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A Few Explicit Examples of Causal Variational Principles

In this chapter, we introduce a few examples of causal variational principles and analyze them in detail. These examples are too simple for being of direct physical interest. Instead, they are chosen in order to illustrate the different mathematical structures introduced previously. It is a specific feature of these examples that a minimizing measure can be given in a closed form, making it possible to analyze the system explicitly. Similar examples were first given in [68].

When constructing simple explicit examples, it is often convenient to choose *non-smooth* Lagrangians, which involve, for example, characteristic functions or are even distributional. In order to treat this non-smooth setting in a mathematically convincing way, one needs to work with additional jet spaces, which we now introduce (for more details, see, e.g., [62, 57]).

Clearly, the fact that a jet $\mathfrak{u}$ is smooth does not imply that the functions ℓ or $\mathcal{L}$ are differentiable in the direction of $\mathfrak{u}$. This must be ensured by additional conditions that are satisfied by suitable subspaces of $\mathfrak{J}$, which we now define. First, we let Γ^{diff} be those vector fields for which the directional derivative of the function ℓ exists,

$$\Gamma^{\text{diff}} = \Big\{ u \in C^\infty(M, T\mathcal{F}) \;\Big|\; D_u \ell(x) \text{ exists for all } x \in M \Big\} . \tag{20.1}$$

This gives rise to the jet space

$$\mathfrak{J}^{\text{diff}} := C^\infty(M, \mathbb{R}) \oplus \Gamma^{\text{diff}} \;\subset\; \mathfrak{J} . \tag{20.2}$$

For the jets in $\mathfrak{J}^{\text{diff}}$, the combination of multiplication and directional derivative in (7.11) is well defined. We choose a linear subspace $\mathfrak{J}^{\text{test}} \subset \mathfrak{J}^{\text{diff}}$ with the property that its scalar and vector components are both vector spaces, that is,

$$\mathfrak{J}^{\text{test}} = C^{\text{test}}(M, \mathbb{R}) \oplus \Gamma^{\text{test}} \;\subseteq\; \mathfrak{J}^{\text{diff}} \tag{20.3}$$

for suitable subspaces $C^{\text{test}}(M, \mathbb{R}) \subset C^\infty(M, \mathbb{R})$ and $\Gamma^{\text{test}} \subset \Gamma^{\text{diff}}$. We then write the *restricted EL equations* (7.13) in the weaker form

$$\nabla_{\mathfrak{u}} \ell|_M = 0 \qquad \text{for all } \mathfrak{u} \in \mathfrak{J}^{\text{test}} . \tag{20.4}$$

Finally, when considering weak solutions of the linearized field equations, it is sometimes useful to restrict attention to jets in a suitably chosen subspace of $\mathfrak{J}^{\text{test}}$, which, in agreement with (14.14), we denote by

$$\mathfrak{J}^{\text{vary}} \subset \mathfrak{J}^{\text{test}} . \tag{20.5}$$

To summarize, we have the inclusions

$$\mathfrak{J}^{\text{vary}} \subset \mathfrak{J}^{\text{test}} \subset \mathfrak{J}^{\text{diff}} \subset \mathfrak{J} \,. \tag{20.6}$$

The compactly supported jets are always denoted by an additional subscript zero.

20.1 A One-Dimensional Gaussian

We let $\mathcal{F} = \mathbb{R}$ and choose the Lagrangian as the Gaussian

$$\mathcal{L}(x,y) = \frac{1}{\sqrt{\pi}}\, e^{-(x-y)^2} \,. \tag{20.7}$$

Lemma 20.1.1 *The Lebesgue measure*

$$d\rho = dx \tag{20.8}$$

is a minimizer of the causal action principle for the Lagrangian (20.7) *in the class of variations of finite volume (see* (6.14) *and* (6.13)*). It is the unique minimizer within this class of variations.*

Proof Writing the difference of the actions as in (6.14), we can carry out the integrals over ρ using that the Gaussian is normalized (see Exercise 20.1),

$$\int_{\mathcal{F}} \mathcal{L}(x,y)\, d\rho(y) = 1 \,. \tag{20.9}$$

We thus obtain

$$\begin{aligned} \mathcal{S}(\rho) - \mathcal{S}(\tilde{\rho}) &= 2 \int_N d(\rho - \tilde{\rho})(x) + \int_N d(\rho - \tilde{\rho})(x) \int_N d(\rho - \tilde{\rho})(y)\, \mathcal{L}(x,y) \\ &= \int_N d(\rho - \tilde{\rho})(x) \int_N d(\rho - \tilde{\rho})(y)\, \mathcal{L}(x,y) \,, \end{aligned} \tag{20.10}$$

where in the last line we used the volume constraint (6.13). In order to show that the last double integral is positive, we take the Fourier transform and use that the Fourier transform of a Gaussian is again a Gaussian. More precisely,

$$\int_N e^{-ip(x-y)}\, \mathcal{L}(x,y)\; dy = e^{-\frac{p^2}{4}} =: f(p) \,. \tag{20.11}$$

Moreover, the estimate

$$\left| \int_N e^{ipx}\; d(\rho - \tilde{\rho})(x) \right| \leq \big|\tilde{\rho} - \rho\big|(\mathcal{F}) < \infty, \tag{20.12}$$

shows that the Fourier transform of the signed measure $\tilde{\rho} - \rho$ is a bounded function $g \in L^\infty(\mathbb{R})$. Approximating this function in $L^2(\mathbb{R})$, we can apply Plancherel's theorem and use the fact that convolution in position space corresponds to multiplication in momentum space. We thus obtain

$$\begin{aligned} &\int_N d(\rho - \tilde{\rho})(x) \int_N d(\rho - \tilde{\rho})(y)\, \mathcal{L}(x,y) \\ &\qquad = \int_N \big(\mathcal{F}^{-1}(fg)\big)(x)\; d(\rho - \tilde{\rho})(x) = \int_{-\infty}^{\infty} \overline{g(p)}\, e^{-\frac{p^2}{4}}\, g(p)\; dp \geq 0 \,, \end{aligned} \tag{20.13}$$

and the inequality is strict unless $\tilde{\rho} = \rho$. This concludes the proof. $\square$

The EL equations read

$$\int_{\mathcal{F}} \mathcal{L}(x,y)\, \mathrm{d}\rho(y) = 1 \qquad \text{for all } x \in \mathbb{R}\,. \tag{20.14}$$

We now specify the jet spaces. Since the Lagrangian is smooth, it is obvious that

$$\mathfrak{J}^{\mathrm{diff}} = \mathfrak{J} = C^\infty(\mathbb{R}) \oplus C^\infty(\mathbb{R}), \tag{20.15}$$

where we identify a vector field $a(x)\,\partial_x$ on $\mathbb{R}$ with the function $a(x)$. The choice of $\mathfrak{J}^{\mathrm{test}}$ is less obvious. For simplicity, we restrict attention to functions that are bounded together with all their derivatives, denoted by

$$C^\infty_{\mathrm{b}} := \left\{ f \in C^\infty(\mathbb{R}) \;\middle|\; f^{(n)} \in L^\infty(\mathbb{R}) \text{ for all } n \in \mathbb{N}_0 \right\}. \tag{20.16}$$

Now different choices are possible. Our first choice is to consider jets whose scalar components are compactly supported,

$$\mathfrak{J}^{\mathrm{test}} = C^\infty_0(\mathbb{R}) \oplus C^\infty_{\mathrm{b}}(\mathbb{R})\,. \tag{20.17}$$

The linearized field equations (8.15) reduce to the scalar equation

$$\int_N \big(\nabla_{1,\mathfrak{v}} + \nabla_{2,\mathfrak{v}}\big)\mathcal{L}(x,y)\, \mathrm{d}\rho(y) - \nabla_{\mathfrak{v}}\, 1 = 0 \qquad \text{for all } x \in \mathbb{R}\,, \tag{20.18}$$

because if this equation holds, then the x-derivative of the left-hand side is also zero. Differentiating the EL equations (20.14) with respect to x, we find that

$$\int_N \nabla_{1,\mathfrak{v}}\mathcal{L}(x,y)\, \mathrm{d}\rho(y) - \nabla_{\mathfrak{v}}\, 1 = 0 \qquad \text{for all } x \in \mathbb{R}\,. \tag{20.19}$$

Subtracting this equation from (20.18), the linearized field simplifies to

$$\int_N \nabla_{2,\mathfrak{v}}\mathcal{L}(x,y)\, \mathrm{d}\rho(y) = 0 \qquad \text{for all } x \in \mathbb{R}\,. \tag{20.20}$$

A specific class of solutions can be given explicitly. Indeed, choosing

$$\mathfrak{u} = (a, A) \quad \text{with} \quad a \in C^\infty_0(\mathbb{R}) \text{ and } A(x) := \int_\infty^x a(t)\; \mathrm{d}t \in C^\infty_{\mathrm{b}}(\mathbb{R})\,, \tag{20.21}$$

integration by parts yields

$$\int_N \nabla_{2,\mathfrak{u}}\mathcal{L}(x,y)\, \mathrm{d}\rho(y) = \int_N \big(A'(y) + A(y)\,\partial_y\big)\mathcal{L}(x,y)\; \mathrm{d}y = 0\,. \tag{20.22}$$

These linearized solutions are referred to as *inner solutions*, as introduced in a more general context in Section 8.3 and [57]. Inner solutions can be regarded as infinitesimal generators of transformations of M that leave the measure ρ unchanged. Therefore, inner solutions do not change the causal fermion system but merely describe symmetry transformations of the measure. With this in mind, inner solutions are not of interest by themselves. But they can be used in order to simplify the form of the jet spaces. For example, by adding suitable inner solutions, one can arrange that the test jets have vanishing scalar components. Indeed, given a jet $\mathfrak{v} = (b, v) \in \mathfrak{J}^{\text{test}}$ (with $\mathfrak{J}^{\text{test}}$ according to (20.17)), taking an indefinite integral of b,

$$B(t) := \int_{-\infty}^{t} b(\tau)\, \mathrm{d}\tau \;\in\; C^\infty_{\mathrm{b}}(\mathbb{R})\,, \tag{20.23}$$

the resulting jet $\mathfrak{u} := (-b, -B)$ is an inner solution (20.21). Adding this jet to $\mathfrak{v}$ gives

$$\tilde{\mathfrak{v}} := \mathfrak{v} + \mathfrak{u} = (0, v - B) \in \mathfrak{J}^{\text{test}}\,, \tag{20.24}$$

which is physically equivalent to $\mathfrak{v}$ and, as desired, has a vanishing scalar component.

In our example, we can use the inner solutions alternatively in order to eliminate the vector component of the test jets. To this end, it is preferable to choose the space of test jets as

$$\mathfrak{J}^{\text{test}} = C^\infty_{\mathrm{b}}(\mathbb{R}) \oplus C^\infty_{\mathrm{b}}(\mathbb{R})\,. \tag{20.25}$$

Now the vector component disappears under the transformation

$$\mathfrak{v} = (b, v) \mapsto \tilde{\mathfrak{v}} := \mathfrak{v} + \mathfrak{u} \qquad \text{with} \qquad \mathfrak{u} = (-v', -v) \in \mathfrak{J}^{\text{test}}\,. \tag{20.26}$$

Therefore, it remains to consider the scalar components of jets. For technical simplicity, we restrict attention to compactly supported functions. Thus, we choose the jet space $\mathfrak{J}^{\text{vary}}$ as

$$\mathfrak{J}^{\text{vary}} = C^\infty_0(\mathbb{R}) \oplus \{0\}\,. \tag{20.27}$$

Then, the linearized field operator in (8.15) reduces to the integral operator with kernel $\mathcal{L}(x, y)$,

$$\big(\Delta(b, 0)\big)(x) = \int_{\mathcal{F}} \mathcal{L}(x, y)\, b(y)\, \mathrm{d}y\,. \tag{20.28}$$

20.2 A Minimizing Measure Supported on a Hyperplane

In the previous example, the support of the minimizing measure was the whole space $\mathcal{F}$. In most examples motivated from the physical applications, however, the minimizing measure will be supported on a low-dimensional subset of $\mathcal{F}$ (see, for instance, the minimizers with singular support for the causal variational principle on the sphere in [74, 10] discussed in Section 6.1). We now give a simple example where the minimizing measure is supported on a hyperplane of $\mathcal{F}$. We let $\mathcal{F} = \mathbb{R}^2$ and choose the Lagrangian as

$$\mathcal{L}(x, y; x', y') = \frac{1}{\sqrt{\pi}}\, \mathrm{e}^{-(x-x')^2} \big(1 + y^2\big) \big(1 + y'^2\big)\,, \tag{20.29}$$

where $(x, y), (x', y') \in \mathcal{F}$.

Lemma 20.2.1 *The measure*

$$\mathrm{d}\rho = \mathrm{d}x \times \delta_y \tag{20.30}$$

(where δ_y is the Dirac measure) is the unique minimizer of the causal action principle for the Lagrangian (20.29) *under variations of finite volume (see* (6.14) *and* (6.13)*).*

Note that this measure is supported on the x-axis,

$$M := \operatorname{supp} \rho = \mathbb{R} \times \{0\} \,. \tag{20.31}$$

Proof of Lemma 20.2.1. Let $\tilde{\rho}$ be a regular Borel measure on $\mathcal{F}$ satisfying (6.13). Then, the difference of actions (6.14) is computed by

$$\mathcal{S}(\tilde{\rho}) - \mathcal{S}(\rho) = \frac{2}{\sqrt{\pi}} \int_{\mathcal{F}} \mathrm{d}(\tilde{\rho} - \rho)(x,y) \int_N \mathrm{d}x' \, \mathrm{e}^{-(x-x')^2} \left(1 + y^2\right) \tag{20.32}$$

$$+ \frac{1}{\sqrt{\pi}} \int_{\mathcal{F}} \mathrm{d}(\tilde{\rho} - \rho)(x,y) \int_{\mathcal{F}} \mathrm{d}(\tilde{\rho} - \rho)(x',y') \, \mathrm{e}^{-(x-x')^2} \left(1 + y^2\right)\left(1 + y'^2\right) . \tag{20.33}$$

Using that the negative part of the measure $\tilde{\rho} - \rho$ is supported on the x-axis, the first term (20.32) can be estimated by

$$\begin{aligned} &\frac{2}{\sqrt{\pi}} \int_{\mathcal{F}} \mathrm{d}(\tilde{\rho} - \rho)(x,y) \int_N \mathrm{d}x' \, \mathrm{e}^{-(x-x')^2} \left(1 + y^2\right) \\ &\quad \overset{(*)}{\geq} \frac{2}{\sqrt{\pi}} \int_{\mathcal{F}} \mathrm{d}(\tilde{\rho} - \rho)(x,y) \int_N \mathrm{d}x' \, \mathrm{e}^{-(x-x')^2} \\ &\quad = \int_{\mathcal{F}} \mathrm{d}(\tilde{\rho} - \rho)(x,y) = 0 \,, \end{aligned} \tag{20.34}$$

where in the last step we used the volume constraint. The second term (20.33), on the other hand, can be rewritten as

$$\frac{1}{\sqrt{\pi}} \int_{\mathcal{F}} \mathrm{d}\mu(x,y) \int_{\mathcal{F}} \mathrm{d}\mu(x',y') \, \mathrm{e}^{-(x-x')^2} , \tag{20.35}$$

with the signed measure ρ defined by

$$\mathrm{d}\mu(x,y) := \left(1 + y^2\right) \, \mathrm{d}(\tilde{\rho} - \rho)(x,y) \,. \tag{20.36}$$

Now we can proceed as in the proof of Lemma 20.1.1 and use that the Fourier transform of the integral kernel is strictly positive. For the uniqueness statement, one uses that the inequality in $(*)$ is strict unless $\tilde{\rho}$ is supported on the x-axis. Then, one can argue as in the proof of Lemma 20.1.1. □

For the minimizing measure (20.30), the function ℓ takes the form

$$\ell(x,y) = \int_{\mathcal{F}} \mathcal{L}(x,y;x',y') \, \mathrm{d}\rho(x',y') - 1 = y^2 \,, \tag{20.37}$$

showing that the EL equations (7.4) are indeed satisfied. We now specify the jet spaces. Since the Lagrangian is smooth, it is obvious that

$$\mathfrak{J}^{\text{diff}} = \mathfrak{J} = C^\infty(\mathbb{R}) \oplus C^\infty(\mathbb{R}, \mathbb{R}^2) \,, \tag{20.38}$$

where $C^\infty(\mathbb{R}, \mathbb{R}^2)$ should be regarded as the space of two-dimensional vector fields along the x-axis. As explained after (20.25), we want to use the inner solutions for simplifying the vector components of the jets. To this end, in analogy to (20.25), we choose

$$\mathfrak{J}^{\text{test}} = C^\infty_{\text{b}}(\mathbb{R}) \oplus C^\infty_{\text{b}}(\mathbb{R}, \mathbb{R}^2) \,. \tag{20.39}$$

The linearized field equations (8.15) read

$$\begin{aligned} 0 &= \nabla_{\mathfrak{u}} \bigg(\int_{-\infty}^{\infty} \big(\nabla_{1,\mathfrak{v}} + \nabla_{2,\mathfrak{v}}\big) \mathrm{e}^{-(x-x')^2} \big(1+y^2\big)\big(1+y'^2\big) \; \mathrm{d}\rho(x',y') \bigg) \bigg|_{y=y'=0} \\ &\qquad - \nabla_{\mathfrak{u}} \Big(\nabla_{\mathfrak{v}} \sqrt{\pi} \Big) \Big|_{y=0} \\ &= \nabla_{\mathfrak{u}} \bigg(\big(1+y^2\big) \int_{-\infty}^{\infty} \big(\nabla_{1,\mathfrak{v}} + \nabla_{2,\mathfrak{v}}\big) \mathrm{e}^{-(x-x')^2} \; \mathrm{d}x' - \nabla_{\mathfrak{v}} \sqrt{\pi} \bigg) \bigg|_{y=y'=0} \,. \end{aligned} \tag{20.40}$$

Now the inner solutions are generated by the vector fields tangential to the x-axis. More precisely, in analogy to (20.21), we consider the jet

$$\begin{aligned} \mathfrak{v} = \big(b, (B,0)\big) \qquad &\text{with} \qquad b \in C^\infty_0(\mathbb{R}) \text{ and} \\ B(x) &:= \int_\infty^x b(t) \; \mathrm{d}t \in C^\infty_{\text{b}}(\mathbb{R}) \,. \end{aligned} \tag{20.41}$$

Exactly as in the example of the one-dimensional Gaussian, integrating by parts as in (20.22), one sees that the jet $\mathfrak{v}$ indeed satisfies the linearized field equations.

By suitably subtracting inner solutions, we can compensate for the tangential components of the jets. This leads us to choose

$$\mathfrak{J}^{\text{vary}} = C^\infty_0(\mathbb{R}) \oplus \big(\{0\} \oplus C^\infty_0(\mathbb{R}) \big) \,. \tag{20.42}$$

Then, the Laplacian simplifies as follows,

$$\begin{aligned} &\langle \mathfrak{u}, \Delta \mathfrak{v} \rangle(x) \\ &= \frac{1}{\sqrt{\pi}} \, \nabla_{\mathfrak{u}} \bigg(\int_{-\infty}^{\infty} \big(\nabla_{1,\mathfrak{v}} + \nabla_{2,\mathfrak{v}}\big) \mathrm{e}^{-(x-x')^2} \; \big(1+y^2\big)\big(1+y'^2\big) \; \mathrm{d}x' \bigg) \bigg|_{y=y'=0} \\ &\quad - \frac{1}{\sqrt{\pi}} \, \nabla_{\mathfrak{u}} \Big(\nabla_{\mathfrak{v}} \sqrt{\pi} \Big) \Big|_{y=0} \\ &= \frac{2}{\sqrt{\pi}} \, u(x) \, v(x) \int_{-\infty}^{\infty} \mathrm{e}^{-(x-x')^2} \; \mathrm{d}x' + \frac{1}{\sqrt{\pi}} \, a(x) \int_{-\infty}^{\infty} \mathrm{e}^{-(x-x')^2} \; b(x') \; \mathrm{d}x' \\ &\quad + a(x) \bigg(\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} b(x) \, \mathrm{e}^{-(x-x')^2} \; \mathrm{d}x' - b(x) \bigg) \\ &= 2 \, u(x) \, v(x) + \frac{1}{\sqrt{\pi}} \, a(x) \int_{-\infty}^{\infty} \mathrm{e}^{-(x-x')^2} \; b(x') \; \mathrm{d}x' \,, \end{aligned} \tag{20.43}$$

where $\mathfrak{u} = (a, (0,u))$ and $\mathfrak{v} = (b, (0,v))$. Hence, the inhomogeneous linearized field equations (8.16) with $\mathfrak{w} = (e, w)$ give rise to separate equations for the scalar and vector components,

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-(x-x')^2} \; b(x') \; \mathrm{d}x' = e(x) \,, \qquad v(x) = \frac{w(x)}{2} \,. \tag{20.44}$$

20.3 A Nonhomogeneous Minimizing Measure

In the previous examples, the minimizing measure ρ was translation invariant in the direction of the x-axis. We now give a general procedure for constructing examples of causal variational principles where the minimizing measure has no translational symmetry. In order to work in a concrete example, our starting point is again the one-dimensional Gaussian (20.7). But our method can be adapted to other kernels in a straightforward way. In view of these generalizations, we begin with the following abstract result.

Lemma 20.3.1 *Let μ be a measure on the m-dimensional manifold $\mathcal{F}$ whose support is the whole manifold,*

$$\operatorname{supp}\mu = \mathcal{F}\,. \tag{20.45}$$

Moreover, let $\mathcal{L}(x,y) \in L^1_{\text{loc}}(\mathcal{F}\times\mathcal{F}, \mathbb{R}^+_0)$ be a symmetric, nonnegative kernel on $\mathcal{F}\times\mathcal{F}$. Next, let $h \in C^0(\mathcal{F}, \mathbb{R}^+)$ be a strictly positive, continuous function on $\mathcal{F}$. Assume that:

(i) $\displaystyle\int_{\mathcal{F}} \mathcal{L}(x,y)\, h(y)\, \mathrm{d}\mu(y) = 1 \qquad$ *for all $x \in \mathcal{F}$.*

(ii) *For all compactly supported bounded functions with zero mean,*

$$g \in L^\infty_0(\mathcal{F}, \mathbb{R}^+) \qquad \textit{and} \qquad \int_{\mathcal{F}} g\, \mathrm{d}\mu = 0\,, \tag{20.46}$$

the following inequality holds:

$$\int_{\mathcal{F}} \mathrm{d}\mu(x) \int_{\mathcal{F}} \mathrm{d}\mu(y)\, \mathcal{L}(x,y)\, g(x)\, g(y) \geq 0\,. \tag{20.47}$$

Then, the measure $\mathrm{d}\rho := h\,\mathrm{d}\mu$ is a minimizer of the causal action principle under variations of finite volume (see (6.14) *and* (6.13)*). If the inequality* (20.47) *is strict for any nonzero g, then the minimizing measure is unique within the class of such variations.*

Proof We consider the variation

$$\tilde{\rho}_\tau = \rho + \tau\, g\, \mathrm{d}\mu = (h + \tau g)\, \mathrm{d}\mu\,. \tag{20.48}$$

Since h is continuous and strictly positive and g is bounded and compactly supported, the function $h + \tau g$ is nonnegative for sufficiently small $|\tau|$. Furthermore, using that g has mean zero, we conclude that (20.48) is an admissible variation of finite volume (6.13). Moreover, the difference of the actions (6.14) is well defined and computed by

$$\begin{aligned} \mathcal{S}(\tilde{\mu}_\tau) - \mathcal{S}(\rho) &= 2\tau \int \mathrm{d}\rho(x)\, g(x) \int_N \mathrm{d}\rho(y)\, h(y)\, \mathcal{L}(x,y) \\ &\quad + \tau^2 \int_N \mathrm{d}\rho(x) \int_N \mathrm{d}\rho(x)\, \mathcal{L}(x,y)\, g(x)\, g(y) \geq 2\tau \int_N g(y)\, \mathrm{d}\rho(y) = 0\,, \end{aligned} \tag{20.49}$$

where in the second step we used the abovementioned assumptions (i) and (ii). The last step follows from the fact that g has mean zero. If the inequality (20.47) is strict, so is the inequality in (20.49), showing that the minimizer ρ is unique.

We conclude that the measure ρ is a minimizer under variations of the form (20.48). In order to treat a general variation of finite volume (6.13), we approximate $\tilde{\rho}$ by a sequence of functions g_n with the property that the measures $g_n \rho$ converge to $\tilde{\rho}$ (here one can work with the notion of vague convergence; for details, see [8, Definition 30.1] or [31]). $\square$

Our goal is to apply this lemma to kernels of the form

$$\mathcal{L}(x,y) = f(x)\, \mathrm{e}^{-(x-y)^2} f(y), \tag{20.50}$$

with a strictly positive function f, which for convenience we again choose as a Gaussian,

$$f(x) = \mathrm{e}^{\alpha x^2} \qquad \text{with } \alpha \in \mathbb{R}\,. \tag{20.51}$$

This kernel has the property (ii) because for all nontrivial $g \in L^\infty_0(\mathcal{F}, \mathbb{R}^+)$,

$$\begin{aligned} &\int_{\mathcal{F}} \mathrm{d}\mu(x) \int_{\mathcal{F}} \mathrm{d}\mu(y)\, \mathcal{L}(x,y)\, g(x)\, g(y) \\ &\quad = \int_{\mathcal{F}} \mathrm{d}\mu(x) \int_{\mathcal{F}} \mathrm{d}\mu(y)\, \mathrm{e}^{-(x-y)^2}\, (fg)(x)\, (fg)(y) > 0\,, \end{aligned} \tag{20.52}$$

where the last step is proved exactly as in the example of the Gaussian (see (20.13)). In order to arrange (i), for h, we make an ansatz again with a Gaussian,

$$h(x) = c\, \mathrm{e}^{\beta x^2}\,. \tag{20.53}$$

Then,

$$\begin{aligned} \int_{\mathcal{F}} \mathcal{L}(x,y)\, h(y)\, \mathrm{d}\mu(y) &= c \int_{-\infty}^{\infty} \mathrm{e}^{\alpha x^2}\, \mathrm{e}^{-(x-y)^2}\, \mathrm{e}^{(\alpha+\beta)y^2}\, \mathrm{d}y \\ &= c \exp\Big(\alpha x^2 - x^2 - \frac{x^2}{\alpha+\beta-1} \Big) \\ &\qquad \times \int_{-\infty}^{\infty} \exp\Big\{ (\alpha+\beta-1) \Big(y - \frac{x}{\alpha+\beta-1} \Big)^2 \Big\}\, \mathrm{d}y \\ &= c \sqrt{\frac{\pi}{1-\alpha-\beta}}\, \exp\Big(\alpha x^2 - x^2 - \frac{x^2}{\alpha+\beta-1} \Big)\,. \end{aligned} \tag{20.54}$$

In order to arrange that this function is constant one, we choose

$$c = \sqrt{\frac{1-\alpha-\beta}{\pi}} \qquad \text{and} \qquad \beta = -\frac{\alpha(2-\alpha)}{1-\alpha}\,. \tag{20.55}$$

For the abovementioned Gaussian integral to converge, we need to ensure that $1-\alpha-\beta > 0$. In view of the formula

$$1 - \alpha - \beta = \frac{1}{1-\alpha}\,, \tag{20.56}$$

this can be arranged simply by choosing $\alpha < 1$. Our finding is summarized as follows.

Proposition 20.3.2 *For any $\alpha < 1$, we let f and h be the Gaussians* (20.51) *and* (20.53) *with c and β according to* (20.55)*. Then, the measure $d\rho = h\,dx$ is the unique minimizer of the causal action corresponding to the Lagrangian* (20.50) *within the class of variations of finite volume.*

As a concrete example, we consider the well-known *Mehler kernel* (see, e.g., [93, Section 1.5])

$$E(x,y) = \frac{1}{\sqrt{1-\mu^2}}\, \exp\bigg(-\frac{\mu^2(x^2+y^2)-2\mu xy}{(1-\mu^2)}\bigg), \tag{20.57}$$

with $\mu > 0$. Rescaling x and y according to

$$x,y \to \sqrt{\frac{1-\mu^2}{\mu}}\; x,y\,, \tag{20.58}$$

the Mehler kernel becomes

$$E(x,y) = \frac{1}{\sqrt{1-\mu^2}}\, \exp\Big(-\mu(x^2+y^2) - 2xy\Big)\,. \tag{20.59}$$

This kernel is of the desired form (20.50) if we choose

$$\alpha = 1-\mu < 1\,, \qquad \beta = \frac{\mu^2-1}{\mu}\,. \tag{20.60}$$

We finally remark that this nonhomogeneous example can be used as the starting point for the construction of higher-dimensional examples with minimizing measures supported on lower-dimensional subsets, exactly as explained for the Gaussian in Section 20.2.

20.4 A Minimizing Measure in Two-Dimensional Minkowski Space

In the previous examples, the Lagrangian was strictly positive (see (20.7), (20.29), (20.50)). Therefore, the causal structure of the resulting spacetime was trivial because all pairs or points were timelike separated. We now give examples where the minimizing measure gives rise to nontrivial causal relations in spacetime. We let $\mathcal{F} = \mathbb{R}^2$, denote the coordinates by (t,x) and choose the Lagrangian

$$\mathcal{L}\big(t,x;t',x\big) = e^{-(t-t')^2}\, \Big(\delta\big((t-t')-(x-x')\big) + \delta\big((t-t')+(x-x')\big)\Big)\,. \tag{20.61}$$

The Lagrangian is nonnegative, and it is strictly positive on the "light rays" $(t-t') = \pm(x-x')$.

Lemma 20.4.1 *The Lebesgue measure*

$$d\rho = dt\; dx \tag{20.62}$$

is a minimizer of the causal action principle for the Lagrangian (20.61) *in the class of variations of finite volume (see* (6.14) *and* (6.13)*). It is the unique minimizer within this class of variations.*

Proof Proceeding as in the proof of Lemma 20.1.1, our task is to show that the Fourier transform of the Lagrangian is strictly positive. To this end, we note that

$$\int_{\mathbb{R}^2} \delta(t-x)\, \mathrm{e}^{\mathrm{i}\omega t - \mathrm{i}kx}\, \mathrm{d}t\, \mathrm{d}x = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega x - \mathrm{i}kx}\, dx = 2\pi\, \delta(\omega - k)\,. \tag{20.63}$$

We thus obtain

$$\begin{aligned}&\int_{\mathbb{R}^2} \Big(\delta\big((t-t')-(x-x')\big) + \delta\big((t-t')+(x-x')\big) \Big)\, \mathrm{e}^{\mathrm{i}\omega t - \mathrm{i}kx}\, \mathrm{d}t\, \mathrm{d}x \\ &\qquad = 2\pi\, \big(\delta(\omega+k) + \delta(\omega-k) \big)\,. \end{aligned}\tag{20.64}$$

Multiplying by the Gaussian in (20.61) corresponds to a convolution in momentum space again by a Gaussian. This convolution gives a strictly positive function, as desired. ☐

The Lagrangian (20.61) has the shortcoming that it is supported only on the boundary of the light cone. In order to improve the situation, we next consider the example

$$\begin{aligned}\mathcal{L}\big(t,x;t',x\big) = \mathrm{e}^{-(t-t')^2} &\Big(\delta\big((t-t')-(x-x')\big) + \delta\big((t-t')+(x-x')\big) \Big) \\ &+ a\, \mathrm{e}^{-\frac{(t-t')^2}{2}}\, \Theta\big((t-t')^2 - (x-x')^2\big)\,. \end{aligned}\tag{20.65}$$

Lemma 20.4.2 *Choosing* $|\alpha| < 1$*, the Lebesgue measure*

$$\mathrm{d}\rho = \mathrm{d}t\, \mathrm{d}x \tag{20.66}$$

is a minimizer of the causal action principle for the Lagrangian (20.65) *in the class of variations of finite volume (see* (6.14) *and* (6.13)*). It is the unique minimizer within this class of variations.*

Proof We compute the Fourier transform of the Heaviside function.

$$\begin{aligned}&\int_{\mathbb{R}^2} \Theta\big(t^2 - x^2\big)\, \mathrm{e}^{\mathrm{i}\omega t - \mathrm{i}kx}\, \mathrm{e}^{-\varepsilon\,|t|}\, \mathrm{d}t\, \mathrm{d}x \\ &\quad = 4 \int_0^\infty \mathrm{d}x \int_{-\infty}^\infty \mathrm{d}t\, \Theta(t-x)\, \cos(\omega t)\, \cos(kx)\, \mathrm{e}^{-\varepsilon t}\, \mathrm{d}t\, \mathrm{d}x \\ &\quad = 2 \int_0^\infty \left(-\frac{\mathrm{e}^{\mathrm{i}\omega x - \varepsilon x}}{\mathrm{i}\omega - \varepsilon} - \frac{\mathrm{e}^{-\mathrm{i}\omega x - \varepsilon x}}{-\mathrm{i}\omega - \varepsilon} \right) \cos(kx)\, \mathrm{d}x \\ &\quad = -\frac{1}{\mathrm{i}\omega - \varepsilon} \left(\frac{1}{\mathrm{i}\omega + k - \varepsilon} + \frac{1}{\mathrm{i}\omega - k - \varepsilon} \right) \\ &\qquad - \frac{1}{-\mathrm{i}\omega - \varepsilon} \left(\frac{1}{-\mathrm{i}\omega + k - \varepsilon} + \frac{1}{-\mathrm{i}\omega - k - \varepsilon} \right). \end{aligned}\tag{20.67}$$

In the limit $\varepsilon \searrow 0$, this converges to a tempered distribution that is singular on the light cone. Taking the convolution with the Gaussian and choosing a sufficiently small, the resulting function is dominated near the light cone by the Fourier transform computed in the proof of Lemma 20.61. Moreover, due to its decay properties at infinity, the same is true away from the light cone. This concludes the proof. ☐

20.5 A Nonlinear Wave Equation in Two-Dimensional Minkowski Space

In the previous examples, the minimizing measures were unique. This means, in particular, that the systems had no dynamical degrees of freedom, and the linearized field equations only admitted trivial solutions. We now explain how one can build in dynamical degrees of freedom. For simplicity, we consider the example of a nonlinear wave equation on a spacetime lattice, but the method can be generalized to many other situations. We choose $\mathcal{F} = \mathbb{R}^2 \times S^1$ and denote the coordinates by $(t,x) \in \mathbb{R}^2$ and $\mathrm{e}^{\mathrm{i}\alpha} \in S^1$. We choose

$$\begin{aligned} \mathcal{L}\big(t,x,\alpha;t',x',\alpha'\big) &= \mathrm{e}^{-(t-t')^2}\,\mathrm{e}^{-(x-x')^2} + \delta(t-t')\,\delta(x-x')\,(\sin\alpha - \sin\alpha')^2 \\ &\quad + g\big(t-t',x-x'\big)\,\sin\alpha\,\sin\alpha'\,, \end{aligned} \tag{20.68}$$

where g is the convolution $g = h * h$ with

$$h(t,x) := \delta(t-1)\,\delta(x) + \delta(t+1)\,\delta(x) - \delta(t)\,\delta(x+1) - \delta(t)\,\delta(x-1), \tag{20.69}$$

thus h is the kernel of a discretized wave operator. We remark that this Lagrangian violates our usual positivity assumption $\mathcal{L}\big(t,x,\alpha;t',x',\alpha'\big) \geq 0$. However, this inequality could be arranged without changing the qualitative properties of the example by mollifying the δ distributions and adding a constant.

Proposition 20.5.1 *Every minimizing measure ρ has the form*

$$\mathrm{d}\rho(t,x,\alpha) = \mathrm{d}t\;\mathrm{d}x\;\delta\big(\alpha - \phi(t,x)\big)\;\mathrm{d}\alpha\,, \tag{20.70}$$

where $\phi(t,x)$ solves the nonlinear discrete wave equation

$$\sin\big(\phi(t+1,x)\big) + \sin\big(\phi(t-1,x)\big) - \sin\big(\phi(t,x+1)\big) - \sin\big(\phi(t,x-1)\big) = 0\,. \tag{20.71}$$

We begin with a preparatory lemma.

Lemma 20.5.2 *Every minimizing measure has the form*

$$\mathrm{d}\rho(t,x,\alpha) = \mathrm{d}\mu(t,x)\,\delta\big(\alpha - \phi(t,x)\big)\;\mathrm{d}\alpha\,, \tag{20.72}$$

with μ the push-forward to the first two variables, that is,

$$\mu = \pi_*\rho \qquad \textit{with} \qquad \pi : \mathbb{R}^2 \times S^1 \to \mathbb{R}^2\,, \quad (t,x,\alpha) \mapsto (t,x)\,, \tag{20.73}$$

and $\phi : \mathbb{R}^2 \to \mathbb{R}$ is a μ-measurable function.

Proof Let ρ be a measure on $\mathcal{F}$. We introduce the function $\phi(t,x)$ by

$$\sin\phi(t,x)\,\mathrm{d}\mu(t,x) = \int_0^{2\pi} \sin\alpha\,\mathrm{d}\rho(t,x,\alpha)\,. \tag{20.74}$$

In words, $\sin\phi(t,x)$ coincides with the mean of $\sin\alpha$ integrated over the circle. The function $\phi(t,x)$ exists because this mean lies in the interval $[-1,1]$ and because the sine takes all values in this interval. Denoting the resulting measure of the form (20.72) by $\tilde{\rho}$, we obtain

$$\mathcal{S}(\rho) - \mathcal{S}(\tilde{\rho}) = \int_{\mathcal{F}} \mathrm{d}\rho(t,x,\alpha) \int_{\mathcal{F}} \mathrm{d}\rho(t',x',\alpha)\, \delta(t-t')\, \delta(x-x')\, \big(\sin\alpha - \phi(t,x)\big)^2 . \tag{20.75}$$

Therefore, ρ is a minimizer if and only if $\rho = \tilde{\rho}$. □

Proof of Proposition 20.5.1. For measures of the form (20.72), the action takes the form

$$\begin{aligned} \mathcal{S} = &\int_{\mathbb{R}^2} \mathrm{d}\mu(t,x) \int_{\mathbb{R}^2} \mathrm{d}\mu(t',x')\, \mathrm{e}^{-(t-t')^2}\, \mathrm{e}^{-(x-x')^2} \\ &+ \int_{\mathbb{R}^2} \mathrm{d}\mu(t,x) \int_{\mathbb{R}^2} \mathrm{d}\mu(t',x')\, g\big(t-t', x-x'\big)\, \sin\phi(t,x)\, \sin\phi(t',x') . \end{aligned} \tag{20.76}$$

Using that g is a convolution,

$$g\big(t-t', x-x'\big) = \int_{\mathbb{R}^2} h(t-\tau, x-z)\, h(t'-\tau, x'-z)\; \mathrm{d}\tau\; \mathrm{d}z , \tag{20.77}$$

the action can be rewritten as

$$\mathcal{S} = \int_{\mathbb{R}^2} \mathrm{d}\mu(t,x) \int_{\mathbb{R}^2} \mathrm{d}\mu(t',x')\, \mathrm{e}^{-(t-t')^2}\, \mathrm{e}^{-(x-x')^2} \tag{20.78}$$

$$+ \int_{\mathbb{R}^2} \bigg(\int_{\mathbb{R}^2} h(t-\tau, x-z)\, \sin\phi(t,x)\, \mathrm{d}\mu(t,x) \bigg)^2\; \mathrm{d}\tau\; \mathrm{d}z . \tag{20.79}$$

Exactly as shown in Section 20.1, the minimizer of (20.78) is given by the Lebesgue measure. The contribution (20.79), on the other hand, is minimal if $\sin\phi(t,x)$ satisfies the discrete wave equation. This concludes the proof. □

20.6 Exercises

Exercise 20.1 *(Functions with self-similar Fourier transform)* The example of Lemma 20.1.1 was based on the fact that the Fourier transform of a Gaussian is again Gaussian (20.11).

(a) Prove (20.11) by direct computation.
(b) Another example of a function that is self-similar under Fourier transforms is the distribution in Minkowski space

$$K_0(p) = \delta(k^2)\, \epsilon(k^0) . \tag{20.80}$$

Show that its Fourier transform indeed gives, up to a constant, the same distribution back. *Hint:* The distribution $K_0(p)$ is the analog of the causal fundamental solution (13.109) for the scalar wave equation (see also (16.29)). Using this fact, one can make use of the explicit form of the causal Green's operators for the scalar wave equation.

(c) Can you think of other functions that are self-similar under the Fourier transform in the above sense? Is there a systematic way to characterize them all?

Exercise 20.2 *(Nonnegative functions with nonnegative Fourier transforms)* Another specific feature of the Gaussian in (20.7), which was used in Lemma 20.1.1, is that it is a positive function whose Fourier transform is again positive.

(a) Show that the same is true for the δ distribution. Can you come up with other functions with this property.
(b) The Lagrangian (20.61) involves a function of two variables with the properties that it is nonnegative and has a nonnegative Fourier transform. How can this idea be used to construct other Lagrangians with the property that the Lebesgue measure is a minimizer?

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