$ be a distinguished element. (The pairing η induces an isomorphism between V and the dual module $\mathsf{V}^* = \mathrm{Hom}(\mathsf{V}, \mathsf{A})$.) The data $(\mathsf{V}, \eta, \mathbf{1})$ is the starting point for defining a cohomological field theory. Given a basis $\{e_i\}$ of V, we write the symmetric form as a matrix

$$\eta_{jk} = \eta(e_j, e_k).$$

The inverse matrix is denoted by η^{jk} as usual.

A cohomological field theory consists of a system $\Omega = (\Omega_{g,r})_{2g-2+r>0}$ of elements

$$\Omega_{g,r} \in H^*(\overline{\mathcal{M}}_{g,r}, \mathsf{A}) \otimes (\mathsf{V}^*)^{\otimes r}.$$

We view $\Omega_{g,r}$ as associating a cohomology class on $\overline{\mathcal{M}}_{g,r}$ to elements of V assigned to the r markings. The CohFT axioms imposed on Ω are:

(i) Each $\Omega_{g,r}$ is S_r-invariant, where the action of the symmetric group S_r permutes both the marked points of $\overline{\mathcal{M}}_{g,r}$ and the copies of V^*.

(ii) Denote the basic gluing maps by

$$q : \overline{\mathcal{M}}_{g-1,r+2} \to \overline{\mathcal{M}}_{g,r},$$
$$\widetilde{q} : \overline{\mathcal{M}}_{g_1,r_1+1} \times \overline{\mathcal{M}}_{g_2,r_2+1} \to \overline{\mathcal{M}}_{g,r}.$$

The pull-backs $q^*(\Omega_{g,r})$ and $\widetilde{q}^*(\Omega_{g,r})$ are equal to the contractions of

$$\Omega_{g-1,r+2} \quad \text{and} \quad \Omega_{g_1,r_1+1} \otimes \Omega_{g_2,r_2+1}$$

by the bivector

$$\sum_{j,k} \eta^{jk} e_j \otimes e_k$$

inserted at the two identified points.

(iii) Let $v_1, \ldots, v_r \in \mathsf{V}$ and let $p : \overline{\mathcal{M}}_{g,r+1} \to \overline{\mathcal{M}}_{g,r}$ be the forgetful map. We require

$$\Omega_{g,r+1}(v_1 \otimes \cdots \otimes v_r \otimes \mathbf{1}) = p^* \Omega_{g,r}(v_1 \otimes \cdots \otimes v_r),$$
$$\Omega_{0,3}(v_1 \otimes v_2 \otimes \mathbf{1}) = \eta(v_1, v_2).$$

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DEFINITION 1.1. A system $\Omega = (\Omega_{g,r})_{2g-2+r>0}$ of elements

$$\Omega_{g,r} \in H^*(\overline{\mathcal{M}}_{g,r}, \mathsf{A}) \otimes (V^*)^{\otimes r}$$

satisfying properties (i) and (ii) is a *cohomological field theory* (*CohFT*). If (iii) is also satisfied, Ω is a *CohFT with unit*.

A CohFT Ω yields a *quantum product* $\star$ on V via

$$\eta(v_1 \star v_2, v_3) = \Omega_{0,3}(v_1 \otimes v_2 \otimes v_3). \tag{1.1}$$

Associativity of $\star$ follows from (ii). The element $\mathbf{1} \in \mathsf{V}$ is the identity for $\star$ by (iii).

A CohFT ω composed only of degree 0 classes,

$$\omega_{g,r} \in H^0(\overline{\mathcal{M}}_{g,r}, \mathsf{A}) \otimes (\mathsf{V}^*)^{\otimes r},$$

is called a *topological field theory*. Via property (ii), $\omega_{g,r}(v_1, \dots, v_r)$ is determined by considering stable curves with a maximal number of nodes. Every irreducible component of such a curve is of genus 0 with 3 special points. The value of $\omega_{g,r}(v_1 \otimes \cdots \otimes v_r)$ is thus uniquely specified by the values of $\omega_{0,3}$ and by the pairing η. In other words, given V and η, a topological field theory is uniquely determined by the associated quantum product.

1.2. Gromov–Witten theory. Let X be a nonsingular projective variety over $\mathbb{C}$. The stack of stable maps $\overline{\mathcal{M}}_{g,r}(X, \beta)$ of genus g curves to X representing the class $\beta \in H_2(X, \mathbb{Z})$ admits an evaluation

$$\mathrm{ev} : \overline{\mathcal{M}}_{g,r}(X, \beta) \to X^r$$

and a forgetful map

$$\rho : \overline{\mathcal{M}}_{g,r}(X, \beta) \to \overline{\mathcal{M}}_{g,r},$$

when $2g-2+r > 0$. (See [10] for an introduction to stable maps.) The *Gromov–Witten* CohFT is constructed from the virtual classes of the moduli of stable maps of X,

$$\Omega^{\mathsf{GW}}_{g,r}(v_1 \otimes \cdots \otimes v_r) = \sum_{\beta \in H_2(X,\mathbb{Z})} q^\beta \rho_*(\mathrm{ev}^*(v_1 \otimes \cdots \otimes v_r) \cap [\overline{\mathcal{M}}_{g,r}(X, \beta)]^{\mathrm{vir}}).$$

The ground ring A is the *Novikov ring*, a suitable completion of the group ring associated to the semigroup of effective curve classes in X,

$$\mathsf{V} = H^*(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathsf{A} = H^*(X, \mathsf{A}),$$

the pairing η is the extension of the Poincaré pairing, and $\mathbf{1} \in \mathsf{V}$ is the unit in cohomology. (We take the Novikov ring with $\mathbb{Q}$-coefficients.)

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1.3. Semisimplicity.

1.3.1. Classification. Let Ω be a CohFT with respect to $(\mathsf{V}, \eta, \mathbf{1})$. We are concerned here with theories for which the algebra V with respect to the quantum product $\star$ defined by (1.1) is semisimple. Such theories are classified in [40]. Specifically, Ω is obtained from the algebra V via the action of an R-matrix

$$\mathsf{R} \in \mathbf{1} + z \cdot \mathrm{End}(\mathsf{V})[[z]],$$

satisfying the symplectic condition

$$\mathsf{R}(z)\mathsf{R}^{\star}(-z) = \mathbf{1}.$$

Here, $\mathsf{R}^{\star}$ denotes the adjoint with respect to η and $\mathbf{1}$ is the identity matrix. The explicit reconstruction of the semisimple CohFT from the R-matrix action will be explained below, following [35].

1.3.2. Actions on CohFTs. Let $\Omega = (\Omega_{g,r})$ be a CohFT with respect to $(\mathsf{V}, \eta, \mathbf{1})$. Fix a symplectic matrix

$$\mathsf{R} \in \mathbf{1} + z \cdot \mathrm{End}\ (\mathsf{V})[[z]]$$

as above. A new CohFT with respect to $(\mathsf{V}, \eta, \mathbf{1})$ is obtained via the cohomology elements

$$\mathsf{R}\Omega = (\mathsf{R}\Omega)_{g,r},$$

defined as sums over stable graphs Γ of genus g with r legs, with contributions coming from vertices, legs, and edges. Specifically,

$$(\mathsf{R}\Omega)_{g,r} = \sum_{\Gamma} \frac{1}{|\mathrm{Aut}(\Gamma)|} (\iota_\Gamma)_\star \left(\prod_v \mathsf{Cont}(v) \prod_l \mathsf{Cont}(l) \prod_e \mathsf{Cont}(e) \right) \tag{1.2}$$

where:

(i) the vertex contribution is

$$\mathsf{Cont}(v) = \Omega_{g(v),r(v)},$$

with $g(v)$ and $r(v)$ denoting the genus and number of half-edges and legs of the vertex,

(ii) the leg contribution is

$$\mathsf{Cont}(l) = \mathsf{R}(\psi_l)$$

where ψ_l is the cotangent class at the marking corresponding to the leg,

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(iii) the edge contribution is

$$\mathsf{Cont}(e) = \frac{\eta^{-1} - \mathsf{R}(\psi_e')\eta^{-1}\mathsf{R}(\psi_e'')^\top}{\psi_e' + \psi_e''}.$$

Here ψ_e' and ψ_e'' are the cotangent classes at the node which represents the edge e. The symplectic condition guarantees that the edge contribution is well defined.

A second action on CohFTs is given by translations. As before, let Ω be a CohFT with respect to $(\mathsf{V}, 1, \eta)$ and consider a power series $\mathsf{T} \in \mathsf{V}[[z]]$ with no terms of degrees 0 and 1:

$$\mathsf{T}(z) = \mathsf{T}_2 z^2 + \mathsf{T}_3 z^3 + \cdots, \mathsf{T}_k \in \mathsf{V}.$$

A new CohFT with respect to $(\mathsf{V}, 1, \eta)$, denoted $\mathsf{T}\Omega$, is defined by setting

$$(\mathsf{T}\Omega)_{g,r}(v_1 \otimes \cdots \otimes v_r) = \sum_{m=0}^{\infty} \frac{1}{m!}(p_m)_\star \Omega_{g,r+m}(v_1 \otimes \cdots \otimes v_r \otimes \mathsf{T}(\psi_{r+1}) \otimes \cdots \otimes \mathsf{T}(\psi_{r+m})) \quad (1.3)$$

where

$$p_m : \overline{\mathcal{M}}_{g,r+m} \to \overline{\mathcal{M}}_{g,r}$$

is the forgetful morphism.

1.3.3. Reconstruction. With the above terminology understood, we can state the Givental–Teleman classification theorem [12, 13, 40]. Fix Ω a semisimple CohFT with respect to $(\mathsf{V}, 1, \eta)$, and write ω for the degree 0 topological part of the theory. Given any symplectic matrix

$$\mathsf{R} \in \mathbf{1} + z \cdot \mathrm{End}(\mathsf{V})[[z]]$$

as above, we form a power series T by plugging $1 \in \mathsf{V}$ into R, removing the free term, and multiplying by z:

$$\mathsf{T}(z) = z(1 - \mathsf{R}(1)) \in \mathsf{V}[[z]].$$

Givental–Teleman classification. There exists a unique symplectic matrix R *for which*

$$\Omega = \mathsf{R}\mathsf{T}\omega.$$

A proof of uniqueness can be found, for example, in [25, Lemma 2.2].

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1.4. Targets $\mathsf{Hilb}^n(\mathbb{C}^2)$ and $\mathsf{Sym}^n(\mathbb{C}^2)$. The CohFTs determined by the T-equivariant Gromov–Witten theories of $\mathsf{Hilb}^n(\mathbb{C}^2)$ and $\mathsf{Sym}^n(\mathbb{C}^2)$ are based on the algebras

$$\mathsf{A} = \mathbb{Q}(t_1, t_2)[[q]], \quad \widetilde{\mathsf{A}} = \mathbb{Q}(t_1, t_2)[[u]], \tag{1.4}$$

and the corresponding free modules

$$\mathsf{V} = \mathcal{F}^n \otimes_{\mathbb{Q}} \mathsf{A}, \quad \widetilde{\mathsf{V}} = \mathcal{F}^n \otimes_{\mathbb{Q}} \widetilde{\mathsf{A}}, \tag{1.5}$$

where $\mathcal{F}^n$ is the Fock space with basis indexed by $\text{Part}(n)$. While the inner product for the CohFT determined by $\mathsf{Hilb}^n(\mathbb{C}^2)$ is the inner product defined in (0.5),

$$\eta(\mu, \nu) = \langle \mu | \nu \rangle = \frac{(-1)^{|\mu|-\ell(\mu)}}{(t_1 t_2)^{\ell(\mu)}} \frac{\delta_{\mu\nu}}{\mathfrak{z}(\mu)},$$

the inner product for the CohFT determined by $\mathsf{Sym}^n(\mathbb{C}^2)$ differs by a sign,

$$\widetilde{\eta}(\mu, \nu) = (-1)^{|\mu|-\ell(\mu)} \langle \mu | \nu \rangle = \frac{1}{(t_1 t_2)^{\ell(\mu)}} \frac{\delta_{\mu\nu}}{\mathfrak{z}(\mu)}.$$

Since the CohFTs are both semisimple, we obtain unique R-matrices from the Givental–Teleman classification,

$$\mathsf{R}^{\mathsf{Hilb}} \quad \text{and} \quad \mathsf{R}^{\mathsf{Sym}}.$$

After rescaling the insertion μ in the T-equivariant Gromov–Witten theory of $\mathsf{Sym}^n(\mathbb{C}^2)$ by the factor $(-i)^{|\mu|-\ell(\mu)}$, the inner products match. Let

$$|\widetilde{\mu}\rangle = (-i)^{\ell(\mu)-|\mu|} |\mu\rangle \in \widetilde{\mathsf{V}}. \tag{1.6}$$

The linear transformation $\mathsf{V} \to \widetilde{\mathsf{V}}$ defined by

$$\mu \mapsto \widetilde{\mu}$$

respects the inner products η and $\widetilde{\eta}$. Moreover, the correspondence claimed by Theorem 4 then simplifies to

$$\langle \mu^1, \mu^2, \dots, \mu^r \rangle_g^{\mathsf{Hilb}^n(\mathbb{C}^2)} = \langle \widetilde{\mu}^1, \widetilde{\mu}^2, \dots, \widetilde{\mu}^r \rangle_g^{\mathsf{Sym}^n(\mathbb{C}^2)}$$

after the variable change $-q = e^{iu}$.

To prove Theorems 2 and 4, we construct an operator series R defined over an open set of values of $q \in \mathbb{C}$ containing $q = 0$ and $q = -1$ with the two following properties:

- $\mathsf{R} = \mathsf{R}^{\text{Hilb}}$ near $q = 0$;
- $\mathsf{R} = \mathsf{R}^{\text{Sym}}$ near $q = -1$ after the variable change $-q = e^{iu}$.

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The operator R is obtained from the asymptotic solution to the QDE of $\mathsf{Hilb}^n(\mathbb{C}^2)$. The existence of R is guaranteed by results of [7, 11]. The asymptotic solution is shown to be the asymptotic expansion of an actual solution to the QDE. The behavior of R near $q=-1$ is then studied by using the solution to the connection problem in [30].

2. The R-matrix for $\mathsf{Hilb}^n(\mathbb{C}^2)$

2.1. The formal Frobenius manifold. We follow the notation of Section 1.4 for the CohFT determined by the T-equivariant Gromov–Witten theory of $\mathsf{Hilb}^n(\mathbb{C}^2)$. The genus 0 Gromov–Witten potential,

$$\mathsf{F}_0^{\mathsf{Hilb}^n(\mathbb{C}^2)}(\gamma) = \sum_{d=0}^{\infty} q^d \sum_{r=0}^{\infty} \frac{1}{r!} \langle \underbrace{\gamma, \ldots, \gamma}_{r} \rangle_{0,d}^{\mathsf{Hilb}^n(\mathbb{C}^2)}, \qquad \gamma \in \mathsf{V}$$

is a formal series in the ring $\mathsf{A}[[\mathsf{V}^*]]$ where

$$\mathsf{A} = \mathbb{Q}(t_1, t_2)[[q]].$$

The T-equivariant genus 0 potential $\mathsf{F}_0^{\mathsf{Hilb}^n(\mathbb{C}^2)}$ defines a formal Frobenius manifold

$$(\mathsf{V}, \star, \eta)$$

at the origin of V.

A basic result of [29] is the *rationality* of the dependence on q. Let

$$\mathsf{Q} = \mathbb{Q}(t_1, t_2, q)$$

be the field of rational functions. By [29, Section 4.2],

$$\mathsf{F}_0^{\mathsf{Hilb}^n(\mathbb{C}^2)}(\gamma) \in \mathsf{Q}[[\mathsf{V}^*]].$$

We often view the formal Frobenius manifold $(\mathsf{V}, \star, \eta)$ as defined over the field Q instead of the ring A.

2.2. Semisimplicity. The formal Frobenius manifold $(\mathsf{V}, \star, \eta)$ is semisimple at the origin after extending Q to the algebraic closure $\overline{\mathsf{Q}}$. The semisimple algebra

$$(\mathrm{Tan}_0 \mathsf{V}, \star_0, \eta)$$

is the small quantum cohomology of $\mathsf{Hilb}^n(\mathbb{C}^2)$. The matrices of quantum multiplication of the algebra $\mathrm{Tan}_0(\mathsf{V})$ have coefficients in Q. The restriction to $q = 0$ is well defined and semisimple with idempotents proportional to the classes of the T-fixed points of $\mathsf{Hilb}^n(\mathbb{C}^2)$. The idempotents of the algebra $\mathrm{Tan}_0(\mathsf{V})$ may be written in the Nakajima basis after extending scalars to $\overline{\mathsf{Q}}$. The algebraic closure is required since the eigenvalues of the matrices of quantum multiplication lie in finite extensions of Q. (More precisely, coefficients lie in the ring $\overline{\mathsf{Q}} \cap \mathsf{A}$ since the eigenvalues lie in A.)

Since the formal Frobenius manifold $(\mathsf{V}, \star, \eta)$ is semisimple at the origin, the full algebra

$$(\mathrm{Tan}_0(\mathsf{V}) \otimes_{\mathsf{Q}} \mathsf{Q}[[\mathsf{V}^*]], \star, \eta) \tag{2.1}$$

is also semisimple (after extending scalars to $\overline{\mathsf{Q}}$). The idempotents of the algebra $\mathrm{Tan}_0(\mathsf{V})$ can be lifted to all orders to obtain idempotents ϵ_μ of the full algebra (2.1) parameterized by partitions μ corresponding to the T-fixed points of $\mathsf{Hilb}^n(\mathbb{C}^2)$.

For the formal Frobenius manifold $(\mathsf{V}, \star, \eta)$, there are two bases of vector fields which play a fundamental role in the theory:

- the *flat* vector fields ∂_μ corresponding the basis elements $|\mu\rangle \in \mathsf{V}$;
- the *normalized idempotents* $\widetilde{\epsilon}_\mu$ of the quantum product $\star$.

The normalized idempotents are proportional to the idempotents and satisfy

$$\eta(\widetilde{\epsilon}_\mu, \widetilde{\epsilon}_\nu) = \delta_{\mu\nu}.$$

Let Ψ be change of basis matrix,

$$\Psi_\mu^\nu = \eta(\widetilde{\epsilon}_\nu, \partial_\mu) \in \overline{\mathsf{Q}}[[\mathsf{V}^*]]$$

between the two frames. Unique *canonical coordinates* at the origin,

$$\{u_\mu \in \overline{\mathsf{Q}}[[\mathsf{V}^*]]\}_{\mu\in\mathrm{Part}(n)}$$

can be found satisfying

$$u_\mu(0) = 0, \qquad \frac{\partial}{\partial u_\mu} = \epsilon_\mu.$$

By [11, Proposition 1.1], the quantum differential equation

$$\nabla_z \widetilde{\mathsf{S}} = 0 \tag{2.2}$$

has a formal fundamental solution in the basis of normalized idempotents of the form

$$\widetilde{\mathsf{S}} = \mathsf{R}(z) e^{\mathsf{u}/z}. \tag{2.3}$$

(We follow the notation of the exposition in [21]. See also [7].) Often the solution is written in the basis of flat vector fields as

$$\mathsf{S} = \Psi^{-1}\mathsf{R}(z)e^{\mathsf{u}/z}.$$

Here, ∇_z is the Dubrovin connection,

$$\mathsf{R}(z) = \mathbf{1} + \mathsf{R}_1 z + \mathsf{R}_2 z^2 + \mathsf{R}_3 z^3 + \dots \tag{2.4}$$

is a series of $|\mathrm{Part}(n)| \times |\mathrm{Part}(n)|$ matrices starting with the identity matrix $\mathbf{1}$, and u is a diagonal matrix with the diagonal entries given by the canonical coordinates u_μ.

We view $\mathsf{R}(z)$ as an $\mathrm{End}(\mathsf{V})$-valued formal power series in z written in the basis of normalized idempotents. The series $\mathsf{R}(z)$ satisfies the symplectic condition

$$\mathsf{R}^\dagger(-z)\mathsf{R}(z) = \mathbf{1}, \tag{2.5}$$

where $\mathsf{R}^\dagger(z)$ is the adjoint of $\mathsf{R}(z)$ with respect to the inner product η. A detailed treatment of the construction and properties of $\mathsf{R}(z)$ can be found in [21, Section 4.6 of Ch. 1].

An R-*matrix* associated to the formal Frobenius manifold $(\mathsf{V}, \star, \eta)$ is a matrix series of the form (2.4) which determines a solution (2.3) of the quantum differential equation (2.2) *and* satisfies the symplectic condition (2.5). The two basic properties of R-matrices which we use are:

(i) There exists a unique R-matrix $\mathsf{R}^{\mathsf{Hilb}}$ associated to $(\mathsf{V}, \star, \eta)$ with coefficients in $\mathsf{A}[[\mathsf{V}^*]]$ which generates the T-equivariant Gromov–Witten theory of $\mathsf{Hilb}^n(\mathbb{C}^2)$ in all higher genus.

(ii) Two R-matrices associated to $(\mathsf{V}, \star, \eta)$ with coefficients in $\mathsf{A}[[\mathsf{V}^*]]$ must differ by right multiplication by

$$\exp\left(\sum_{j=1}^{\infty} \mathsf{a}_{2j-1} z^{2j-1}\right)$$

where each a_{2j-1} is a *diagonal* matrix with coefficients in A.

Property (i) is a statement of the Givental–Teleman classification of semisimple CohFTs applied to the T-equivariant Gromov–Witten theory of $\mathsf{Hilb}^n(\mathbb{C}^2)$. For properties (i) and (ii), we have left the field $\overline{\mathsf{Q}}$ and returned to the ring A because the CohFT associated to $\mathsf{Hilb}^n(\mathbb{C}^2)$ is defined over A. In Section 2.5, the unique R-matrix of property (i) will be shown to actually have coefficients in $\overline{\mathsf{Q}}[[\mathsf{V}^*]]$.

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2.3. Divisor equation. The formal Frobenius manifold $(\mathsf{V}, \star, \eta)$ is actually well defined away from the origin along the line with coordinate t determined by the vector

$$|2, 1^{n-2}\rangle \in \mathsf{V}.$$

At the point $-t|2, 1^{n-2}\rangle \in \mathsf{V}$, the potential of the Frobenius manifold is

$$\begin{aligned}\mathsf{F}_0^{\mathsf{Hilb}^n(\mathbb{C}^2)}(-t|2, 1^{n-2}\rangle + \gamma) &= \sum_{d=0}^{\infty} q^d \sum_{r=0}^{\infty}\sum_{m=0}^{\infty} \frac{t^m}{r!m!} \langle \underbrace{\gamma, \ldots, \gamma}_{r}, \underbrace{D, \ldots, D}_{m} \rangle_{0,d}^{\mathsf{Hilb}^n(\mathbb{C}^2)} \\ &= \sum_{d=0}^{\infty} q^d \sum_{r=0}^{\infty}\sum_{m=0}^{\infty} \frac{(dt)^m}{r!m!} \langle \underbrace{\gamma, \ldots, \gamma}_{r} \rangle_{0,d}^{\mathsf{Hilb}^n(\mathbb{C}^2)}, \\ &= \sum_{d=0}^{\infty} q^d e^{dt} \sum_{r=0}^{\infty} \frac{1}{r!} \langle \underbrace{\gamma, \ldots, \gamma}_{r} \rangle_{0,d}^{\mathsf{Hilb}^n(\mathbb{C}^2)}\end{aligned}$$

for $\gamma \in \mathsf{V}$ and $D = -|2, 1^{n-2}\rangle$ as in (0.4). We have used the *divisor equation* of Gromov–Witten theory in the second equality. The potential near the point $-t|2, 1^{n-2}\rangle \in \mathsf{V}$ is obtained from the potential at $0 \in \mathsf{V}$ by the substitution

$$\mathsf{F}_0^{\mathsf{Hilb}^n(\mathbb{C}^2)}(-t|2, 1^{n-2}\rangle + \gamma) = \mathsf{F}_0^{\mathsf{Hilb}^n(\mathbb{C}^2)}(\gamma)|_{q \mapsto qe^t}.$$

The Frobenius manifold is semisimple at all the points $-t|2, 1^{n-2}\rangle \in \mathsf{V}$.

Let Ω be the CohFT associated to the T-equivariant Gromov–Witten theory of $\mathsf{Hilb}^n(\mathbb{C}^2)$. The genus 0 data of Ω is exactly given by the formal Frobenius manifold $(\mathsf{V}, \star, \eta)$ at the origin. Define the $-t|2, 1^{n-2}\rangle$-shifted CohFT by

$$\begin{aligned}\Omega_{g,r}^{-t|2,1^{n-2}\rangle}(\gamma \otimes \cdots \otimes \gamma) &= \sum_{m \geqslant 0} \frac{t^m}{m!} \rho_{r*}^{r+m}(\Omega_{g,r+m}(\gamma \otimes \cdots \otimes \gamma \otimes D^{\otimes m})) \\ &= \Omega_{g,r}(\gamma \otimes \cdots \otimes \gamma)|_{q \mapsto qe^t}.\end{aligned}$$

Here, ρ_r^{r+m} is the forgetful map which drops the last m markings,

$$\rho_r^{r+m} : \overline{\mathcal{M}}_{g,r+m} \to \overline{\mathcal{M}}_{g,r}. \tag{2.6}$$

We have again used the divisor equation of Gromov–Witten theory in the second equality.

Let $\mathsf{R}^{\mathsf{Hilb}}$ be the unique R-matrix associated to the T-equivariant Gromov–Witten theory of $\mathsf{Hilb}^n(\mathbb{C}^2)$. The shifted CohFT $\Omega_{g,r}^{-t|2,1^{n-2}\rangle}$ is obtained from the semisimple genus 0 data

$$\mathsf{F}_0^{\mathsf{Hilb}^n(\mathbb{C}^2)}(-t|2, 1^{n-2}\rangle + \gamma)$$

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by the unique R-matrix

$$\mathsf{R}^{\mathsf{Hilb}}(-t|2,1^{n-2}\rangle+\gamma).$$

On the other hand, the R-matrix

$$\mathsf{R}^{\mathsf{Hilb}}|_{q\mapsto qe^t}$$

also generates $\Omega_{g,r}^{-t|2,1^{n-2}\rangle}$ from the same semisimple genus 0 data. By the uniqueness of the R-matrix in the Givental–Teleman classification,

$$\mathsf{R}^{\mathsf{Hilb}}(-t|2,1^{n-2}\rangle+\gamma)=\mathsf{R}^{\mathsf{Hilb}}|_{q\mapsto qe^t}.$$

We find the following differential equation:

$$-\frac{\partial}{\partial t}\mathsf{R}^{\mathsf{Hilb}}=q\frac{\partial}{\partial q}\mathsf{R}^{\mathsf{Hilb}}. \tag{2.7}$$

The differential equation is an *extra* condition satisfied by $\mathsf{R}^{\mathsf{Hilb}^n(\mathbb{C}^2)}$ in *addition* to determining a solution (2.3) of the quantum differential equation (2.2) and satisfying the symplectic condition (2.5).

PROPOSITION 6. *Two* R*-matrices associated to* $(\mathsf{V},\star,\eta)$ *with coefficients in* $\mathsf{A}[[\mathsf{V}^*]]$ *which* both *satisfy the differential equation*

$$-\frac{\partial}{\partial t}\mathsf{R}=q\frac{\partial}{\partial q}\mathsf{R}$$

must differ by right multiplication by

$$\exp\left(\sum_{i=1}^{\infty}\mathsf{a}_{2j-1}z^{2j-1}\right)$$

where each a_{2j-1} *is a* diagonal *matrix with coefficients in* $\mathbb{Q}(t_1,t_2)$.

Proof. Let $\widetilde{\mathsf{R}}$ and $\widehat{\mathsf{R}}$ be two R-matrices associated to $(\mathsf{V},\star,\eta)$ which both satisfy the additional differential equation. By property (ii) of R-matrices associated to $(\mathsf{V},\star,\eta)$,

$$\widetilde{\mathsf{R}}=\widehat{\mathsf{R}}\cdot\exp\left(\sum_{i=1}^{\infty}\mathsf{a}_{2j-1}z^{2j-1}\right)$$

where each a_{2j-1} is a diagonal matrix with coefficients in

$$\mathsf{A}=\mathbb{Q}(t_1,t_2)[[q]].$$

Differentiation yields

$$
\begin{aligned}
-\frac{\partial}{\partial t}\widetilde{\mathsf{R}} &= -\frac{\partial}{\partial t}\widehat{\mathsf{R}}\cdot\exp\left(\sum_{i=1}^{\infty}\mathsf{a}_{2j-1}z^{2j-1}\right) - \widehat{\mathsf{R}}\cdot\frac{\partial}{\partial t}\exp\left(\sum_{i=1}^{\infty}\mathsf{a}_{2j-1}z^{2j-1}\right)\\
&= -\frac{\partial}{\partial t}\widehat{\mathsf{R}}\cdot\exp\left(\sum_{i=1}^{\infty}\mathsf{a}_{2j-1}z^{2j-1}\right)\\
&= q\frac{\partial}{\partial q}\widehat{\mathsf{R}}\cdot\exp\left(\sum_{i=1}^{\infty}\mathsf{a}_{2j-1}z^{2j-1}\right).
\end{aligned}
$$

The second equality uses the independence of a_{2j-1} with respect to the coordinate t. The third equality uses the differential equation for $\widehat{\mathsf{R}}$. Application of the operator $q(\partial/\partial q)$ to $\widetilde{\mathsf{R}}$ yields

$$
q\frac{\partial}{\partial q}\widetilde{\mathsf{R}} = q\frac{\partial}{\partial q}\widehat{\mathsf{R}}\cdot\exp\left(\sum_{i=1}^{\infty}\mathsf{a}_{2j-1}z^{2j-1}\right) + \widehat{\mathsf{R}}\cdot q\frac{\partial}{\partial q}\exp\left(\sum_{i=1}^{\infty}\mathsf{a}_{2j-1}z^{2j-1}\right).
$$

Finally, using the differential equation for $\widetilde{\mathsf{R}}$, we find

$$
\widehat{\mathsf{R}}\cdot q\frac{\partial}{\partial q}\exp\left(\sum_{i=1}^{\infty}\mathsf{a}_{2j-1}z^{2j-1}\right) = 0.
$$

By the invertibility of $\widehat{\mathsf{R}}$,

$$
q\frac{\partial}{\partial q}\exp\left(\sum_{i=1}^{\infty}\mathsf{a}_{2j-1}z^{2j-1}\right) = 0.
$$

Hence, the matrices a_{2j-1} are independent of q. □

2.4. Proof of Theorem 1. Since the CohFT $\Omega_{g,r}$ associated to the T-equivariant Gromov–Witten theory of $\mathsf{Hilb}^n(\mathbb{C}^2)$ is defined over the ring

$$
\mathsf{A} = \mathbb{Q}(t_1, t_2)[[q]],
$$

we can consider the CohFT $\Omega_{g,r}|_{q=0}$ defined over the ring $\mathbb{Q}(t_1, t_2)$. The CohFT $\Omega_{g,r}|_{q=0}$ is semisimple with R-matrix given by the $q = 0$ restriction,

$$
\mathsf{R}^{\mathsf{Hilb}}|_{q=0},
$$

of the R-matrix of $\Omega_{g,r}$. As a corollary of Proposition 6, we obtain the following characterization of $\mathsf{R}^{\mathsf{Hilb}}$.

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PROPOSITION 7. *$\mathsf{R}^{\mathrm{Hilb}}$ is the unique R-matrix associated to $(\mathsf{V}, \star, \eta)$ which satisfies*

$$-\frac{\partial}{\partial t}\mathsf{R}^{\mathsf{Hilb}} = q\frac{\partial}{\partial q}\mathsf{R}^{\mathsf{Hilb}}$$

and *has $q = 0$ restriction which equals the R-matrix of the restricted* CohFT $\Omega_{g,r}|_{q=0}$.

Since $\Omega_{g,r}|_{q=0}$ concerns constant maps of genus g curves to $\mathsf{Hilb}^n(\mathbb{C}^2)$, the CohFT can be written explicitly in terms of Hodge integrals. The moduli space of maps in degree 0 is

$$\overline{\mathcal{M}}_{g,r}(\mathsf{Hilb}^n(\mathbb{C}^2), 0) = \overline{\mathcal{M}}_{g,r} \times \mathsf{Hilb}^n(\mathbb{C}^2)$$

with virtual class

$$\mathbb{E}_g^* \otimes \mathrm{Tan}(\mathsf{Hilb}^n(\mathbb{C}^2)).$$

Since the Hodge bundle is pulled back from $\overline{\mathcal{M}}_{1,1}$ in genus 1 and $\overline{\mathcal{M}}_g$ in higher genera, all invariants in positive genus vanish other than

$$\langle \mu \rangle_{1,0}^{\mathsf{Hilb}^n(\mathbb{C}^2)} = \sum_{\eta\in\mathrm{Part}(n)} \mu|_\eta \int_{\overline{\mathcal{M}}_{1,1}} \frac{e(\mathbb{E}_1^* \otimes \mathrm{Tan}_\eta(\mathsf{Hilb}^n(\mathbb{C}^2)))}{e(\mathrm{Tan}_\eta(\mathsf{Hilb}^n(\mathbb{C}^2)))}, \tag{2.8}$$

$$\langle\, \rangle_{g\geqslant 2,0}^{\mathsf{Hilb}^n(\mathbb{C}^2)} = \sum_{\eta\in\mathrm{Part}(n)} \int_{\overline{\mathcal{M}}_g} \frac{e(\mathbb{E}_g^* \otimes \mathrm{Tan}_\eta(\mathsf{Hilb}^n(\mathbb{C}^2)))}{e(\mathrm{Tan}_\eta(\mathsf{Hilb}^n(\mathbb{C}^2)))}.$$

Theorem 1 follows from Proposition 7 together with the fact that (2.8) is the complete list of degree 0 invariants. □

The unique R-matrix $\mathsf{R}^{\mathsf{Hilb}}|_{q=0}$ of the CohFT $\Omega_{g,r}|_{q=0}$ can be explicitly written after the coordinates on V are set to 0. The formula is presented in Section 6.1.

2.5. Proof of Theorem 5. As we have seen in Section 2.3 using the divisor equation, the dependence of the potential of the formal Frobenius manifold $(\mathsf{V}, \star, \eta)$ at the origin,

$$\mathsf{F}_0^{\mathsf{Hilb}^n(\mathbb{C}^2)} \in \mathsf{Q}[[\mathsf{V}^*]],$$

along $-t|2, 1^{n-2}\rangle$ can be expressed as

$$\mathsf{F}_0^{\mathsf{Hilb}^n(\mathbb{C}^2)} = (\mathsf{F}_0^{\mathsf{Hilb}^n(\mathbb{C}^2)}|_{t=0})_{q\mapsto qe^t}.$$

The same dependence on t then also holds for the matrices of quantum multiplication for $(\mathsf{V}, \star, \eta)$ and their common eigenvalues.

In the procedure for constructing an R-matrix associated to $(\mathsf{V}, \star, \eta)$, we can take all the undetermined diagonal constants for R_{2j-1} equal to 0 for all j. (See [21, Section 4.6 of Ch. 1].) The resulting associated R-matrix $\mathsf{R}^{\#}$ will satisfy

$$-\frac{\partial}{\partial t}\mathsf{R}^{\#} = q\frac{\partial}{\partial q}\mathsf{R}^{\#}$$

since the same $q \mapsto qe^t$ dependence on t holds for all terms in the procedure. By Proposition 6,

$$\mathsf{R}^{\mathsf{Hilb}} = \mathsf{R}^{\#} \cdot \exp\left(\sum_{i=1}^{\infty} \mathsf{a}_{2j-1} z^{2j-1}\right), \tag{2.9}$$

where each a_{2j-1} is a *diagonal* matrix with coefficients in $\mathbb{Q}(t_1, t_2)$.

By [29, Section 4.2], the third derivatives of the potential of $(\mathsf{V}, \star, \eta)$ are defined over Q,

$$\frac{\partial^3}{\partial t_{\mu^1}\partial t_{\mu^2}\partial t_{\mu^3}} \mathsf{F}_0^{\mathsf{Hilb}^n(\mathbb{C}^2)} \in \mathsf{Q}[[\mathsf{V}^*]].$$

Hence, the matrices of quantum multiplication also have coefficients in $\mathsf{Q}[[\mathsf{V}^*]]$. As we have remarked in Section 2.2, the common eigenvalues require finite extensions of Q. Using the procedure for the construction of $\mathsf{R}^{\mathsf{Hilb}}$ with undetermined diagonal constants in $\mathbb{Q}(t_1, t_2)$ fixed by (2.9), we see that the coefficients of $\mathsf{R}^{\mathsf{Hilb}}$ lie in the ring $\overline{\mathsf{Q}}[[\mathsf{V}^*]]$.

The definition of the R-matrix action then yields the rationality of Theorem 5 after applying Galois invariance: for all genera $g \geqslant 0$ and $\mu^1, \mu^2, \ldots, \mu^r \in \mathrm{Part}(n)$, the series

$$\langle \mu^1, \mu^2, \ldots, \mu^r \rangle_g^{\mathsf{Hilb}^n(\mathbb{C}^2)} \in \mathbb{Q}(t_1, t_2)[[q]]$$

is the Taylor expansion in q of a rational function in $\mathbb{Q}(t_1, t_1, q)$. □

3. The R-matrix for $\mathsf{Sym}^n(\mathbb{C}^2)$

3.1. The formal Frobenius manifold. The T-equivariant Gromov–Witten potential in genus 0,

$$\mathsf{F}_0^{\mathsf{Sym}^n(\mathbb{C}^2)}(\gamma) = \sum_{b=0}^{\infty} u^b \sum_{n=0}^{\infty} \frac{1}{r!} \langle \underbrace{\gamma, \ldots, \gamma}_{r} \rangle_{0,b}^{\mathsf{Sym}^n(\mathbb{C}^2)}, \quad \gamma \in \widetilde{\mathsf{V}},$$

is a formal series in the ring $\widetilde{\mathsf{A}}[[\widetilde{\mathsf{V}}^*]]$, where $\widetilde{\mathsf{A}}$ is given in (1.4). The potential $\mathsf{F}_0^{\mathsf{Sym}^n(\mathbb{C}^2)}$ defines a formal Frobenius manifold

$$(\widetilde{\mathsf{V}}, \widetilde{\star}, \widetilde{\eta})$$

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at the origin of $\widetilde{\mathsf{V}}$. The Frobenius algebra

$$(\mathrm{Tan}_0\widetilde{\mathsf{V}}, \widetilde{\star}_0, \widetilde{\eta})$$

is the small quantum cohomology of $\mathsf{F}_0^{\mathsf{Sym}^n(\mathbb{C}^2)}$, which is calculated in [1, Section 3.3]. (The term *small* here refers to deformations by *twisted divisors*, introduced in [1, Section 2.2].) The formal Frobenius manifold $(\widetilde{\mathsf{V}}, \widetilde{\star}, \widetilde{\eta})$ is semisimple at the origin. The idempotents of $\mathrm{Tan}_0\widetilde{\mathsf{V}}$ can be written in terms of the standard basis of the Chen–Ruan cohomology of $\mathsf{Sym}^n(\mathbb{C}^2)$ after extension of scalars. Again, the idempotents of $\mathrm{Tan}_0\widetilde{\mathsf{V}}$ are indexed by partitions μ.

We write $\{\widetilde{u}_\mu \in \widetilde{\mathsf{A}}^{\mathrm{cl}}[[\widetilde{\mathsf{V}}^*]]\}_{\mu\in\mathrm{Part}(n)}$ for the unique canonical coordinates of $(\widetilde{\mathsf{V}}, \widetilde{\star}, \widetilde{\eta})$ satisfying that $\widetilde{u}_\mu(0) = 0$ and $\partial/\partial\widetilde{u}_\mu$ is an idempotent. ($\widetilde{\mathsf{A}}^{\mathrm{cl}}$ denotes the algebraic closure of the field of fractions of $\widetilde{\mathsf{A}}$.) By [11, Proposition 1.1], the quantum differential equation associated to the formal Frobenius manifold $(\widetilde{\mathsf{V}}, \widetilde{\star}, \widetilde{\eta})$,

$$\widetilde{\nabla}_z\widetilde{\mathsf{S}} = 0,$$

admits a formal fundamental solution of the form

$$\widetilde{\mathsf{S}} = \widetilde{\mathsf{R}}(z)e^{\widetilde{\mathsf{u}}/z},$$

written in the basis of normalized idempotents. Here $\widetilde{\nabla}_z$ is the Dubrovin connection associated to $(\widetilde{\mathsf{V}}, \widetilde{\star}, \widetilde{\eta})$, $\widetilde{\mathsf{u}}$ is the diagonal matrix with diagonal entries given by $\widetilde{u}_\mu$, and

$$\widetilde{\mathsf{R}}(z) = \mathbf{1} + \widetilde{\mathsf{R}}_1 z + \widetilde{\mathsf{R}}_2 z^2 + \cdots$$

is an $\mathrm{End}(\widetilde{\mathsf{V}})$-valued formal power series in z written in the basis of normalized idempotents. The symplectic condition

$$\widetilde{\mathsf{R}}^\dagger(-z)\widetilde{\mathsf{R}}(z) = \mathbf{1},$$

taken with respect to the inner product $\widetilde{\eta}$, is required.

By [11, Proposition 1.1], two R-matrices satisfying the quantum differential equation associated to $(\widetilde{\mathsf{V}}, \widetilde{\star}, \widetilde{\eta})$ and the symplectic condition must differ by right multiplication by

$$\exp\left(\sum_{j=1}^{\infty} \widetilde{\mathsf{a}}_{2j-1} z^{2j-1}\right),$$

where each $\widetilde{\mathsf{a}}_{2j-1}$ is a diagonal matrix with coefficients in $\widetilde{\mathsf{A}}$.

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3.2. Divisor equation. Let $\overline{\mathcal{M}}_{g,r}(\mathrm{Sym}^n(\mathbb{C}^2))$ be the moduli space of n-pointed genus g stable maps to $\mathrm{Sym}^n(\mathbb{C}^2)$. (The moduli stack, which parameterizes stable maps *with sections to all marked gerbes*, is also used in [41].) Let

$$\mathrm{ev}_i : \overline{\mathcal{M}}_{g,r}(\mathrm{Sym}^n(\mathbb{C}^2)) \to I\mathrm{Sym}^n(\mathbb{C}^2)$$

be the T-equivariant evaluation map at the ith marked point with values in the inertia stack

$$I\mathrm{Sym}^n(\mathbb{C}^2)$$

of $\mathrm{Sym}^n(\mathbb{C}^2)$. The inertia stack $I\mathrm{Sym}^n(\mathbb{C}^2)$ is a disjoint union indexed by conjugacy classes of the symmetric group S_n. For $\mu \in \mathrm{Part}(n)$, the component $I_\mu \subset I\mathrm{Sym}^n(\mathbb{C}^2)$ indexed by the conjugacy class of cycle type μ is isomorphic to the stack quotient

$$[\mathbb{C}^{2n}_\sigma / C(\sigma)],$$

where $\sigma \in S_n$ has cycle type μ, $\mathbb{C}^{2n}_\sigma$ is the invariant part of $\mathbb{C}^{2n}$ under the action of σ, and $C(\sigma)$ is the centralizer of $\sigma \in S_n$. Let

$$[I_\mu] \in H^0_{\mathsf{T}}(I_\mu) \subset H^*_{\mathsf{T}}(I\mathrm{Sym}^n(\mathbb{C}^2))$$

be the fundamental class. There is an additive isomorphism

$$H^*_{\mathsf{T}}(I\mathrm{Sym}^n(\mathbb{C}^2)) \simeq \widetilde{\mathsf{V}}$$

given by sending $[I_\mu]$ to $|\mu\rangle$.

The *unramified* T*-equivariant Gromov–Witten invariants* are defined by

$$\langle \mu^1, \ldots, \mu^r \rangle^{\mathrm{Sym}^n(\mathbb{C}^2)}_{g,0} = \int_{[\overline{\mathcal{M}}_{g,r}(\mathrm{Sym}^n(\mathbb{C}^2))]^{\mathrm{vir}}} \prod_{i=1}^r \mathrm{ev}_i^*([I_{\mu^i}]).$$

Consider

$$\mathrm{ev}^{-1}_{r+1}(I_{(2,1^{n-2})}) \cap \cdots \cap \mathrm{ev}^{-1}_{r+b}(I_{(2,1^{n-2})}) \subset \overline{\mathcal{M}}_{g,r+b}(\mathrm{Sym}^n(\mathbb{C}^2)),$$

and let

$$\overline{\mathcal{M}}_{g,r,b}(\mathrm{Sym}^n(\mathbb{C}^2)) = [(\mathrm{ev}^{-1}_{r+1}(I_{(2,1^{n-2})}) \cap \cdots \cap \mathrm{ev}^{-1}_{r+b}(I_{(2,1^{n-2})}))/S_b]$$

where the symmetric group S_b acts by permuting the last b marked points. The Gromov–Witten invariants with b free ramification points are defined by

$$\langle \mu^1, \ldots, \mu^r \rangle^{\mathrm{Sym}^n(\mathbb{C}^2)}_{g,b} = \int_{[\overline{\mathcal{M}}_{g,r,b}(\mathrm{Sym}^n(\mathbb{C}^2))]^{\mathrm{vir}}} \prod_{i=1}^r \mathrm{ev}_i^*([I_{\mu^i}]).$$

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The above definition can be rewritten as

$$\langle \mu^1, \ldots, \mu^r \rangle_{g,b}^{\mathsf{Sym}^n(\mathbb{C}^2)} = \frac{1}{b!} \langle \mu^1, \ldots, \mu^r, \underbrace{(2, 1^{n-2}), \ldots, (2, 1^{n-2})}_{b} \rangle_{g,0}^{\mathsf{Sym}^n(\mathbb{C}^2)}. \quad (3.1)$$

(See [1, Section 2.2] for a parallel discussion.)

Property (3.1) will be important for our study of the Frobenius manifold $(\widetilde{\mathsf{V}}, \widetilde{\star}, \widetilde{\eta})$. In particular, the *divisor equation* holds:

$$\langle \mu^1, \ldots, \mu^r, (2, 1^{n-2}) \rangle_{g,b}^{\mathsf{Sym}^n(\mathbb{C}^2)} = (b+1) \langle \mu^1, \ldots, \mu^r \rangle_{g,b+1}^{\mathsf{Sym}^n(\mathbb{C}^2)},$$

or equivalently,

$$\langle \mu^1, \ldots, \mu^r, (2, 1^{n-2}) \rangle_{g}^{\mathsf{Sym}^n(\mathbb{C}^2)} = \frac{\partial}{\partial u} \langle \mu^1, \ldots, \mu^r \rangle_{g}^{\mathsf{Sym}^n(\mathbb{C}^2)},$$

for the generating series

$$\langle \mu^1, \ldots, \mu^r, (2, 1^{n-2}) \rangle_{g}^{\mathsf{Sym}^n(\mathbb{C}^2)} = \sum_{b=0}^{\infty} \langle \mu^1, \ldots, \mu^r, (2, 1^{n-2}) \rangle_{g,b}^{\mathsf{Sym}^n(\mathbb{C}^2)} u^b,$$

$$\langle \mu^1, \ldots, \mu^r \rangle_{g}^{\mathsf{Sym}^n(\mathbb{C}^2)} = \sum_{b=0}^{\infty} \langle \mu^1, \ldots, \mu^r \rangle_{g,b}^{\mathsf{Sym}^n(\mathbb{C}^2)} u^b.$$

3.3. Shifted CohFT. Let $\widetilde{t}$ be the coordinate of the vector $[I_{(2,1^{n-2})}] \in \widetilde{\mathsf{V}}$. The formal Frobenius manifold $(\widetilde{\mathsf{V}}, \widetilde{\star}, \widetilde{\eta})$ is well defined at $\widetilde{t}[I_{(2,1^{n-2})}] \in \widetilde{\mathsf{V}}$, at which the potential of the Frobenius manifold is

$$\begin{aligned}
&\mathsf{F}_0^{\mathsf{Sym}^n(\mathbb{C}^2)}(\widetilde{t}[I_{(2,1^{n-2})}] + \gamma) \\
&= \sum_{b=0}^{\infty} u^b \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \frac{\widetilde{t}^m}{r!m!} \langle \underbrace{\gamma, \ldots, \gamma}_{r}, \underbrace{(2, 1^{n-2}), \ldots, (2, 1^{n-2})}_{m} \rangle_{0,b}^{\mathsf{Sym}^n(\mathbb{C}^2)} \\
&= \sum_{b=0}^{\infty} \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \frac{u^b \widetilde{t}^m}{r!m!b!} \langle \underbrace{\gamma, \ldots, \gamma}_{n}, \underbrace{(2, 1^{n-2}), \ldots, (2, 1^{n-2})}_{m+b} \rangle_{0,0}^{\mathsf{Sym}^n(\mathbb{C}^2)} \\
&= \sum_{b=0}^{\infty} \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \frac{u^b \widetilde{t}^m (m+b)!}{r!m!b!(m+b)!} \langle \underbrace{\gamma, \ldots, \gamma}_{r}, \underbrace{(2, 1^{n-2}), \ldots, (2, 1^{n-2})}_{m+b} \rangle_{0,0}^{\mathsf{Sym}^n(\mathbb{C}^2)} \\
&= \sum_{b=0}^{\infty} \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \frac{u^b \widetilde{t}^m (m+b)!}{r!m!b!} \langle \underbrace{\gamma, \ldots, \gamma}_{r} \rangle_{0,m+b}^{\mathsf{Sym}^n(\mathbb{C}^2)}
\end{aligned}$$

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$$= \sum_{b=0}^{\infty}\sum_{r=0}^{\infty}\sum_{m=0}^{\infty}\frac{u^b\widetilde{t}^m}{r!}\binom{m+b}{m}\langle\underbrace{\gamma,\ldots,\gamma}_{r}\rangle_{0,m+b}^{\mathsf{Sym}^n(\mathbb{C}^2)}$$

$$= \sum_{d=0}^{\infty}(u+\widetilde{t})^d\sum_{r=0}^{\infty}\frac{1}{r!}\langle\underbrace{\gamma,\ldots,\gamma}_{r}\rangle_{0,b}^{\mathsf{Sym}^n(\mathbb{C}^2)}.$$

In the above calculation above, we have used (3.1) in the second and fourth equalities. We conclude

$$\mathsf{F}_0^{\mathsf{Sym}^n(\mathbb{C}^2)}(\widetilde{t}[I_{(2,1^{n-2})}]+\gamma) = \mathsf{F}_0^{\mathsf{Sym}^n(\mathbb{C}^2)}(\gamma)|_{u\mapsto u+\widetilde{t}}.$$

Certainly the Frobenius manifold is semisimple at $\widetilde{t}[I_{(2,1^{n-2})}] \in \widetilde{\mathsf{V}}$.

Let $\widetilde{\Omega}_{g,r}$ be the CohFT associated to the T-equivariant Gromov–Witten theory of $\mathsf{Sym}^n(\mathbb{C}^2)$:

$$\widetilde{\Omega}_{g,r}(\gamma\otimes\cdots\otimes\gamma) = \sum_{b=0}^{\infty}\frac{u^b}{b!}\rho_*(\mathrm{ev}^*(\gamma^{\otimes r}\otimes[I_{(2,1^{n-2})}]^{\otimes b})\cap[\overline{\mathcal{M}}_{g,r+b}(\mathrm{Sym}^n(\mathbb{C}^2))]^{\mathrm{vir}}).$$

The genus 0 data of $\widetilde{\Omega}_{g,r}$ is exactly given by the formal Frobenius manifold $(\widetilde{\mathsf{V}}, \widetilde{\star}, \widetilde{\eta})$ at the origin. Define the $\widetilde{t}[I_{(2,1^{n-2})}]$-shifted CohFT by

$$\widetilde{\Omega}_{g,r}^{\widetilde{t}[I_{(2,1^{n-2})}]}(\gamma\otimes\cdots\otimes\gamma) = \sum_{m\geqslant 0}\frac{\widetilde{t}^m}{m!}\rho_{r*}^{r+m}(\widetilde{\Omega}_{g,r+m}(\gamma\otimes\cdots\otimes\gamma\otimes[I_{(2,1^{n-2})}]^{\otimes m}))$$
$$= \widetilde{\Omega}_{g,n}(\gamma\otimes\cdots\otimes\gamma)|_{u\mapsto u+\widetilde{t}},$$

where ρ_r^{r+m} is given by (2.6).

Let $\mathsf{R}^{\mathsf{Sym}}$ be the unique R-matrix associated to the T-equivariant Gromov–Witten theory of $\mathrm{Sym}^n(\mathbb{C}^2)$. The shifted CohFT $\widetilde{\Omega}_{g,r}^{\widetilde{t}[I_{(2,1^{n-2})}]}$ is obtained from the semisimple genus 0 data

$$\mathsf{F}_0^{\mathsf{Sym}^n(\mathbb{C}^2)}(\widetilde{t}[I_{(2,1^{n-2})}]+\gamma)$$

by the unique R-matrix

$$\mathsf{R}^{\mathsf{Sym}}(\widetilde{t}[I_{(2,1^{n-2})}]+\gamma).$$

On the other hand, the R-matrix

$$\mathsf{R}^{\mathsf{Sym}}|_{u\mapsto u+\widetilde{t}}$$

also generates $\widetilde{\Omega}_{g,r}^{\widetilde{t}[I_{(2,1^{n-2})}]}$ from the same semisimple genus 0 data. By the uniqueness of the R-matrix in the Givental–Teleman classification,

$$\mathsf{R}^{\mathsf{Sym}}(\widetilde{t}[I_{(2,1^{n-2})}]+\gamma) = \mathsf{R}^{\mathsf{Sym}}|_{u\mapsto u+\widetilde{t}}.$$

Hence, we obtain the following differential equation:

$$\frac{\partial}{\partial \widetilde{t}} \mathsf{R}^{\mathsf{Sym}} = \frac{\partial}{\partial u} \mathsf{R}^{\mathsf{Sym}}. \tag{3.2}$$

PROPOSITION 8. *Two* R*-matrices associated to* $(\widetilde{\mathsf{V}}, \widetilde{\star}, \widetilde{\eta})$ *with coefficients in* $\widetilde{\mathsf{A}}[[\widetilde{\mathsf{V}}^*]]$ *which both satisfy the differential equation* (3.2) *must differ by right multiplication by*

$$\exp\left(\sum_{j=1}^{\infty} \widetilde{\mathsf{a}}_{2j-1} z^{2j-1}\right),$$

where each $\widetilde{\mathsf{a}}_{2j-1}$ *is a diagonal matrix with coefficients in* $\mathbb{Q}(t_1, t_2)$.

Proof. Let $\widetilde{\mathsf{R}}$ and $\widehat{\mathsf{R}}$ be two R-matrices associated to $(\widetilde{\mathsf{V}}, \widetilde{\star}, \widetilde{\eta})$ which both satisfy (3.2). Then

$$\widetilde{\mathsf{R}} = \widehat{\mathsf{R}} \cdot \exp\left(\sum_{j=1}^{\infty} \widetilde{\mathsf{a}}_{2j-1} z^{2j-1}\right),$$

where each $\widetilde{\mathsf{a}}_{2j-1}$ is a diagonal matrix with coefficients in $\widetilde{\mathsf{A}} = \mathbb{Q}(t_1, t_2)[[u]]$. We show

$$\frac{\partial}{\partial u} \widetilde{\mathsf{a}}_{2j-1} = 0, \quad j \geqslant 1. \tag{3.3}$$

By the product rule, we have

$$\frac{\partial}{\partial \widetilde{t}} \widetilde{\mathsf{R}} = \frac{\partial}{\partial \widetilde{t}} \widehat{\mathsf{R}} \cdot \exp\left(\sum_{j=1}^{\infty} \widetilde{\mathsf{a}}_{2j-1} z^{2j-1}\right) + \widehat{\mathsf{R}} \cdot \frac{\partial}{\partial \widetilde{t}} \exp\left(\sum_{j=1}^{\infty} \widetilde{\mathsf{a}}_{2j-1} z^{2j-1}\right).$$

Since $\widetilde{\mathsf{a}}_{2j-1}$ is independent of $\widetilde{t}$, the right side is

$$\frac{\partial}{\partial \widetilde{t}} \widehat{\mathsf{R}} \cdot \exp\left(\sum_{j=1}^{\infty} \widetilde{\mathsf{a}}_{2j-1} z^{2j-1}\right).$$

Since $\widehat{\mathsf{R}}$ satisfies (3.2), we obtain

$$\frac{\partial}{\partial u} \widetilde{\mathsf{R}} = \frac{\partial}{\partial u} \widehat{\mathsf{R}} \cdot \exp\left(\sum_{j=1}^{\infty} \widetilde{\mathsf{a}}_{2j-1} z^{2j-1}\right).$$

On the other hand, product rule also yields

$$\frac{\partial}{\partial u} \widetilde{\mathsf{R}} = \frac{\partial}{\partial u} \widehat{\mathsf{R}} \cdot \exp\left(\sum_{j=1}^{\infty} \widetilde{\mathsf{a}}_{2j-1} z^{2j-1}\right) + \widehat{\mathsf{R}} \cdot \frac{\partial}{\partial u} \exp\left(\sum_{j=1}^{\infty} \widetilde{\mathsf{a}}_{2j-1} z^{2j-1}\right).$$

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Comparing the two equations above, we find

$$\widehat{\mathsf{R}} \cdot \frac{\partial}{\partial u} \exp\left(\sum_{j=1}^{\infty} \widetilde{\mathsf{a}}_{2j-1} z^{2j-1}\right) = 0.$$

Since $\widetilde{\mathsf{R}}$ is invertible, we conclude

$$\frac{\partial}{\partial u} \exp\left(\sum_{j=1}^{\infty} \widetilde{\mathsf{a}}_{2j-1} z^{2j-1}\right) = 0$$

which implies (3.3). □

As a corollary of Proposition 8, we obtain the following characterization of $\mathsf{R}^{\mathsf{Sym}}$ parallel to Proposition 7.

PROPOSITION 9. $\mathsf{R}^{\mathsf{Sym}}$ *is the unique* R*-matrix associated to* $(\widetilde{\mathsf{V}}, \widetilde{\star}, \widetilde{\eta})$ *which satisfies*

$$\frac{\partial}{\partial \widetilde{t}} \mathsf{R}^{\mathsf{Sym}} = \frac{\partial}{\partial u} \mathsf{R}^{\mathsf{Sym}}$$

and *has* $u = 0$ *restriction which equals the* R*-matrix of the restricted* CohFT $\widetilde{\Omega}_{g,r}\big|_{u=0}$.

3.4. Proof of Theorem 3.

The restricted CohFT $\widetilde{\Omega}_{g,r}|_{u=0}$ is defined over $\mathbb{Q}(t_1, t_2)$ and is semisimple with the R-matrix given by

$$\mathsf{R}^{\mathsf{Sym}}|_{u=0}.$$

By Proposition 9, the unique R-matrix $\mathsf{R}^{\mathsf{Sym}}$ that generates the CohFT $\widetilde{\Omega}_{g,r}$ is the unique R-matrix associated to the Frobenius manifold $(\widetilde{\mathsf{V}}, \widetilde{\star}, \widetilde{\eta})$ which satisfies

$$\frac{\partial}{\partial \widetilde{t}} \mathsf{R}^{\mathsf{Sym}} = \frac{\partial}{\partial u} \mathsf{R}^{\mathsf{Sym}}$$

and has $u = 0$ restriction equal to the R-matrix of the CohFT $\widetilde{\Omega}_{g,r}|_{u=0}$. By the definition of $\widetilde{\Omega}_{g,r}$, the $u = 0$ restriction $\widetilde{\Omega}_{g,r}|_{u=0}$ is equivalent to the collection of the invariants

$$\langle \mu^1, \ldots, \mu^r \rangle_{g,0}^{\mathrm{Sym}^n(\mathbb{C}^2)}$$

for $g \geqslant 0$ and $\mu^1, \ldots, \mu^r \in \mathrm{Part}(n)$. □

The R-matrix $\mathsf{R}^{\mathsf{Sym}}|_{u=0}$ is written explicitly (after the coordinates of $\widetilde{\mathsf{V}}$ are set to 0) in Section 6.2.

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4. Local theories of curves: stable maps

4.1. Local theories of curves. Let $\overline{\mathcal{M}}_{g,r}$ be the moduli space of Deligne–Mumford stable curves of genus g with r markings. (We always assume g and r satisfy the *stability* condition $2g - 2 + r > 0$.) Let

$$\mathcal{C} \to \overline{\mathcal{M}}_{g,r}$$

be the universal curve with sections

$$\mathsf{p}_1, \ldots, \mathsf{p}_r : \overline{\mathcal{M}}_{g,r} \to \mathcal{C}$$

associated to the markings. Let

$$\pi : \mathbb{C}^2 \times \mathcal{C} \to \overline{\mathcal{M}}_{g,r} \tag{4.1}$$

be the universal *local curve* over $\overline{\mathcal{M}}_{g,r}$. The torus T acts on the $\mathbb{C}^2$ factor. The Gromov–Witten and Donaldson–Thomas theories of the morphism π are defined by the π-relative T-equivariant virtual class of the universal π-relative moduli spaces of stable maps and stable pairs.

4.2. Stable maps. We define a CohFT $\widetilde{\Lambda}$ via the moduli space of π-relative stable maps to the universal local curve (4.1) based on the algebra

$$\widetilde{\mathsf{A}} = \mathbb{Q}(t_1, t_2)[[u]]$$

and the corresponding free module

$$\widetilde{\mathsf{V}} = \mathcal{F}^n \otimes_{\mathbb{Q}} \widetilde{\mathsf{A}},$$

where $\mathcal{F}^n$ is the Fock space with basis indexed by $\mathrm{Part}(n)$. The inner product for the CohFT is

$$\widetilde{\eta}(\mu, \nu) = \frac{1}{(t_1 t_2)^{\ell(\mu)}} \frac{\delta_{\mu\nu}}{\mathfrak{z}(\mu)}.$$

The algebra, free module, and inner product for $\widetilde{\Lambda}$ are exactly the same as for the CohFT $\widetilde{\Omega}$ obtained from the orbifold Gromov–Witten theory of $\mathsf{Sym}^n(\mathbb{C}^2)$.

Let $\mu^1, \ldots, \mu^r \in \mathrm{Part}(n)$, and let $\overline{\mathcal{M}}_h^\bullet(\pi, \mu^1, \ldots, \mu^r)$ be the moduli space of stable relative maps to the fibers of π,

$$\epsilon : \overline{\mathcal{M}}_h^\bullet(\pi, \mu^1, \ldots, \mu^r) \to \overline{\mathcal{M}}_{g,r}.$$

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(The subscript $\bullet$ indicates the domain curves is possibly disconnected (but no connected component is contracted to a point). See [2].) The fiber of ϵ over the moduli point

$$(C, p_1, \ldots, p_r) \in \overline{\mathcal{M}}_{g,r}$$

is the moduli space of stable maps of genus h to $\mathbb{C}^2 \times C$ relative to the divisors determined by the nodes and the markings of C with boundary condition μ^i over the divisor $\mathbb{C}^2 \times p_i$. (The cohomology weights of the boundary condition are all the identity class.) The moduli space $\overline{\mathcal{M}}^{\bullet}_h(\pi, \mu^1, \ldots, \mu^r)$ has π-relative virtual dimension

$$-n(2g - 2 + r) + \sum_{i=1}^{r} \ell(\mu^i).$$

The CohFT $\widetilde{\Lambda}$ is defined via the π-relative T-equivariant virtual class by

$$\widetilde{\Lambda}_{g,r}(\mu^1 \otimes \cdots \otimes \mu^r) = \sum_{b \geqslant 0} \frac{u^b}{b!} \epsilon_*([\overline{\mathcal{M}}^{\bullet}_{h[b]}(\pi, \mu^1, \ldots, \mu^r)]^{\mathrm{vir}_\pi}).$$

Here, summation index b is the branch point number, so

$$2h[b] - 2 = b + n(2g - 2 + r) - \sum_{i=1}^{r} \ell(\mu^i).$$

The moduli space of stable maps is empty unless $h[b]$ is an integer.

4.3. Axioms. The defining axioms of a CohFT are listed in Section 1. Axiom (i) for $\widetilde{\Lambda}$ follows from the symmetry of the construction. Axiom (ii) is a consequence of the degeneration formula of relative Gromov–Witten theory. Axiom (iii) requires a proof.

PROPOSITION 10. *The identity axiom holds:*

$$\widetilde{\Lambda}_{g,r+1}(\mu^1 \otimes \cdots \otimes \mu^r \otimes \mathbf{1}) = p^* \widetilde{\Lambda}_{g,r}(\mu^1 \otimes \cdots \otimes \mu^r)$$

where $p : \overline{\mathcal{M}}_{g,r+1} \to \overline{\mathcal{M}}_{g,r}$ is the forgetful map.

Proof. Consider first the standard identity equation in Gromov–Witten theory

$$[\overline{\mathcal{M}}^{\bullet}_{h,1}(\pi, \mu^1, \ldots, \mu^r)]^{\mathrm{vir}_\pi} = p_\pi^* [\overline{\mathcal{M}}^{\bullet}_h(\pi, \mu^1, \ldots, \mu^r)]^{\mathrm{vir}_\pi}, \tag{4.2}$$

where 1 is a new marking on the domain curve and p_π is the map forgetting 1,

$$p_\pi : \overline{\mathcal{M}}^{\bullet}_{h,1}(\pi, \mu^1, \ldots, \mu^r) \to \overline{\mathcal{M}}^{\bullet}_h(\pi, \mu^1, \ldots, \mu^r).$$

By stability, the image of the marking 1 together with the r relative points yields a stable $r+1$ pointed curve,

$$\epsilon_1 : \overline{\mathcal{M}}^{\bullet}_{h,1}(\pi, \mu^1, \ldots, \mu^r) \to \overline{\mathcal{M}}_{g,r+1}.$$

Since the stable maps are of degree n, we obtain

$$\epsilon_{1*}([\overline{\mathcal{M}}^{\bullet}_{h,1}(\pi, \mu^1, \ldots, \mu^r)]^{\mathrm{vir}_\pi}) = np^*\epsilon_*([\overline{\mathcal{M}}^{\bullet}_{h}(\pi, \mu^1, \ldots, \mu^r)]^{\mathrm{vir}_\pi}). \tag{4.3}$$

The Proposition then follows from (4.3) and the relation

$$\epsilon_{1*}([\overline{\mathcal{M}}^{\bullet}_{h,1}(\pi, \mu^1, \ldots, \mu^r)]^{\mathrm{vir}_\pi}) = n\epsilon_*([\overline{\mathcal{M}}^{\bullet}_{h}(\pi, \mu^1, \ldots, \mu^r, (1^n))]^{\mathrm{vir}_\pi})$$

obtained from the degeneration formula by universally bubbling off the image of the marking 1 in the target. □

Finally, since $\widetilde{\Lambda}_{0,3}$ is nothing more than the Gromov–Witten theory of the local 3-fold $\mathbb{C}^2 \times \mathbb{P}^1$ with 3 relative divisors appearing in the original triangle of equivalence of Section 0.5, the property

$$\widetilde{\Lambda}_{0,3}(\mu^1 \otimes \mu^2 \otimes \mathbf{1}) = \widetilde{\eta}(v_1, v_2)$$

holds.

4.4. The divisor equation. The divisor equation for the CohFT $\widetilde{\Lambda}$ will play an important role.

PROPOSITION 11. *The divisor equation holds:*

$$\widetilde{\Lambda}_{g,r+1}(\mu^1 \otimes \cdots \otimes \mu^r \otimes (2, 1^{n-2})) = \frac{\partial}{\partial u} \widetilde{\Lambda}_{g,r}(\mu^1 \otimes \cdots \otimes \mu^r).$$

Proof. Proposition 11 is equivalent to the following statement:

$$p_*\epsilon_*([\overline{\mathcal{M}}^{\bullet}_{h}(\pi, \mu^1, \ldots, \mu^r, (2, 1^{n-2}))]^{\mathrm{vir}_\pi}) = b[h] \cdot \epsilon_*([\overline{\mathcal{M}}^{\bullet}_{h}(\pi, \mu^1, \ldots, \mu^r)]^{\mathrm{vir}_\pi}), \tag{4.4}$$

where $b[h]$ is defined by the relation

$$2h - 2 = b[h] + n(2g - 2 + r) - \sum_{i=1}^{r} \ell(\mu^i).$$

In order to prove (4.4), consider first the standard dilaton equation

$$\begin{aligned} &\epsilon_* p_{\pi*}(\psi_1[\overline{\mathcal{M}}^{\bullet}_{h,1}(\pi, \mu^1, \ldots, \mu^r)]^{\mathrm{vir}_\pi}) \\ &\quad = \left(2h - 2 + \sum_{i=1}^{r} \ell(\mu^i)\right) \cdot \epsilon_*([\overline{\mathcal{M}}^{\bullet}_{h}(\pi, \mu^1, \ldots, \mu^r)]^{\mathrm{vir}_\pi}). \end{aligned} \tag{4.5}$$

Here, ψ_1 is the cotangent line on the domain curve of the map. Equation (4.4) then follows from (4.5) and the degeneration relation

$$\begin{aligned}\epsilon_{1*}(\psi_1[\overline{\mathcal{M}}^{\bullet}_{h,1}(\pi,\mu^1,\ldots,\mu^r)]^{\mathrm{vir}_\pi})\\ = \epsilon_*([\overline{\mathcal{M}}^{\bullet}_{h}(\pi,\mu^1,\ldots,\mu^r,(2,1^{n-2}))]^{\mathrm{vir}_\pi})\\ + n\psi\cdot\epsilon_*([\overline{\mathcal{M}}^{\bullet}_{h}(\pi,\mu^1,\ldots,\mu^r,(1^n))]^{\mathrm{vir}_\pi})\end{aligned}$$

obtained by universally bubbling off the image of the marking 1 in the target. In the second term on the right in the degeneration relation, ψ is the cotangent line at relative marking associated to the boundary condition (1^n). There is no higher genus distribution to the bubble as can be seen from calculation of the equivariant cap. (The equivariant cap invariant for (1^n) has been calculated in [**34**, Section 2.5]. The crucial point is that s_3 occurs *only* in the leading u^{-2} summand. The leading summand contributes the right most term in the above degeneration formula. All other summands of the equivariant cap invariant for (1^n) have vanishing contributions.)

We now apply p_* to the degeneration relation. Since $\epsilon p_\pi = p\epsilon_1$, we have

$$\epsilon_* p_{\pi*}(\psi_1[\overline{\mathcal{M}}^{\bullet}_{h,1}(\pi,\mu^1,\ldots,\mu^r)]^{\mathrm{vir}_\pi}) = p_*\epsilon_{1*}(\psi_1[\overline{\mathcal{M}}^{\bullet}_{h,1}(\pi,\mu^1,\ldots,\mu^r)]^{\mathrm{vir}_\pi}).$$

Using the identity axiom, we see

$$\begin{aligned}p_*(n\psi\cdot\epsilon_*([\overline{\mathcal{M}}^{\bullet}_{h}(\pi,\mu^1,\ldots,\mu^r,(1^n))]^{\mathrm{vir}_\pi}))\\ = n(2g-2+r)\cdot\epsilon_*([\overline{\mathcal{M}}^{\bullet}_{h}(\pi,\mu^1,\ldots,\mu^r)]^{\mathrm{vir}_\pi}).\end{aligned}$$

Equation (4.4) then follows. □

4.5. Equivalence. The CohFT $\widetilde{\Omega}$ obtained from the orbifold Gromov–Witten theory of $\mathsf{Sym}^n(\mathbb{C}^2)$ and the CohFT $\widetilde{\Lambda}$ are equal in genus 0 (and hence both semisimple) by [**1**, **2**]. The two CohFTs satisfy identical divisor equations. The equality after restriction to $u = 0$,

$$\widetilde{\Omega}_{g,r}(\mu^1,\ldots,\mu^r)|_{u=0} = \widetilde{\Lambda}_{g,r}(\mu^1,\ldots,\mu^r)|_{u=0}, \quad g \geqslant 0, \mu^1,\ldots,\mu^r \in \mathrm{Part}(n),$$

follows from a simple matching of moduli spaces and obstruction theories. By Theorem 3, we conclude the following equivalence.

PROPOSITION 12. *We have* $\widetilde{\Omega} = \widetilde{\Lambda}$.

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5. Local theories of curves: stable pairs

5.1. Stable pairs. We define a CohFT Λ via the moduli space of π-relative T-equivariant stable pairs on the universal local curve,

$$\pi : \mathbb{C}^2 \times \mathcal{C} \to \overline{\mathcal{M}}_{g,r}, \tag{5.1}$$

based on the algebra

$$\mathsf{A} = \mathbb{Q}(t_1, t_2)[[q]]$$

and the corresponding free module

$$\mathsf{V} = \mathcal{F}^n \otimes_{\mathbb{Q}} \mathsf{A},$$

where $\mathcal{F}^n$ is the Fock space with basis indexed by $\mathrm{Part}(n)$. The inner product for the CohFT is

$$\eta(\mu, \nu) = \frac{(-1)^{|\mu|-\ell(\mu)}}{(t_1 t_2)^{\ell(\mu)}} \frac{\delta_{\mu\nu}}{\mathfrak{z}(\mu)}.$$

The algebra, free module, and inner product for Λ are exactly the same as for the CohFT Ω obtained from Gromov–Witten theory of $\mathsf{Hilb}^n(\mathbb{C}^2)$.

Let $\mu^1, \ldots, \mu^r \in \mathrm{Part}(n)$, and let $\mathsf{P}_k(\pi, \mu^1, \ldots, \mu^r)$ be the moduli space of stable pairs on the fibers of π,

$$\epsilon : \mathsf{P}_k(\pi, \mu^1, \ldots, \mu^r) \to \overline{\mathcal{M}}_{g,r}.$$

The fiber of ϵ over the moduli point

$$(C, p_1, \ldots, p_r) \in \overline{\mathcal{M}}_{g,r}$$

is the moduli space of stable pairs of Euler characteristic k on $\mathbb{C}^2 \times C$ relative to the divisors determined by the nodes and the markings of C with boundary condition μ^i over the divisor $\mathbb{C}^2 \times p_i$. (The cohomology weights of the boundary condition are all the identity class.) The moduli space $\mathsf{P}_k(\pi, \mu^1, \ldots, \mu^r)$ has π-relative virtual dimension

$$-n(2g - 2 + r) + \sum_{i=1}^{r} \ell(\mu^i).$$

The CohFT Λ is defined via the π-relative T-equivariant virtual class by

$$\Lambda_{g,r}(\mu^1 \otimes \cdots \otimes \mu^r) = \sum_{d \geqslant 0} q^d \epsilon_*([\mathsf{P}_{d+n(1-g)}(\pi, \mu^1, \ldots, \mu^r)]^{\mathrm{vir}_\pi}).$$

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5.2. Axioms and the divisor equation. The defining axioms of a CohFT are listed in Section 1. Axiom (i) for Λ follows from the symmetry of the construction. Axiom (ii) is a consequence of the degeneration formula of relative stable pairs theory. The proof of Axiom (iii) is parallel to the proof of Proposition 10.

PROPOSITION 13. *The identity axiom holds:*

$$\Lambda_{g,r+1}(\mu^1 \otimes \cdots \otimes \mu^r \otimes \mathbf{1}) = p^* \Lambda_{g,r}(\mu^1 \otimes \cdots \otimes \mu^r)$$

where $p : \overline{\mathcal{M}}_{g,r+1} \to \overline{\mathcal{M}}_{g,r}$ *is the forgetful map.*

Instead of using the 1-pointed stable map space, we consider the universal target

$$p_\pi : \mathcal{T}_k \to \mathsf{P}_k(\pi, \mu^1, \ldots, \mu^r) \tag{5.2}$$

with the universal curve expressed as the descendent $\mathrm{ch}_2(\mathcal{F})$ of the universal sheaf $\mathcal{F}$ on $\mathcal{T}_k$. Otherwise, the proof is identical.

The divisor equation for Λ is identical to the divisor equation for the Gromov–Witten theory of $\mathsf{Hilb}^n(\mathbb{C}^2)$. The proof is parallel to the proof of Proposition 11.

PROPOSITION 14. *The divisor equation holds:*

$$-\Lambda_{g,r+1}(\mu^1 \otimes \cdots \otimes \mu^r \otimes (2, 1^{n-2})) = q\frac{\partial}{\partial q}\Lambda_{g,r}(\mu^1 \otimes \cdots \otimes \mu^r).$$

Proof. Instead of using the 1-pointed stable map space with ψ_1, we consider the universal target (5.2) with the descendent $\mathrm{ch}_3(\mathcal{F})$ of the universal sheaf $\mathcal{F}$ on $\mathcal{T}_k$. The notation here will be parallel to the stable maps case,

$$\epsilon_1 : \mathcal{T}_k \to \overline{\mathcal{M}}_{g,r+1}.$$

Proposition 14 is equivalent to the following statement:

$$\begin{aligned} &p_*\epsilon_*([\mathsf{P}_{d+n(1-g)}(\pi, \mu^1, \ldots, \mu^r, (2, 1^{n-2}))]^{\mathrm{vir}_\pi}) \\ &\quad = d \cdot \epsilon_*([\mathsf{P}_{d+n(1-g)}(\pi, \mu^1, \ldots, \mu^r)]^{\mathrm{vir}_\pi}). \end{aligned}$$

Consider first the dilaton equation in the theory of stable pairs

$$\begin{aligned} &\epsilon_* p_{\pi*}(\mathrm{ch}_3(\mathcal{F})[\mathcal{T}_{d+n(1-g)}]^{\mathrm{vir}_\pi}) \\ &\quad = \left(d + n(1-g) - \frac{n}{2}(2-2g)\right) \cdot \epsilon_*([\mathsf{P}_{d+n(1-g)}(\pi, \mu^1, \ldots, \mu^r)]^{\mathrm{vir}_\pi}). \end{aligned}$$

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(See [32, Section 3.2].) We have also the degeneration relation

$$\epsilon_{1*}(\mathrm{ch}_3(\mathcal{F})[\mathcal{T}_{d+n(1-g)}]^{\mathrm{vir}_\pi}) = -\epsilon_*([\mathsf{P}_{d+n(1-g)}(\pi, \mu^1, \ldots, \mu^r, (2, 1^{n-2}))]^{\mathrm{vir}_\pi})$$

obtained by universally bubbling off the descendent in the target. The contribution of the boundary condition (1^n) vanishes as can be seen from calculation of the equivariant cap. (The equivariant cap invariant for (1^n) has been calculated in [34, Section 2.5]. For stable pairs, s_3 does not occur. All summands have vanishing contributions.) We conclude

$$\begin{aligned} p_*\epsilon_*([\mathsf{P}_{d+n(1-g)}(\pi, \mu^1, \ldots, \mu^r, (2, 1^{n-2}))]^{\mathrm{vir}_\pi}) \\ = p_*\epsilon_{1*}(\mathrm{ch}_3(\mathcal{F})[\mathcal{T}_{d+n(1-g)}]^{\mathrm{vir}_\pi}) \\ = \epsilon_* p_{\pi*}(\mathrm{ch}_3(\mathcal{F})[\mathcal{T}_{d+n(1-g)}]^{\mathrm{vir}_\pi}) \\ = d \cdot \epsilon_*([\mathsf{P}_{d+n(1-g)}(\pi, \mu^1, \ldots, \mu^r)]^{\mathrm{vir}_\pi}), \end{aligned}$$

completing the proof of the divisor equation. □

5.3. Equivalence. The CohFT Ω obtained from the Gromov–Witten theory of $\mathsf{Hilb}^n(\mathbb{C}^2)$ and the CohFT Λ are equal in genus 0 (and hence both semisimple) by [30, 31]. The two CohFTs satisfy identical divisor equations. The equality after restriction to $q = 0$,

$$\Omega_{g,r}(\mu^1, \ldots, \mu^r)|_{q=0} = \Lambda_{g,r}(\mu^1, \ldots, \mu^r)|_{q=0}, \quad g \geqslant 0, \mu^1, \ldots, \mu^r \in \mathrm{Part}(n),$$

follows from a simple matching of moduli spaces and obstruction theories. By Theorem 1, we conclude the following equivalence.

PROPOSITION 15. *We have* $\Omega = \Lambda$.

Since the matching of obstruction theories here in the $q = 0$ case is interesting, we discuss the matter in more detail. The moduli spaces

$$\prod_{i=1}^{r} \mathrm{ev}_i^{-1}(\mu^i) \subset \overline{\mathcal{M}}_{g,r}(\mathsf{Hilb}^n(\mathbb{C}^2), 0) \quad \text{and} \quad \mathsf{P}_{n(1-g)}(\pi, \mu^1, \ldots, \mu^r)$$

are isomorphic. The T-equivariant obstruction bundle for the Gromov–Witten theory of $\mathsf{Hilb}^n(\mathbb{C}^2)$ in the $q = 0$ case is

$$\mathsf{Tan}(\mathsf{Hilb}^n(\mathbb{C}^2)) \otimes \mathbb{E}_g^*. \tag{5.3}$$

The T-equivariant obstruction bundle for π-relative stable pairs theory in the $q = 0$ case is

$$\mathsf{Tan}(\mathsf{Hilb}^n(\mathbb{C}^2))^* \otimes \mathbb{E}_g^* \otimes [t_1 + t_2], \tag{5.4}$$

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where $[t_1 + t_2]$ denotes a trivial bundle of weight $t_1 + t_2$. (The calculation is by Serre duality and standard techniques.) The matching of (5.3) and (5.4) follows from the holomorphic symplectic condition which takes the following T-equivariant form:

$$\mathsf{Tan}(\mathsf{Hilb}^n(\mathbb{C}^2))^* = \mathsf{Tan}(\mathsf{Hilb}^n(\mathbb{C}^2)) \otimes [-t_1 - t_2].$$

6. The restricted CohFTs

6.1. The Hilbert scheme $\mathsf{Hilb}^n(\mathbb{C}^2)$. Following [23, Ch. VI, Section 10], let

$$J_\lambda \in \mathcal{F} \otimes_{\mathbb{C}} \mathbb{C}(t_1, t_2)$$

be an integral form of the Jack symmetric function depending on the parameter $\alpha = 1/\theta$. Define

$$\mathsf{J}^\lambda = t_2^{|\lambda|} t_1^{\ell(\cdot)} J_\lambda|_{\alpha=-t_1/t_2}.$$

The vector J^λ in Fock space corresponds to the T-equivariant class of the T-fixed point of $\mathsf{Hilb}^n(\mathbb{C}^2)$ associated to λ. See also [30, Section 2.2].

The Bernoulli numbers B_m are defined by

$$\frac{t}{e^t - 1} = \sum_{m=0}^{\infty} \frac{B_m}{m!} t^m. \tag{6.1}$$

For $\lambda \in \mathrm{Part}(n)$, define

$$N_{2m-1,\lambda}(t_1, t_2) = \sum_{s \in \mathsf{D}_\lambda} \left(\frac{1}{(a(s)t_2 - (l(s)+1)t_1)^{2m-1}} + \frac{1}{(l(s)t_1 - (a(s)+1)t_2)^{2m-1}} \right).$$

Here, the sum is over all boxes s in the Young diagram D_λ corresponding to λ. The standard arm and leg lengths are $a(s)$ and $l(s)$, respectively.

PROPOSITION 16. *After the coordinates of V are set to 0, the matrix $\mathsf{R}^{\mathsf{Hilb}}|_{q=0}$ in the basis*

$$\{\mathsf{J}^\lambda \mid \lambda \in \mathrm{Part}(n)\}$$

of V is diagonal with entries

$$\exp\left(\sum_{m>0} \frac{B_{2m}}{2m(2m-1)} z^{2m-1} N_{2m-1,\lambda}(t_1, t_2) \right).$$

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Proof. The R-matrix $\mathsf{R}^{\mathsf{Hilb}}|_{q=0}$ is the R-matrix classifying the CohFT $\Omega|_{q=0}$. The CohFT $\Omega|_{q=0}$ concerns the T-equivariant Gromov–Witten theory of $\mathsf{Hilb}^n(\mathbb{C}^2)$ in degree 0. The T-fixed locus

$$\mathsf{Hilb}^n(\mathbb{C}^2)^{\mathsf{T}} \subset \mathsf{Hilb}^n(\mathbb{C}^2)$$

is a set of isolated points indexed by partitions of n (see, for example, [28, Section 5.2]). A partition

$$\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots) \in \mathrm{Part}(n)$$

defines a Young diagram D_λ whose ith row has λ_i boxes. The point in $\mathsf{Hilb}^n(\mathbb{C}^2)^{\mathsf{T}}$ indexed by λ is defined by the monomial ideal $\mathcal{I}_\lambda \subset \mathbb{C}[x, y]$ generated by

$$(y^{\lambda_1}, xy^{\lambda_2}, x^2y^{\lambda_3}, \ldots, x^{i-1}y^{\lambda_i}, \ldots).$$

The Young diagram D_λ also determines the conjugate partition

$$\lambda' = (\lambda'_1, \lambda'_2, \lambda'_3, \ldots)$$

obtained by setting λ'_i to be the number of boxes in the ith column of D_λ.

For $s = (i, j) \in \mathsf{D}_\lambda$, the box on the ith row and the jth column, define the arm and leg lengths by

$$a(s) = \lambda'_j - i \quad \text{and} \quad l(s) = \lambda_i - j,$$

respectively. By [28, Proposition 5.8], the weights associated to $s \in \mathsf{D}_\lambda$ of the T-action on the tangent space $\mathrm{Tan}_\lambda(\mathsf{Hilb}^n(\mathbb{C}^2))$ are

$$-a(s)t_2 + (l(s) + 1)t_1, \quad -l(s)t_1 + (a(s) + 1)t_2. \tag{6.2}$$

As s varies in D_λ, we obtain $2n$ tangent weights.

By virtual localization, the T-equivariant Gromov–Witten theory of $\mathsf{Hilb}^n(\mathbb{C}^2)$ in degree 0 equals the Gromov–Witten theory of $\mathsf{Hilb}^n(\mathbb{C}^2)^{\mathsf{T}}$ twisted (In the sense of [4].) by the inverse T-equivariant Euler class of the rank-$2n$ bundle on which T acts with weights (6.2).

The R-matrix $\mathsf{R}^{\mathsf{Hilb}}|_{q=0}$ is the unique matrix which transforms the Gromov–Witten theory of $\mathsf{Hilb}^n(\mathbb{C}^2)^{\mathsf{T}}$ to the aforementioned twisted Gromov–Witten theory of $\mathsf{Hilb}^n(\mathbb{C}^2)^{\mathsf{T}}$. The latter theory consists of Hodge integrals over $\overline{\mathcal{M}}_{g,r}$, which have been calculated in [9, 27]. To identify the R-matrix, we consider the twisted theory in the setting of [4] and apply the quantum Riemann–Roch theorem. Then, $\mathsf{R}^{\mathsf{Hilb}}|_{q=0}$ coincides with the symplectic transformation in the quantum Riemann–Roch theorem, which is the claim of the Proposition. (Since $\mathsf{R}^{\mathsf{Hilb}}|_{q=0} = \mathrm{Id} + O(z)$, we may discard the scalar factors in quantum Riemann–Roch theorem.) □

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6.2. The symmetric product $\mathsf{Sym}^n(\mathbb{C}^2)$. For $\mathsf{Sym}^n(\mathbb{C}^2)$, the vector $|\mu\rangle \in \widetilde{\mathsf{V}}$ in Fock space corresponds to the fundamental class $[I_\mu]$ of the component

$$I_\mu \subset I\mathsf{Sym}^n(\mathbb{C}^2)$$

of the inertial stack.

PROPOSITION 17. *After the coordinates of $\widetilde{\mathsf{V}}$ are set to 0, the matrix $\mathsf{R}^{\mathsf{Sym}}|_{u=0}$ in the basis*

$$\{|\mu\rangle \mid \mu \in \mathrm{Part}(n)\}$$

of $\widetilde{\mathsf{V}}$ is diagonal with entries

$$\exp\left(-\sum_{m>0} \frac{B_{2m}}{2m(2m-1)} \sum_{i=1}^{\ell(\mu)} \left(\frac{1}{(\mu_i t_1)^{2m-1}} + \frac{1}{(\mu_i t_2)^{2m-1}}\right) z^{2m-1}\right).$$

Proof. The R-matrix $\mathsf{R}^{\mathsf{Sym}}|_{u=0}$ classifies the restricted CohFT $\widetilde{\Omega}_{g,r}|_{u=0}$. By definition, the CohFT $\widetilde{\Omega}_{g,r}|_{u=0}$ is obtained from the T-equivariant Gromov–Witten theory of $\mathrm{Sym}^n(\mathbb{C}^2)$—the Gromov–Witten theory of the classifying orbifold BS_n twisted (In the sense of [41].) by the inverse T-equivariant Euler class of the rank-$2n$ trivial bundle on which T acts with weights t_1 and t_2 (each appearing n times). By the orbifold quantum Riemann–Roch theorem of [41], the twisted theory is obtained from the Gromov–Witten theory of BS_n by the action of a symplectic operator Q. The operator Q coincides with $\mathsf{R}^{\mathsf{Sym}}|_{u=0}$ in the basis $\{[I_\mu]\}_{\mu\in\mathrm{Part}(n)}$ of

$$H^*_{\mathsf{T}}(IBS_n) = H^*_{\mathsf{T}}(I\mathsf{Sym}^n(\mathbb{C}^2)).$$

Next, we identify Q. Consider the conjugacy class $\mathrm{Conj}(\mu)$ of S_n corresponding to the partition

$$\mu = (\mu_1, \mu_2, \mu_3, \ldots).$$

Let $\sigma \in S_n$ be an element of $\mathrm{Conj}(\mu)$. Then, σ can be written as a product of disjoint cycles of lengths μ_i. The vector space $(\mathbb{C}^2)^{\oplus n}$ decomposes into a direct sum of σ-eigenspaces. The eigenvalues and ranks of these eigenspaces depend only on the conjugacy class, not σ. The eigenvalues are

$$\exp\left(2\pi\sqrt{-1}\frac{l}{\mu_i}\right), \quad 0 \leqslant l \leqslant \mu_i - 1, i = 1, 2, 3, \ldots.$$

Each such eigenvalue has a rank-2 eigenspace with T-weights t_1, t_2. (If μ has equal parts, the eigenspaces increase by the multiplicity factor.)

By [41], the operator Q is diagonal in the basis $\{[I_\mu]\}$ of $H^*_{\mathsf{T}}(IBS_n)$ with entry at $[I_\mu]$ given by

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$$\exp\Bigg(\sum_{m>1} \frac{-1}{m(m-1)} \sum_{i=1}^{\ell(\mu)} \Bigg\{ B_m(0) + B_m\left(\frac{1}{\mu_i}\right) + \cdots + B_m\left(\frac{\mu_i - 1}{\mu_i}\right) \Bigg\}$$
$$\times z^{m-1}\left(\frac{1}{t_1^{m-1}} + \frac{1}{t_2^{m-1}}\right)\Bigg).$$

Here, $B_m(x)$ is the Bernoulli polynomial,

$$\frac{te^{xt}}{e^t - 1} = \sum_{m=0}^{\infty} B_m(x) \frac{t^m}{m!}.$$

From the identity

$$\sum_{l=0}^{r-1} \frac{te^{tl/r}}{e^t - 1} = \frac{t}{e^t - 1} \frac{e^t - 1}{e^{t/r} - 1} = \frac{t/r}{e^{t/r} - 1} \cdot r,$$

and the definition (6.1) of the Bernoulli numbers, we see

$$\sum_{l=0}^{r-1} B_m\left(\frac{l}{r}\right) = \frac{B_m}{r^{m-1}}.$$

The above expression for the diagonal elements of Q can be written as

$$\exp\left(-\sum_{m>0} \frac{B_{2m}}{2m(2m-1)} \sum_{i=1}^{\ell(\mu)} \left(\frac{1}{(\mu_i t_1)^{2m-1}} + \frac{1}{(\mu_i t_2)^{2m-1}}\right) z^{2m-1}\right)$$

which completes the proof. □

The product structure on $H_{\mathsf{T}}^*(I\mathrm{Sym}^n\mathbb{C}^2)$ is described in [1, Section 3.3] in terms of the representation theory of the symmetric group. The *normalized* idempotents of $H_{\mathsf{T}}^*(I\mathrm{Sym}^n\mathbb{C}^2)$ are

$$\mathsf{I}^\lambda = \sum_{\mu \in \mathrm{Part(n)}} \chi_\lambda(\mu) I_\mu (t_1 t_2)^{\ell(\mu)/2}, \tag{6.3}$$

where $\chi_\lambda(\mu)$ is the character of S_n. The action of the matrix $\mathsf{R}^{\mathsf{Sym}}|_{u=0}$ in the idempotent basis of $H_{\mathsf{T}}^*(I\mathrm{Sym}^n\mathbb{C}^2)$ is given by

$$\mathsf{R}^{\mathsf{Sym}}|_{u=0}(\mathsf{I}^\lambda)$$
$$= \sum_{\mu} \chi_\lambda(\mu) \exp\Bigg(-\sum_{m>0} \frac{B_{2m}}{2m(2m-1)}$$
$$\times \sum_{i=1}^{\ell(\mu)} \left(\frac{1}{(\mu_i t_1)^{2m-1}} + \frac{1}{(\mu_i t_2)^{2m-1}}\right) z^{2m-1}\Bigg) (t_1 t_2)^{\ell(\mu)/2} |\mu\rangle \tag{6.4}$$

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written in $\widetilde{\mathsf{V}}$. Under the identification $\mathsf{V} \to \widetilde{\mathsf{V}}$ given by (1.6), we have

$$\begin{aligned}
&\mathsf{R}^{\mathsf{Sym}}|_{u=0}(\mathsf{I}^{\lambda}) \\
&= \sum_{\mu} \chi_{\lambda}(\mu) \exp\Bigg(-\sum_{m>0} \frac{B_{2m}}{2m(2m-1)} \\
&\quad \times \sum_{i=1}^{\ell(\mu)} \bigg(\frac{1}{(\mu_i t_1)^{2m-1}} + \frac{1}{(\mu_i t_2)^{2m-1}}\bigg) z^{2m-1}\Bigg) \\
&\quad \cdot \sqrt{-1}^{\ell(\mu)-|\mu|} (t_1 t_2)^{\ell(\mu)/2} |\mu\rangle
\end{aligned} \tag{6.5}$$

written in V. (We use $\sqrt{-1}$ here instead of i for clarity in the formulas (since i also occurs as an index of summation).)

7. Quantum differential equations

We recall the quantum differential equation for $\mathsf{Hilb}^n(\mathbb{C}^2)$ calculated in [**29**] and further studied in [**30**]. We follow here the exposition [**29**, **30**].

Consider the Fock space introduced in Section 0.2 (after extension of scalars to $\mathbb{C}$),

$$\mathcal{F} \otimes_{\mathbb{C}} \mathbb{C}[t_1, t_2] \cong \bigoplus_{n \geqslant 0} H_{\mathsf{T}}^*(\mathsf{Hilb}^n(\mathbb{C}^2), \mathbb{C}), \tag{7.1}$$

freely generated by commuting creation operators α_{-k} for $k \in \mathbb{Z}_{>0}$ acting on the vacuum vector $v_{\emptyset}$. The intersection pairing on the T-equivariant cohomology of $\mathsf{Hilb}^n(\mathbb{C}^2)$ induces a pairing on Fock space,

$$\eta(\mu, \nu) = \frac{(-1)^{|\mu|-\ell(\mu)}}{(t_1 t_2)^{\ell(\mu)}} \frac{\delta_{\mu\nu}}{\mathfrak{z}(\mu)}.$$

The paper [**30**] also uses a Hermitian pairing $\langle -, - \rangle_H$ on the Fock space (7.1) defined by the three following properties:

- $\langle \mu | \nu \rangle_H = (1/(t_1 t_2)^{\ell(\mu)})(\delta_{\mu\nu}/\mathfrak{z}(\mu))$;
- $\langle af, g \rangle_H = a\langle f, g \rangle_H$, $a \in \mathbb{C}(t_1, t_2)$;
- $\langle f, g \rangle_H = \overline{\langle g, f \rangle_H}$, where $\overline{a(t_1, t_2)} = a(-t_1, -t_2)$.

The quantum differential equation (QDE) for the Hilbert schemes of points on $\mathbb{C}^2$ is given by

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$$q\frac{d}{dq}\Phi = \mathsf{M}_D\Phi, \quad \Phi \in \mathcal{F} \otimes_{\mathbb{C}} \mathbb{C}(t_1, t_2), \tag{7.2}$$

where M_D is the operator of quantum multiplication by D,

$$\begin{aligned}\mathsf{M}_D = (t_1 + t_2) \sum_{k>0} \frac{k}{2} \frac{(-q)^k + 1}{(-q)^k - 1} \alpha_{-k}\alpha_k - \frac{t_1 + t_2}{2} \frac{(-q) + 1}{(-q) - 1} |\cdot| \\ + \frac{1}{2} \sum_{k,l>0} [t_1 t_2 \alpha_{k+l}\alpha_{-k}\alpha_{-l} - \alpha_{-k-l}\alpha_k\alpha_l].\end{aligned} \tag{7.3}$$

While the quantum differential equation (7.2) has a regular singular point at $q = 0$, the point $q = -1$ is regular.

The quantum differential equation considered in Givental's theory contains a parameter z. In the case of the Hilbert schemes of points on $\mathbb{C}^2$, the QDE with parameter z is

$$zq\frac{d}{dq}\Phi = \mathsf{M}_D\Phi, \quad \Phi \in \mathcal{F} \otimes_{\mathbb{C}} \mathbb{C}(t_1, t_2). \tag{7.4}$$

For $\Phi \in \mathcal{F} \otimes_{\mathbb{C}} \mathbb{C}(t_1, t_2)$, define

$$\Phi_z = \Phi\left(\frac{t_1}{z}, \frac{t_2}{z}, q\right).$$

Define $\Theta \in \mathrm{Aut}(\mathcal{F})$ by

$$\Theta|\mu\rangle = z^{\ell(\mu)}|\mu\rangle.$$

The following Proposition allows us to use the results in [**30**].

PROPOSITION 18. *If Φ is a solution of (7.2), then $\Theta\Phi_z$ is a solution of (7.4).*

Proposition 18 follow immediately from the following direct computation.

LEMMA 19. *For $k > 0$, we have $\Theta\alpha_k = (1/z)\alpha_k\Theta$ and $\Theta\alpha_{-k} = z\alpha_{-k}\Theta$.*

8. Solutions of the QDE

8.1. Preparations. In what follows, we fix an integer $n \geqslant 1$ and consider the solutions of the QDE for $\mathsf{Hilb}^n(\mathbb{C}^2)$. The equivariant parameters t_1, t_2 are treated as complex numbers varying in a Euclidean open domain of $\mathbb{C}$. We work with the energy n subspace of Fock space. For notational simplicity, we often omit n from the formulas.

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8.2. At the origin. At $q=0$, the operator $\mathsf{M}_D(0)$ has distinct eigenvalues

$$-c(\lambda; t_1, t_2) = -\sum_{(i,j)\in\lambda} [(j-1)t_1 + (i-1)t_2].$$

So $\mathsf{M}_D(q)$ has distinct eigenvalues for small for $q \in \mathbb{C}$ with $|q|$ small. Furthermore, the values of q for which $\mathsf{M}_D(q)$ has repeated eigenvalues are roots of polynomial equations, so there are only finitely many such q. Therefore, we can find a path

$$\gamma : [0,1] \to \mathbb{C}, \quad \gamma(0) = 0, \quad \gamma(1) = -1,$$

and an open neighborhood $\mathcal{U} \subset \mathbb{C}$ containing γ such that:

(i) $\mathsf{M}_D(q)$ has distinct eigenvalues for $q \in \mathcal{U}$.

(ii) As q moves along γ to $q=-1$, the function $q^{-c(\lambda;t_1,t_2)}$ is transported to $\exp(\pi\sqrt{-1}c(\lambda; t_1, t_2))$.

For $q \in \mathcal{U}$, the eigenvalues of $\mathsf{M}_D(q)$, denoted by $v(\lambda; q)$, are analytic in q. Near $q=0$, we have a power series expansion

$$v(\lambda; q) = -c(\lambda; t_1, t_2) + O(q).$$

Therefore, the canonical coordinates

$$u(\lambda; q) = \int v(\lambda; q)\, d\log q,$$

are analytic in q. We choose the integration constants so that in $q \to 0$ we have the asymptotics

$$u(\lambda; q) = -c(\lambda; t_1, t_2)\log q + O(q). \tag{8.1}$$

Let u be the diagonal matrix with entries $u(\lambda; q)$.

8.3. Formal solutions. By [11, Proposition 1.1], we have the following results (see also [7] and [21, Theorem 1]).

PROPOSITION 20. *There is an open neighborhood $\mathcal{U}_0 \subset \mathcal{U}$ of* 0 *on which the QDE* (7.4) *has a formal solution of the form*

$$\mathsf{R}^{\mathsf{Hilb}} e^{\mathsf{u}/z}. \tag{8.2}$$

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In [11, Proposition 1.1], the asymptotics takes the form

$$\Psi \mathsf{R} e^{\mathsf{u}/z} \tag{8.3}$$

where $\Psi(q)$ is the matrix whose columns are length-1 eigenvectors of $\mathsf{M}_D(q)$. (In [21, Theorem 1], a different convention is followed: the same transition matrix is denoted Ψ^{-1}.) In other word, Ψ is the transition matrix from the canonical basis to the flat basis. In (8.3), the matrix R is in canonical coordinates. Together $\Psi\mathsf{R}$ is the R-matrix in flat coordinates.

After the restriction $q = 0$, canonical and flat coordinates are the given by the respective bases

$$\{\mathsf{J}^\lambda \mid \lambda \in \mathrm{Part}(n)\} \quad \text{and} \quad \{|\mu\rangle \mid \mu \in \mathrm{Part}(n)\}$$

of Fock space.

The asymptotics in the $z \to 0$ limit will play a crucial role. For our study, we must specify how z approaches $0 \in \mathbb{C}$. Let

$$\mathfrak{R} \subset \mathbb{C}$$

be a ray emanating from 0 satisfying the following four conditions:

- For $z \in \mathfrak{R}$ or $z \in -\mathfrak{R}$;

$$\left|\arg\left(\frac{t_1}{z}\right)\right| < \pi, \quad \left|\arg\left(\frac{t_2}{z}\right)\right| < \pi. \tag{8.4}$$

- For $z \in \mathfrak{R}$ or $z \in -\mathfrak{R}$, and for any partition λ of n, and $s \in \mathsf{D}_\lambda$, we have

$$\left|\arg\left(\frac{(l(s)+1)t_1 - a(s)t_2}{z}\right)\right| < \pi, \quad \left|\arg\left(\frac{-l(s)t_1 + (a(s)+1)t_2}{z}\right)\right| < \pi. \tag{8.5}$$

- For $z \in \mathfrak{R}$ or $z \in -\mathfrak{R}$, and for any two partitions λ, λ' of n, we have

$$\arg\left(\frac{-c(\lambda; t_1, t_2) + c(\lambda'; t_1, t_2)}{z}\right) \notin \frac{\pi}{2} + \mathbb{Z}\pi. \tag{8.6}$$

- For $z \in \mathfrak{R}$,

$$\mathrm{Re}(\sqrt{-1}t_1/z) < 0, \quad \mathrm{Re}(\sqrt{-1}t_2/z) > 0. \tag{8.7}$$

For suitable t_1, t_2 varying in small enough domains, we can find $\mathfrak{R} \subset \mathbb{C}$ satisfying these conditions.

8.4. Solutions. We recall the solution of QDE (7.2) constructed in [30].

As in Section 6.1, let $J_\lambda \in \mathcal{F} \otimes_{\mathbb{C}} \mathbb{C}(t_1, t_2)$ be the integral form of the Jack symmetric function depending on the parameter $\alpha = 1/\theta$ of [23, 30]. Then

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$$\mathsf{J}^\lambda = t_2^{|\lambda|} t_1^{\ell(\cdot)} J_\lambda|_{\alpha=-t_1/t_2}$$

is an eigenfunction of $\mathsf{M}_D(0)$ with eigenvalue $-c(\lambda; t_1, t_2)$. The coefficient of

$$|\mu\rangle \in \mathcal{F} \otimes_{\mathbb{C}} \mathbb{C}(t_1, t_2)$$

in the expansion of J^λ is $(t_1 t_2)^{\ell(\mu)}$ times a polynomial in t_1 and t_2 of degree $|\lambda| - \ell(\mu)$.

By direct calculation, detailed in Section 8.6, we find

$$\langle \mathsf{J}^\lambda, \mathsf{J}^\mu \rangle_H = \eta(\mathsf{J}^\lambda, \mathsf{J}^\mu). \tag{8.8}$$

Since J^λ corresponds to the T-equivariant class of the T-fixed point of $\mathsf{Hilb}^n(\mathbb{C}^2)$ associated to λ,

$$\|\mathsf{J}^\lambda\|^2 = \|\mathsf{J}^\lambda\|_H^2 = \prod_{\mathsf{w}:\text{ tangent weights at }\lambda} \mathsf{w} \tag{8.9}$$

see [30]. The tangent weights are given by (6.2).

There are solutions to (7.2) of the form

$$\mathsf{Y}^\lambda(q) q^{-c(\lambda; t_1, t_2)}, \quad \mathsf{Y}^\lambda(q) \in \mathcal{F} \otimes_{\mathbb{C}} \mathbb{C}(t_1, t_2)[[q]],$$

which converge for $|q| < 1$ and satisfy $\mathsf{Y}^\lambda(0) = \mathsf{J}^\lambda$. (See, for example, [19, Ch. XIX] for a discussion of how these solutions are constructed.)

By [30, Corollary 1],

$$\langle \mathsf{Y}^\lambda(q), \mathsf{Y}^\mu(q) \rangle_H = \delta_{\lambda\mu} \|\mathsf{J}^\lambda\|_H^2 = \langle \mathsf{J}^\lambda, \mathsf{J}^\mu \rangle_H. \tag{8.10}$$

As in [30, Section 3.1.3], let Y be the matrix whose column vectors are Y^λ. Let J be the matrix whose column vectors are J^λ. Let $\mathsf{G}_{\mathrm{DT}}(t_1, t_2)$ be the diagonal matrix with eigenvalues

$$q^{-c(\lambda; t_1, t_2)} \prod_{\mathsf{w}:\text{ tangent weights at }\lambda} \frac{1}{\Gamma(\mathsf{w}+1)}.$$

Define the following further diagonal matrices:

Matrix	Eigenvalues
L	$z^{-\lvert\lambda\rvert} \prod_{\mathsf{w}:\text{ tangent weights at }\lambda} \mathsf{w}^{1/2}$
L_0	$q^{-c(\lambda; t_1, t_2)/z}$
A	$\prod_{\mathsf{w}:\text{ tangent weights at }\lambda} (\mathsf{w}/z)^{\mathsf{w}/z} e^{-\mathsf{w}/z}$

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Consider the following solution to (7.4),

$$S = (2\pi)^{|\cdot|}\Theta \mathsf{Y}_z \mathsf{G}_{\mathrm{DT}z} A, \tag{8.11}$$

where S is defined over $\mathcal{U}$. As before,

$$\mathsf{Y}_z = \mathsf{Y}\left(\frac{t_1}{z}, \frac{t_2}{z}, q\right), \quad \mathsf{G}_{\mathrm{DT}z} = \mathsf{G}_{\mathrm{DT}}\left(\frac{t_1}{z}, \frac{t_2}{z}, q\right).$$

PROPOSITION 21. *As $z \to 0$ along $\mathfrak{R}$, the operator $Se^{-\mathsf{u}/z}|_{q=0}$ has the asymptotics*

$$\mathsf{R}^{\mathrm{Hilb}}|_{q=0}.$$

Proof. We write S as

$$\begin{aligned}(2\pi)^{|\cdot|}\Theta \mathsf{Y}_z \mathsf{G}_{\mathrm{DT}z} A &= (2\pi)^{|\cdot|}\Theta \mathsf{Y}_z L^{-1} L L_0 L_0^{-1} \mathsf{G}_{\mathrm{DT}z} A.\\ &= (\Theta \mathsf{Y}_z L^{-1})((2\pi)^{|\cdot|} L L_0^{-1} \mathsf{G}_{\mathrm{DT}z} A) L_0.\end{aligned}$$

At $q = 0$, the columns of $\Theta \mathsf{Y}_z L^{-1}$ are

$$\begin{aligned}&\Theta \mathsf{Y}^\lambda\left(0; \frac{t_1}{z}, \frac{t_2}{z}\right) z^{|\lambda|} \prod_{\mathsf{w}:\text{ tangent weights at } \lambda} \mathsf{w}^{-1/2}\\ &= \Theta \mathsf{J}^\lambda\left(\frac{t_1}{z}, \frac{t_2}{z}\right) z^{|\lambda|} \prod_{\mathsf{w}:\text{ tangent weights at } \lambda} \mathsf{w}^{-1/2}\\ &= \mathsf{J}^\lambda(t_1, t_2) \prod_{\mathsf{w}:\text{ tangent weights at } \lambda} \mathsf{w}^{-1/2}.\end{aligned}$$

So $\Psi|_{q=0} = \Theta \mathsf{Y}_z L^{-1}$. By (8.1), $L_0 e^{-\mathsf{u}/z}|_{q=0} = \mathbf{1}$, where $\mathbf{1}$ is the identity matrix.

It remains to calculate the asymptotics of $(2\pi)^{|\cdot|} L L_0^{-1} \mathsf{G}_{\mathrm{DT}z} A$. Recall the Stirling asymptotics for Gamma function (see, for example, [43]):

$$\frac{1}{\Gamma(x+1)} \sim \frac{x^{-1/2} x^{-x} e^x}{\sqrt{2\pi}} \exp\left(\sum_{m>0} \frac{B_{2m}}{2m(2m-1)} \left(\frac{-1}{x}\right)^{2m-1}\right), \quad |x| \to \infty, |\arg(x)| < \pi. \tag{8.12}$$

By condition (8.5), the formula (8.12) is applicable to G_{DT} as $z \to 0$ along $\mathfrak{R}$. We conclude that the asymptotics of $L_0^{-1} \mathsf{G}_{\mathrm{DT}z}$ is a diagonal matrix with eigenvalues

$$\prod_{\mathsf{w}:\text{ tangent weights at }\lambda} \frac{(\mathsf{w}/z)^{-1/2}(\mathsf{w}/z)^{-(\mathsf{w}/z)}e^{\mathsf{w}/z}}{\sqrt{2\pi}}$$

$$\times \exp\left(\sum_{m>0} \frac{B_{2m}}{2m(2m-1)}\left(\frac{-1}{(\mathsf{w}/z)}\right)^{2m-1}\right)$$

$$= \frac{z^{|\lambda|}}{(2\pi)^{|\lambda|}} \prod_{\mathsf{w}:\text{ tangent weights at }\lambda} \mathsf{w}^{-1/2}(\mathsf{w}/z)^{-\mathsf{w}/z}e^{\mathsf{w}/z}$$

$$\times \exp\left(\sum_{m>0} \frac{B_{2m}}{2m(2m-1)}\left(\frac{-z}{\mathsf{w}}\right)^{2m-1}\right).$$

Therefore, $(2\pi)^{|\cdot|}LL_0^{-1}\mathsf{G}_{\mathrm{DT}_z}A$ has asymptotics given by a diagonal matrix with eigenvalues

$$\prod_{\mathsf{w}:\text{ tangent weights at }\lambda} \exp\left(\sum_{m>0} \frac{B_{2m}}{2m(2m-1)}\left(\frac{-z}{\mathsf{w}}\right)^{2m-1}\right),$$

which coincides, by Proposition 16, with $\mathsf{R}^{\mathrm{Hilb}}|_{q=0}$ written as a matrix with both domain and range in the canonical basis.

Taken all together, $Se^{-\mathsf{u}/z}|_{q=0}$ is

$$(\Theta\mathsf{Y}_z L^{-1})((2\pi)^{|\cdot|}LL_0^{-1}\mathsf{G}_{\mathrm{DT}_z}A)L_0\exp(-\mathsf{u}/z)|_{q=0}$$

and has the asymptotics $\mathsf{R}^{\mathrm{Hilb}}|_{q=0}$ written as a matrix with domain in the canonical basis and range in the flat basis. □

8.5. Asymptotics of solutions.

PROPOSITION 22. *For every $p \in \gamma$, there exists an open neighborhood*

$$\mathcal{U}_p \subset \mathcal{U}$$

on which the system (7.4) has a fundamental solution $\mathcal{S}$ satisfying the following property: in the $z \to 0$ limit along $\mathfrak{R}$, $\mathcal{S}$ has asymptotics of the form

$$\mathcal{S} \sim \mathcal{R}e^{\mathsf{u}/z}$$

where $\mathcal{R} = \mathbf{1} + \mathcal{R}_1 z + \mathcal{R}_2 z^2 + \cdots$ is an operator-valued z-series with coefficients analytic in q.

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Proof. For $p \neq 0$, the result follows immediately from [42, Theorem 26.3]. More precisely, we use the change of variables $q = pe^{-x}$ to transform the system (7.4) into a system of the form

$$z\frac{d}{dx}\Phi = M\Phi.$$

We use an analytical simplification, which exists according to [42, Theorem 26.1], to transform the latter system to a collection of 1-dimensional ODEs of the form

$$z\frac{df}{dx} = a(x, t_1, t_2)f,$$

which can be easily solved. Combining the solutions of these ODEs with the analytical simplification gives the solution $\mathcal{S}$. The $z \to 0$ limit can be taken along $\mathfrak{R}$ because the proof of [42, Theorem 26.1] allows the $z \to 0$ limit to be taken along any direction.

For $p = 0$ the above argument is still valid. The needed analytical simplification exists because of the condition (8.6) allows us to apply [37, Theorem 3], whose proof was completed in [38]. We then transform (7.4) to a collection of 1-dimensional ODEs of the form

$$zq\frac{df}{dq} = a(q, t_1, t_2)f.$$

Combining the solutions of these ODEs with the analytical simplification gives the solution $\mathcal{S}$. By condition (8.6), the $z \to 0$ limit can be taken along $\mathfrak{R}$. □

PROPOSITION 23. *There exists an open neighborhood $\mathcal{U}' \subset \mathcal{U}$ of γ on which the solution S has the asymptotics*

$$S \sim \mathsf{R}e^{\mathsf{u}/z} \quad \text{as } z \to 0 \text{ along } \mathfrak{R}, \tag{8.13}$$

where $\mathsf{R} = \mathbf{1} + \mathsf{R}_1 z + \mathsf{R}_2 z^2 + \cdots$ is an operator-valued z-series with coefficients analytic in q and $\mathsf{R} = \mathsf{R}^{\mathsf{Hilb}}$ in a neighborhood of $q = 0$. Moreover, R is symplectic,

$$\mathsf{R}^\dagger(-z)\mathsf{R}(z) = \mathbf{1}$$

with the adjoint taken with respect to the pairing η.

Proof. After intersection, we may assume the two open sets $\mathcal{U}_0$ in Propositions 20 and 22 are the same.

Consider the open sets $\mathcal{U}_q$ of Proposition 22. Certainly $\gamma \subset \bigcup_{q\in\gamma}\mathcal{U}_q$. Since γ is compact, there exist finitely many points

$$t_0 = 0 < t_1 < t_2 < \cdots < t_k = 1$$

for which $\gamma \subset \bigcup_{i=0}^k \mathcal{U}_{\gamma(t_i)}$. Define

$$\mathcal{U}_i = \mathcal{U}_{\gamma(t_i)}.$$

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Consider the solution $\mathcal{S}$ over $\mathcal{U}_0$ constructed in Proposition 22. The asymptotics of $\mathcal{S}$ as $z \to 0$ along $\mathfrak{R}$, $\mathcal{R}e^{\mathsf{u}/z}$ is an asymptotical solution of (7.4). We compare $\mathcal{R}e^{\mathsf{u}/z}$ with the asymptotical solution (8.2). By [11, Remark 1 after the proof of Proposition 1.1], $\mathcal{R}$ and $\mathsf{R}^{\mathsf{Hilb}}$ differ by a z-series with coefficients given by diagonal matrices independent of q. (Here, all operators are represented by matrices in the canonical basis. Since both $e^{\mathsf{u}/z}$ and $(\mathsf{R}^{\mathsf{Hilb}}|_{q=0})^{-1}(\mathcal{R}|_{q=0})$ are diagonal, these matrices commute.) Therefore,

$$\mathcal{R} = \mathsf{R}^{\mathsf{Hilb}}(\mathsf{R}^{\mathsf{Hilb}}|_{q=0})^{-1}(\mathcal{R}|_{q=0})$$

and $(\mathsf{R}^{\mathsf{Hilb}}|_{q=0})^{-1}(\mathcal{R}|_{q=0})$ is diagonal.

The solution S in (8.11) is a fundamental solution to (7.4). Hence, there exists a matrix C, independent of q, satisfying

$$SC = \mathcal{S}$$

on $\mathcal{U}_0$. The asymptotics of $S|_{q=0}$ as $z \to 0$ along $\mathfrak{R}$ were calculated in Proposition 21. After comparing with the asymptotics of $\mathcal{S}|_{q=0}$, we find

$$C \sim (\mathsf{R}^{\mathsf{Hilb}}|_{q=0})^{-1}(\mathcal{R}|_{q=0})$$

as $z \to 0$ along $\mathfrak{R}$. Therefore, on $\mathcal{U}_0$, we find

$$\begin{aligned} S = \mathcal{S}C^{-1} &\sim \mathcal{R}e^{\mathsf{u}/z}(\mathcal{R}|_{q=0})^{-1}(\mathsf{R}^{\mathsf{Hilb}}|_{q=0}) \\ &= \mathcal{R}(\mathcal{R}|_{q=0})^{-1}(\mathsf{R}^{\mathsf{Hilb}}|_{q=0})e^{\mathsf{u}/z} = \mathsf{R}^{\mathsf{Hilb}}e^{\mathsf{u}/z} \end{aligned}$$

as $z \to 0$ along $\mathfrak{R}$.

Suppose now (8.13) is proven over $\mathcal{U}_{<l} = \bigcup_{i=0}^{l-1}\mathcal{U}_i$. Consider the solution $\mathcal{S}$ over $\mathcal{U}_l$ constructed in Proposition 22. The asymptotics of $\mathcal{S}$ as $z \to 0$ along $\mathfrak{R}$, $\mathcal{R}e^{\mathsf{u}/z}$, is an asymptotical solution of (7.4) on $\mathcal{U}_l$. We compare $\mathcal{R}e^{\mathsf{u}/z}$ with the asymptotical solution (8.13) over $\mathcal{U}_{<l} \cap \mathcal{U}_l$. As before, $\mathcal{R}$ and R differ by a z-series with coefficients diagonal matrices independent of q. Let $p_l \in \mathcal{U}_{<l} \cap \mathcal{U}_l$. Then, over $\mathcal{U}_{<l} \cap \mathcal{U}_l$, we have

$$\mathcal{R} = \mathsf{R}(\mathsf{R}|_{q=p_l})^{-1}(\mathcal{R}|_{q=p_l}).$$

Moreover, $(R|_{q=p_l})^{-1}(\mathcal{R}|_{q=p_l})$ is a diagonal matrix. As before, there exists a matrix C, independent of q, satisfying

$$SC = \mathcal{S}$$

on $\mathcal{U}_{<l} \cap \mathcal{U}_l$. Comparing asymptotics at $q = p_l$, we find

$$C \sim (\mathsf{R}|_{q=p_l})^{-1}(\mathcal{R}|_{q=p_l})$$

as $z \to 0$ along $\mathfrak{R}$.

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On $\mathcal{U}_l$, define $\mathsf{R} = \mathcal{R}(\mathcal{R}|_{q=p_l})^{-1}(\mathsf{R}|_{q=p_l})$. Over $\mathcal{U}_{<l} \cap \mathcal{U}_l$, we have

$$S = \mathcal{S}C^{-1} \sim \mathcal{R}e^{\mathsf{u}/z}(\mathcal{R}|_{q=p_l})^{-1}(\mathsf{R}|_{q=p_l}) = \mathcal{R}(\mathcal{R}|_{q=p_l}^{-1})(\mathsf{R}|_{q=p_l})e^{\mathsf{u}/z} = \mathsf{R}e^{\mathsf{u}/z}$$

as $z \to 0$ along $\mathfrak{R}$. We have proven (8.13) over $\mathcal{U}_{<l} \cup \mathcal{U}_l$.

Finally, we prove the constructed R is symplectic. (An easier way to see the symplectic property of R is the following: being symplectic is a closed condition in q, and R is symplectic on $\mathcal{U}_0$ because R is $\mathsf{R}^{\mathrm{Hilb}}$ on $\mathcal{U}_0$. However, we include a detailed calculation to verify the symplectic condition to show how all the formula fit together.) We compute $S^\dagger(-z)S(z)$ in two ways. By (8.13), as $z \to 0$ along $\mathfrak{R}$, we find

$$S^\dagger(-z)S(z) \sim e^{-\mathsf{u}^\dagger/z}\mathsf{R}^\dagger(-z)\mathsf{R}(z)e^{\mathsf{u}/z} = e^{-\mathsf{u}/z}\mathsf{R}^\dagger(-z)\mathsf{R}(z)e^{\mathsf{u}/z}.$$

On the other hand, using the definition of S in (8.11) and the matrix L, we find

$$S(z) = (2\pi)^{|\cdot|}\Theta\mathsf{Y}_z\mathsf{G}_{\mathrm{DT}_z}A = (\Theta\mathsf{Y}_z L^{-1})(2\pi)^{|\cdot|}L\mathsf{G}_{\mathrm{DT}_z}A.$$

By direct calculation, detailed in Section 8.6, we have

$$\langle \Theta\mathsf{Y}_z^\lambda z^{|\lambda|}, \Theta\mathsf{Y}_z^\mu z^{|\mu|}\rangle_H = \eta(\Theta\mathsf{Y}_z^\lambda z^{|\lambda|}, (\Theta\mathsf{Y}_z^\mu z^{|\mu|})|_{z\mapsto -z}) = \delta_{\lambda\mu} \prod_{\mathsf{w}:\ \text{tangent weight at } \lambda} \mathsf{w}. \tag{8.14}$$

Hence,

$$(\Theta\mathsf{Y}_z L^{-1})^\dagger|_{z\mapsto -z}(\Theta\mathsf{Y}_z L^{-1}) = \mathbf{1}.$$

Then,

$$\begin{aligned} &S^\dagger(-z)S(z)\\ &= ((2\pi)^{|\cdot|}L\mathsf{G}_{\mathrm{DT}_z}A)^\dagger|_{z\mapsto -z}(\Theta\mathsf{Y}_z L^{-1})^\dagger|_{z\mapsto -z}(\Theta\mathsf{Y}_z L^{-1})((2\pi)^{|\cdot|}L\mathsf{G}_{\mathrm{DT}_z}A)\\ &= ((2\pi)^{|\cdot|}L\mathsf{G}_{\mathrm{DT}_z}A)^\dagger|_{z\mapsto -z}((2\pi)^{|\cdot|}L\mathsf{G}_{\mathrm{DT}_z}A). \end{aligned}$$

By the analysis of asymptotics of G_{DT} in the proof of Proposition 21, we have the following as $z \to 0$ along $\mathfrak{R}$:

$$\begin{aligned} &(2\pi)^{|\cdot|}L\mathsf{G}_{\mathrm{DT}_z}A\\ &\sim \mathrm{Diag}\left(q^{-c(\lambda;t_1,t_2)/z} \prod_{\mathsf{w}:\ \text{tangent weights at } \lambda} \exp\left(\sum_{m>0} \frac{B_{2m}}{2m(2m-1)}\left(\frac{-z}{\mathsf{w}}\right)^{2m-1}\right)\right),\\ &((2\pi)^{|\cdot|}L\mathsf{G}_{\mathrm{DT}_z}A)^\dagger|_{z\mapsto -z}\\ &\sim \mathrm{Diag}\left(q^{c(\lambda;t_1,t_2)/z} \prod_{\mathsf{w}:\ \text{tangent weights at } \lambda} \exp\left(\sum_{m>0} \frac{B_{2m}}{2m(2m-1)}\left(\frac{z}{\mathsf{w}}\right)^{2m-1}\right)\right). \end{aligned}$$

We conclude $S^\dagger(-z)S(z) \sim \mathbf{1}$.

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Comparing the two asymptotics of $S^\dagger(-z)S(z)$, we find

$$\exp(-\mathsf{u}/z)\mathsf{R}^\dagger(-z)\mathsf{R}(z)\exp(\mathsf{u}/z) = \mathbf{1},$$

which implies $\mathsf{R}^\dagger(-z)\mathsf{R}(z) = \mathbf{1}$. □

8.6. Calculations of scalar products. We first check (8.8). Write

$$\mathsf{J}^\lambda = \sum_\epsilon \mathsf{J}^\lambda_\epsilon(t_1,t_2)|\epsilon\rangle, \quad \mathsf{J}^\mu = \sum_{\epsilon'} \mathsf{J}^\mu_{\epsilon'}(t_1,t_2)|\epsilon\rangle,$$

where $\mathsf{J}^\lambda_\epsilon(t_1,t_2), \mathsf{J}^\mu_{\epsilon'}(t_1,t_2) \in \mathbb{C}(t_1,t_2)[[q]]$. Then

$$\begin{aligned}\langle \mathsf{J}^\lambda, \mathsf{J}^\mu\rangle_H &= \sum_{\epsilon,\epsilon'} \mathsf{J}^\lambda_\epsilon(t_1,t_2)\overline{\mathsf{J}^\mu_{\epsilon'}(t_1,t_2)}\langle\epsilon|\epsilon'\rangle_H \\ &= \sum_{\epsilon,\epsilon'} \mathsf{J}^\lambda_\epsilon(t_1,t_2)\mathsf{J}^\mu_{\epsilon'}(-t_1,-t_2)\frac{1}{(t_1t_2)^{\ell(\epsilon)}}\frac{\delta_{\epsilon\epsilon'}}{\mathfrak{z}(\epsilon)}. \end{aligned} \tag{8.15}$$

Since $\mathsf{J}^\mu_{\epsilon'}(t_1,t_2)$ is $(t_1t_2)^{\ell(\epsilon')}$ times a polynomial in t_1 and t_2 of degree $|\mu| - \ell(\epsilon')$, we have

$$\mathsf{J}^\mu_{\epsilon'}(-t_1,-t_2) = (-1)^{2\ell(\epsilon')}(-1)^{|\mu|-\ell(\epsilon')}\mathsf{J}^\mu_{\epsilon'}(t_1,t_2).$$

We can therefore write (8.15) as

$$\begin{aligned}&\sum_{\epsilon,\epsilon'} \mathsf{J}^\lambda_\epsilon(t_1,t_2)\mathsf{J}^\mu_{\epsilon'}(t_1,t_2)\frac{(-1)^{|\mu|-\ell(\epsilon')}}{(t_1t_2)^{\ell(\epsilon)}}\frac{\delta_{\epsilon\epsilon'}}{\mathfrak{z}(\epsilon)} \\ &\quad = \sum_{\epsilon,\epsilon'} \mathsf{J}^\lambda_\epsilon(t_1,t_2)\mathsf{J}^\mu_{\epsilon'}(t_1,t_2)\frac{(-1)^{|\epsilon|-\ell(\epsilon)}}{(t_1t_2)^{\ell(\epsilon)}}\frac{\delta_{\epsilon\epsilon'}}{\mathfrak{z}(\epsilon)} \\ &\quad = \eta(\mathsf{J}^\lambda, \mathsf{J}^\mu),\end{aligned}$$

where, in the first equality, we have used $|\mu| = |\epsilon|$.

We now check (8.14). Write

$$\mathsf{Y}^\lambda = \sum_\epsilon \mathsf{Y}^\lambda_\epsilon(t_1,t_2)|\epsilon\rangle, \quad \mathsf{Y}^\mu = \sum_{\epsilon'} \mathsf{Y}^\mu_{\epsilon'}(t_1,t_2)|\epsilon'\rangle,$$

where $\mathsf{Y}^\lambda_\epsilon(t_1,t_2), \mathsf{Y}^\mu_{\epsilon'}(t_1,t_2) \in \mathbb{C}(t_1,t_2)[[q]]$. Then,

$$\langle \Theta \mathsf{Y}^\lambda_z z^{|\lambda|}, \Theta \mathsf{Y}^\mu_z z^{|\mu|}\rangle_H = z^{|\lambda|+|\mu|}\sum_{\epsilon,\epsilon'} \mathsf{Y}^\lambda_\epsilon\left(\frac{t_1}{z},\frac{t_2}{z}\right)\overline{\mathsf{Y}^\mu_{\epsilon'}\left(\frac{t_1}{z},\frac{t_2}{z}\right)}\langle\Theta\epsilon|\Theta\epsilon'\rangle_H$$

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$$= z^{|\lambda|+|\mu|} \sum_{\epsilon,\epsilon'} \mathsf{Y}^\lambda_\epsilon\left(\frac{t_1}{z}, \frac{t_2}{z}\right)\mathsf{Y}^\mu_{\epsilon'}\left(-\frac{t_1}{z}, -\frac{t_2}{z}\right)\langle\epsilon|\epsilon'\rangle_H z^{\ell(\epsilon)+\ell(\epsilon')}$$

$$= z^{|\lambda|+|\mu|} \sum_{\epsilon,\epsilon'} \mathsf{Y}^\lambda_\epsilon\left(\frac{t_1}{z}, \frac{t_2}{z}\right)\mathsf{Y}^\mu_{\epsilon'}\left(-\frac{t_1}{z}, -\frac{t_2}{z}\right)\frac{z^{\ell(\epsilon)+\ell(\epsilon')}}{(t_1t_2)^{\ell(\epsilon)}}\frac{\delta_{\epsilon\epsilon'}}{\mathfrak{z}(\epsilon)}.$$

We have

$$\begin{aligned}
&\sum_{\epsilon,\epsilon'} \mathsf{Y}^\lambda_\epsilon\left(\frac{t_1}{z}, \frac{t_2}{z}\right)\mathsf{Y}^\mu_{\epsilon'}\left(-\frac{t_1}{z}, -\frac{t_2}{z}\right)\frac{z^{\ell(\epsilon)+\ell(\epsilon')}}{(t_1t_2)^{\ell(\epsilon)}}\frac{\delta_{\epsilon\epsilon'}}{\mathfrak{z}(\epsilon)} \\
&= \sum_{\epsilon,\epsilon'} \mathsf{Y}^\lambda_\epsilon\left(\frac{t_1}{z}, \frac{t_2}{z}\right)\mathsf{Y}^\mu_{\epsilon'}\left(-\frac{t_1}{z}, -\frac{t_2}{z}\right)\frac{1}{((t_1/z)(t_2/z))^{\ell(\epsilon)}}\frac{\delta_{\epsilon\epsilon'}}{\mathfrak{z}(\epsilon)} \\
&= \langle \mathsf{Y}^\lambda, \mathsf{Y}^\mu\rangle_H|_{t_i\mapsto t_i/z} \\
&= \delta_{\lambda\mu} \prod_{\mathsf{w}:\text{ tangent weight at }\lambda} \mathsf{w}\Big|_{t_i\mapsto t_i/z} \\
&= \delta_{\lambda\mu} \prod_{\mathsf{w}:\text{ tangent weight at }\lambda} \mathsf{w}/z^{2|\lambda|}.
\end{aligned} \tag{8.16}$$

In the second equality of (8.16), we have used the definition of $\langle -, -\rangle_H$. In the third equality, we have used (8.9) and (8.10). Since $|\lambda| = |\mu|$, we have

$$\langle \Theta\mathsf{Y}^\lambda_z z^{|\lambda|}, \Theta\mathsf{Y}^\mu_z z^{|\mu|}\rangle_H = \delta_{\lambda\mu} \prod_{\mathsf{w}:\text{ tangent weight at }\lambda} \mathsf{w}.$$

On the other hand, we have

$$\begin{aligned}
&\eta(\Theta\mathsf{Y}^\lambda_z z^{|\lambda|}, (\Theta\mathsf{Y}^\mu_z z^{|\mu|})|_{z\mapsto -z}) \\
&= z^{|\lambda|+|\mu|}(-1)^{|\mu|} \sum_{\epsilon,\epsilon'} \mathsf{Y}^\lambda_\epsilon\left(\frac{t_1}{z}, \frac{t_2}{z}\right)\mathsf{Y}^\mu_{\epsilon'}\left(-\frac{t_1}{z}, -\frac{t_2}{z}\right)\eta(\epsilon|\epsilon')z^{\ell(\epsilon)+\ell(\epsilon')}(-1)^{\ell(\epsilon')} \\
&= z^{|\lambda|+|\mu|} \sum_{\epsilon,\epsilon'} \mathsf{Y}^\lambda_\epsilon\left(\frac{t_1}{z}, \frac{t_2}{z}\right)\mathsf{Y}^\mu_{\epsilon'}\left(-\frac{t_1}{z}, -\frac{t_2}{z}\right) \\
&\quad\times \frac{(-1)^{|\epsilon|-\ell(\epsilon)}}{(t_1t_2)^{\ell(\epsilon)}}\frac{\delta_{\epsilon\epsilon'}}{\mathfrak{z}(\epsilon)} z^{\ell(\epsilon)+\ell(\epsilon')}(-1)^{\ell(\epsilon')}(-1)^{|\mu|}.
\end{aligned}$$

Since the factors of (-1) cancel, the above is

$$z^{|\lambda|+|\mu|} \sum_{\epsilon,\epsilon'} \mathsf{Y}^\lambda_\epsilon\left(\frac{t_1}{z}, \frac{t_2}{z}\right)\mathsf{Y}^\mu_{\epsilon'}\left(-\frac{t_1}{z}, -\frac{t_2}{z}\right)\frac{z^{\ell(\epsilon)+\ell(\epsilon')}}{(t_1t_2)^{\ell(\epsilon)}}\frac{\delta_{\epsilon\epsilon'}}{\mathfrak{z}(\epsilon)} = \delta_{\lambda\mu} \prod_{\mathsf{w}:\text{ tangent weight at }\lambda} \mathsf{w},$$

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where we have used (8.16). We conclude

$$\eta(\Theta \mathsf{Y}^{\lambda}_{z} z^{|\lambda|}, (\Theta \mathsf{Y}^{\mu}_{z} z^{|\mu|})|_{z\mapsto -z}) = \delta_{\lambda\mu} \prod_{\mathsf{w}:\ \text{tangent weight at } \lambda} \mathsf{w}.$$

9. Analytic continuation

9.1. Asymptotics near $q = -1$. We study the value of S at $q = -1$, using the solution to the connection problem in [30, Section 4]. Let

$$\mathsf{H}^{\lambda}(q, t)$$

be the integral form of the Macdonald polynomial as in [30, Equation (33)]. More precisely,

$$\mathsf{H}^{\lambda}(q, t) = t^{\mathsf{n}(\lambda)} \prod_{\square\in\lambda} (1 - q^{a(\square)} t^{-l(\square)-1}) \varUpsilon \mathsf{P}^{\lambda}(q, t^{-1}),$$

where

$$\varUpsilon|\mu\rangle = \prod_{i=1}^{\ell(\mu)} (1 - t^{-\mu_i})^{-1} |\mu\rangle$$

and

$$\mathsf{n}(\lambda) = \sum_{i=1}^{\ell(\lambda)} (i-1)\lambda_i.$$

Let H be the matrix with columns H^{λ} and the following identification of parameters:

$$(q, t) = (T_1, T_2), \quad T_i = \exp(2\pi\sqrt{-1} t_i). \tag{9.1}$$

Define the operators G_{GW} and $\boldsymbol{\varGamma}$ by

$$\mathsf{G}_{\mathrm{GW}}(t_1, t_2)|\mu\rangle = \prod_i g(\mu_i, t_1) g(\mu_i, t_2) |\mu\rangle,$$

$$\boldsymbol{\varGamma}|\mu\rangle = \frac{(2\pi\sqrt{-1})^{\ell(\mu)}}{\prod_i \mu_i} \mathsf{G}_{\mathrm{GW}}(t_1, t_2)|\mu\rangle,$$

where $g(x, t) = x^{tx}/\varGamma(tx)$.

By [30, Theorem 4], $\mathsf{Y}\mathsf{G}_{\mathrm{DT}}|_{q=-1} = 1/(2\pi\sqrt{-1})^{|\cdot|} \boldsymbol{\varGamma}\mathsf{H}$. Therefore, the solution S in (8.11) satisfies

$$S|_{q=-1} = \frac{1}{\sqrt{-1}^{|\cdot|}} \Theta \boldsymbol{\varGamma}_z \mathsf{H}_z A. \tag{9.2}$$

By Proposition 23,

$$S|_{q=-1} \sim \mathsf{R}|_{q=-1} e^{\mathsf{u}/z}|_{q=-1} \tag{9.3}$$

as $z \to 0$ along $\mathfrak{R}$. We determine $\mathsf{R}|_{q=-1}$ by studying the asymptotics of the right side of (9.2).

As we see, as $z \to 0$ along $\mathfrak{R}$, the right side of (9.2) admits an asymptotical expansion of the following form

$$\frac{1}{\sqrt{-1}^{|\cdot|}} \Theta \boldsymbol{\Gamma}_z \mathsf{H}_z A \sim \mathsf{Z}^+ \mathsf{Z}^-, \tag{9.4}$$

where $\mathsf{Z}^+ = \mathbf{1} + O(z)$ is a z-series and $\mathsf{Z}^- = \mathbf{1} + O(1/z)$ is a $1/z$-series.

We write $A = A_0 A_1$, where

Matrix	Eigenvalues
A_0	$\prod_{\mathsf{w}:\text{ tangent weights at }\lambda} z^{-\mathsf{w}/z} e^{-\mathsf{w}/z}$
A_1	$\prod_{\mathsf{w}:\text{ tangent weights at }\lambda} \mathsf{w}^{\mathsf{w}/z}$

The operator A_0 is a scalar multiple of the identity matrix since

$$\sum_{\mathsf{w}:\text{ tangent weights}} \mathsf{w} = |\lambda|(t_1 + t_2).$$

As a result,

$$S|_{q=-1} = \frac{1}{\sqrt{-1}^{|\cdot|}} \Theta \boldsymbol{\Gamma}_z A_0 \mathsf{H}_z A_1. \tag{9.5}$$

Since we have

$$\prod_{\mathsf{w}:\text{ tangent weights at }\lambda} \mathsf{w}^{\mathsf{w}/z} = \exp\left(\frac{1}{z} \sum_{\mathsf{w}:\text{ tangent weights at }\lambda} \mathsf{w} \log \mathsf{w}\right),$$

A_1 contributes to Z^-.

By [23, Ch. VI, equation (8.19)], matrix coefficients of H are polynomials in T_1, T_2. As mentioned in [30], our H^λ is the same as $\widetilde{H}_\lambda$ in [17, Definition 3.5.2]. To see the equivalence, the first step is

$$\widetilde{H}_\lambda(z; q, t) = t^{\mathsf{n}(\lambda)} J_\lambda[Z/(1 - t^{-1}); q, t^{-1}],$$

by the equation just below Theorem/Definition 6.1 of [18]. Here, $Z/(1 - t^{-1})$ stands for the plethystic substitution defined in [17, Section 3.3]. The function $J_\lambda(z; q, t)$, defined by [17, equation (54)], is the same as that defined in [23, Ch. VI, Section 8, (8.3)], as remarked just above [17, Section 3.5.2]. (Note there

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is a typo in [17, equation (54)]: the plethystic substitution should be $(1-t)Z$ instead of $(1-t^{-1})Z$.) By [23, Ch. VI, Section 8, (8.3)],

$$J_\lambda(z;q,t)=\prod_{\square\in\lambda}(1-q^{a(\square)}t^{l(\square)+1})\mathsf{P}^\lambda(z;q,t).$$

Working through the definition given in [17, Section 3.3], we find that the plethystic substitution

$$Z\mapsto Z/(1-t^{-1})$$

is equivalent to the map $|\mu\rangle \mapsto (1/\prod_i(1-t^{-\mu_i}))|\mu\rangle$. Thus

$$\mathsf{H}^\lambda(q,t)=\widetilde{H}_\lambda(z;q,t).$$

By the identification of parameters (9.1) and condition (8.7), we see that as $z\to 0$ along $\mathfrak{R}$, we have $q\to 0$, $t^{-1}\to 0$. By [17, Definition 3.5.3], we have

$$\mathsf{H}^\lambda(q,t)=\sum_\mu \widetilde{K}_{\mu\lambda}(q,t)s_\mu,$$

where $\widetilde{K}_{\mu\lambda}(q,t)$ are the *Kostka–Macdonald polynomials* (or q,t-Kostka coefficients) and

$$s_\mu=\sum_\nu \chi_\mu(\nu)|\nu\rangle$$

is the Schur function.

By the discussion below [17, Definition 3.5.3], we can define $K_{\mu\lambda}(q,t)$ by

$$J_\lambda(z;q,t)=\sum_\mu K_{\mu\lambda}(q,t)s_\mu[Z/(1-t)].$$

As noted in the discussion below [17, Definition 3.5.3],

$$\widetilde{K}_{\mu\lambda}(q,t)=t^{\mathsf{n}(\lambda)}K_{\mu\lambda}(q,t^{-1}). \tag{9.6}$$

Therefore, we can write

$$\begin{aligned}\sum_\mu \widetilde{K}_{\mu\lambda}(q,t)s_\mu &= \sum_\mu \widetilde{K}_{\mu\lambda}(q,t)\sum_\nu \chi_\mu(\nu)|\nu\rangle\\ &=\sum_\nu\left(\sum_\mu \widetilde{K}_{\mu\lambda}(q,t)\chi_\mu(\nu)\right)|\nu\rangle,\end{aligned}$$

and

$$\sum_\mu \widetilde{K}_{\mu\lambda}(q,t)\chi_\mu(\nu)=\sum_\mu t^{\mathsf{n}(\lambda)}K_{\mu\lambda}(q,t^{-1})\chi_\mu(\nu).$$

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As $z \to 0$ along $\mathfrak{R}$, we have $q, t^{-1} \to 0$. By [17, equation (59)],

$$\widetilde{K}_{\mu\lambda}(0,t) = \widetilde{K}_{\mu\lambda}(t),$$

where $\widetilde{K}_{\mu\lambda}(t)$ is the *cocharge Kostka–Foulkes polynomial* given in [17, Definition 3.4.13] in terms of the Kostka–Foulkes polynomials $K_{\mu\lambda}(t)$,

$$\widetilde{K}_{\mu\lambda}(t) = t^{\mathsf{n}(\lambda)} K_{\mu\lambda}(t^{-1}).$$

Comparing this with (9.6), we see that

$$K_{\mu\lambda}(0,t) = K_{\mu\lambda}(t).$$

By [17, Corollary 3.4.12 (vi)],

$$K_{\mu\lambda}(0) = \delta_{\mu\lambda}.$$

Therefore,

$$\lim_{z\to 0 \text{ along } \mathfrak{R}} K_{\mu\lambda}(q, t^{-1}) = \delta_{\mu\lambda}.$$

Thus as $z \to 0$ along $\mathfrak{R}$, we have $\mathsf{X}^{-1}\mathsf{H}_z \exp(-2\pi\sqrt{-1}t_2\mathsf{n}/z)$ tends to $\mathbf{1}$, where X is the matrix with entries $\chi_\lambda(\mu)$ and n is the diagonal matrix with diagonal entries $\mathsf{n}(\lambda)$. Hence,

$$\mathsf{H}_z \sim \mathsf{X}\exp(2\pi\sqrt{-1}t_2\mathsf{n}/z), \tag{9.7}$$

as $z \to 0$ along $\mathfrak{R}$. So H_z contributes X to Z^+ and $\exp(2\pi\sqrt{-1}t_2\mathsf{n}/z)$ to Z^-.

It remains to study the asymptotics of $(1/\sqrt{-1}^{|\cdot|})\Theta\boldsymbol{\Gamma}_z A_0$. By condition (8.4), the Stirling asymptotics (8.12) is applicable to $\boldsymbol{\Gamma}_z$ as $z \to 0$ along $\mathfrak{R}$. We find $\boldsymbol{\Gamma}_z$ has the asymptotics a diagonal matrix with entries:

$$\begin{aligned}&\sqrt{-1}^{\ell(\mu)}(t_1t_2)^{\ell(\mu)/2} z^{-\ell(\mu)} e^{(|\mu|(t_1+t_2))/z}\\ &\quad\times \exp\left(-|\mu|\frac{t_1\log t_1 + t_2\log t_2}{z} + |\mu|(t_1+t_2)\frac{\log z}{z}\right)\\ &\quad\times \prod_{i=1}^{\ell(\mu)} \exp\left(\sum_{m>0}\frac{B_{2m}}{2m(2m-1)}\left(\left(\frac{-z}{\mu_i t_1}\right)^{2m-1} + \left(\frac{-z}{\mu_i t_2}\right)^{2m-1}\right)\right).\end{aligned}$$

There are now several cancelations:

- The term $e^{(|\mu|(t_1+t_2))/z}$ cancels with $\prod_{\mathsf{w}\text{: tangent weights at } \mu} e^{-\mathsf{w}/z}$ in A_0.
- The term $\exp(|\mu|(t_1+t_2)(\log z/z))$ cancels with $\prod_{\mathsf{w}\text{: tangent weights } t\mu} z^{-\mathsf{w}/z}$ in A_0.
- The factor $z^{-\ell(\mu)}$ cancels with Θ.

Also, the term $\exp(-|\mu|(t_1\log t_1 + t_2\log t_2)/z)$ contributes to Z^-.

Therefore, the part of Z^+ coming from $(1/\sqrt{-1}^{|\cdot|})\Theta\boldsymbol{\Gamma}_z A_0$ is the diagonal matrix with entries

$$\sqrt{-1}^{\ell(\mu)-|\mu|}(t_1t_2)^{\ell(\mu)/2} \times \prod_{i=1}^{\ell(\mu)} \exp\left(\sum_{m>0}\frac{B_{2m}}{2m(2m-1)}\left(\left(\frac{-z}{\mu_i t_1}\right)^{2m-1}+\left(\frac{-z}{\mu_i t_2}\right)^{2m-1}\right)\right). \quad (9.8)$$

Comparing the two asymptotical expansions of $S|_{q=-1}$ in (9.3) and (9.4), we find

$$\mathsf{R}|_{q=-1}=\mathsf{Z}^+, \quad e^{\mathsf{u}/z}|_{q=-1}=\mathsf{Z}^-.$$

(What we need here is that if a function of the form $\exp(A/z)$, with A a diagonal matrix independent of z, has an asymptotical expansion into a z-series starting with $\mathbf{1}$, then $A=0$. The result follows by computing the asymptotical coefficients using their definitions.) Hence, $\mathsf{u}|_{q=-1}$ is the diagonal matrix with diagonal entries

$$-|\lambda|(t_1\log t_1+t_2\log t_2)+\sum_{\mathsf{w}:\text{ tangent weights at }\lambda}\mathsf{w}\log\mathsf{w}+2\pi\sqrt{-1}t_2\mathsf{n}(\lambda).$$

By construction, the columns of $\mathsf{R}|_{q=-1,z=0}$ are normalized idempotents of the ring

$$H^*_{\mathsf{T}}(I\mathrm{Sym}^n(\mathbb{C}^2))$$

written in the basis $\{|\mu\rangle\ \mu\in\mathrm{Part}(n)\}$ of V. Since $\mathsf{R}|_{q=-1}=\mathsf{Z}^+$, by looking at $\mathsf{Z}^+|_{z=0}$, we find the idempotent appearing on the column indexed by λ is I^λ written in the basis $\{|\mu\rangle\ \mu\in\mathrm{Part}(n)\}$ of V. Moreover, the action of $\mathsf{R}|_{q=-1}$ on I^λ is given by the column of Z^+ indexed by λ, which is computed by combining (9.7) and (9.8). More precisely,

$$\mathsf{R}|_{q=-1}(\mathsf{I}^\lambda)=\sum_\mu \chi_\lambda(\mu)\exp\left(-\sum_{m>0}\frac{B_{2m}}{2m(2m-1)} \times\sum_i\left(\frac{1}{(\mu_i t_1)^{2m-1}}+\frac{1}{(\mu_i t_2)^{2m-1}}\right)z^{2m-1}\right) \cdot\sqrt{-1}^{\ell(\mu)-|\mu|}(t_1t_2)^{\ell(\mu)/2}|\mu\rangle. \quad (9.9)$$

Comparing (6.5) with (9.9), we find

$$\mathsf{R}^{\mathrm{Sym}}|_{u=0}=\mathsf{R}|_{q=-1},$$

after $-q=e^{iu}$. As a consequence, in a neighborhood of $q=-1$, $\mathsf{R}^{\mathrm{Sym}}=\mathsf{R}$.

We have proven the following result parallel to Proposition 21.

PROPOSITION 24. *As $z \to 0$ along $\mathfrak{R}$, the operator $Se^{-\mathsf{u}/z}|_{q=-1}$ has the asymptotics*

$$\mathsf{R}^{\mathsf{Sym}}|_{u=0}.$$

9.2. Proof of Theorem 4. We already have a match of the complete genus 0 theory for $\mathsf{Hilb}^n(\mathbb{C}^2)$ and $\mathsf{Sym}^n(\mathbb{C}^2)$ under the identification

$$\mathsf{V} \to \widetilde{\mathsf{V}}, \quad |\mu\rangle \mapsto |\widetilde{\mu}\rangle. \tag{9.10}$$

To prove Theorem 4, we need only match the R-matrices $\mathsf{R}^{\mathsf{Hilb}}$ and $\mathsf{R}^{\mathsf{Sym}}$ via (9.10) and the variable change

$$-q = e^{iu}. \tag{9.11}$$

The coordinate $\widetilde{t}$ along $|2, 1^{n-2}\rangle \in \widetilde{\mathsf{V}}$ is related to the coordinate t along $|2, 1^{n-2}\rangle \in \mathsf{V}$ via (9.10) by

$$\tilde{t} = (-i)^{-1} t = it.$$

By the chain rule,

$$-\frac{\partial}{\partial t} = -i\frac{\partial}{\partial \widetilde{t}} \quad \text{and} \quad q\frac{\partial}{\partial q} = -i\frac{\partial}{\partial u}.$$

The differential equations

$$-\frac{\partial}{\partial t}\mathsf{R}^{\mathsf{Hilb}} = q\frac{\partial}{\partial q}\mathsf{R}^{\mathsf{Hilb}} \quad \text{and} \quad \frac{\partial}{\partial \widetilde{t}}\mathsf{R}^{\mathsf{Sym}} = \frac{\partial}{\partial u}\mathsf{R}^{\mathsf{Sym}}$$

therefore exactly match via (9.10) and the variable change (9.11). Hence, by Proposition 9, we need only match

$$\mathsf{R}^{\mathsf{Hilb}}|_{q=-1} = [\mathsf{R}^{\mathsf{Hilb}}|_{-q=e^{iu}}]_{u=0}$$

with $\mathsf{R}^{\mathsf{Sym}}|_{u=0}$.

The matching $\mathsf{R}^{\mathsf{Hilb}}|_{q=-1} = \mathsf{R}^{\mathsf{Sym}}|_{u=0}$ is a nontrivial assertion. The difficulty can be summarized as follows. While we have closed forms for

$$\mathsf{R}^{\mathsf{Hilb}}|_{q=0} \quad \text{and} \quad \mathsf{R}^{\mathsf{Sym}}|_{u=0},$$

by Propositions 16 and 17, respectively, we must control the $q = -1$ evaluation of $\mathsf{R}^{\mathsf{Hilb}}$ which is far away from $q = 0$.

The issue is resolved by the analytic continuation of the solution to the QDE of $\mathsf{Hilb}^n(\mathbb{C}^2)$ computed in [30]. The study of the QDE of $\mathsf{Hilb}^n(\mathbb{C}^2)$ in [30] concerns only the *small* quantum cohomology (all the coordinates of V are set to 0). The results of Propositions 21 and 24 show the R-matrix associated to the solution S of the QDE has asymptotics

$$\mathsf{R}^{\mathsf{Hilb}}|_{q=0} = \mathsf{R}|_{q=0} \quad \text{and} \quad \mathsf{R}^{\mathsf{Sym}}|_{u=0} = \mathsf{R}|_{q=-1}.$$

Since R matches $\mathsf{R}^{\mathsf{Hilb}}$ for small q by Proposition 23 and is analytic along the path γ connecting 0 to -1, we conclude

$$\mathsf{R}^{\mathsf{Hilb}}|_{q=-1} = \mathsf{R}^{\mathsf{Sym}}|_{u=0} \tag{9.12}$$

at least when all the coordinates of V are set to 0. By Proposition 8, any difference between two operators (9.12) persists after setting the coordinates of V to 0. Hence, the equality (9.12) is valid when the dependence on V is included. □

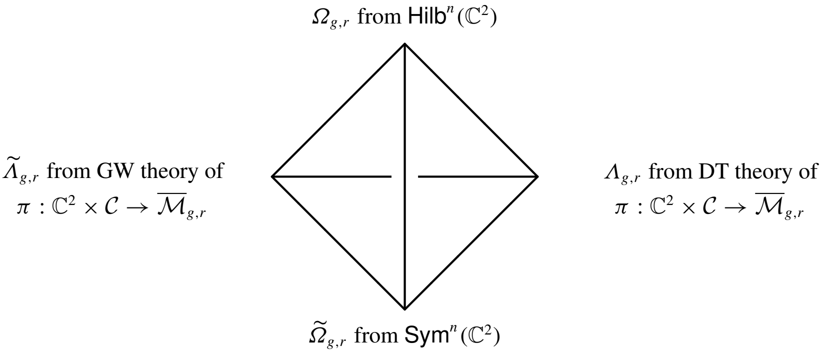
The proof of Theorem 4 not only yields the series result

$$\begin{aligned} &\langle \mu^1, \mu^2, \dots, \mu^r \rangle_g^{\mathsf{Hilb}^n(\mathbb{C}^2)} \\ &\quad = (-i)^{\sum_{i=1}^r \ell(\mu^i) - |\mu^i|} \langle \mu^1, \mu^2, \dots, \mu^r \rangle_g^{\mathsf{Sym}^n(\mathbb{C}^2)} \quad \text{after} \quad -q = e^{iu} \end{aligned}$$

but matches the full CohFTs

$$\Omega_{g,r} = \widetilde{\Omega}_{g,r} \quad \text{after } \mathsf{V} \to \widetilde{\mathsf{V}} \quad \text{and} \quad -q = e^{iu}.$$

9.3. Proof of Theorem 2. We obtain a tetrahedron of equivalences of CohFTs:



Both the CohFTs Ω and Λ are based on

$$\mathsf{A} = \mathbb{Q}(t_1, t_2)[[q]]$$

and (V, η). The exact matching

$$\Omega = \Lambda$$

is Proposition 15. Similarly, $\widetilde{\Omega}$ and $\widetilde{\Lambda}$ are based on

$$\widetilde{\mathsf{A}} = \mathbb{Q}(t_1, t_2)[[u]]$$

and $(\widetilde{\mathsf{V}}, \widetilde{\eta})$. The exact matching

$$\widetilde{\Omega} = \widetilde{\Lambda}$$

is Proposition 12. Finally, we have the crepant resolution matching

$$\Omega_{g,r} = \widetilde{\Omega}_{g,r} \quad \text{after } \mathsf{V} \rightarrow \widetilde{\mathsf{V}} \quad \text{and} \quad -q = e^{iu}$$

from Section 9.2. As a result, we also obtain the GW/DT matching

$$\widetilde{\Lambda}_{g,r} = \Lambda_{g,r} \quad \text{after } \widetilde{\mathsf{V}} \leftarrow \mathsf{V} \quad \text{and} \quad -q = e^{iu}.$$

The proof of Theorem 2 is complete. □

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