ON THE SECOND COHOMOLOGY OF $GL(n, 2)$

Dedicated to the memory of Hanna Neumann

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The purpose of this note is to prove the following result.

THEOREM A. *Let G be a finite group with the following properties*

(i) $V \lhd G, |V| = 2^n$ *and V is elementary abelian,*

(ii) $G/V \simeq GL(n,2)$,

(iii) $C_G(V) \subseteq V$.

If $n \geqq 6$ *then G splits over V.*

We also may state this result in terms of the second cohomology group $H^2(GL(n,2), V)$ where V is the standard n-dimensional F_2-module for $GL(n,2)$.

THEOREM B. $H^2(GL(n,2), V) = 0$ *if* $n \geqq 6$.

REMARK. By [4; p. 124] we know that $H^i(GL(n,q), V) = 0$ for $1 \leqq i \leqq 2$ and $q > 2$ where V is the standard F_q-module for $GL(n,q)$. A simple counting argument shows $H^1(GL(n,2), V) = 0$ with the sole exeption $n = 3$ where $\dim F_2 H^1(GL(3,2), V) = 1$.

It is known that there is a unique nonsplit extension of V by $GL(n,2)$ for $n = 3$ and 4 with S_2-subgroups of type $G_2(3)$ and $\cdot 3$ respectively. The case of a faithful extension of V of order 2^5 by $GL(5,2)$ will be treated somewhere else.

For results concerning $H^i(G, V)$ where $1 \leqq i \leqq 2$ and G is either a symplectic or an orthogonal group the reader may consult [3] and [5].

Proof of the theorem

By the assumptions of theorem A we may think about V as an F_2-vectorspace acted upon G/V as the full automorphism group. We prove the assertion by a series of lemmas.

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(1) *If $\tau \in G - V$ acts as a transvection on V then there is a $t \in \tau V$ such that $t^2 = 1$.*

PROOF. Choose a suitable basis $v_1, \cdots, v_n$ of V such that $C_V(\tau) = \langle v_2, \cdots, v_n \rangle$ and $v_1^\tau = v_1 + v_2$. If n is even choose elements $\rho_1, \cdots, \rho_{n/2}$ of order 3 in G such that $C_V(\rho_i) = \langle v_1, \cdots, v_{2i-3}, v_{2i-2}, v_{2i+1}, v_{2i+2}, \cdots, v_n \rangle$ and permutes the non trivial elements in $\langle v_{2i-1}, v_{2i} \rangle$. If $n - 1$ is even choose ρ_i for $1 \leqq i \leqq (n-1)/2 - 1$ as above and choose $\rho_{(n-1)/2}$ as an element of order 7 with $C_V(\rho_{(n-1)/2}) = \langle v_1, \cdots, v_{n-3} \rangle$ and which acts irreducibly on $\langle v_{n-2}, v_{n-1}, v_n \rangle$. Let X be the group generated by V and the ρ_i's. Then X/V acts fixed-point-free on V and τ normalizes X. A Frattini-argument gives us the assertion.

(2) *Let v_1 be a nontrivial element in V and H be its stabilizer in G. Then $O_2(H)$ is an extraspecial group of width $n-1$ and type $(+)$* (this means $|O_2(H)| = 2^{2n-1}$ and $O_2(H)$ possesses an elementary abelian subgroup of order 2^n) *extended faithfully by a group isomorphic to $GL(n-1, 2)$.*

PROOF. Set $A = O_2(H)$ and fix a basis $v_1, \cdots, v_n$ of V. Then the action of H/V on V in respect to this basis is described by matrices of the form $\begin{bmatrix} 1 & 0 \\ F & L \end{bmatrix}$ where L is a regular $(n-1) \times (n-1)$-matrix over F_2 and F is a $(n-1) \times 1$-matrix over F_2. The elements of A/V correspond to those matrices where L is the identity matrix. Hence $[A, V] = \langle v_1 \rangle$. By (1) we know that for $a \in A^*$ either $a^2 = 1$ or $a^2 = v_1$ holds. But then $A/\langle v_1 \rangle$ is elementary abelian and as $Z(A) = D(A) = A' = \langle v_1 \rangle$ it follows that A is extraspecial. As A contains the elementary abelian group V of order 2^n it follows that A is of type $(+)$ and we are done.

(3) *G splits over V.*

PROOF. We use the same notation as in (2). A result in [1; (2.2)] tells us: *Assume $\mathscr{V}$ is a $2m$-dimensional orthogonal F_2-vectorspace of type $(+)$ and $X \simeq GL(m, 2)$ is a subgroup of $O(\mathscr{V})$ such that X normalizes an isotropic subspace $\mathscr{U}$ of dimension m. If $m \geqq 5$ then there is a X-invariant, isotropic subspace $\mathscr{W}$ of $\mathscr{V}$ such that $\mathscr{V} = \mathscr{U} \oplus \mathscr{W}$.*

So we can find an elementary abelian subgroup W of A, $|W| = 2^n$ such that W is H/A-admissible, $VW = A$ and $V \cap W = \langle v_1 \rangle$. As H/W acts faithfully as a subgroup of $GL(n, 2)$ on W there is a subgroup $H_1 \subset H$ such that $H_1 \supset W$, $H_1/W \simeq GL(n-1, 2)$ and $H_1 \cap A = W$. Similarly, we have a subgroup H_2 such that $H_2 \supset V$, $H_2/V \simeq GL(n-1, 2)$ and $H_2 \cap A = V$. Then by the modular law:

$$H_1 = H_1 \cap H = H_1 \cap H_2 W = (H_1 \cap H_2)W$$

and so

$$GL(n-1, 2) \simeq H_1/W \simeq (H_1 \cap H_2)/\langle v_1 \rangle.$$

Set $H_3 = H_1 \cap H_2$. Then H_3 is an extension of the group $\langle v_1 \rangle$ by $GL(n-1,2)$. By [2] this extension splits. So there is a group $H_4 \subset H$, $H_4 \simeq GL(n-1,2)$ and $H_4A = H$. Furthermore there is a H_4-admissible subgroup W_0 of W with $W = \langle v_1 \rangle \times W_0$. So $W_0H_4 \cap V = 1$ and by a result of Gaschütz the assertion follows (see [4; I, 17.4]).

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