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Abstract

Let G be a connected split reductive group over a finite field $\mathbb{F}_q$ and X a smooth projective geometrically connected curve over $\mathbb{F}_q$. The ℓ-adic cohomology of stacks of G-shtukas is a generalization of the space of automorphic forms with compact support over the function field of X. In this paper, we construct a constant term morphism on the cohomology of stacks of shtukas which is a generalization of the constant term morphism for automorphic forms. We also define the cuspidal cohomology which generalizes the space of cuspidal automorphic forms. Then we show that the cuspidal cohomology has finite dimension and that it is equal to the (rationally) Hecke-finite cohomology defined by V. Lafforgue.

Contents

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Introduction

Let X be a smooth projective geometrically connected curve over a finite field $\mathbb{F}_q$. We denote by F its function field, by $\mathbb{A}$ the ring of adèles of F and by $\mathbb{O}$ the ring of integral adèles.

Let G be a connected split reductive group over $\mathbb{F}_q$. For simplicity, we assume in the introduction that the center of G is finite.

We consider the space of automorphic forms $C_c(G(F)\backslash G(\mathbb{A})/G(\mathbb{O}), \mathbb{C})$. On the one hand, there is the notion of cuspidal automorphic form. An automorphic form is said to be cuspidal if its image under the constant term morphism along any proper parabolic subgroup of G is zero. A theorem of Harder [Har74, Theorem 1.2.1] says that the space of cuspidal automorphic forms has finite dimension. The proof uses the Harder–Narasimhan truncations and the contractibility of deep enough strata.

On the other hand, the space of automorphic forms is equipped with an action of the Hecke algebra $C_c(G(\mathbb{O})\backslash G(\mathbb{A})/G(\mathbb{O}), \mathbb{Q})$ by convolution on the right. An automorphic form is said to be (rationally) Hecke-finite if it belongs to a finite-dimensional subspace that is stable under the action of the Hecke algebra.

In [Laf18, Proposition 8.23], Vincent Lafforgue proved that the space of cuspidal automorphic forms and the space of Hecke-finite automorphic forms are equal. In fact, the space of cuspidal

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automorphic forms is stable under the action of the Hecke algebra and is finite-dimensional, and thus it is included in the space of Hecke-finite automorphic forms. The converse direction follows from the following fact: any non-zero image of the constant term morphism along a proper parabolic subgroup P with Levi quotient M is supported on the components indexed by a cone in the lattice of the cocharacters of the center of M. Hence it generates an infinite-dimensional vector space under the action of the Hecke algebra of M. Thus a non-cuspidal automorphic form can not be Hecke-finite for the Hecke algebra of M.

Let ℓ be a prime number not dividing q. In [Dri78] and [Dri87], Drinfeld introduced the stacks classifying GL_n-shtukas for the representation $\mathrm{St} \boxtimes \mathrm{St}^*$ of $\mathrm{GL}_n \times \mathrm{GL}_n$, where St is the standard representation of GL_n and St^* is its dual, and considered their ℓ-adic cohomology. These were also used by Laurent Lafforgue in [Laf97]. Later in [Var04], Varshavsky defined the stacks classifying G-shtukas $\mathrm{Cht}_{G,I,W}$ for general G and for an arbitrary representation W of $\widehat{G}^I$, where $\widehat{G}$ is the Langlands dual group of G over $\mathbb{Q}_\ell$ and I is a finite set (Drinfeld considered the case $G = \mathrm{GL}_n$, $I = \{1, 2\}$ and $W = \mathrm{St} \boxtimes \mathrm{St}^*$). Varshavsky also defined the degree j cohomology group with compact support $H^j_{G,I,W}$ of the ℓ-adic intersection complex of $\mathrm{Cht}_{G,I,W}$ (this stack is smooth in the case of Drinfeld but not in general). In particular, when $I = \emptyset$ and $W = \mathbf{1}$ is the one-dimensional trivial representation of the trivial group $\widehat{G}^{\emptyset}$, the cohomology group $H^0_{G,\emptyset,\mathbf{1}}$ coincides with $C_c(G(F)\backslash G(\mathbb{A})/G(\mathbb{O}), \mathbb{Q}_\ell)$.

The Hecke algebra $C_c(G(\mathbb{O})\backslash G(\mathbb{A})/G(\mathbb{O}), \mathbb{Z}_\ell)$ acts on the cohomology group $H^j_{G,I,W}$. In [Laf18], Vincent Lafforgue defined the subspace $H^{j,\,\mathrm{Hf}}_{G,I,W}$ of $H^j_{G,I,W}$ which consists of the cohomology classes c for which $C_c(G(\mathbb{O})\backslash G(\mathbb{A})/G(\mathbb{O}), \mathbb{Z}_\ell) \cdot c$ is a finitely generated $\mathbb{Z}_\ell$-submodule of $H^j_{G,I,W}$. When $I = \emptyset$ and $W = \mathbf{1}$, the space $H^{0,\,\mathrm{Hf}}_{G,\emptyset,\mathbf{1}}$ coincides with the space of Hecke-finite automorphic forms, and thus coincides with the space of cuspidal automorphic forms. Vincent Lafforgue used $H^{0,\,\mathrm{Hf}}_{G,I,W}$ to construct the excursion operators on the space of cuspidal automorphic forms and obtained a canonical decomposition of this space indexed by the Langlands parameters.

We can also define a subspace $H^{j,\,\mathrm{Hf\text{-}rat}}_{G,I,W}$ of $H^j_{G,I,W}$ which consists of the cohomology classes c for which $C_c(G(\mathbb{O})\backslash G(\mathbb{A})/G(\mathbb{O}), \mathbb{Q}_\ell) \cdot c$ is a finite-dimensional $\mathbb{Q}_\ell$-vector subspace of $H^j_{G,I,W}$. By definition, we have $H^{j,\,\mathrm{Hf}}_{G,I,W} \subset H^{j,\,\mathrm{Hf\text{-}rat}}_{G,I,W}$. When $I = \emptyset$ and $W = \mathbf{1}$, it is easy to see that they are equal.

In this paper, we are interested in the constant term morphism of the cohomology of stacks of shtukas, analogous to the case of automorphic forms. For any parabolic subgroup P of G, let M be its Levi quotient. As in [Var04], we can define the stack of P-shtukas $\mathrm{Cht}_{P,I,W}$ and the stack of M-shtukas $\mathrm{Cht}_{M,I,W}$. The morphisms $G \hookleftarrow P \twoheadrightarrow M$ induce a correspondence

$$\mathrm{Cht}_{G,I,W} \leftarrow \mathrm{Cht}_{P,I,W} \rightarrow \mathrm{Cht}_{M,I,W}.$$

From this we construct a constant term morphism

$$C^{P,\,j}_G : H^j_{G,I,W} \rightarrow H^j_{M,I,W}.$$

Then we define the cuspidal cohomology $H^{j,\,\mathrm{cusp}}_{G,I,W} \subset H^j_{G,I,W}$ as the intersection of the kernels of the constant term morphisms for all proper parabolic subgroups.

This construction was suggested by Vincent Lafforgue. He also conjectured the following.

- The cuspidal cohomology is of finite dimension.
- The following three $\mathbb{Q}_\ell$-vector subspaces of $H^j_{G,I,W}$ are equal:

$$H^{j,\,\mathrm{Hf}}_{G,I,W} = H^{j,\,\mathrm{Hf\text{-}rat}}_{G,I,W} = H^{j,\,\mathrm{cusp}}_{G,I,W}.$$

In this paper, we prove these conjectures except for the equality with $H_{G,I,W}^{j,\,\mathrm{Hf}}$, which we plan to treat in a future paper. The main results are as follows.

Theorem 0.0.1 (Theorem 5.0.1). *The $\mathbb{Q}_\ell$-vector space $H_{G,I,W}^{j,\,\mathrm{cusp}}$ has finite dimension.*

Proposition 0.0.2 (Proposition 6.0.1). *The two $\mathbb{Q}_\ell$-vector subspaces $H_{G,I,W}^{j,\,\mathrm{cusp}}$ and $H_{G,I,W}^{j,\,\mathrm{Hf\text{-}rat}}$ of $H_{G,I,W}^{j}$ are equal.*

As a consequence, $H_{G,I,W}^{j,\,\mathrm{Hf}}$ has finite dimension.

In particular, when $I=\emptyset$ and $W=\mathbf{1}$, the constant term morphism $C_G^{P,\,0}$ coincides with the usual constant term morphism for automorphic forms. In this case, Theorem 0.0.1 coincides with Theorem 1.2.1 in [Har74], and Proposition 0.0.2 coincides with [Laf18, Proposition 8.23] mentioned before.

Let $N\subset X$ be a finite subscheme. Theorem 0.0.1 and Proposition 0.0.2 are still true for the cohomology with level structure on N.

Structure of the paper

In § 1 we construct the parabolic induction diagram and define Harder–Narasimhan truncations which are compatible with the parabolic induction. In § 2 we recall the cohomology of the stacks of G-shtukas and define the cohomology of the stacks of M-shtukas. In § 3 we construct the constant term morphism using the compatibility of the geometric Satake equivalence with the constant term functors for the Beilinson–Drinfeld affine grassmannians.

The idea of the proofs of Theorem 0.0.1 and Proposition 0.0.2 is analogous to the case of automorphic forms. The goal of §§ 4 and 5 is to prove Theorem 0.0.1. In § 4 we prove the contractibility of deep enough horospheres. In § 5 we use this result and an argument by induction on the semisimple rank to prove the finiteness of cuspidal cohomology. In § 6 we show that the constant term morphism commutes with the action of the Hecke algebra, and we prove Proposition 0.0.2.

Notation and conventions

0.0.3 Let G be a connected split reductive group over $\mathbb{F}_q$. Let G^{der} be the derived group of G and $G^{\mathrm{ab}}:=G/G^{\mathrm{der}}$ the abelianization of G. Let Z_G be the center of G and G^{ad} the adjoint group of G (equal to G/Z_G).

0.0.4 We fix a discrete subgroup Ξ_G of $Z_G(\mathbb{A})$ such that $\Xi_G\cap Z_G(\mathbb{O})Z_G(F)=\{1\}$, the quotient $Z_G(F)\backslash Z_G(\mathbb{A})/Z_G(\mathbb{O})\Xi_G$ is finite and the composition of morphisms $\Xi_G\hookrightarrow Z_G(\mathbb{A})\hookrightarrow G(\mathbb{A})\twoheadrightarrow G^{\mathrm{ab}}(\mathbb{A})$ is injective. Note that the volume of $G(F)\backslash G(\mathbb{A})/G(\mathbb{O})\Xi_G$ is finite. We write $\Xi:=\Xi_G$.

0.0.5 We fix a Borel subgroup $B\subset G$. By a parabolic subgroup we will mean a standard parabolic subgroup (i.e. a parabolic subgroup containing B), unless explicitly stated otherwise.

0.0.6 Let H be a connected split reductive group over $\mathbb{F}_q$ with a fixed Borel subgroup. Let Λ_H (respectively $\widehat{\Lambda}_H$) denote the weight (respectively coweight) lattice of H. Let $\langle\ ,\ \rangle:\widehat{\Lambda}_H\times\Lambda_H\to\mathbb{Z}$ denote the natural pairing between the two.

Let $\widehat{\Lambda}_H^{+}\subset\widehat{\Lambda}_H$ denote the monoid of dominant coweights and $\widehat{\Lambda}_H^{\mathrm{pos}}\subset\widehat{\Lambda}_H$ the monoid generated by positive simple coroots. Let $\widehat{\Lambda}_H^{\mathbb{Q}}:=\widehat{\Lambda}_H\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$. Let $\widehat{\Lambda}_H^{\mathrm{pos},\mathbb{Q}}$ and $\widehat{\Lambda}_H^{+,\mathbb{Q}}$ denote the rational cones of $\widehat{\Lambda}_H^{\mathrm{pos}}$ and $\widehat{\Lambda}_H^{+}$. We use analogous notation for the weight lattice.

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We use the partial order on $\widehat{\Lambda}_H^{\mathbb{Q}}$ defined by $\mu_1 \leqslant^H \mu_2 \Leftrightarrow \mu_2 - \mu_1 \in \widehat{\Lambda}_H^{\mathrm{pos},\mathbb{Q}}$ (i.e. $\mu_2 - \mu_1$ is a linear combination of simple coroots of H with coefficients in $\mathbb{Q}_{\geqslant 0}$).

We will apply these notations to $H = G$, $H = G^{\mathrm{ad}}$ or $H =$ some Levi quotient M of G.

0.0.7 We denote by Γ_G the set of simple roots of G and by $\widehat{\Gamma}_G$ the set of simple coroots. The standard parabolic subgroups of G are in bijection with the subsets of Γ_G in the following way. To a parabolic subgroup P with Levi quotient M, we associate the subset Γ_M in Γ_G equal to the set of simple roots of M.

0.0.8 Let $N \subset X$ be a finite subscheme. We denote by $\mathcal{O}_N$ the ring of functions on N and write $K_{G,N} := \mathrm{Ker}(G(\mathbb{O}) \to G(\mathcal{O}_N))$.

Let H be an algebraic group over $\mathbb{F}_q$. We denote by H_N the Weil restriction $\mathrm{Res}_{\mathcal{O}_N/\mathbb{F}_q} H$.

0.0.9 If not specified, all schemes are defined over $\mathbb{F}_q$ and all the fiber products are taken over $\mathbb{F}_q$.

0.0.10 For any scheme S over $\mathbb{F}_q$ and x an S-point of X, we denote by $\Gamma_x \subset X \times S$ the graph of x.

0.0.11 For any scheme S over $\mathbb{F}_q$, we denote by $\mathrm{Frob}_S : S \to S$ the Frobenius morphism over $\mathbb{F}_q$. For any G-bundle $\mathcal{G}$ on $X \times S$, we denote by ${}^{\tau}\mathcal{G}$ the G-bundle $(\mathrm{Id}_X \times_{\mathbb{F}_q} \mathrm{Frob}_S)^*\mathcal{G}$.

0.0.12 We use [LMB99, Definitions 3.1 and 4.1] for prestacks, stacks and algebraic stacks.

0.0.13 As in [LMB99, § 18], [LO08] and [LO09], for $\mathcal{X}$ an algebraic stack locally of finite type over $\mathbb{F}_q$, we denote by $D_c^b(\mathcal{X}, \mathbb{Q}_\ell)$ the bounded derived category of constructible ℓ-adic sheaves on $\mathcal{X}$. We have the notion of six operators and perverse sheaves.

If $f : \mathcal{X}_1 \to \mathcal{X}_2$ is a morphism of finite type of schemes (respectively algebraic stacks) locally of finite type, we will denote by $f_!$, f_*, f^*, $f^!$ the corresponding functors between $D_c^b(\mathcal{X}_1, \mathbb{Q}_\ell)$ and $D_c^b(\mathcal{X}_2, \mathbb{Q}_\ell)$, always understood in the derived sense.

0.0.14 We will work with étale cohomology. So for any stack (respectively scheme) (for example $\mathrm{Cht}_{G,N,I,W}$ and $\mathrm{Gr}_{G,I,W}$), we consider only the reduced substack (respectively subscheme) associated to it.

1. Parabolic induction diagram of stacks of shtukas

The goal of this section is to introduce the parabolic induction diagram of stacks of shtukas without a bound on the modifications at paws in §§ 1.1–1.3 and to introduce the Harder–Narasimhan stratification for the parabolic induction diagram in §§ 1.4–1.7.

In §§ 1.1–1.3 we work in the context of prestacks (see 0.0.12).

1.1 Reminder of stacks of shtukas and Beilinson–Drinfeld affine grassmannians

This subsection is based on [Var04, § 2] and [Laf18, §§ 1 and 2]. All the results are well known.

DEFINITION 1.1.1. We define $\mathrm{Bun}_{G,N}$ to be the prestack that associates to any affine scheme S over $\mathbb{F}_q$ the groupoid

$$\begin{aligned}\mathrm{Bun}_{G,N}(S) := \{(\mathcal{G}, \psi),\ &\text{where } \mathcal{G} \text{ is a } G\text{-bundle on } X \times S,\\ &\psi \text{ is an isomorphism of } G\text{-bundles} : \mathcal{G}\big|_{N \times S} \xrightarrow{\sim} G\big|_{N \times S}\}.\end{aligned}$$

1.1.2 $\mathrm{Bun}_{G,N}$ is a smooth algebraic stack over $\mathbb{F}_q$, locally of finite type.

Definition 1.1.3. We define $\mathrm{Hecke}_{G,N,I}$ to be the prestack that associates to any affine scheme S over $\mathbb{F}_q$ the groupoid $\mathrm{Hecke}_{G,N,I}(S)$ that classifies the following data:

(i) $(x_i)_{i\in I} \in (X\smallsetminus N)^I(S)$;
(ii) $(\mathcal{G},\psi), (\mathcal{G}',\psi') \in \mathrm{Bun}_{G,N}(S)$;
(iii) an isomorphism of G-bundles $\phi : \mathcal{G}\big|_{(X\times S)\smallsetminus(\bigcup_{i\in I}\Gamma_{x_i})} \xrightarrow{\sim} \mathcal{G}'\big|_{(X\times S)\smallsetminus(\bigcup_{i\in I}\Gamma_{x_i})}$ which preserves the N-level structure, i.e. $\psi'\circ\phi\big|_{N\times S} = \psi$.

1.1.4 The prestack $\mathrm{Hecke}_{G,N,I}$ is an inductive limit of algebraic stacks over $(X\smallsetminus N)^I$. We define the morphism of paws $\mathrm{Hecke}_{G,N,I} \to (X\smallsetminus N)^I$ by sending $((x_i)_{i\in I}, (\mathcal{G},\psi) \xrightarrow{\phi} (\mathcal{G}',\psi'))$ to $(x_i)_{i\in I}$.

1.1.5 We denote by pr_0 (respectively pr_1) the projection $\mathrm{Hecke}_{G,N,I} \to \mathrm{Bun}_{G,N}$ which sends $((x_i)_{i\in I}, (\mathcal{G},\psi) \xrightarrow{\phi} (\mathcal{G}',\psi'))$ to $(\mathcal{G},\psi)$ (respectively to $(\mathcal{G}',\psi')$).

Let $\mathrm{Frob} : \mathrm{Bun}_{G,N} \to \mathrm{Bun}_{G,N}$ be the Frobenius morphism over $\mathbb{F}_q$. With the notation in 0.0.11, for any affine scheme S over $\mathbb{F}_q$, the morphism $\mathrm{Frob} : \mathrm{Bun}_{G,N}(S) \to \mathrm{Bun}_{G,N}(S)$ is given by $(\mathcal{G},\psi) \to ({}^\tau\mathcal{G}, {}^\tau\psi)$.

Definition 1.1.6. We define the prestack of shtukas $\mathrm{Cht}_{G,N,I}$ to be the following fiber product.

$$\begin{tikzcd}
\mathrm{Cht}_{G,N,I} \arrow[r] \arrow[d] & \mathrm{Hecke}_{G,N,I} \arrow[d, "(\mathrm{pr}_0,\, \mathrm{pr}_1)"] \\
\mathrm{Bun}_{G,N} \arrow[r, "(\mathrm{Id},\, \mathrm{Frob})"] & \mathrm{Bun}_{G,N} \times_{\mathbb{F}_q} \mathrm{Bun}_{G,N}
\end{tikzcd} \tag{1.1}$$

1.1.7 Concretely, $\mathrm{Cht}_{G,N,I}$ is the prestack which associates to any affine scheme S over $\mathbb{F}_q$ the groupoid $\mathrm{Cht}_{G,N,I}(S)$ classifying the following data:

(i) $(x_i)_{i\in I} \in (X\smallsetminus N)^I(S)$;
(ii) $(\mathcal{G},\psi) \in \mathrm{Bun}_{G,N}(S)$;
(iii) an isomorphism of G-bundles $\phi : \mathcal{G}\big|_{(X\times S)\smallsetminus(\bigcup_{i\in I}\Gamma_{x_i})} \xrightarrow{\sim} {}^\tau\mathcal{G}\big|_{(X\times S)\smallsetminus(\bigcup_{i\in I}\Gamma_{x_i})}$ which preserves the N-level structure, i.e. ${}^\tau\psi\circ\phi\big|_{N\times S} = \psi$.

We define the morphism of paws $\mathfrak{p}_G : \mathrm{Cht}_{G,N,I} \to (X\smallsetminus N)^I$ by sending $((x_i)_{i\in I}, (\mathcal{G},\psi) \xrightarrow{\phi} ({}^\tau\mathcal{G}, {}^\tau\psi))$ to $(x_i)_{i\in I}$.

1.1.8 The prestack $\mathrm{Cht}_{G,N,I}$ is an inductive limit of algebraic stacks over $(X\smallsetminus N)^I$.

1.1.9 We will omit the index N if $N = \emptyset$.

We will need a local model of $\mathrm{Cht}_{G,N,I}$. For this, we recall the definition of Beilinson–Drinfeld affine grassmannians.

1.1.10 For $(x_i)_{i\in I} \in X^I(S)$, $d \in \mathbb{N}$, we denote by $\Gamma_{\sum dx_i}$ the closed subscheme of $X \times S$ whose ideal is generated by $(\prod_{i\in I} t_i)^d$ locally for the Zariski topology, where t_i is an equation of the graph Γ_{x_i}. We define $\Gamma_{\sum \infty x_i} := \varinjlim_d \Gamma_{\sum dx_i}$ to be the formal neighborhood of $\bigcup_{i\in I} \Gamma_{x_i}$ in $X \times S$.

A G-bundle on $\Gamma_{\sum \infty x_i}$ is a projective limit of G-bundles on $\Gamma_{\sum dx_i}$ as $d \to \infty$.

DEFINITION 1.1.11. We define the Beilinson–Drinfeld affine grassmannian $\mathrm{Gr}_{G,I}$ to be the ind-scheme that associates to any affine scheme S over $\mathbb{F}_q$ the set $\mathrm{Gr}_{G,I}(S)$ classifying the following data:

(i) $(x_i)_{i\in I} \in X^I(S)$;
(ii) $\mathcal{G}, \mathcal{G}'$ two G-bundles on $\Gamma_{\sum \infty x_i}$;
(iii) an isomorphism of G-bundles $\phi : \mathcal{G}|_{\Gamma_{\sum \infty x_i} \smallsetminus (\bigcup_{i\in I} \Gamma_{x_i})} \xrightarrow{\sim} \mathcal{G}'|_{\Gamma_{\sum \infty x_i} \smallsetminus (\bigcup_{i\in I} \Gamma_{x_i})}$ where the precise meaning is given in [Laf18, Notation 1.7];
(iv) a trivialization $\theta : \mathcal{G}' \xrightarrow{\sim} G$ on $\Gamma_{\sum \infty x_i}$.

1.1.12 We have the morphism of paws: $\mathrm{Gr}_{G,I} \to X^I$. The fiber over $(x_i)_{i\in I} \in X^I_{\mathbb{F}_q}$ is $\prod_{y\in\{x_i|i\in I\}} \mathrm{Gr}_{G,y}$, where $\mathrm{Gr}_{G,y}$ is the usual affine grassmannian, i.e. the fpqc quotient $G_{\mathcal{K}_y}/G_{\mathcal{O}_y}$, where $\mathcal{O}_y$ is the complete local ring on y and $\mathcal{K}_y$ is its field of fractions.

DEFINITION 1.1.13. (a) For any $d \in \mathbb{N}$, we define $G_{I,d}$ to be the group scheme over X^I that associates to any affine scheme S over $\mathbb{F}_q$ the set consisting of pairs $((x_i)_{i\in I}, f)$, where $(x_i)_{i\in I} \in X^I(S)$ and f is an automorphism of the trivial G-bundle on $\Gamma_{\sum dx_i}$.

(b) We define the group scheme $G_{I,\infty} := \varprojlim G_{I,d}$.

1.1.14 The fiber of $G_{I,\infty}$ over $(x_i)_{i\in I} \in X^I_{\mathbb{F}_q}$ is $\prod_{y\in\{x_i|i\in I\}} G_{\mathcal{O}_y}$.

1.1.15 The group scheme $G_{I,\infty}$ acts on $\mathrm{Gr}_{G,I}$ by changing the trivialization θ. We denote by $[G_{I,\infty}\backslash \mathrm{Gr}_{G,I}]$ the quotient prestack. For any affine scheme S over $\mathbb{F}_q$, $[G_{I,\infty}\backslash \mathrm{Gr}_{G,I}](S)$ is the groupoid classifying the data (i), (ii) and (iii) in Definition 1.1.11.

1.1.16 We have a morphism of prestacks:

$$\begin{gathered}\epsilon_{G,N,I,\infty} : \mathrm{Cht}_{G,N,I} \to [G_{I,\infty}\backslash \mathrm{Gr}_{G,I}] \\ ((x_i)_{i\in I}, (\mathcal{G}, \psi) \xrightarrow{\phi} ({}^\tau\mathcal{G}, {}^\tau\psi)) \mapsto ((x_i)_{i\in I}, \mathcal{G}|_{\Gamma_{\sum\infty x_i}} \xrightarrow{\phi} {}^\tau\mathcal{G}|_{\Gamma_{\sum\infty x_i}}).\end{gathered} \tag{1.2}$$

Remark 1.1.17. The prestack $[G_{I,\infty}\backslash \mathrm{Gr}_{G,I}]$ is not an inductive limit of algebraic stacks. But we can still use it for the construction in §§ 1.2 and 1.3. We will construct a variant of morphism (1.2) for algebraic stacks in 2.4.1.

The following definition will be used in § 4.

DEFINITION 1.1.18. (a) We define $\mathrm{Bun}_{G,N,I,d}$ to be the prestack that associates to any affine scheme S over $\mathbb{F}_q$ the groupoid classifying the following data:

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(i) $(x_i)_{i\in I} \in (X\smallsetminus N)^I(S)$;
(ii) $\mathcal{G}$: a G-bundle over $X \times S$;
(iii) a level structure on the divisor $(N \times S) + \Gamma_{\sum dx_i}$, i.e. an isomorphism of G-bundles: $\psi : \mathcal{G}\big|_{(N\times S)+\Gamma_{\sum dx_i}} \xrightarrow{\sim} G\big|_{(N\times S)+\Gamma_{\sum dx_i}}$.

(b) We define $\mathrm{Bun}_{G,N,I,\infty} := \varprojlim \mathrm{Bun}_{G,N,I,d}$.

1.1.19 $\mathrm{Bun}_{G,N,I,d}$ is a smooth algebraic stack over $(X\smallsetminus N)^I$. Its fiber over a point $(x_i)_{i\in I} \in (X\smallsetminus N)^I(\mathbb{F}_q)$ is $\mathrm{Bun}_{G,N+\sum dx_i}$.

1.1.20 The definitions and constructions in this subsection work for all affine smooth geometrically connected algebraic groups over $\mathbb{F}_q$ (we will use these for parabolic subgroups of G and their Levi quotients).

1.2 Parabolic induction diagrams

1.2.1 Let P be a parabolic subgroup of G and let M be its Levi quotient. Applying the definitions and constructions in § 1.1 to P and M, respectively, we define $\mathrm{Bun}_{P,N}$, $\mathrm{Cht}_{P,N,I}$, $\mathrm{Gr}_{P,I}$, $P_{I,\infty}$, $\epsilon_{P,N,I,\infty}$ and $\mathrm{Bun}_{M,N}$, $\mathrm{Cht}_{M,N,I}$, $\mathrm{Gr}_{M,I}$, $M_{I,\infty}$, $\epsilon_{M,N,I,\infty}$.

Remark 1.2.2. When N is non-empty, the prestack $\mathrm{Cht}_{P,N,I}$ defined above is not the same as the one defined in [Var04, 2.28]. We will describe the difference in Remark 3.4.4.

1.2.3 The morphisms of groups $G \hookleftarrow P \twoheadrightarrow M$ induce morphisms of prestacks over $\mathrm{Spec}\,\mathbb{F}_q$:

$$\mathrm{Bun}_{G,N} \xleftarrow{i^{\mathrm{Bun}}} \mathrm{Bun}_{P,N} \xrightarrow{\pi^{\mathrm{Bun}}} \mathrm{Bun}_{M,N}. \tag{1.3}$$

CONSTRUCTION 1.2.4. The morphisms of groups $G \hookleftarrow P \twoheadrightarrow M$ induce morphisms of prestacks over $(X\smallsetminus N)^I$.

$$\begin{tikzcd} & \mathrm{Cht}_{P,N,I} \arrow[dl, "i"'] \arrow[dr, "\pi"] \arrow[dd, "\mathfrak{p}_P"'] & \\ \mathrm{Cht}_{G,N,I} \arrow[dr, "\mathfrak{p}_G"'] & & \mathrm{Cht}_{M,N,I} \arrow[dl, "\mathfrak{p}_M"] \\ & (X\smallsetminus N)^I & \end{tikzcd} \tag{1.4}$$

More concretely, for any affine scheme S over $\mathbb{F}_q$:

$i : \mathrm{Cht}_{P,N,I}(S) \to \mathrm{Cht}_{G,N,I}(S)$ is given by $(\mathcal{P} \to {}^\tau\mathcal{P}) \mapsto (\mathcal{P} \overset{P}{\times} G \to {}^\tau\mathcal{P} \overset{P}{\times} G)$ where the level structure $\psi : \mathcal{P}\big|_{N\times S} \xrightarrow{\sim} P\big|_{N\times S}$ is sent to $\psi \overset{P}{\times} G$;

$\pi : \mathrm{Cht}_{P,N,I}(S) \to \mathrm{Cht}_{M,N,I}(S)$ is given by $(\mathcal{P} \to {}^\tau\mathcal{P}) \mapsto (\mathcal{P} \overset{P}{\times} M \to {}^\tau\mathcal{P} \overset{P}{\times} M)$ where the level structure ψ is sent to $\psi \overset{P}{\times} M$.

1.2.5 The morphisms of groups $G \hookleftarrow P \twoheadrightarrow M$ induce morphisms of ind-schemes over X^I:

$$\mathrm{Gr}_{G,I} \xleftarrow{i^0} \mathrm{Gr}_{P,I} \xrightarrow{\pi^0} \mathrm{Gr}_{M,I}. \tag{1.5}$$

1.2.6 Let $\mathcal{X}$ (respectively $\mathcal{Y}$) be an (ind-)scheme over a base S that is equipped with an action of a group scheme A (respectively B) over S from the right. Let $A \to B$ be a morphism of group schemes over S. Let $\mathcal{X} \to \mathcal{Y}$ be a morphism of (ind-)schemes over S which is A-equivariant (where A acts on $\mathcal{Y}$ via $A \to B$). This morphism induces a morphism of quotient prestacks

$$[A \backslash \mathcal{X}] \to [B \backslash \mathcal{Y}].$$

1.2.7 Applying 1.2.6 to $i^0 : \mathrm{Gr}_{P,I} \to \mathrm{Gr}_{G,I}$ and $P_{I,\infty} \hookrightarrow G_{I,\infty}$, we obtain a morphism of prestacks:

$$\overline{i^0} : [P_{I,\infty} \backslash \mathrm{Gr}_{P,I}] \to [G_{I,\infty} \backslash \mathrm{Gr}_{G,I}].$$

Applying 1.2.6 to $\pi^0 : \mathrm{Gr}_{P,I} \to \mathrm{Gr}_{M,I}$ and $P_{I,\infty} \twoheadrightarrow M_{I,\infty}$, we obtain a morphism of prestacks:

$$\overline{\pi^0} : [P_{I,\infty} \backslash \mathrm{Gr}_{P,I}] \to [M_{I,\infty} \backslash \mathrm{Gr}_{M,I}].$$

1.2.8 The following diagram of prestacks is commutative.

$$\begin{tikzcd}
\mathrm{Cht}_{G,N,I} \arrow[d, "\epsilon_{G,N,I,\infty}"] & \mathrm{Cht}_{P,N,I} \arrow[l, "i"'] \arrow[r, "\pi"] \arrow[d, "\epsilon_{P,N,I,\infty}"] & \mathrm{Cht}_{M,N,I} \arrow[d, "\epsilon_{M,N,I,\infty}"] \\
{[G_{I,\infty} \backslash \mathrm{Gr}_{G,I}]} & {[P_{I,\infty} \backslash \mathrm{Gr}_{P,I}]} \arrow[l, "\overline{i^0}"'] \arrow[r, "\overline{\pi^0}"] & {[M_{I,\infty} \backslash \mathrm{Gr}_{M,I}]}
\end{tikzcd} \tag{1.6}$$

1.3 Quotient by Ξ

1.3.1 Let Z_G be the center of G as defined in 0.0.3. We have an action of Bun_{Z_G} on $\mathrm{Bun}_{G,N}$ by twisting a G-bundle by a Z_G-bundle, i.e. the action of $\mathcal{T}_Z \in \mathrm{Bun}_{Z_G}$ is given by $\mathcal{G} \mapsto (\mathcal{G} \times \mathcal{T}_Z)/Z_G$. Similarly, Bun_{Z_G} acts on $[G_{I,\infty} \backslash \mathrm{Gr}_{G,I}]$, i.e. the action of $\mathcal{T}_Z \in \mathrm{Bun}_{Z_G}$ is given by

$$(\mathcal{G} \xrightarrow{\phi} \mathcal{G}') \mapsto ((\mathcal{G} \times \mathcal{T}_G|_{\Gamma_{\sum \infty x_i}})/Z_G \xrightarrow{\phi} (\mathcal{G}' \times \mathcal{T}_G|_{\Gamma_{\sum \infty x_i}})/Z_G).$$

For $\mathcal{T}_Z \in \mathrm{Bun}_{Z_G}(\mathbb{F}_q)$, we have a canonical identification $\mathcal{T}_Z \simeq {}^{\tau}\mathcal{T}_Z$. Thus $\mathrm{Bun}_{Z_G}(\mathbb{F}_q)$ acts on $\mathrm{Cht}_{G,N,I}$ by twisting a G-bundle by a Z_G-bundle, i.e. the action of $\mathcal{T}_Z \in \mathrm{Bun}_{Z_G}(\mathbb{F}_q)$ is given by $(\mathcal{G} \xrightarrow{\phi} {}^{\tau}\mathcal{G}) \mapsto ((\mathcal{G} \times \mathcal{T}_Z)/Z_G \xrightarrow{\phi} {}^{\tau}(\mathcal{G} \times \mathcal{T}_Z)/Z_G)$.

The group Ξ defined in 0.0.4 acts on $\mathrm{Bun}_{G,N}$, $\mathrm{Cht}_{G,N,I}$ and $[G_{I,\infty} \backslash \mathrm{Gr}_{G,I}]$ via $\Xi \to Z_G(\mathbb{A}) \to \mathrm{Bun}_{Z_G}(\mathbb{F}_q)$.

1.3.2 Note that the morphism $\epsilon_{G,N,I,\infty}$ defined in (1.2) is Ξ-equivariant.

Now applying Definition 1.1.13 to Z_G (respectively G^{ad}), we define a group scheme $(Z_G)_{I,\infty}$ (respectively $G^{\mathrm{ad}}_{I,\infty}$) over X^I. We have $G^{\mathrm{ad}}_{I,\infty} = G_{I,\infty}/(Z_G)_{I,\infty}$. The group scheme $(Z_G)_{I,\infty}$ acts trivially on $\mathrm{Gr}_{G,I}$, so the action of $G_{I,\infty}$ on $\mathrm{Gr}_{G,I}$ factors through $G^{\mathrm{ad}}_{I,\infty}$. We use this action to define the quotient prestack $[G^{\mathrm{ad}}_{I,\infty} \backslash \mathrm{Gr}_{G,I}]$. The morphism $G_{I,\infty} \twoheadrightarrow G^{\mathrm{ad}}_{I,\infty}$ induces a morphism $[G_{I,\infty} \backslash \mathrm{Gr}_{G,I}] \to [G^{\mathrm{ad}}_{I,\infty} \backslash \mathrm{Gr}_{G,I}]$, which is Ξ-equivariant for the trivial action of Ξ on $[G^{\mathrm{ad}}_{I,\infty} \backslash \mathrm{Gr}_{G,I}]$.

Hence the composition of morphisms

$$\mathrm{Cht}_{G,N,I} \xrightarrow{\epsilon_{G,N,I,\infty}} [G_{I,\infty} \backslash \mathrm{Gr}_{G,I}] \to [G^{\mathrm{ad}}_{I,\infty} \backslash \mathrm{Gr}_{G,I}]$$

is Ξ-equivariant. Thus it factors through

$$\epsilon^{\Xi}_{G,N,I,\infty} : \mathrm{Cht}_{G,N,I} / \Xi \to [G^{\mathrm{ad}}_{I,\infty} \backslash \mathrm{Gr}_{G,I}]. \tag{1.7}$$

We will construct a variant of morphism (1.7) for algebraic stacks in 2.4.1.

1.3.3 Z_G acts on a P-bundle via $Z_G \hookrightarrow P$. Just as in 1.3.1, we have an action of Bun_{Z_G} on $\mathrm{Bun}_{P,N}$ by twisting a P-bundle by a Z_G-bundle. This leads to an action of Ξ on $\mathrm{Bun}_{P,N}$, $\mathrm{Cht}_{P,N,I}$ and $[P_{I,\infty}\backslash \mathrm{Gr}_{P,I}]$ via $\Xi \to Z_G(\mathbb{A}) \to \mathrm{Bun}_{Z_G}(\mathbb{F}_q)$.

Using the morphism $Z_G \hookrightarrow M$, we similarly obtain an action of Ξ on $\mathrm{Bun}_{M,N}$, $\mathrm{Cht}_{M,N,I}$ and $[M_{I,\infty}\backslash \mathrm{Gr}_{M,I}]$.

1.3.4 Applying Definition 1.1.13 to $\overline{P} := P/Z_G$ (respectively $\overline{M} := M/Z_G$), we define a group scheme $\overline{P}_{I,\infty}$ (respectively $\overline{M}_{I,\infty}$) over X^I. We have $\overline{P}_{I,\infty} = P_{I,\infty}/(Z_G)_{I,\infty}$ and $\overline{M}_{I,\infty} = M_{I,\infty}/(Z_G)_{I,\infty}$.

The morphism $\epsilon_{P,N,I,\infty}$ defined in 1.2.1 is Ξ-equivariant. Since the group scheme $(Z_G)_{I,\infty}$ acts trivially on $\mathrm{Gr}_{P,I}$, the action of $P_{I,\infty}$ on $\mathrm{Gr}_{P,I}$ factors through $\overline{P}_{I,\infty}$. We denote by $[\overline{P}_{I,\infty}\backslash \mathrm{Gr}_{P,I}]$ the resulting quotient prestack. The morphism $P_{I,\infty} \twoheadrightarrow \overline{P}_{I,\infty}$ induces a morphism $[P_{I,\infty}\backslash \mathrm{Gr}_{P,I}] \to [\overline{P}_{I,\infty}\backslash \mathrm{Gr}_{P,I}]$, which is Ξ-equivariant for the trivial action of Ξ on $[\overline{P}_{I,\infty}\backslash \mathrm{Gr}_{P,I}]$. Hence the composition of morphisms $\mathrm{Cht}_{P,N,I} \xrightarrow{\epsilon_{P,N,I,\infty}} [P_{I,\infty}\backslash \mathrm{Gr}_{P,I}] \to [\overline{P}_{I,\infty}\backslash \mathrm{Gr}_{P,I}]$ is Ξ-equivariant. Thus it factors through

$$\epsilon^{\Xi}_{P,N,I,\infty} : \mathrm{Cht}_{P,N,I}/\Xi \to [\overline{P}_{I,\infty}\backslash \mathrm{Gr}_{P,I}]. \tag{1.8}$$

Similarly, the composition of morphisms $\mathrm{Cht}_{M,N,I} \xrightarrow{\epsilon_{M,N,I,\infty}} [M_{I,\infty}\backslash \mathrm{Gr}_{M,I}] \to [\overline{M}_{I,\infty}\backslash \mathrm{Gr}_{M,I}]$ is Ξ-equivariant for the trivial action of Ξ on $[\overline{M}_{I,\infty}\backslash \mathrm{Gr}_{M,I}]$. Thus it factors through

$$\epsilon^{\Xi}_{M,N,I,\infty} : \mathrm{Cht}_{M,N,I}/\Xi \to [\overline{M}_{I,\infty}\backslash \mathrm{Gr}_{M,I}]. \tag{1.9}$$

1.3.5 The morphisms i and π in (1.6) are Ξ-equivariant. Diagram (1.6) induces a commutative diagram of prestacks.

$$\begin{tikzcd}
\mathrm{Cht}_{G,N,I}/\Xi \arrow[d, "\epsilon^{\Xi}_{G,N,I,\infty}"] & \mathrm{Cht}_{P,N,I}/\Xi \arrow[l, "i"'] \arrow[r, "\pi"] \arrow[d, "\epsilon^{\Xi}_{P,N,I,\infty}"] & \mathrm{Cht}_{M,N,I}/\Xi \arrow[d, "\epsilon^{\Xi}_{M,N,I,\infty}"] \\
{[G^{\mathrm{ad}}_{I,\infty}\backslash \mathrm{Gr}_{G,I}]} & {[\overline{P}_{I,\infty}\backslash \mathrm{Gr}_{P,I}]} \arrow[l, "\overline{i}^0"'] \arrow[r, "\overline{\pi}^0"] & {[\overline{M}_{I,\infty}\backslash \mathrm{Gr}_{M,I}]}
\end{tikzcd} \tag{1.10}$$

In the remaining part of § 1, we introduce the Harder–Narasimhan stratification (compatible with the action of Ξ) for the parabolic induction diagram (1.4). In order to do so, we use the Harder–Narasimhan stratification for the parabolic induction diagram (1.3). From now on we work in the context of algebraic (ind-)stacks.

In § 1.4, we recall the usual Harder–Narasimhan stratification $\mathrm{Bun}_G^{\leqslant^G \mu} \subset \mathrm{Bun}_G$ and a variant $\mathrm{Bun}_G^{\leqslant^{G^{\mathrm{ad}}} \mu} \subset \mathrm{Bun}_G$ which is compatible with the action by Ξ.

In § 1.5, we introduce the Harder–Narasimhan stratification $\mathrm{Bun}_M^{\leqslant^{G^{\mathrm{ad}}} \mu} \subset \mathrm{Bun}_M$, which allows us to construct in § 1.6 the truncated parabolic induction diagrams (1.26):

$$\mathrm{Bun}_G^{\leqslant^{G^{\mathrm{ad}}} \mu}/\Xi \leftarrow \mathrm{Bun}_P^{\leqslant^{G^{\mathrm{ad}}} \mu}/\Xi \to \mathrm{Bun}_M^{\leqslant^{G^{\mathrm{ad}}} \mu}/\Xi.$$

In § 1.7, we define the Harder–Narasimhan stratification on the stacks of shtukas using §§ 1.4–1.6.

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1.4 Harder–Narasimhan stratification of Bun_G

In 1.4.1–1.4.10, we recall the Harder–Narasimhan stratification of Bun_G defined in [Sch15] and [DG15, § 7]. (In these papers, the group is reductive over an algebraically closed field. Since our group G is split over $\mathbb{F}_q$, we use Galois descent to obtain the stratification over $\mathbb{F}_q$.)

In 1.4.11–1.4.17, we recall a variant of the Harder–Narasimhan stratification of Bun_G which is compatible with the quotient by Ξ, as in [Var04, § 2] and [Laf18, § 1].

1.4.1 Applying 0.0.6 to group G, we define $\widehat{\Lambda}_G$, $\widehat{\Lambda}_G^+$, $\widehat{\Lambda}_G^{\mathrm{pos}}$, $\widehat{\Lambda}_G^{\mathbb{Q}}$, $\widehat{\Lambda}_G^{+,\mathbb{Q}}$, $\widehat{\Lambda}_G^{\mathrm{pos},\mathbb{Q}}$ and the partial order '$\leqslant^G$' on $\widehat{\Lambda}_G^{\mathbb{Q}}$.

1.4.2 [Sch15, 2.1.2] Let P be a parabolic subgroup of G and M its Levi quotient. Consider the sublattice $\widehat{\Lambda}_{[M,M]_{\mathrm{sc}}} \subset \widehat{\Lambda}_G$ spanned by the simple coroots of M. We define

$$\widehat{\Lambda}_{G,P} := \widehat{\Lambda}_G / \widehat{\Lambda}_{[M,M]_{\mathrm{sc}}}. \tag{1.11}$$

Let $\widehat{\Lambda}_{G,P}^{\mathbb{Q}} := \widehat{\Lambda}_{G,P} \otimes_{\mathbb{Z}} \mathbb{Q}$. We denote by $\widehat{\Lambda}_{G,P}^{\mathrm{pos}}$ the image of $\widehat{\Lambda}_G^{\mathrm{pos}}$ in $\widehat{\Lambda}_{G,P}$, and by $\widehat{\Lambda}_{G,P}^{\mathrm{pos},\mathbb{Q}}$ the image of $\widehat{\Lambda}_G^{\mathrm{pos},\mathbb{Q}}$ in $\widehat{\Lambda}_{G,P}^{\mathbb{Q}}$. We introduce the partial order on $\widehat{\Lambda}_{G,P}$ by

$$\mu_1 \leqslant^G \mu_2 \Leftrightarrow \mu_2 - \mu_1 \in \widehat{\Lambda}_{G,P}^{\mathrm{pos}}.$$

1.4.3 [Sch15, 2.1.3], [DG15, 7.1.3, 7.1.5] Let Z_M be the center of M. Let $\widehat{\Lambda}_{Z_M}$ be the coweight lattice of Z_M, i.e. $\mathrm{Hom}(\mathbb{G}_m, Z_M)$. Note that it equals to $\widehat{\Lambda}_{Z_M^0} = \mathrm{Hom}(\mathbb{G}_m, Z_M^0)$, where Z_M^0 is the neutral connected component of Z_M.

We have a natural inclusion $\widehat{\Lambda}_{Z_M} \subset \widehat{\Lambda}_G$ (because Z_M is included in the image of $B \hookrightarrow P \twoheadrightarrow M$). The composition $\widehat{\Lambda}_{Z_M}^{\mathbb{Q}} \hookrightarrow \widehat{\Lambda}_G^{\mathbb{Q}} \twoheadrightarrow \widehat{\Lambda}_{G,P}^{\mathbb{Q}}$ is an isomorphism:

$$\widehat{\Lambda}_{Z_M}^{\mathbb{Q}} \xrightarrow{\sim} \widehat{\Lambda}_{G,P}^{\mathbb{Q}}. \tag{1.12}$$

We define the slope map to be the composition

$$\phi_P : \widehat{\Lambda}_{G,P} \to \widehat{\Lambda}_{G,P}^{\mathbb{Q}} \cong \widehat{\Lambda}_{Z_M}^{\mathbb{Q}} \hookrightarrow \widehat{\Lambda}_G^{\mathbb{Q}}. \tag{1.13}$$

We define pr_P to be the composition

$$\mathrm{pr}_P : \widehat{\Lambda}_G^{\mathbb{Q}} \twoheadrightarrow \widehat{\Lambda}_{G,P}^{\mathbb{Q}} \simeq \widehat{\Lambda}_{Z_M}^{\mathbb{Q}}. \tag{1.14}$$

By definition, we have $\widehat{\Lambda}_{G,G}^{\mathbb{Q}} = \widehat{\Lambda}_{Z_G}^{\mathbb{Q}}$, $\widehat{\Lambda}_{G,P} = \widehat{\Lambda}_{M,M}$ and $\widehat{\Lambda}_{G,B} = \widehat{\Lambda}_G$. So ϕ_B is just the inclusion $\widehat{\Lambda}_G \hookrightarrow \widehat{\Lambda}_G^{\mathbb{Q}}$.

LEMMA 1.4.4 [Sch15, Proposition 3.1]. *The slope map ϕ_P preserves the partial orders '$\leqslant^G$' on $\widehat{\Lambda}_{G,P}$ and $\widehat{\Lambda}_G^{\mathbb{Q}}$ in the sense that it maps $\widehat{\Lambda}_{G,P}^{\mathrm{pos}}$ to $\widehat{\Lambda}_G^{\mathrm{pos},\mathbb{Q}}$.*

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1.4.5 [Var04, Lemma 2.2], [Sch15, 2.2.1, 2.2.2], [DG15, 7.2.3] The map $\mathrm{Bun}_P \to \mathrm{Bun}_M$ in 1.2.3 induces a bijection on the set of connected components of Bun_P and Bun_M. We have $\pi_0(\mathrm{Bun}_P) \cong \pi_0(\mathrm{Bun}_M) \cong \widehat{\Lambda}_{G,P}$. Let $\deg_M : \mathrm{Bun}_M \to \pi_0(\mathrm{Bun}_M) \cong \widehat{\Lambda}_{G,P}$ and $\deg_P : \mathrm{Bun}_P \to \mathrm{Bun}_M \to \widehat{\Lambda}_{G,P}$.

Definition 1.4.6 [DG15, 7.3.3, 7.3.4]. For any $\mu \in \widehat{\Lambda}_G^{+,\mathbb{Q}}$, we define $\mathrm{Bun}_G^{\leqslant^G \mu}$ to be the stack that associates to any affine scheme S over $\mathbb{F}_q$ the groupoid

$$\begin{aligned}\mathrm{Bun}_G^{\leqslant^G \mu}(S) := \{\mathcal{G} \in \mathrm{Bun}_G(S)|\ &\text{for each geometric point } s \in S,\ \text{each parabolic} \\ &\text{subgroup } P \text{ and each } P\text{-structure } \mathcal{P} \text{ of } \mathcal{G}_s,\ \text{we have } \phi_P \circ \deg_P(\mathcal{P}) \leqslant^G \mu\},\end{aligned}$$

where a P-structure of $\mathcal{G}_s$ is a P-bundle $\mathcal{P}$ on X_s such that $\mathcal{P} \overset{P}{\times} G \simeq \mathcal{G}_s$.

Remark 1.4.7. (a) By [Sch15, Lemma 3.3], the above Definition 1.4.6 is equivalent to

$$\begin{aligned}\mathrm{Bun}_G^{\leqslant^G \mu}(S) := \{\mathcal{G} \in \mathrm{Bun}_G(S)|\ &\text{for each geometric point } s \in S, \\ &\text{and each } B\text{-structure } \mathcal{B} \text{ of } \mathcal{G}_s,\ \text{we have } \deg_B(\mathcal{B}) \leqslant^G \mu\}\end{aligned}$$

(the argument repeats the proof in [Sch15, Lemma 3.3] by replacing $\phi_G(\check{\lambda}_G)$ by μ).

(b) By [Sch15, Proposition 3.2 and Remark 3.2.4], the definition of $\mathrm{Bun}_G^{\leqslant^G \mu}$ in (a) is equivalent to the Tannakian description:

$$\begin{aligned}\mathrm{Bun}_G^{\leqslant^G \mu}(S) := \{\mathcal{G} \in \mathrm{Bun}_G(S)|\ &\text{for each geometric point } s \in S, \\ &\text{each } B\text{-structure } \mathcal{B} \text{ of } \mathcal{G}_s \text{ and each } \lambda \in \Lambda_G,\ \text{we have } \deg \mathcal{B}_\lambda \leqslant \langle \mu, \lambda \rangle\},\end{aligned}$$

where $\mathcal{B}_\lambda$ is the line bundle associated to $\mathcal{B}$ and $B \to T \xrightarrow{\lambda} \mathbb{G}_m$.

(c) The reason why we use Definition 1.4.6 (rather than its equivalent forms) is that it will be useful for non-split groups in future works.

Lemma 1.4.8 [DG15, 7.3.4, Proposition 7.3.5]. (a) *For any* $\mu \in \widehat{\Lambda}_G^{+,\mathbb{Q}}$, *the stack* $\mathrm{Bun}_G^{\leqslant^G \mu}$ *is an open substack of* Bun_G.

(b) *For any* $\mu_1 \leqslant^G \mu_2$, *we have an open immersion* $\mathrm{Bun}_G^{\leqslant^G \mu_1} \hookrightarrow \mathrm{Bun}_G^{\leqslant^G \mu_2}$.

(c) *We have* $\mathrm{Bun}_G = \bigcup_{\mu \in \widehat{\Lambda}_G^{+,\mathbb{Q}}} \mathrm{Bun}_G^{\leqslant^G \mu}$.

(d) *The open substack* $\mathrm{Bun}_G^{\leqslant^G \mu}$ *is of finite type.*

Definition 1.4.9. For any $\lambda \in \widehat{\Lambda}_G^{+,\mathbb{Q}}$, let $\mathrm{Bun}_G^{(\lambda)} \subset \mathrm{Bun}_G$ be the quasi-compact locally closed reduced substack defined in [Sch15, Theorem 2.1] and [DG15, Theorem 7.4.3]. It is called a Harder–Narasimhan stratum of Bun_G.

1.4.10 [DG15, Corollary 7.4.5] We have

$$\mathrm{Bun}_G^{(\lambda)} \neq \emptyset \Rightarrow \lambda \in \bigcup_{P \subsetneq G} \iota \circ \mathrm{pr}_P(\widehat{\Lambda}_G),$$

where pr_P is defined in (1.14) and $\iota : \widehat{\Lambda}_{Z_M}^{\mathbb{Q}} \hookrightarrow \widehat{\Lambda}_G^{\mathbb{Q}}$ is the inclusion. For any $\mu \in \widehat{\Lambda}_G^{+,\mathbb{Q}}$, we have

$$\mathrm{Bun}_G^{\leqslant^G \mu} = \bigcup_{\lambda \in \widehat{\Lambda}_G^{+,\mathbb{Q}},\ \lambda \leqslant^G \mu} \mathrm{Bun}_G^{(\lambda)}.$$

The set $\{\lambda \in \widehat{\Lambda}_G^{+,\mathbb{Q}} \mid \lambda \leqslant^G \mu \text{ and } \mathrm{Bun}_G^{(\lambda)} \neq \emptyset\}$ is finite. This gives another proof of Lemma 1.4.8(d).

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The above open substack $\mathrm{Bun}_G^{\leqslant^G \mu}$ is not preserved by the action of Ξ on Bun_G. Now we introduce open substacks which are preserved by the action of Ξ.

1.4.11 Applying 0.0.6 to group G^{ad}, we define $\widehat{\Lambda}_{G^{\mathrm{ad}}}$, $\widehat{\Lambda}^{+}_{G^{\mathrm{ad}}}$, $\widehat{\Lambda}^{\mathrm{pos}}_{G^{\mathrm{ad}}}$, $\widehat{\Lambda}^{\mathbb{Q}}_{G^{\mathrm{ad}}}$, $\widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}}$, $\widehat{\Lambda}^{\mathrm{pos},\mathbb{Q}}_{G^{\mathrm{ad}}}$ and the partial order '$\leqslant^{G^{\mathrm{ad}}}$' on $\widehat{\Lambda}_{G^{\mathrm{ad}}}$.

The morphism $G \twoheadrightarrow G/Z_G = G^{\mathrm{ad}}$ induces a morphism

$$\Upsilon_G : \widehat{\Lambda}^{\mathbb{Q}}_G \to \widehat{\Lambda}^{\mathbb{Q}}_{G^{\mathrm{ad}}}. \tag{1.15}$$

It maps $\widehat{\Lambda}^{\mathrm{pos},\mathbb{Q}}_G$ to $\widehat{\Lambda}^{\mathrm{pos},\mathbb{Q}}_{G^{\mathrm{ad}}}$.

DEFINITION 1.4.12. For any $\mu \in \widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}}$, we define $\mathrm{Bun}_G^{\leqslant^{G^{\mathrm{ad}}} \mu}$ to be the stack that associates to any affine scheme S over $\mathbb{F}_q$ the groupoid

$$\begin{aligned}\mathrm{Bun}_G^{\leqslant^{G^{\mathrm{ad}}} \mu}(S) := \{\mathcal{G} \in \mathrm{Bun}_G(S) \mid &\text{ for each geometric point } s \in S, \text{ each parabolic subgroup } P \\ &\text{and each } P\text{-structure } \mathcal{P} \text{ on } \mathcal{G}_s, \text{ we have } \Upsilon_G \circ \phi_P \circ \deg_P(\mathcal{P}) \leqslant^{G^{\mathrm{ad}}} \mu\}.\end{aligned}$$

Remark 1.4.13. For the same reason as in Remark 1.4.7, Definition 1.4.12 is equivalent to [Var04, Notation 2.1(b)] and [Laf18, (1.3)].

1.4.14 Just as in 1.4.10, for $\mu \in \widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}}$, we have

$$\mathrm{Bun}_G^{\leqslant^{G^{\mathrm{ad}}} \mu} = \bigcup_{\lambda \in \widehat{\Lambda}^{+,\mathbb{Q}}_G,\ \Upsilon_G(\lambda) \leqslant^{G^{\mathrm{ad}}} \mu} \mathrm{Bun}_G^{(\lambda)}.$$

The set $\{\lambda \in \widehat{\Lambda}^{+,\mathbb{Q}}_G \mid \Upsilon_G(\lambda) \leqslant^{G^{\mathrm{ad}}} \mu \text{ and } \mathrm{Bun}_G^{(\lambda)} \neq \emptyset\}$ is finite modulo $\widehat{\Lambda}_{Z_G}$.

1.4.15 The action of Ξ on Bun_G preserves $\mathrm{Bun}_G^{\leqslant^{G^{\mathrm{ad}}} \mu}$. We define the quotient $\mathrm{Bun}_G^{\leqslant^{G^{\mathrm{ad}}} \mu}/\Xi$.

LEMMA 1.4.16. (a) *For any $\mu \in \widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}}$, the stack $\mathrm{Bun}_G^{\leqslant^{G^{\mathrm{ad}}} \mu}$ is an open substack of Bun_G.*

(b) *For any $\mu_1 \leqslant^{G^{\mathrm{ad}}} \mu_2$, we have an open immersion $\mathrm{Bun}_G^{\leqslant^{G^{\mathrm{ad}}} \mu_1} \hookrightarrow \mathrm{Bun}_G^{\leqslant^{G^{\mathrm{ad}}} \mu_2}$.*

(c) *The stack Bun_G is the inductive limit of these open substacks: $\mathrm{Bun}_G = \bigcup_{\mu \in \widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}}} \mathrm{Bun}_G^{\leqslant^{G^{\mathrm{ad}}} \mu}$.*

(d) *The stack $\mathrm{Bun}_G^{\leqslant^{G^{\mathrm{ad}}} \mu}/\Xi$ is of finite type.*

Proof. Parts (a), (b) and (c) are induced by Lemma 1.4.8 (see also [Var04, Lemme A.3)]. Part (d) follows from 1.4.14. □

Remark 1.4.17. See [Var04, Lemmas 3.1 and 3.7] for another proof of Lemma 1.4.8(d) and Lemma 1.4.16(d).

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1.5 Harder–Narasimhan stratification of Bun_M

Let P be a proper parabolic subgroup of G and M its Levi quotient.

1.5.1 Applying 0.0.6 to group M, we define $\widehat{\Lambda}_M$, $\widehat{\Lambda}_M^+$, $\widehat{\Lambda}_M^{\mathrm{pos}}$, $\widehat{\Lambda}_M^{\mathbb{Q}}$, $\widehat{\Lambda}_M^{+,\mathbb{Q}}$, $\widehat{\Lambda}_M^{\mathrm{pos},\mathbb{Q}}$ and the partial order '$\leqslant^M$' on $\widehat{\Lambda}_M^{\mathbb{Q}}$.

1.5.2 Sections 1.4.2–1.4.10 work also for M. In particular, let P' be a parabolic subgroup of M; we have the slope map $\phi_{P'} : \widehat{\Lambda}_{M,P'} \to \widehat{\Lambda}_M^{\mathbb{Q}}$ and $\deg_{P'} : \mathrm{Bun}_{P'} \to \widehat{\Lambda}_{M,P'}$.

Definition 1.5.3. Applying Definition 1.4.9 to M, for any $\lambda \in \widehat{\Lambda}_M^{+,\mathbb{Q}}$, we define a quasi-compact locally closed substack $\mathrm{Bun}_M^{(\lambda)} \subset \mathrm{Bun}_M$, called a Harder–Narasimhan stratum of Bun_M.

Now we introduce $\mathrm{Bun}_M^{\leqslant^{G^{\mathrm{ad}}}\mu} \subset \mathrm{Bun}_M$ which will be used to construct diagram (1.26).

Definition 1.5.4. For any $\mu \in \widehat{\Lambda}_{G^{\mathrm{ad}}}^{+,\mathbb{Q}}$, we define $\mathrm{Bun}_M^{\leqslant^{G^{\mathrm{ad}}}\mu}$ to be the stack that associates to any affine scheme S over $\mathbb{F}_q$ the groupoid $\mathrm{Bun}_M^{\leqslant^{G^{\mathrm{ad}}}\mu}(S) :=$

$$\{\mathcal{M} \in \mathrm{Bun}_M(S) | \text{ for each geometric point } s \in S, \text{ each parabolic subgroup } P' \text{ of } M \text{ and each } P'\text{-structure } \mathcal{P}' \text{ of } \mathcal{M}_s, \text{ we have } \Upsilon_G \circ \phi_{P'} \circ \deg_{P'}(\mathcal{P}') \leqslant^{G^{\mathrm{ad}}} \mu\},$$

where $\Upsilon_G : \widehat{\Lambda}_M^{\mathbb{Q}} = \widehat{\Lambda}_G^{\mathbb{Q}} \to \widehat{\Lambda}_{G^{\mathrm{ad}}}^{\mathbb{Q}}$ is defined in (1.15).

Similarly to Lemma 1.4.16, we have

Lemma 1.5.5. (a) *For any $\mu \in \widehat{\Lambda}_{G^{\mathrm{ad}}}^{+,\mathbb{Q}}$, the stack $\mathrm{Bun}_M^{\leqslant^{G^{\mathrm{ad}}}\mu}$ is an open substack of Bun_M.*

(b) *For any $\mu_1 \leqslant^{G^{\mathrm{ad}}} \mu_2$, we have an open immersion $\mathrm{Bun}_M^{\leqslant^{G^{\mathrm{ad}}}\mu_1} \hookrightarrow \mathrm{Bun}_M^{\leqslant^{G^{\mathrm{ad}}}\mu_2}$.*

(c) *The stack Bun_M is the inductive limit of these open substacks: $\mathrm{Bun}_M = \bigcup_{\mu \in \widehat{\Lambda}_{G^{\mathrm{ad}}}^{+,\mathbb{Q}}} \mathrm{Bun}_M^{\leqslant^{G^{\mathrm{ad}}}\mu}$.*

1.5.6 The action of Ξ on Bun_M preserve $\mathrm{Bun}_M^{\leqslant^{G^{\mathrm{ad}}}\mu}$. We define the quotient $\mathrm{Bun}_M^{\leqslant^{G^{\mathrm{ad}}}\mu}/\Xi$. Note that Ξ is a lattice in $Z_G(F)\backslash Z_G(\mathbb{A})$. However, Ξ is only a discrete subgroup but not a lattice in $Z_M(F)\backslash Z_M(\mathbb{A})$ (since $P \subsetneq G$). We will see that $\mathrm{Bun}_M^{\leqslant^{G^{\mathrm{ad}}}\mu}/\Xi$ is locally of finite type but not necessarily of finite type.

1.5.7 Note that $\widehat{\Lambda}_{G,P} = \widehat{\Lambda}_{M,M}$. Consider the composition of morphisms

$$\mathrm{Bun}_M \xrightarrow{\deg_M} \widehat{\Lambda}_{M,M} \to \widehat{\Lambda}_{M,M}^{\mathbb{Q}} \simeq \widehat{\Lambda}_{Z_M}^{\mathbb{Q}} \twoheadrightarrow \widehat{\Lambda}_{Z_M/Z_G}^{\mathbb{Q}}, \tag{1.16}$$

where $\deg_M$ is defined in 1.4.5. We denote by A_M the image of $\widehat{\Lambda}_{M,M}$ in $\widehat{\Lambda}_{Z_M/Z_G}^{\mathbb{Q}}$. For any $\nu \in \widehat{\Lambda}_{Z_M/Z_G}^{\mathbb{Q}}$, we denote by Bun_M^{ν} its inverse image in Bun_M. It is non-empty if and only if $\nu \in A_M$.

Definition 1.5.8. We define $\mathrm{Bun}_M^{\leqslant^{G^{\mathrm{ad}}}\mu,\nu}$ to be the intersection of $\mathrm{Bun}_M^{\leqslant^{G^{\mathrm{ad}}}\mu}$ and Bun_M^{ν}.

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1.5.9 The stack $\mathrm{Bun}_M^{\leqslant^{G^{\mathrm{ad}}}\mu,\nu}$ is open and closed in $\mathrm{Bun}_M^{\leqslant^{G^{\mathrm{ad}}}\mu}$ and is open in Bun_M^{ν}. We have a decomposition

$$\mathrm{Bun}_M^{\leqslant^{G^{\mathrm{ad}}}\mu} = \bigsqcup_{\nu\in\widehat{\Lambda}^{\mathbb{Q}}_{Z_M/Z_G}} \mathrm{Bun}_M^{\leqslant^{G^{\mathrm{ad}}}\mu,\nu}. \tag{1.17}$$

1.5.10 Just as in 1.4.14, we have

$$\mathrm{Bun}_M^{\leqslant^{G^{\mathrm{ad}}}\mu} = \bigcup_{\lambda\in\widehat{\Lambda}^{+,\mathbb{Q}}_M,\ \Upsilon_G(\lambda)\leqslant^{G^{\mathrm{ad}}}\mu} \mathrm{Bun}_M^{(\lambda)}.$$

1.5.11 Similarly to (1.14), we define

$$\mathrm{pr}_P^{\mathrm{ad}} : \widehat{\Lambda}^{\mathbb{Q}}_{G^{\mathrm{ad}}} \to \widehat{\Lambda}^{\mathbb{Q}}_{Z_M/Z_G}. \tag{1.18}$$

Taking into account that $\widehat{\Lambda}_G=\widehat{\Lambda}_M$ and $\widehat{\Lambda}_{G,P}=\widehat{\Lambda}_{M,M}$, for any $\lambda\in\widehat{\Lambda}^{+,\mathbb{Q}}_M$, we deduce that $\mathrm{Bun}_M^{(\lambda)}\subset\mathrm{Bun}_M^{\nu}$ if and only if $\nu=\mathrm{pr}_P^{\mathrm{ad}}\circ\Upsilon_G(\lambda)$.

1.5.12 We deduce from 1.5.10 and 1.5.11 that

$$\mathrm{Bun}_M^{\leqslant^{G^{\mathrm{ad}}}\mu,\nu} = \bigcup_{\lambda\in\widehat{\Lambda}^{+,\mathbb{Q}}_M,\ \Upsilon_G(\lambda)\leqslant^{G^{\mathrm{ad}}}\mu,\ \mathrm{pr}_P^{\mathrm{ad}}\circ\Upsilon_G(\lambda)=\nu} \mathrm{Bun}_M^{(\lambda)}. \tag{1.19}$$

1.5.13 We denote by $\widehat{\Lambda}^{\mathrm{pos},\mathbb{Q}}_{Z_M/Z_G}:=\mathrm{pr}_P^{\mathrm{ad}}(\widehat{\Lambda}^{\mathrm{pos},\mathbb{Q}}_{G^{\mathrm{ad}}})$. We introduce the partial order on $\widehat{\Lambda}^{\mathbb{Q}}_{Z_M/Z_G}$ by

$$\mu_1\leqslant^{G^{\mathrm{ad}}}\mu_2 \Leftrightarrow \mu_2-\mu_1\in\widehat{\Lambda}^{\mathrm{pos},\mathbb{Q}}_{Z_M/Z_G}.$$

By definition, for $\check{\gamma}\in\widehat{\Gamma}_M$, we have $\mathrm{pr}_P^{\mathrm{ad}}\circ\Upsilon_G(\check{\gamma})=0$. By [Sch15, Proposition 3.1], for $\check{\gamma}\in\widehat{\Gamma}_G-\widehat{\Gamma}_M$ we have $\mathrm{pr}_P^{\mathrm{ad}}\circ\Upsilon_G(\check{\gamma})>0$ and these $\mathrm{pr}_P^{\mathrm{ad}}\circ\Upsilon_G(\check{\gamma})$ are linearly independent. Thus for $\lambda_1,\lambda_2\in\widehat{\Lambda}^{\mathbb{Q}}_{G^{\mathrm{ad}}}$ and $\lambda_1\leqslant^{G^{\mathrm{ad}}}\lambda_2$, we have $\mathrm{pr}_P^{\mathrm{ad}}(\lambda_1)\leqslant^{G^{\mathrm{ad}}}\mathrm{pr}_P^{\mathrm{ad}}(\lambda_2)$. Also, the inclusion $\widehat{\Lambda}^{\mathbb{Q}}_{Z_M/Z_G}\subset\widehat{\Lambda}^{\mathbb{Q}}_{G^{\mathrm{ad}}}$ maps $\widehat{\Lambda}^{\mathrm{pos},\mathbb{Q}}_{Z_M/Z_G}$ to $\widehat{\Lambda}^{\mathrm{pos},\mathbb{Q}}_{G^{\mathrm{ad}}}$.

Lemma 1.5.14. *Let $\mu\in\widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}}$. Then the stack $\mathrm{Bun}_M^{\leqslant^{G^{\mathrm{ad}}}\mu,\nu}$ is empty unless $\nu\in A_M$ defined in 1.5.7 and $\nu\leqslant^{G^{\mathrm{ad}}}\mathrm{pr}_P^{\mathrm{ad}}(\mu)$.*

Proof. The first condition follows from 1.5.7. To prove the second condition, note that for the set $\{\lambda\in\widehat{\Lambda}^{+,\mathbb{Q}}_M \mid \Upsilon_G(\lambda)\leqslant^{G^{\mathrm{ad}}}\mu,\ \mathrm{pr}_P^{\mathrm{ad}}\circ\Upsilon_G(\lambda)=\nu\}$ to be non-empty, by 1.5.13 we must have $\nu\leqslant^{G^{\mathrm{ad}}}\mathrm{pr}_P^{\mathrm{ad}}(\mu)$. □

1.5.15 Let $\overline{M}=M/Z_G$ as in 1.3.4. For $\lambda,\mu\in\widehat{\Lambda}^{\mathbb{Q}}_{G^{\mathrm{ad}}}$, we define $\lambda\leqslant^{\overline{M}}\mu$ if and only if $\mu-\lambda$ is a linear combination of simple coroots of M with coefficients in $\mathbb{Q}_{\geqslant 0}$ modulo $\widehat{\Lambda}^{\mathbb{Q}}_{Z_G}$.

1.5.16 Let $\lambda,\mu\in\widehat{\Lambda}^{\mathbb{Q}}_{G^{\mathrm{ad}}}$ and $\lambda\leqslant^{G^{\mathrm{ad}}}\mu$. We write $\lambda=\mu-\sum_{\check{\gamma}\in\widehat{\Gamma}_G}c_{\check{\gamma}}\Upsilon_G(\check{\gamma})$ for some $c_{\check{\gamma}}\in\mathbb{Q}_{\geqslant 0}$. We deduce from 1.5.13 that $\mathrm{pr}_P^{\mathrm{ad}}(\lambda)=\mathrm{pr}_P^{\mathrm{ad}}(\mu)$ if and only if $c_{\check{\gamma}}=0$ for all $\check{\gamma}\in\widehat{\Gamma}_G-\widehat{\Gamma}_M$. Hence

$$\lambda\leqslant^{G^{\mathrm{ad}}}\mu \quad\text{and}\quad \mathrm{pr}_P^{\mathrm{ad}}(\lambda)=\mathrm{pr}_P^{\mathrm{ad}}(\mu) \;\Leftrightarrow\; \lambda\leqslant^{\overline{M}}\mu. \tag{1.20}$$

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1.5.17 Let $\mu \in \widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}}$ and $\nu \leqslant^{G^{\mathrm{ad}}} \mathrm{pr}^{\mathrm{ad}}_P(\mu)$. For every $\check{\gamma} \in \widehat{\Gamma}_G - \widehat{\Gamma}_M$, let $c_\gamma \in \mathbb{Q}_{\geqslant 0}$ be the unique coefficient such that

$$\mathrm{pr}^{\mathrm{ad}}_P(\mu) - \sum_{\check{\gamma}\in\widehat{\Gamma}_G-\widehat{\Gamma}_M} c_\gamma \, \mathrm{pr}^{\mathrm{ad}}_P \circ \Upsilon_G(\check{\gamma}) = \nu.$$

We define $\mu_\nu := \mu - \sum_{\check{\gamma}\in\widehat{\Gamma}_G-\widehat{\Gamma}_M} c_\gamma \Upsilon_G(\check{\gamma})$. As in 1.5.16, we deduce that

$$\lambda \leqslant^{G^{\mathrm{ad}}} \mu \quad \text{and} \quad \mathrm{pr}^{\mathrm{ad}}_P(\lambda) = \nu \;\Leftrightarrow\; \lambda \leqslant^{\overline{M}} \mu_\nu. \tag{1.21}$$

1.5.18 The action of Ξ on Bun_M preserves $\mathrm{Bun}_M^{\leqslant^{G^{\mathrm{ad}}}\mu,\,\nu}$. We define the quotient $\mathrm{Bun}_M^{\leqslant^{G^{\mathrm{ad}}}\mu,\,\nu}/\Xi$.

Lemma 1.5.19. *The stack* $\mathrm{Bun}_M^{\leqslant^{G^{\mathrm{ad}}}\mu,\,\nu}/\Xi$ *is of finite type.*

Proof. By (1.21), we have

$$\{\lambda \in \widehat{\Lambda}^{+,\mathbb{Q}}_M \mid \Upsilon_G(\lambda) \leqslant^{G^{\mathrm{ad}}} \mu,\ \mathrm{pr}^{\mathrm{ad}}_P \circ \Upsilon_G(\lambda) = \nu\} = \{\lambda \in \widehat{\Lambda}^{+,\mathbb{Q}}_M \mid \Upsilon_G(\lambda) \leqslant^{\overline{M}} \mu_\nu\}.$$

We deduce from 1.4.10 (applied to M) that the set $\{\lambda \in \widehat{\Lambda}^{+,\mathbb{Q}}_M \mid \Upsilon_G(\lambda) \leqslant^{G^{\mathrm{ad}}} \mu,\ \mathrm{pr}^{\mathrm{ad}}_P \circ \Upsilon_G(\lambda) = \nu,$ $\mathrm{Bun}_M^{(\lambda)} \neq \emptyset\}$ is finite modulo $\widehat{\Lambda}_{Z_G}$. By Definition 1.5.3, $\mathrm{Bun}_M^{(\lambda)}$ is of finite type. From 1.5.12 we deduce the lemma. □

1.5.20 By Lemma 1.5.14, the decomposition (1.17) is in fact indexed by a translated cone in $\widehat{\Lambda}^{\mathbb{Q}}_{Z_M/Z_G}$:

$$\widehat{\Lambda}^{\mu}_{Z_M/Z_G} := \{\nu \in \widehat{\Lambda}^{\mathbb{Q}}_{Z_M/Z_G},\ \nu \leqslant^{G^{\mathrm{ad}}} \mathrm{pr}^{\mathrm{ad}}_P(\mu)\}. \tag{1.22}$$

We deduce that

$$\mathrm{Bun}_M^{\leqslant^{G^{\mathrm{ad}}}\mu} = \bigsqcup_{\nu\in\widehat{\Lambda}^{\mu}_{Z_M/Z_G}} \mathrm{Bun}_M^{\leqslant^{G^{\mathrm{ad}}}\mu,\,\nu} \tag{1.23}$$

and

$$\mathrm{Bun}_M^{\leqslant^{G^{\mathrm{ad}}}\mu}/\Xi = \bigsqcup_{\nu\in\widehat{\Lambda}^{\mu}_{Z_M/Z_G}} \mathrm{Bun}_M^{\leqslant^{G^{\mathrm{ad}}}\mu,\,\nu}/\Xi. \tag{1.24}$$

1.6 Harder–Narasimhan stratification of parabolic induction

Recall that we have morphisms (1.3): $\mathrm{Bun}_G \xleftarrow{i^{\mathrm{Bun}}} \mathrm{Bun}_P \xrightarrow{\pi^{\mathrm{Bun}}} \mathrm{Bun}_M$.

Definition 1.6.1. Let $\mu \in \widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}}$. We define $\mathrm{Bun}_P^{\leqslant^{G^{\mathrm{ad}}}\mu}$ to be the inverse image of $\mathrm{Bun}_G^{\leqslant^{G^{\mathrm{ad}}}\mu}$ in Bun_P.

Lemma 1.6.2. *The image of* $\mathrm{Bun}_P^{\leqslant^{G^{\mathrm{ad}}}\mu}$ *in* Bun_M *is included in* $\mathrm{Bun}_M^{\leqslant^{G^{\mathrm{ad}}}\mu}$.

Proof. Let $\mathcal{P} \in \mathrm{Bun}_P^{\leqslant^{G^{\mathrm{ad}}}\mu}$ and let $\mathcal{M}$ be its image in Bun_M. We will check that $\mathcal{M} \in \mathrm{Bun}_M^{\leqslant^{G^{\mathrm{ad}}}\mu}$. For any parabolic subgroup P' of M, let M' be its Levi quotient. Let $\mathcal{P}'$ be a P'-structure of $\mathcal{M}$ and $\mathcal{M}' := \mathcal{P}' \overset{P'}{\times} M'$. By Definition 1.5.4, we need to prove that $\Upsilon_G \circ \phi_{P'} \circ \deg_{P'}(\mathcal{P}') \leqslant^{G^{\mathrm{ad}}} \mu$.

Let $P'' := P \underset{M}{\times} P'$. It is a parabolic subgroup of G with Levi quotient M'. We have the following.

$$\begin{tikzcd}[column sep=small] & & P'' \arrow[dl] \arrow[dr] & & \\ & P \arrow[dl] \arrow[dr] & & P' \arrow[dl] \arrow[dr] & \\ G & & M & & M' \end{tikzcd}$$

By [DG16, Lemma 2.5.8], we can define a P''-bundle $\mathcal{P}'' := \mathcal{P} \underset{\mathcal{M}}{\times} \mathcal{P}'$. We have $\deg_{P'} \mathcal{P}' = \deg_{M'} \mathcal{M}' = \deg_{P''} \mathcal{P}''$. Taking into account that $\widehat{\Lambda}_G^{\mathbb{Q}} = \widehat{\Lambda}_M^{\mathbb{Q}}$, we deduce that $\Upsilon_G \circ \phi_{P'} \circ \deg_{P'}(\mathcal{P}') = \Upsilon_G \circ \phi_{P''} \circ \deg_{P''}(\mathcal{P}'') \leqslant^{G^{\mathrm{ad}}} \mu$, where the last inequality follows from the definition of $\mathrm{Bun}_P^{\leqslant^{G^{\mathrm{ad}}}\mu}$. $\square$

1.6.3 By Lemma 1.6.2, morphisms (1.3) induce morphisms:

$$\mathrm{Bun}_G^{\leqslant^{G^{\mathrm{ad}}}\mu} \leftarrow \mathrm{Bun}_P^{\leqslant^{G^{\mathrm{ad}}}\mu} \rightarrow \mathrm{Bun}_M^{\leqslant^{G^{\mathrm{ad}}}\mu}. \tag{1.25}$$

The group Ξ acts on all these stacks. All the morphisms are Ξ-equivariant. Thus morphisms (1.25) induce morphisms:

$$\mathrm{Bun}_G^{\leqslant^{G^{\mathrm{ad}}}\mu}/\Xi \leftarrow \mathrm{Bun}_P^{\leqslant^{G^{\mathrm{ad}}}\mu}/\Xi \rightarrow \mathrm{Bun}_M^{\leqslant^{G^{\mathrm{ad}}}\mu}/\Xi. \tag{1.26}$$

1.6.4 For any $\nu \in \widehat{\Lambda}_{Z_M/Z_G}^{\mathbb{Q}}$, we define Bun_P^{ν} to be the inverse image of Bun_M^{ν} in Bun_P. We define $\mathrm{Bun}_P^{\leqslant^{G^{\mathrm{ad}}}\mu,\nu} := \mathrm{Bun}_P^{\leqslant^{G^{\mathrm{ad}}}\mu} \cap \mathrm{Bun}_P^{\nu}$. Morphisms (1.26) induce morphisms:

$$\mathrm{Bun}_G^{\leqslant^{G^{\mathrm{ad}}}\mu}/\Xi \leftarrow \mathrm{Bun}_P^{\leqslant^{G^{\mathrm{ad}}}\mu,\nu}/\Xi \rightarrow \mathrm{Bun}_M^{\leqslant^{G^{\mathrm{ad}}}\mu,\nu}/\Xi. \tag{1.27}$$

1.7 Harder–Narasimhan stratification of stack of shtukas

Notation 1.7.1. In the remaining part of the paper, we will only use the truncations indexed by '$\leqslant^{G^{\mathrm{ad}}}$' (rather than '$\leqslant^{G}$'). To simplify the notation, from now on, '$\leqslant$' means '$\leqslant^{G^{\mathrm{ad}}}$'.

DEFINITION 1.7.2. Let $\mu \in \widehat{\Lambda}_{G^{\mathrm{ad}}}^{+,\mathbb{Q}}$ (respectively $\lambda \in \widehat{\Lambda}_G^{+,\mathbb{Q}}$). We define $\mathrm{Cht}_{G,N,I}^{\leqslant\mu}$ (respectively $\mathrm{Cht}_{G,N,I}^{(\lambda)}$) to be the inverse image of $\mathrm{Bun}_G^{\leqslant\mu}$ (respectively $\mathrm{Bun}_G^{(\lambda)}$) by the morphism

$$\mathrm{Cht}_{G,N,I} \rightarrow \mathrm{Bun}_G, \quad ((x_i)_{i\in I}, (\mathcal{G}, \psi) \xrightarrow{\phi} ({}^{\tau}\mathcal{G}, {}^{\tau}\psi)) \mapsto \mathcal{G}.$$

Similarly, we define $\mathrm{Cht}_{M,N,I}^{\leqslant\mu}$ (respectively $\mathrm{Cht}_{M,N,I}^{\leqslant\mu,\nu}$, $\mathrm{Cht}_{M,N,I}^{(\lambda)}$) using the morphism $\mathrm{Cht}_{M,N,I} \rightarrow \mathrm{Bun}_M$ and $\mathrm{Cht}_{P,N,I}^{\leqslant\mu}$ (respectively $\mathrm{Cht}_{P,N,I}^{\leqslant\mu,\nu}$) using the morphism $\mathrm{Cht}_{P,N,I} \rightarrow \mathrm{Bun}_P$.

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1.7.3 The following diagram is commutative

$$\begin{tikzcd} \mathrm{Cht}_{G,N,I} \arrow[d] & \mathrm{Cht}_{P,N,I} \arrow[l, "i"'] \arrow[r, "\pi"] \arrow[d] & \mathrm{Cht}_{M,N,I} \arrow[d] \\ \mathrm{Bun}_G & \mathrm{Bun}_P \arrow[l, "i^{\mathrm{Bun}}"'] \arrow[r, "\pi^{\mathrm{Bun}}"] & \mathrm{Bun}_M \end{tikzcd} \tag{1.28}$$

where the first line is defined in (1.4). We deduce that $\mathrm{Cht}_{P,N,I}^{\leqslant\mu}$ is the inverse image of $\mathrm{Cht}_{G,N,I}^{\leqslant\mu}$ in $\mathrm{Cht}_{P,N,I}$.

Lemma 1.7.4. *The image of* $\mathrm{Cht}_{P,N,I}^{\leqslant\mu}$ *in* $\mathrm{Cht}_{M,N,I}$ *is included in* $\mathrm{Cht}_{M,N,I}^{\leqslant\mu}$.

Proof. This follows from Lemma 1.6.2 and the commutativity of (1.28). □

1.7.5 Just as in 1.6.3 and 1.6.4, morphisms (1.4) induce morphisms:

$$\mathrm{Cht}_{G,N,I}^{\leqslant\mu}/\Xi \leftarrow \mathrm{Cht}_{P,N,I}^{\leqslant\mu}/\Xi \rightarrow \mathrm{Cht}_{M,N,I}^{\leqslant\mu}/\Xi, \tag{1.29}$$

$$\mathrm{Cht}_{G,N,I}^{\leqslant\mu}/\Xi \leftarrow \mathrm{Cht}_{P,N,I}^{\leqslant\mu,\,\nu}/\Xi \rightarrow \mathrm{Cht}_{M,N,I}^{\leqslant\mu,\,\nu}/\Xi. \tag{1.30}$$

We deduce from (1.24) a decomposition:

$$\mathrm{Cht}_{M,N,I}^{\leqslant\mu}/\Xi = \bigsqcup_{\nu\in\widehat{\Lambda}^{\mu}_{Z_M/Z_G}} \mathrm{Cht}_{M,N,I}^{\leqslant\mu,\,\nu}/\Xi. \tag{1.31}$$

2. Cohomology of stacks of shtukas

In §§ 2.1–2.5 we recall the definition of the cohomology of stacks of G-shtukas with values in perverse sheaves coming from $[G_{I,\infty}\backslash \mathrm{Gr}_{G,I}]$ via $\epsilon_{G,N,I,\infty}$, i.e. coming from $G_{I,\infty}$-equivariant perverse sheaves over $\mathrm{Gr}_{G,I}$. These sections are based on [Laf18, §§ 1, 2 and 4].

In § 2.6 we define the cohomology of stacks of M-shtukas.

Notation 2.0.1. Our results are of geometric nature, i.e. we will not consider the action of $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$. From now on, we pass to the base change over $\overline{\mathbb{F}_q}$. We keep the same notations X, $\mathrm{Bun}_{G,N}$, $\mathrm{Cht}_{G,N,I}$, $\mathrm{Gr}_{G,I}$, etc., but now everything is over $\overline{\mathbb{F}_q}$ and the fiber products are taken over $\overline{\mathbb{F}_q}$.

2.1 Reminder of a generalization of the geometric Satake equivalence

2.1.1 The geometric Satake equivalence for the affine grassmannian is established in [MV07] over the ground field $\mathbb{C}$. By [MV07, § 14], [Gai07, § 1.6] and [Zhu17], the constructions in [MV07] carries over to the case of an arbitrary algebraically closed ground field of characteristic prime to ℓ.

2.1.2 Let $\widehat{G}$ be the Langlands dual group of G over $\mathbb{Q}_\ell$ defined by the geometric Satake equivalence for the affine grassmannian, as in [MV07, Theorem 7.3] and [Gai07, Theorem 2.2].

2.1.3 [MV07, § 2], [Gai01, 1.1.1 and § 6] The Beilinson–Drinfeld affine grassmannian $\mathrm{Gr}_{G,I}$ is an ind-scheme. Every finite-dimensional closed subscheme of $\mathrm{Gr}_{G,I}$ is contained in some finite-dimensional closed subscheme of $\mathrm{Gr}_{G,I}$ stable under the action of $G_{I,\infty}$.

We denote by $\mathrm{Perv}_{G_{I,\infty}}(\mathrm{Gr}_{G,I}, \mathbb{Q}_\ell)$ the category of $G_{I,\infty}$-equivariant perverse sheaves with $\mathbb{Q}_\ell$-coefficients on $\mathrm{Gr}_{G,I}$ (for the perverse normalization relative to X^I).

2.1.4 As in [Gai07, 2.5], we denote by $P^{\widehat{G},I}$ the category of perverse sheaves with $\mathbb{Q}_\ell$-coefficients on X^I (for the perverse normalization relative to X^I) endowed with an extra structure given in [Gai07].

THEOREM 2.1.5 [Gai07, Theorem 2.6]. *There is a canonical equivalence of categories*

$$\mathrm{Perv}_{G_{I,\infty}}(\mathrm{Gr}_{G,I}, \mathbb{Q}_\ell) \xrightarrow{\sim} P^{\widehat{G},I},$$

compatible with the tensor structures defined in [Gai07].

2.1.6 We denote by $\mathrm{Rep}_{\mathbb{Q}_\ell}(\widehat{G}^I)$ the category of finite-dimensional $\mathbb{Q}_\ell$-linear representations of $\widehat{G}^I$. We have a fully faithful functor $\mathrm{Rep}_{\mathbb{Q}_\ell}(\widehat{G}^I) \to P^{\widehat{G},I} : W \mapsto W \otimes \mathbb{Q}_{\ell X^I}$. The composition of this functor and the inverse functor $P^{\widehat{G},I} \xrightarrow{\sim} \mathrm{Perv}_{G_{I,\infty}}(\mathrm{Gr}_{G,I}, \mathbb{Q}_\ell)$ in Theorem 2.1.5 gives the following.

COROLLARY 2.1.7. *We have a canonical natural fully faithful $\mathbb{Q}_\ell$-linear fiber functor:*

$$\mathrm{Sat}_{G,I} : \mathrm{Rep}_{\mathbb{Q}_\ell}(\widehat{G}^I) \to \mathrm{Perv}_{G_{I,\infty}}(\mathrm{Gr}_{G,I}, \mathbb{Q}_\ell).$$

DEFINITION 2.1.8. For any $W \in \mathrm{Rep}_{\mathbb{Q}_\ell}(\widehat{G}^I)$, we define $\mathcal{S}_{G,I,W} := \mathrm{Sat}_{G,I}(W)$. We define $\mathrm{Gr}_{G,I,W}$ to be the support of $\mathcal{S}_{G,I,W}$.

2.1.9 When $W = W_1 \oplus W_2$, by the functoriality of $\mathrm{Sat}_{G,I}$, we have $\mathcal{S}_{G,I,W} = \mathcal{S}_{G,I,W_1} \oplus \mathcal{S}_{G,I,W_2}$. Then $\mathrm{Gr}_{G,I,W} = \mathrm{Gr}_{G,I,W_1} \cup \mathrm{Gr}_{G,I,W_2}$.

2.1.10 By [Laf18, Théroème 1.17], the above definition of $\mathrm{Gr}_{G,I,W}$ is equivalent to [Laf18, Définition 1.12 and the definition after (1.14)] (which describes $\mathrm{Gr}_{G,I,W}$ as a generalization of the Zariski closure of the Schubert cell in affine grassmannian). It is well known that $\mathrm{Gr}_{G,I,W}$ is a closed subscheme of $\mathrm{Gr}_{G,I}$ and that it is projective (see [MV07, §§ 2–3], [Zhu17, Proposition 2.1.5]). The ind-scheme $\mathrm{Gr}_{G,I}$ is an inductive limit of $\mathrm{Gr}_{G,I,W}$. ‘

Remark 2.1.11. By [Laf18, Théroème 1.17], when W is irreducible, the perverse sheaf $\mathcal{S}_{G,I,W}$ is (not canonically) isomorphic to the intersection complex (with coefficient in $\mathbb{Q}_\ell$ and the perverse normalization relative to X^I) of $\mathrm{Gr}_{G,I,W}$.

2.2 Satake perverse sheaves on quotient stacks

The stacks $[G_{I,\infty} \backslash \mathrm{Gr}_{G,I}]$ or $[G_{I,\infty} \backslash \mathrm{Gr}_{G,I,W}]$ are not algebraic because the group scheme $G_{I,\infty}$ is of infinite dimension. For technical reasons, we will need algebraic stacks.

PROPOSITION 2.2.1 [Gai01, 1.1.1]. *For $d \in \mathbb{Z}_{\geqslant 0}$ large enough depending on W, the action of* $\mathrm{Ker}(G_{I,\infty} \to G_{I,d})$ *on* $\mathrm{Gr}_{G,I,W}$ *is trivial. Thus the action of $G_{I,\infty}$ on* $\mathrm{Gr}_{G,I,W}$ *factors through* $G_{I,d}$.

2.2.2 For d as in Proposition 2.2.1, we define the quotient stack $[G_{I,d} \backslash \mathrm{Gr}_{G,I,W}]$. Since the group scheme $G_{I,d}$ is of finite dimension, the stack $[G_{I,d} \backslash \mathrm{Gr}_{G,I,W}]$ is algebraic.

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2.2.3 Let $\mathcal{S}_{G,I,W}$ be the $G_{I,\infty}$-equivariant perverse sheaf on $\mathrm{Gr}_{G,I,W}$ defined in Definition 2.1.8. By Proposition 2.2.1, the action of $G_{I,\infty}$ on $\mathcal{S}_{G,I,W}$ factors through $G_{I,d}$. Since the kernel of $G_{I,\infty} \twoheadrightarrow G_{I,d}$ is connected, by [BBDG82, Proposition 4.2.5], we deduce that $\mathcal{S}_{G,I,W}$ is also $G_{I,d}$-equivariant.

Let $\xi_{G,I,d} : \mathrm{Gr}_{G,I,W} \to [G_{I,d}\backslash \mathrm{Gr}_{G,I,W}]$ be the canonical morphism. It is smooth of dimension $\dim G_{I,d}$. By [BBDG82, Corollaire 4.2.6.2] and the discussion after it, there exists a perverse sheaf (up to shift $[\dim G_{I,d}]$) (for the perverse normalization relative to X^I) $\mathcal{S}^d_{G,I,W}$ on $[G_{I,d}\backslash \mathrm{Gr}_{G,I,W}]$ such that $\mathcal{S}_{G,I,W} = \xi^*_{G,I,d}\mathcal{S}^d_{G,I,W}$.

2.2.4 Let $d \leqslant d'$ be two integers large enough as in Proposition 2.2.1. Then the morphisms $G_{I,\infty} \twoheadrightarrow G_{I,d'} \twoheadrightarrow G_{I,d}$ induce a commutative diagram.

$$\begin{tikzcd}
 & \mathrm{Gr}_{G,I,W} \arrow[dl] \arrow[d, "\xi_{G,I,d'}"'] \arrow[dr, "\xi_{G,I,d}"] & \\
{[G_{I,\infty}\backslash \mathrm{Gr}_{G,I,W}]} \arrow[r] & {[G_{I,d'}\backslash \mathrm{Gr}_{G,I,W}]} \arrow[r, "\mathrm{pr}^d_{d'}"] & {[G_{I,d}\backslash \mathrm{Gr}_{G,I,W}]}
\end{tikzcd} \tag{2.1}$$

We have $(\xi_{G,I,d'})^*\mathcal{S}^{d'}_{G,I,W} = \mathcal{S}_{G,I,W} = (\xi_{G,I,d})^*\mathcal{S}^d_{G,I,W} = (\xi_{G,I,d'})^*(\mathrm{pr}^d_{d'})^*\mathcal{S}^d_{G,I,W}$. By [BBDG82, Proposition 4.2.5], the functor $(\xi_{G,I,d'})^*$ (up to shift) is fully faithful. We deduce that $\mathcal{S}^{d'}_{G,I,W} = (\mathrm{pr}^d_{d'})^*\mathcal{S}^d_{G,I,W}$.

2.2.5 By Proposition 2.2.1, the action of $G^{\mathrm{ad}}_{I,\infty}$ on $\mathrm{Gr}_{G,I,W}$ factors through $G^{\mathrm{ad}}_{I,d}$. We define the quotient stack $[G^{\mathrm{ad}}_{I,d}\backslash \mathrm{Gr}_{G,I,W}]$.

As in the discussion after [Laf18, Définition 2.14], since $(Z_G)_{I,\infty}$ acts trivially on $\mathrm{Gr}_{G,I,W}$, the $G_{I,\infty}$-equivariant perverse sheaf $\mathcal{S}_{G,I,W}$ on $\mathrm{Gr}_{G,I,W}$ is also $G^{\mathrm{ad}}_{I,\infty}$-equivariant and $G^{\mathrm{ad}}_{I,d}$-equivariant. Indeed, by 2.1.9 it is enough to prove this for W irreducible. By Remark 2.1.11, in this case $\mathcal{S}_{G,I,W}$ is isomorphic to the intersection complex of $\mathrm{Gr}_{G,I,W}$, hence is $G^{\mathrm{ad}}_{I,\infty}$-equivariant.

Just as in 2.2.3, let $\xi^{\mathrm{ad}}_{G,I,d} : \mathrm{Gr}_{G,I,W} \to [G^{\mathrm{ad}}_{I,d}\backslash \mathrm{Gr}_{G,I,W}]$ be the canonical morphism. There exists a perverse sheaf (up to shift $[\dim G^{\mathrm{ad}}_{I,d}]$) (for the perverse normalization relative to X^I) $\mathcal{S}^{\mathrm{ad},\,d}_{G,I,W}$ on $[G^{\mathrm{ad}}_{I,d}\backslash \mathrm{Gr}_{G,I,W}]$ such that $\mathcal{S}_{G,I,W} = (\xi^{\mathrm{ad}}_{G,I,d})^*\mathcal{S}^{\mathrm{ad},\,d}_{G,I,W}$.

2.3 Representability of stacks of shtukas

Definition 2.3.1. We define $\mathrm{Cht}_{G,N,I,W}$ to be the inverse image of $[G_{I,\infty}\backslash \mathrm{Gr}_{G,I,W}]$ in $\mathrm{Cht}_{G,N,I}$ by $\epsilon_{G,N,I,\infty}$.

2.3.2 $\mathrm{Cht}_{G,N,I}$ is an inductive limit of closed subtacks $\mathrm{Cht}_{G,N,I,W}$.

2.3.3 Let $\mu \in \widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}}$. We define $\mathrm{Cht}^{\leqslant\mu}_{G,N,I,W} := \mathrm{Cht}_{G,N,I,W} \cap \mathrm{Cht}^{\leqslant\mu}_{G,N,I}$, where $\mathrm{Cht}^{\leqslant\mu}_{G,N,I}$ is defined in Definition 1.7.2. We define the quotient $\mathrm{Cht}_{G,N,I,W}/\Xi$ and $\mathrm{Cht}^{\leqslant\mu}_{G,N,I,W}/\Xi$.

Proposition 2.3.4 [Var04, Proposition 2.16]. *The stack* $\mathrm{Cht}_{G,N,I,W}$ *is a Deligne–Mumford stack locally of finite type. The stack* $\mathrm{Cht}^{\leqslant\mu}_{G,N,I,W}/\Xi$ *is a Deligne–Mumford stack of finite type.*

2.3.5 The stack $\mathrm{Cht}_{G,N,I,W}/\Xi = \varinjlim_{\mu\in\widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}}} \mathrm{Cht}^{\leqslant\mu}_{G,N,I,W}/\Xi$ is locally of finite type.

2.4 Satake perverse sheaf on stacks of shtukas

2.4.1 For any $d \in \mathbb{Z}_{\geqslant 0}$ large enough as in Proposition 2.2.1, we define $\epsilon_{G,N,I,d}$ to be the composition of morphisms

$$\epsilon_{G,N,I,d} : \mathrm{Cht}_{G,N,I,W} \xrightarrow{\epsilon_{G,N,I,\infty}} [G_{I,\infty} \backslash \mathrm{Gr}_{G,I,W}] \to [G_{I,d} \backslash \mathrm{Gr}_{G,I,W}]. \tag{2.2}$$

This is morphism (2.3) in [Laf18].

Just as in 1.3.2, we define a morphism

$$\epsilon^{\Xi}_{G,N,I,d} : \mathrm{Cht}_{G,N,I,W} /\Xi \to [G^{\mathrm{ad}}_{I,d} \backslash \mathrm{Gr}_{G,I,W}]. \tag{2.3}$$

This is morphism (2.10) in [Laf18].

2.4.2 We denote by $\dim_{X^I} G_{I,d}$ the relative dimension of $G_{I,d}$ over X^I and by $|I|$ the cardinal of I. We have $\dim_{X^I} G_{I,d} = d \cdot |I| \cdot \dim G$.

PROPOSITION 2.4.3 [Laf18, Proposition 2.8]. *The morphisms $\epsilon_{G,N,I,d}$ (respectively $\epsilon^{\Xi}_{G,N,I,d}$) is smooth of dimension $\dim_{X^I} G_{I,d}$ (respectively $\dim_{X^I} G^{\mathrm{ad}}_{I,d}$).*

2.4.4 For all $d \in \mathbb{Z}_{\geqslant 0}$ large enough as in Proposition 2.2.1, we have morphisms over $(X \smallsetminus N)^I$.

$$\begin{tikzcd}
\mathrm{Cht}_{G,N,I,W} \arrow[dr, "\epsilon_{G,N,I,d}"] & & \mathrm{Gr}_{G,I,W} \arrow[dl, "\xi_{G,I,d}"'] \\
& {[G_{I,d} \backslash \mathrm{Gr}_{G,I,W}]} &
\end{tikzcd}$$

We deduce from Proposition 2.4.3 that $\dim \mathrm{Cht}_{G,N,I,W} = \dim \mathrm{Gr}_{G,I,W}$. We refer to [Laf18, Proposition 2.11] for the fact that $\mathrm{Cht}_{G,N,I,W}$ is locally isomorphic to $\mathrm{Gr}_{G,I,W}$ for the étale topology. We will not use this result in this paper.

DEFINITION 2.4.5. Let $d \in \mathbb{Z}_{\geqslant 0}$ large enough as in Proposition 2.2.1. We define $\mathcal{F}_{G,N,I,W} := (\epsilon_{G,N,I,d})^* \mathcal{S}^d_{G,I,W}$.

Remark 2.4.6. As in 2.2.4, let $d, d' \in \mathbb{Z}_{\geqslant 0}$ both large enough with $d \leqslant d'$. Then we have $\epsilon_{G,N,I,d} = \mathrm{pr}^d_{d'} \circ \epsilon_{G,N,I,d'}$. Thus $(\epsilon_{G,N,I,d})^* \mathcal{S}^d_{G,I,W} = (\epsilon_{G,N,I,d'})^* (\mathrm{pr}^d_{d'})^* \mathcal{S}^d_{G,I,W} = (\epsilon_{G,N,I,d'})^* \mathcal{S}^{d'}_{G,I,W}$. Hence $\mathcal{F}_{G,N,I,W}$ is independent of d.

DEFINITION 2.4.7. We define $\mathcal{F}^{\Xi}_{G,N,I,W} := (\epsilon^{\Xi}_{G,N,I,d})^* \mathcal{S}^{\mathrm{ad},\, d}_{G,I,W}$.

Just as in Remark 2.4.6, $\mathcal{F}^{\Xi}_{G,N,I,W}$ is independent of d.

LEMMA 2.4.8. *The complex $\mathcal{F}_{G,N,I,W}$ (respectively $\mathcal{F}^{\Xi}_{G,N,I,W}$) is a perverse sheaf (for the perverse normalization relative to $(X \smallsetminus N)^I$) on $\mathrm{Cht}_{G,N,I}$ (respectively $\mathrm{Cht}_{G,N,I} /\Xi$) supported on $\mathrm{Cht}_{G,N,I,W}$ (respectively $\mathrm{Cht}_{G,N,I,W} /\Xi$) (in the context of 0.0.13). When W is irreducible, $\mathcal{F}_{G,N,I,W}$ (respectively $\mathcal{F}^{\Xi}_{G,N,I,W}$) is (not canonically) isomorphic to the intersection complex (with coefficient in $\mathbb{Q}_\ell$ and the perverse normalization relative to $(X \smallsetminus N)^I$) of $\mathrm{Cht}_{G,N,I,W}$ (respectively $\mathrm{Cht}_{G,N,I,W} /\Xi$).*

Proof. The lemma follows from Corollary 2.1.7, Remark 2.1.11 and Proposition 2.4.3. □

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2.5 Cohomology of stacks of G-shtukas

Recall that we have the morphism of paws $\mathfrak{p}_G : \mathrm{Cht}_{G,I,N} /\Xi \to (X \smallsetminus N)^I$.

Definition 2.5.1 [Laf18, Definitions 4.1 and 4.7]. For any $\mu \in \widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}}$, we define

$$\mathcal{H}^{\leqslant\mu}_{G,N,I,W} := R(\mathfrak{p}_G)_!(\mathcal{F}^{\Xi}_{G,N,I,W}|_{\mathrm{Cht}^{\leqslant\mu}_{G,N,I,W} /\Xi}) \in D^b_c((X \smallsetminus N)^I, \mathbb{Q}_\ell).$$

For any $j \in \mathbb{Z}$, we define degree j cohomology sheaf (for the ordinary t-structure)

$$\mathcal{H}^{j,\leqslant\mu}_{G,N,I,W} := R^j(\mathfrak{p}_G)_!(\mathcal{F}^{\Xi}_{G,N,I,W}|_{\mathrm{Cht}^{\leqslant\mu}_{G,N,I,W} /\Xi}).$$

This is a $\mathbb{Q}_\ell$-constructible sheaf on $(X \smallsetminus N)^I$.

The complex $\mathcal{H}^{\leqslant\mu}_{G,N,I,W}$ and the sheaf $\mathcal{H}^{j,\leqslant\mu}_{G,N,I,W}$ depend on Ξ. We do not write Ξ in the index to simplify the notations.

2.5.2 Let $\mu_1, \mu_2 \in \widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}}$ and $\mu_1 \leqslant \mu_2$. We have an open immersion:

$$\mathrm{Cht}^{\leqslant\mu_1}_{G,N,I,W} /\Xi \hookrightarrow \mathrm{Cht}^{\leqslant\mu_2}_{G,N,I,W} /\Xi. \tag{2.4}$$

For any j, morphism (2.4) induces a morphism of sheaves:

$$\mathcal{H}^{j,\leqslant\mu_1}_{G,N,I,W} \to \mathcal{H}^{j,\leqslant\mu_2}_{G,N,I,W}.$$

Definition 2.5.3. We define

$$\mathcal{H}^{j}_{G,N,I,W} := \varinjlim_{\mu} \mathcal{H}^{j,\leqslant\mu}_{G,N,I,W}$$

as an inductive limit in the category of constructible sheaves on $(X \smallsetminus N)^I$.

2.5.4 Let $\overline{\eta^I}$ be a geometric point over the generic point η^I of X^I.

Definition 2.5.5. We define

$$H^{j,\leqslant\mu}_{G,N,I,W} := \mathcal{H}^{j,\leqslant\mu}_{G,N,I,W}\Big|_{\overline{\eta^I}}, \quad H^{j}_{G,N,I,W} := \mathcal{H}^{j}_{G,N,I,W}\Big|_{\overline{\eta^I}}. \tag{2.5}$$

By definition $H^{j,\leqslant\mu}_{G,N,I,W}$ is a $\mathbb{Q}_\ell$-vector space of finite dimension. We have $H^{j}_{G,N,I,W} = \varinjlim_{\mu} H^{j,\leqslant\mu}_{G,N,I,W}$.

2.6 Cohomology of stacks of M-shtukas

Let P be a proper parabolic subgroup of G and let M be its Levi quotient.

2.6.1 Let $\widehat{M}$ be the Langlands dual group of M over $\mathbb{Q}_\ell$ defined by the geometric Satake equivalence. The compatibility between the geometric Satake equivalence and the constant term functor along P (that we will recall in Theorem 3.2.6 below) induces a canonical inclusion $\widehat{M} \hookrightarrow \widehat{G}$ (compatible with pinning).

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2.6.2 We view $W \in \mathrm{Rep}_{\mathbb{Q}_\ell}(\widehat{G}^I)$ as a representation of $\widehat{M}^I$ via $\widehat{M}^I \hookrightarrow \widehat{G}^I$. As in §§ 2.1–2.4, we define $\mathrm{Gr}_{M,I,W}$ and $\mathrm{Cht}_{M,N,I,W}$. For $d \in \mathbb{Z}_{\geqslant 0}$ large enough such that the action of $M_{I,\infty}$ on $\mathrm{Gr}_{M,I,W}$ factors through $M_{I,d}$, we define

$$\epsilon_{M,N,I,d} : \mathrm{Cht}_{M,N,I,W} \to [M_{I,d}\backslash \mathrm{Gr}_{M,I,W}],$$
$$\epsilon^{\Xi}_{M,N,I,d} : \mathrm{Cht}_{M,N,I,W}/\Xi \to [\overline{M}_{I,d}\backslash \mathrm{Gr}_{M,I,W}].$$

We define perverse sheaf $\mathcal{S}_{M,I,W}$ on $\mathrm{Gr}_{M,I,W}$, perverse sheaves (up to shift) $\mathcal{S}^d_{M,I,W}$ on $[M_{I,d}\backslash \mathrm{Gr}_{M,I,W}]$ and $\mathcal{S}^{\mathrm{ad},d}_{M,I,W}$ on $[\overline{M}_{I,d}\backslash \mathrm{Gr}_{M,I,W}]$. We define

$$\mathcal{F}_{M,N,I,W} := \epsilon^*_{M,N,I,d}\mathcal{S}^d_{M,I,W} \quad \text{and} \quad \mathcal{F}^{\Xi}_{M,N,I,W} := (\epsilon^{\Xi}_{M,N,I,d})^*\mathcal{S}^{\mathrm{ad},d}_{M,I,W}.$$

2.6.3 Applying [Var04, Proposition 2.16] to M, we deduce that $\mathrm{Cht}_{M,I,N,W}$ is a Deligne–Mumford stack locally of finite type and that for $\lambda \in \widehat{\Lambda}^{+,\mathbb{Q}}_M$, the Deligne–Mumford stack $\mathrm{Cht}^{(\lambda)}_{M,I,N,W}$ (defined in Definition 1.7.2) is of finite type.

Let $\mu \in \widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}}$. We define $\mathrm{Cht}^{\leqslant\mu}_{M,N,I,W} := \mathrm{Cht}_{M,N,I,W} \cap \mathrm{Cht}^{\leqslant\mu}_{M,N,I}$, where $\mathrm{Cht}^{\leqslant\mu}_{M,N,I}$ is defined in Definition 1.7.2. We define the quotient $\mathrm{Cht}_{M,N,I,W}/\Xi$ and $\mathrm{Cht}^{\leqslant\mu}_{M,N,I,W}/\Xi$. As in 1.5.6, Ξ is a lattice in $Z_G(F)\backslash Z_G(\mathbb{A})$ but only a discrete subgroup in $Z_M(F)\backslash Z_M(\mathbb{A})$. The decomposition (1.31) induces a decomposition

$$\mathrm{Cht}^{\leqslant\mu}_{M,N,I,W}/\Xi = \bigsqcup_{\nu \in \widehat{\Lambda}^{\mu}_{Z_M/Z_G}} \mathrm{Cht}^{\leqslant\mu,\,\nu}_{M,N,I,W}/\Xi, \tag{2.6}$$

where each $\mathrm{Cht}^{\leqslant\mu,\,\nu}_{M,I,N,W}/\Xi$ is of finite type (just as in Lemma 1.5.19).

Recall that we have the morphism of paws $\mathfrak{p}_M : \mathrm{Cht}_{M,I,N}/\Xi \to (X \smallsetminus N)^I$.

DEFINITION 2.6.4. For any $\mu \in \widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}}$ and $\nu \in \widehat{\Lambda}^{\mathbb{Q}}_{Z_M/Z_G}$, we define

$$\mathcal{H}^{\leqslant\mu,\,\nu}_{M,N,I,W} := R(\mathfrak{p}_M)_!(\mathcal{F}^{\Xi}_{M,I,N,W}\big|_{\mathrm{Cht}^{\leqslant\mu,\,\nu}_{M,N,I,W}/\Xi}) \in D^b_c((X \smallsetminus N)^I, \mathbb{Q}_\ell).$$

For any $j \in \mathbb{Z}$, we define degree j cohomology sheaf

$$\mathcal{H}^{j,\,\leqslant\mu,\,\nu}_{M,N,I,W} := R^j(\mathfrak{p}_M)_!(\mathcal{F}^{\Xi}_{M,I,N,W}\big|_{\mathrm{Cht}^{\leqslant\mu,\,\nu}_{M,N,I,W}/\Xi}).$$

2.6.5 If $\nu \notin \widehat{\Lambda}^{\mu}_{Z_M/Z_G}$, by Lemma 1.5.14, $\mathrm{Cht}^{\leqslant\mu,\,\nu}_{M,I,N,W}/\Xi = \emptyset$. In this case $\mathcal{H}^{\leqslant\mu,\,\nu}_{M,N,I,W} = 0$.

DEFINITION 2.6.6. Let $\overline{\eta^I}$ be the geometric generic point of X^I fixed in 2.5.4. We define

$$H^{j,\,\leqslant\mu,\,\nu}_{M,N,I,W} := \mathcal{H}^{j,\,\leqslant\mu,\,\nu}_{M,N,I,W}\Big|_{\overline{\eta^I}}. \tag{2.7}$$

This is a finite-dimensional $\mathbb{Q}_\ell$-vector space. We define

$$H^{j,\,\leqslant\mu}_{M,N,I,W} := \prod_{\nu \in \widehat{\Lambda}^{\mu}_{Z_M/Z_G}} H^{j,\,\leqslant\mu,\,\nu}_{M,N,I,W}. \tag{2.8}$$

2.6.7 Let $\mu_1, \mu_2 \in \widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}}$ and $\mu_1 \leqslant \mu_2$. We have an open immersion:

$$\mathrm{Cht}^{\leqslant \mu_1}_{M,N,I,W} / \Xi \hookrightarrow \mathrm{Cht}^{\leqslant \mu_2}_{M,N,I,W} / \Xi. \tag{2.9}$$

For any j, morphism (2.9) induces a morphism of vector spaces:

$$H^{j, \leqslant \mu_1}_{M,N,I,W} \to H^{j, \leqslant \mu_2}_{M,N,I,W}.$$

Definition 2.6.8. We define

$$H^{j}_{M,N,I,W} := \varinjlim_{\mu} H^{j, \leqslant \mu}_{M,N,I,W}$$

as an inductive limit in the category of $\mathbb{Q}_\ell$-vector spaces.

Definition 2.6.9. For any $\nu \in \widehat{\Lambda}^{\mathbb{Q}}_{Z_M/Z_G}$, we define $H^{j,\nu}_{M,N,I,W} := \varinjlim_{\mu} H^{j, \leqslant \mu, \nu}_{M,N,I,W}$ as an inductive limit in the category of $\mathbb{Q}_\ell$-vector spaces.

3. Constant term morphisms and cuspidal cohomology

Let P be a parabolic subgroup of G and M its Levi quotient. Let $W \in \mathrm{Rep}_{\mathbb{Q}_\ell}(\widehat{G}^I)$. The goal of this section is to construct a constant term morphism from $H^j_{G,N,I,W}$ to $H^j_{M,N,I,W}$ (in fact, to a variant $H'^{j}_{M,N,I,W}$ of $H^j_{M,N,I,W}$ defined in § 3.4 below). There are two steps.

First, we will construct a commutative diagram

$$\begin{tikzcd}
 & \mathrm{Cht}_{P,N,I,W}/\Xi \arrow[dl, "i"'] \arrow[dr, "\pi"] \arrow[dd, "\mathfrak{p}_P"] & \\
\mathrm{Cht}_{G,N,I,W}/\Xi \arrow[dr, "\mathfrak{p}_G"'] & & \mathrm{Cht}_{M,N,I,W}/\Xi \arrow[dl, "\mathfrak{p}_M"] \\
 & (X \smallsetminus N)^I &
\end{tikzcd} \tag{3.1}$$

where the morphism π is of finite type. Therefore the complex $\pi_! i^* \mathcal{F}^{\Xi}_{G,N,I,W}$ on $\mathrm{Cht}_{M,N,I,W}/\Xi$ is well defined in $D^b_c(\mathrm{Cht}_{M,N,I,W}/\Xi, \mathbb{Q}_\ell)$ (in the context of 0.0.13). We will construct a canonical morphism of complexes on $\mathrm{Cht}_{M,N,I,W}/\Xi$:

$$\pi_! i^* \mathcal{F}^{\Xi}_{G,N,I,W} \to \mathcal{F}^{\Xi}_{M,N,I,W}. \tag{3.2}$$

Second, the cohomological correspondence given by (3.1) and (3.2) will give a morphism from $H^j_{G,N,I,W}$ to $H^j_{M,N,I,W}$.

3.1 Some geometry of the parabolic induction diagram

Recall that we have morphisms over X^I in (1.5): $\mathrm{Gr}_{G,I} \xleftarrow{i^0} \mathrm{Gr}_{P,I} \xrightarrow{\pi^0} \mathrm{Gr}_{M,I}$.

Proposition 3.1.1. *We have* $(i^0)^{-1}(\mathrm{Gr}_{G,I,W}) \subset (\pi^0)^{-1}(\mathrm{Gr}_{M,I,W})$, *where the inverse images are in the sense of reduced subschemes in* $\mathrm{Gr}_{P,I}$.

Proof. It is enough to prove the inclusion for each fiber over X^I. By 1.1.12, we reduce the case of the Beilinson–Drinfeld affine grassmannian with paws indexed by I to the case of the usual affine grassmannian $\mathrm{Gr}_G = G_{\mathcal{K}}/G_{\mathcal{O}}$.

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When $P = B$, the statement follows from [MV07, Theorem 3.2]. More concretely, for ω a dominant coweight of G, we denote by $\mathrm{Gr}_{G,\omega}$ the Zariski closure of the Schubert cell defined by ω in Gr_G. For ν a coweight of T, we denote by $\mathrm{Gr}_{T,\nu}$ the component of Gr_T (which is discrete) associated to ν. We denote by C_ω the set of coweights of G which are W-conjugated to a dominant coweight $\leqslant \omega$ (where the order is taken in the coweight lattice of G). By [MV07, Theorem 3.2] the subscheme $(i^0)^{-1}(\mathrm{Gr}_{G,\omega}) \cap (\pi^0)^{-1}(\mathrm{Gr}_{T,\nu})$ in Gr_B is non-empty if and only if $\nu \in C_\omega$. Hence

$$\pi^0((i^0)^{-1}(\mathrm{Gr}_{G,\omega})) = \bigsqcup_{\nu \in C_\omega} \mathrm{Gr}_{T,\nu}. \tag{3.3}$$

For general P with Levi quotient M, we denote by B' the Borel subgroup of M. We use the following diagram, where the square is Cartesian.

$$\begin{tikzcd}
& & \mathrm{Gr}_B \arrow[dl, "i_P^B"'] \arrow[dr, "\pi^B_{B'}"] & & \\
& \mathrm{Gr}_P \arrow[dl, "i_G^P"'] \arrow[dr, "\pi^P_M"] & & \mathrm{Gr}_{B'} \arrow[dl, "i^{B'}_M"'] \arrow[dr, "\pi^{B'}_T"] & \\
\mathrm{Gr}_G & & \mathrm{Gr}_M & & \mathrm{Gr}_T
\end{tikzcd}$$

Since the square is Cartesian, we have

$$(\pi_T^{B'})(i_M^{B'})^{-1}(\pi_M^P)(i_G^P)^{-1}\mathrm{Gr}_{G,\omega} = (\pi_T^{B'} \circ \pi_{B'}^B)(i_G^P \circ i_P^B)^{-1}\mathrm{Gr}_{G,\omega} \overset{(3.3)}{=} \bigsqcup_{\nu \in C_\omega} \mathrm{Gr}_{T,\nu}. \tag{3.4}$$

For any dominant coweight λ of M, we denote by $\mathrm{Gr}_{M,\lambda}$ the Zariski closure of the Schubert cell defined by λ in Gr_M. Applying [MV07, Theorem 3.2] to $\mathrm{Gr}_M \leftarrow \mathrm{Gr}_{B'} \rightarrow \mathrm{Gr}_T$, we have

$$(\pi_T^{B'})(i_M^{B'})^{-1}\mathrm{Gr}_{M,\lambda} = \bigsqcup_{\nu \in C_\lambda} \mathrm{Gr}_{T,\nu}. \tag{3.5}$$

The subscheme $(i_G^P)^{-1}\mathrm{Gr}_{G,\omega}$ in Gr_P is stable under the action of $P_{\mathcal{O}}$. The subscheme $(\pi_M^P)(i_G^P)^{-1}\mathrm{Gr}_{G,\omega}$ in Gr_M is stable under the action of $M_{\mathcal{O}}$, so is a union of strata in Gr_M. We deduce from (3.4) and (3.5) that $\mathrm{Gr}_{M,\lambda}$ can be in $(\pi_M^P)(i_G^P)^{-1}\mathrm{Gr}_{G,\omega}$ only if $\lambda \in C_\omega$. Thus

$$(\pi_M^P)(i_G^P)^{-1}\mathrm{Gr}_{G,\omega} \subset \bigcup_{\lambda \in C_\omega \cap \widehat{\Lambda}_M^+} \mathrm{Gr}_{M,\lambda}. \qquad \square$$

3.1.2 We define $\mathrm{Gr}_{P,I,W} := (i^0)^{-1}(\mathrm{Gr}_{G,I,W})$. As a consequence of Proposition 3.1.1, morphisms (1.5) induce morphisms over X^I:

$$\mathrm{Gr}_{G,I,W} \xleftarrow{i^0} \mathrm{Gr}_{P,I,W} \xrightarrow{\pi^0} \mathrm{Gr}_{M,I,W}. \tag{3.6}$$

3.1.3 We deduce from the commutative diagram (1.6) that

$$i^{-1}(\mathrm{Cht}_{G,N,I,W}) \subset \pi^{-1}(\mathrm{Cht}_{M,N,I,W}),$$

where the inverse images are in the sense of reduced substacks in $\mathrm{Cht}_{P,N,I}$. We define $\mathrm{Cht}_{P,N,I,W} := i^{-1}(\mathrm{Cht}_{G,N,I,W})$. Morphisms in (1.4) induce morphisms over $(X \smallsetminus N)^I$:

$$\mathrm{Cht}_{G,N,I,W} \xleftarrow{i} \mathrm{Cht}_{P,N,I,W} \xrightarrow{\pi} \mathrm{Cht}_{P,N,I,W}. \tag{3.7}$$

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3.1.4 Let $d \in \mathbb{Z}_{\geqslant 0}$ large enough depending on W as in Proposition 2.2.1 applied to $\mathrm{Gr}_{G,I,W}$ and to $\mathrm{Gr}_{M,I,W}$. To simplify the notations, we write $\epsilon_{G,d}$ for $\epsilon_{G,N,I,d}$ defined in 2.4.1 and $\epsilon_{M,d}$ for $\epsilon_{M,N,I,d}$ defined in 2.6.2. Similarly we define $\epsilon_{P,d}$ to be the composition

$$\mathrm{Cht}_{P,N,I,W} \to [P_{I,\infty}\backslash \mathrm{Gr}_{P,I,W}] \to [P_{I,d}\backslash \mathrm{Gr}_{P,I,W}].$$

We deduce from the commutative diagram (1.6), morphisms (3.6) and (3.7) a commutative diagram of algebraic stacks.

$$\begin{tikzcd}
\mathrm{Cht}_{G,N,I,W} \arrow[d, "\epsilon_{G,d}"] & \mathrm{Cht}_{P,N,I,W} \arrow[l, "i"'] \arrow[r, "\pi"] \arrow[d, "\epsilon_{P,d}"] & \mathrm{Cht}_{M,N,I,W} \arrow[d, "\epsilon_{M,d}"] \\
{[G_{I,d}\backslash \mathrm{Gr}_{G,I,W}]} & {[P_{I,d}\backslash \mathrm{Gr}_{P,I,W}]} \arrow[l, "\overline{i_d^0}"'] \arrow[r, "\overline{\pi_d^0}"] & {[M_{I,d}\backslash \mathrm{Gr}_{M,I,W}]}
\end{tikzcd} \tag{3.8}$$

3.1.5 The right square in (3.8) is not Cartesian. We have a commutative diagram, where the square is Cartesian.

$$\begin{tikzcd}
\mathrm{Cht}_{P,N,I,W} \arrow[dr, "\pi_d"] \arrow[drr, "\pi"] \arrow[ddr, "\epsilon_{P,d}"'] & & \\
& \widetilde{\mathrm{Cht}}_{M,N,I,W} \arrow[r, "\widetilde{\pi_d^0}"'] \arrow[d, "\widetilde{\epsilon_{M,d}}"] & \mathrm{Cht}_{M,N,I,W} \arrow[d, "\epsilon_{M,d}"] \\
& {[P_{I,d}\backslash \mathrm{Gr}_{P,I,W}]} \arrow[r, "\pi_d^0"'] & {[M_{I,d}\backslash \mathrm{Gr}_{M,I,W}]}
\end{tikzcd} \tag{3.9}$$

Remark 3.1.6. Note that $\widetilde{\mathrm{Cht}}_{M,N,I,W}$ depends on the choice of d. We do not write d in index to shorten the notation.

DEFINITION 3.1.7. Let U be the unipotent radical of P. We have $P/U = M$. Applying Definition 1.1.13 to U, we define the group scheme $U_{I,d}$ over X^I.

LEMMA 3.1.8. *The morphism π_d is smooth of relative dimension* $\dim_{X^I} U_{I,d}$.

The following proof was suggested to the author by a referee.

Proof. Proposition 2.4.3 works also for P and M. Hence the morphism $\epsilon_{P,d}$ is smooth of relative dimension $\dim_{X^I} P_{I,d}$ and the morphism $\epsilon_{M,d}$ (hence $\widetilde{\epsilon_{M,d}}$) is smooth of relative dimension $\dim_{X^I} M_{I,d}$. Thus to prove that π_d is smooth, it is enough to show that it induces a surjective map between relative tangent spaces.

For any closed point $x_P = ((x_i), \mathcal{P} \to {}^{\tau}\mathcal{P})$ of $\mathrm{Cht}_{P,N,I,W}$, let $x_M := \pi_d(x_P)$. We have the canonical morphism

$$T_{\epsilon_{P,d}}(x_P) \to T_{\widetilde{\epsilon_{M,d}}}(x_M), \tag{3.10}$$

where $T_{\epsilon_{P,d}}(x_P)$ (respectively $T_{\widetilde{\epsilon_{M,d}}}(x_M)$) is the tangent space of $\mathrm{Cht}_{P,N,I,W}$ (respectively $\widetilde{\mathrm{Cht}}_{M,N,I,W}$) at x_P (respectively x_M) relative to $[P_{I,d}\backslash \mathrm{Gr}_{P,I,W}]$.

Let $y = \epsilon_{P,d}(x_P)$. By the proof of [Laf18, Proposition 2.8], we have a Cartesian square

$$\begin{tikzcd} \epsilon_{P,d}^{-1}(y) \arrow[r] \arrow[d] & \mathrm{Bun}_{P,N+d\sum x_i} \arrow[d, "(b_1^P,\, b_2^P)"] \\ \mathrm{Bun}_{P,N} \arrow[r, "(\mathrm{Id},\mathrm{Id})"] & \mathrm{Bun}_{P,N} \times \mathrm{Bun}_{P,N} \end{tikzcd} \tag{3.11}$$

where b_1^P is a smooth morphism (which is the forgetful morphism of the level structure on $d\sum x_i$) and b_2^P has zero differential (because it is the composition of the Frobenius morphism with some other morphism). We have $T_{\epsilon_{P,d}}(x_P) = T_{b_1^p}(x_P)$ (see for example [Laf97, I. 2. Proposition 1]). It is well known that $\mathrm{Bun}_{P,N+d\sum x_i} \xrightarrow{b_1^P} \mathrm{Bun}_{P,N}$ is a $P_{d\sum x_i}$-torsor, where $P_{d\sum x_i}$ is defined in 0.0.8. We deduce that $T_{b_1^p}(x_P) = \mathrm{Lie}(P_{d\sum x_i})$.

Similarly, we have a Cartesian square (taking into account that $\widetilde{\epsilon_{M,d}}^{-1}(y) = \epsilon_{M,d}^{-1}(\overline{\pi_d^0}(y))$)

$$\begin{tikzcd} \widetilde{\epsilon_{M,d}}^{-1}(y) \arrow[r] \arrow[d] & \mathrm{Bun}_{M,N+d\sum x_i} \arrow[d, "(b_1^M,\, b_2^M)"] \\ \mathrm{Bun}_{M,N} \arrow[r, "(\mathrm{Id},\mathrm{Id})"] & \mathrm{Bun}_{M,N} \times \mathrm{Bun}_{M,N} \end{tikzcd} \tag{3.12}$$

where b_1^M is a smooth morphism (which is the forgetful morphism of the level structure on $d\sum x_i$) and b_2^M has zero differential. We deduce that $T_{\widetilde{\epsilon_{M,d}}}(x_M) = T_{b_1^M}(x_M) = \mathrm{Lie}(M_{d\sum x_i})$, where $M_{d\sum x_i}$ is defined in 0.0.8.

Morphism (3.10) is the canonical morphism $\mathrm{Lie}(P_{d\sum x_i}) \to \mathrm{Lie}(M_{d\sum x_i})$ induced by $P \twoheadrightarrow M$. Hence it is surjective. We deduce also that the relative tangent space of π_d is $\mathrm{Lie}(U_{d\sum x_i})$. □

3.2 Compatibility of the geometric Satake equivalence and parabolic induction

The goal of this section is to recall (3.17) and deduce (3.20), which is the key ingredient for the next section.

3.2.1 We apply Definition 1.1.11 to $\mathbb{G}_m$ and denote by $\mathrm{Gr}_{\mathbb{G}_m,I}$ the associated reduced ind-scheme. We denote by ρ_G (respectively ρ_M) the half sum of positive roots of G (respectively M). Since $2(\rho_G - \rho_M)$ is a character of M, the morphism $2(\rho_G - \rho_M) : M \to \mathbb{G}_m$ induces a morphism $\mathrm{Gr}_{M,I} \to \mathrm{Gr}_{\mathbb{G}_m,I}$ by sending a M-bundle $\mathcal{M}$ to the $\mathbb{G}_m$-bundle $\mathcal{M} \overset{M}{\times} \mathbb{G}_m$. We have a morphism $\deg : \mathrm{Gr}_{\mathbb{G}_m,I} \to \mathbb{Z}$ by taking the degree of a $\mathbb{G}_m$-bundle. We have the composition of morphisms

$$\mathrm{Gr}_{M,I} \to \mathrm{Gr}_{\mathbb{G}_m,I} \xrightarrow{\deg} \mathbb{Z}. \tag{3.13}$$

We define $\mathrm{Gr}_{M,I}^n$ as the inverse image of $n \in \mathbb{Z}$. It is open and closed in $\mathrm{Gr}_{M,I}$. We define $\mathrm{Gr}_{P,I}^n := (\pi^0)^{-1}\mathrm{Gr}_{M,I}^n$. Morphism (1.5) induces a morphism

$$\mathrm{Gr}_{G,I} \xleftarrow{i_n^0} \mathrm{Gr}_{P,I}^n \xrightarrow{\pi_n^0} \mathrm{Gr}_{M,I}^n. \tag{3.14}$$

3.2.2 Recall that we have defined $\widehat{\Lambda}_{G,P}$ in 1.4.2. As in [Sch15, 2.1.2], we define $\Lambda_{G,P} := \{\lambda \in \Lambda_G | \langle \check{\alpha}, \lambda \rangle = 0 \text{ for all } \check{\alpha} \in \widehat{\Gamma}_M\}$. The pairing $\langle\ ,\ \rangle$ in 1.4.1 induces a pairing $\langle\ ,\ \rangle : \widehat{\Lambda}_{G,P} \times \Lambda_{G,P} \to \mathbb{Z}$.

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3.2.3 We denote by $\mathrm{Rep}_{\mathbb{Q}_\ell}(\widehat{M}^I)$ the category of finite-dimensional $\mathbb{Q}_\ell$-linear representations of $\widehat{M}^I$. Let $W \in \mathrm{Rep}_{\mathbb{Q}_\ell}(\widehat{M}^I)$. Then $Z_{\widehat{M}}$ acts on W via $Z_{\widehat{M}} \hookrightarrow \widehat{M}^I$ diagonally. We have the decomposition as $Z_{\widehat{M}}$ representation: $W = \bigoplus_{\theta \in \Lambda_{Z_{\widehat{M}}}} W^\theta$.

Since $\theta \in \Lambda_{Z_{\widehat{M}}} = \widehat{\Lambda}_{G,P}$ and $2(\rho_G - \rho_M) \in \Lambda_{G,P}$, we can consider $\langle \theta, 2(\rho_G - \rho_M)\rangle$. Let $W_n = \bigoplus_{\langle \theta, 2(\rho_G-\rho_M)\rangle = n} W^\theta$. We have $W = \bigoplus_{n \in \mathbb{Z}} W_n$.

Let $\mathrm{Rep}_{\mathbb{Q}_\ell}(\widehat{M}^I)^\theta$ be the category of finite-dimensional $\mathbb{Q}_\ell$-linear representations of $\widehat{M}^I$ such that $Z_{\widehat{M}}$ acts by θ. We have $\mathrm{Rep}_{\mathbb{Q}_\ell}(\widehat{M}^I) = \bigoplus_{\theta \in \Lambda_{Z_{\widehat{M}}}} \mathrm{Rep}_{\mathbb{Q}_\ell}(\widehat{M}^I)^\theta$. Let

$$\mathrm{Rep}_{\mathbb{Q}_\ell}(\widehat{M}^I)_n = \bigoplus_{\theta \in \Lambda_{Z_{\widehat{M}}},\, \langle \theta, 2(\rho_G-\rho_M)\rangle = n} \mathrm{Rep}_{\mathbb{Q}_\ell}(\widehat{M}^I)^\theta.$$

We have

$$\mathrm{Rep}_{\mathbb{Q}_\ell}(\widehat{M}^I) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Rep}_{\mathbb{Q}_\ell}(\widehat{M}^I)_n.$$

We define $(\mathrm{Res}^{\widehat{G}^I}_{\widehat{M}^I})_n$ to be the composition of morphisms $\mathrm{Rep}_{\mathbb{Q}_\ell}(\widehat{G}^I) \xrightarrow{\mathrm{Res}^{\widehat{G}^I}_{\widehat{M}^I}} \mathrm{Rep}_{\mathbb{Q}_\ell}(\widehat{M}^I) \twoheadrightarrow \mathrm{Rep}_{\mathbb{Q}_\ell}(\widehat{M}^I)_n$.

3.2.4 In morphism (3.13), $\mathrm{Gr}_{M,I,W^\theta}$ is sent to $\langle \theta, 2(\rho_G - \rho_M)\rangle$. We deduce that $\mathrm{Gr}^n_{M,I} \cap \mathrm{Gr}_{M,I,W} = \mathrm{Gr}_{M,I,W_n}$.

3.2.5 In Corollary 2.1.7, we defined a fully faithful functor

$$\mathrm{Sat}_{G,I} : \mathrm{Rep}_{\mathbb{Q}_\ell}(\widehat{G}^I) \to \mathrm{Perv}_{G_{I,\infty}}(\mathrm{Gr}_{G,I}, \mathbb{Q}_\ell)$$

which sends W to $\mathcal{S}_{G,I,W}$. We denote by $\mathrm{Perv}_{G_{I,\infty}}(\mathrm{Gr}_{G,I}, \mathbb{Q}_\ell)^{\mathrm{MV}}$ the subcategory of essential image of this functor. Similarly, we define

$$\mathrm{Sat}_{M,I} : \mathrm{Rep}_{\mathbb{Q}_\ell}(\widehat{M}^I) \to \mathrm{Perv}_{M_{I,\infty}}(\mathrm{Gr}_{M,I}, \mathbb{Q}_\ell)^{\mathrm{MV}}.$$

Let $\mathrm{Sat}_{M,I,n}$ be the restriction of $\mathrm{Sat}_{M,I}$ to $\mathrm{Rep}_{\mathbb{Q}_\ell}(\widehat{M}^I)_n$.

Theorem 3.2.6 ([BD99, 5.3.29], [BG02, Theorem 4.3.4], [MV07, Theorem 3.6] (for $M = T$), [BR18, Proposition 15.2]).

(a) *For any $n \in \mathbb{Z}$, the complex*

$$(\pi_n^0)_!(i_n^0)^* \mathcal{S}_{G,I,W} \otimes \big(\mathbb{Q}_\ell[1](\tfrac{1}{2})\big)^{\otimes n}$$

is in $\mathrm{Perv}_{M_{I,\infty}}(\mathrm{Gr}^n_{M,I}, \mathbb{Q}_\ell)^{\mathrm{MV}}$.

(b) *We denote by $((\pi_n^0)_!(i_n^0)^*)^\sim$ the shifted functor $(\pi_n^0)_!(i_n^0)^* \otimes (\mathbb{Q}_\ell[1](\frac{1}{2}))^{\otimes n}$. Then there is a canonical isomorphism of fiber functors*

$$\mathrm{Sat}_{M,I,n} \circ (\mathrm{Res}^{\widehat{G}^I}_{\widehat{M}^I})_n = ((\pi_n^0)_!(i_n^0)^*)^\sim \circ \mathrm{Sat}_{G,I}. \tag{3.15}$$

In other words, the following diagram of categories canonically commutes.

$$\begin{tikzcd} \mathrm{Perv}_{G_{I,\infty}}(\mathrm{Gr}_{G,I}, \mathbb{Q}_\ell)^{\mathrm{MV}} \arrow[r, "((\pi_n^0)_!(i_n^0)^*)^\sim"] & \mathrm{Perv}_{M_{I,\infty}}(\mathrm{Gr}^n_{M,I}, \mathbb{Q}_\ell)^{\mathrm{MV}} \\ \mathrm{Rep}_{\mathbb{Q}_\ell}(\widehat{G}^I) \arrow[u, "\mathrm{Sat}_{G,I}"] \arrow[r, "(\mathrm{Res}^{\widehat{G}^I}_{\widehat{M}^I})_n"] & \mathrm{Rep}_{\mathbb{Q}_\ell}(\widehat{M}^I)_n \arrow[u, "\mathrm{Sat}_{M,I,n}"'] \end{tikzcd} \tag{3.16}$$

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Remark 3.2.7. The references cited above in Theorem 3.2.6 are for the case of affine grassmannians (i.e. I is a singleton). The general case (i.e. I is arbitrary) can be deduced from the case of affine grassmannians using the fact that the constant term functor commutes with fusion (i.e. convolution). The proof for $I = \{1,2\}$ is already included in the proof of [BR18, Proposition 15.2]. For general I the proof is similar.

Corollary 3.2.8. *There is a canonical isomorphism*

$$\mathcal{S}_{M,I,W_n} \simeq (\pi_n^0)_!(i_n^0)^*\mathcal{S}_{G,I,W}[n](n/2). \tag{3.17}$$

Proof. Applying (3.15) to W and taking into account that $\mathcal{S}_{M,I,W_n} = \mathrm{Sat}_{M,I,n}(W_n)$ and $\mathcal{S}_{G,I,W} = \mathrm{Sat}_{G,I}(W)$, we deduce (3.17). □

3.2.9 For any n, denote by $\mathrm{Gr}^n_{P,I,W} = \mathrm{Gr}^n_{P,I} \cap \mathrm{Gr}_{P,I,W}$. We have a commutative diagram, where the first line is induced by (3.6).

$$\begin{tikzcd}
\mathrm{Gr}_{G,I,W} \arrow[d, "\xi_{G,d}"] & \mathrm{Gr}^n_{P,I,W} \arrow[l, "i_n^0"'] \arrow[r, "\pi_n^0"] \arrow[d, "\xi_{P,d}"] & \mathrm{Gr}_{M,I,W_n} \arrow[d, "\xi_{M,d}"] \\
{[G_{I,d}\backslash \mathrm{Gr}_{G,I,W}]} & {[P_{I,d}\backslash \mathrm{Gr}^n_{P,I,W}]} \arrow[l, "\overline{i^0_{d,n}}"'] \arrow[r, "\overline{\pi^0_{d,n}}"] & {[M_{I,d}\backslash \mathrm{Gr}_{M,I,W_n}]}
\end{tikzcd}$$

The morphism

$$\mathrm{Gr}^n_{P,I,W} \to [P_{I,d}\backslash \mathrm{Gr}^n_{P,I,W}] \underset{[M_{I,d}\backslash \mathrm{Gr}_{M,I,W_n}]}{\times} \mathrm{Gr}_{M,I,W_n} = [U_{I,d}\backslash \mathrm{Gr}^n_{P,I,W}]$$

is a $U_{I,d}$-torsor. Since the group scheme $U_{I,d}$ is unipotent over X^I, we deduce that

$$(\pi_n^0)_!(\xi_{P,d})^* \simeq (\xi_{M,d})^*(\overline{\pi^0_{d,n}})_![-2m](-m), \tag{3.18}$$

where $m = \dim \xi_{P,d} - \dim \xi_{M,d} = \dim_{X^I} U_{I,d}$.

Corollary 3.2.8 implies

$$\mathcal{S}^d_{M,I,W_n} \xrightarrow{\sim} (\overline{\pi^0_{d,n}})_!(\overline{i^0_{d,n}})^*\mathcal{S}^d_{G,I,W}[n-2m](n/2-m). \tag{3.19}$$

3.2.10 Let $(\omega_i)_{i\in I} \in (\widehat{\Lambda}^+_M)^I$. Let V^{ω_i} be the irreducible representation of $\widehat{M}$ of highest weight ω_i. Note that $\widehat{\Lambda}_{G,P} = \widehat{\Lambda}_{M,M}$ (defined in 1.4.2). By definition, it coincides with $\pi_1(M)$ defined in [Var04, Lemma 2.2]. We denote by $[\sum_{i\in I}\omega_i]$ the image of $\sum_{i\in I}\omega_i$ by the projection $\widehat{\Lambda}_M \twoheadrightarrow \widehat{\Lambda}_{M,M}$.

Lemma 3.2.11 [Var04, Proposition 2.16(d)]. *The stack* $\mathrm{Cht}_{M,N,I,\boxtimes_{i\in I}V^{\omega_i}}$ *is non-empty if and only if* $[\sum_{i\in I}\omega_i]$ *is zero.*

3.2.12 Let W and W^θ as in 3.2.3. Then W has a unique decomposition of the form

$$W = \bigoplus_{(\omega_i)_{i\in I}\in(\widehat{\Lambda}_M^+)^I} (\boxtimes_{i\in I} V^{\omega_i}) \otimes_{\mathbb{Q}_\ell} \mathfrak{M}_{(\omega_i)_{i\in I}},$$

where $\mathfrak{M}_{(\omega_i)_{i\in I}}$ are finite-dimensional $\mathbb{Q}_\ell$-vector spaces, all but a finite number of them are zero. We have

$$W^\theta = \bigoplus_{(\omega_i)_{i\in I}\in(\widehat{\Lambda}_M^+)^I,\, [\sum_{i\in I}\omega_i]=\theta} (\boxtimes_{i\in I} V^{\omega_i}) \otimes_{\mathbb{Q}_\ell} \mathfrak{M}_{(\omega_i)_{i\in I}}.$$

Lemma 3.2.11 implies that $\mathrm{Cht}_{M,N,I,W^\theta}$ is non-empty if and only if θ is zero. For such θ, we have $\langle\theta, 2(\rho_G-\rho_M)\rangle = 0$. We deduce that $\mathrm{Cht}_{M,N,I,W} = \bigcup_{n\in\mathbb{Z}} \mathrm{Cht}_{M,N,I,W_n} = \mathrm{Cht}_{M,N,I,W_0}$. So the image of

$$\epsilon_{M,d} : \mathrm{Cht}_{M,N,I,W} \to [M_{I,d}\backslash \mathrm{Gr}_{M,I,W}]$$

is in $[M_{I,d}\backslash \mathrm{Gr}_{M,I,W_0}]$.

3.2.13 With the notations of diagram (3.8), we have

$$\begin{aligned}(\epsilon_{M,d})^*\mathcal{S}^d_{M,I,W} &= (\epsilon_{M,d})^*\mathcal{S}^d_{M,I,W_0}\\ &\xrightarrow{\sim} (\epsilon_{M,d})^*(\overline{\pi^0_{d,0}})_!(\overline{i^0_{d,0}})^*\mathcal{S}^d_{G,I,W}[-2m](-m)\\ &= (\epsilon_{M,d})^*(\overline{\pi^0_d})_!(\overline{i^0_d})^*\mathcal{S}^d_{G,I,W}[-2m](-m).\end{aligned} \tag{3.20}$$

The first and third equality follows from 3.2.12. The second isomorphism follows from (3.19) applied to $n = 0$.

3.3 Construction of the morphism (3.2)

3.3.1 Consider diagrams (3.8) and (3.9). Let $m = \dim_{X^I} U_{I,d}$ as in 3.2.9. By Lemma 3.1.8, $m = \dim \pi_d$. We construct a canonical map of functors from $D^b_c([P_{I,d}\backslash\mathrm{Gr}_{P,I,W}],\mathbb{Q}_\ell)$ to $D^b_c(\mathrm{Cht}_{M,N,I,W},\mathbb{Q}_\ell)$,

$$\pi_!(\epsilon_{P,d})^* \to (\epsilon_{M,d})^*(\overline{\pi^0_d})_![-2m](-m), \tag{3.21}$$

as the composition

$$\begin{aligned}\pi_!(\epsilon_{P,d})^* &\simeq (\widetilde{\pi^0_d})_!(\pi_d)_!(\pi_d)^*(\widetilde{\epsilon_{M,d}})^*\\ &\to (\widetilde{\pi^0_d})_!(\widetilde{\epsilon_{M,d}})^*[-2m](-m) \xleftarrow{\sim} (\epsilon_{M,d})^*(\overline{\pi^0_d})_![-2m](-m).\end{aligned} \tag{3.22}$$

The second morphism in (3.22) is induced by the isomorphism $(\pi_d)^*[2m](m) \simeq (\pi_d)^!$ (because π_d is smooth) and the counit map $\mathrm{Co} : (\pi_d)_!(\pi_d)^! \to \mathrm{Id}$. (The composition $(\pi_d)_!(\pi_d)^*[2m](m) \xrightarrow{\sim} (\pi_d)_!(\pi_d)^! \to \mathrm{Id}$ is the trace map in [SGA4, XVIII 2].)

The third morphism is the proper base change ([SGA4, XVII 5], [LO08, § 12]).

3.3.2 Now we construct a morphism of complexes in $D^b_c(\mathrm{Cht}_{M,N,I,W},\mathbb{Q}_\ell)$:

$$\begin{aligned}\pi_! i^*\mathcal{F}_{G,N,I,W} &= \pi_! i^*(\epsilon_{G,d})^*\mathcal{S}^d_{G,I,W}\\ &\overset{(a)}{\simeq} \pi_!(\epsilon_{P,d})^*(\overline{i^0_d})^*\mathcal{S}^d_{G,I,W}\\ &\overset{(b)}{\to} (\epsilon_{M,d})^*(\overline{\pi^0_d})_!(\overline{i^0_d})^*\mathcal{S}^d_{G,I,W}[-2m](-m)\\ &\overset{(c)}{\xleftarrow{\sim}} (\epsilon_{M,d})^*\mathcal{S}^d_{M,I,W} = \mathcal{F}_{M,N,I,W},\end{aligned} \tag{3.23}$$

where (a) is induced by the commutativity of diagram (3.8), (b) is induced by morphism (3.21), and (c) is (3.20).

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3.3.3 All the constructions in 3.1–3.3 are compatible with the quotient by Ξ. In particular, just as in 3.1.4, diagram (1.10) induces a commutative diagram.

$$\begin{tikzcd} \mathrm{Cht}_{G,N,I,W}/\Xi \arrow[d, "\epsilon^{\Xi}_{G,N,I,d}"] & \mathrm{Cht}_{P,N,I,W}/\Xi \arrow[l, "i"'] \arrow[r, "\pi"] \arrow[d, "\epsilon^{\Xi}_{P,N,I,d}"] & \mathrm{Cht}_{M,N,I,W}/\Xi \arrow[d, "\epsilon^{\Xi}_{M,N,I,d}"] \\ {[G^{\mathrm{ad}}_{I,d}\backslash \mathrm{Gr}_{G,I,W}]} & {[\overline{P}_{I,d}\backslash \mathrm{Gr}_{P,I,W}]} \arrow[l, "\overline{i^0_d}"'] \arrow[r, "\overline{\pi^0_d}"] & {[\overline{M}_{I,d}\backslash \mathrm{Gr}_{M,I,W}]} \end{tikzcd} \tag{3.24}$$

CONSTRUCTION 3.3.4. Just as in 3.3.2 (using (3.24) instead of (3.8)), we construct a canonical morphism of complexes in $D^b_c(\mathrm{Cht}_{M,N,I,W}/\Xi, \mathbb{Q}_\ell)$:

$$\pi_! i^* \mathcal{F}^{\Xi}_{G,N,I,W} \to \mathcal{F}^{\Xi}_{M,N,I,W}. \tag{3.25}$$

3.4 More on cohomology groups

When the level structure N is non-empty, to construct the constant term morphism of cohomology groups, we need a variant of $H^j_{M,N,I,W}$.

3.4.1 Let $\mathcal{O}_N$ be the ring of functions on N as in 0.0.8. The finite group $G(\mathcal{O}_N)$ (respectively $P(\mathcal{O}_N)$ and $M(\mathcal{O}_N)$) acts on $\mathrm{Cht}_{G,N,I,W}$ (respectively $\mathrm{Cht}_{P,N,I,W}$ and $\mathrm{Cht}_{M,N,I,W}$) by changing the level structure on N: $g \in G(\mathcal{O}_N)$ sends a level structure ψ_G to $g^{-1} \circ \psi_G$.

By [Var04, Proposition 2.16(b)], $\mathrm{Cht}_{G,N,I,W}$ (respectively $\mathrm{Cht}_{P,N,I,W}$ and $\mathrm{Cht}_{M,N,I,W}$) is a finite étale Galois cover of $\mathrm{Cht}_{G,I,W}\big|_{(X\smallsetminus N)^I}$ (respectively $\mathrm{Cht}_{P,I,W}\big|_{(X\smallsetminus N)^I}$ and $\mathrm{Cht}_{M,I,W}\big|_{(X\smallsetminus N)^I}$) with Galois group $G(\mathcal{O}_N)$ (respectively $P(\mathcal{O}_N)$ and $M(\mathcal{O}_N)$).

DEFINITION 3.4.2. We define

$$\mathrm{Cht}'_{P,N,I,W} := \mathrm{Cht}_{P,N,I,W} \overset{P(\mathcal{O}_N)}{\times} G(\mathcal{O}_N), \quad \mathrm{Cht}'_{M,N,I,W} := \mathrm{Cht}_{M,N,I,W} \overset{P(\mathcal{O}_N)}{\times} G(\mathcal{O}_N),$$

where $P(\mathcal{O}_N)$ acts on $G(\mathcal{O}_N)$ by left action (by left multiplication) and $P(\mathcal{O}_N)$ acts on $\mathrm{Cht}_{M,N,I,W}$ via the quotient $P(\mathcal{O}_N) \twoheadrightarrow M(\mathcal{O}_N)$.

3.4.3 Morphisms (3.7) induce morphisms

$$\mathrm{Cht}_{G,N,I,W} \xleftarrow{i'} \mathrm{Cht}'_{P,N,I,W} \xrightarrow{\pi'} \mathrm{Cht}'_{M,N,I,W}. \tag{3.26}$$

Indeed, the morphism i' is giving by

$$((\mathcal{P}, \psi_P) \to ({}^\tau\mathcal{P}, {}^\tau\psi_P), g \in G(\mathcal{O}_N)) \mapsto ((\mathcal{G}, g^{-1} \circ \psi_G) \to ({}^\tau\mathcal{G}, g^{-1} \circ {}^\tau\psi_G)),$$

where $\mathcal{G} = \mathcal{P} \overset{P}{\times} G$ and $\psi_G = \psi_P \overset{P}{\times} G$. The morphism π' is induced by π, which is $P(\mathcal{O}_N)$-equivariant (because $P(\mathcal{O}_N)$ acts on $\mathrm{Cht}_{P,N,I,W}$ and $\mathrm{Cht}_{M,N,I,W}$ by changing the level structure on N).

Remark 3.4.4. The morphism $\mathrm{Cht}'_{P,N,I,W} \to \mathrm{Cht}_{P,I,W} \underset{\mathrm{Cht}_{G,I,W}}{\times} \mathrm{Cht}_{G,N,I,W}$ is a $G(\mathcal{O}_N)$-equivariant morphism of $G(\mathcal{O}_N)$-torsors over $\mathrm{Cht}_{P,I,W}$, and thus it is an isomorphism. In [Var04, 2.28], the stack $\mathrm{Cht}'_{P,N,I,W}$ is denoted by $FBun_{P,D,n,\bar{\omega}}$. The reason why we will need $\mathrm{Cht}'_{P,N,I,W}$ instead of $\mathrm{Cht}_{P,N,I,W}$ is justified in Example 3.5.15 and Theorem 4.2.1.

DEFINITION 3.4.5. We define

$$\mathrm{Cht}'^{\leqslant\mu}_{P,N,I,W} := \mathrm{Cht}^{\leqslant\mu}_{P,N,I,W} \overset{P(\mathcal{O}_N)}{\times} G(\mathcal{O}_N),$$
$$\mathrm{Cht}'^{\leqslant\mu}_{M,N,I,W} := \mathrm{Cht}^{\leqslant\mu}_{M,N,I,W} \overset{P(\mathcal{O}_N)}{\times} G(\mathcal{O}_N), \quad \mathrm{Cht}'^{\leqslant\mu,\nu}_{M,N,I,W} := \mathrm{Cht}^{\leqslant\mu,\nu}_{M,N,I,W} \overset{P(\mathcal{O}_N)}{\times} G(\mathcal{O}_N).$$

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3.4.6 We have a commutative diagram of algebraic stacks.

$$\begin{tikzcd} & \mathrm{Cht}_{P,N,I,W} \arrow[dl, "i"'] \arrow[r, "\pi"] \arrow[d] & \mathrm{Cht}_{M,N,I,W} \arrow[d] \\ \mathrm{Cht}_{G,N,I,W} \arrow[d] & \mathrm{Cht}'_{P,N,I,W} \arrow[l, "i'"'] \arrow[r, "\pi'"] \arrow[d] & \mathrm{Cht}'_{M,N,I,W} \arrow[d] \\ \mathrm{Bun}_G & \mathrm{Bun}_P \arrow[l] \arrow[r] & \mathrm{Bun}_M \end{tikzcd} \tag{3.27}$$

We deduce that $\mathrm{Cht}'^{\leqslant\mu}_{P,N,I,W}$ is also the inverse image of $\mathrm{Bun}_P^{\leqslant\mu}$ by $\mathrm{Cht}'_{P,N,I,W} \to \mathrm{Bun}_P$ and $\mathrm{Cht}'^{\leqslant\mu}_{M,N,I,W}$ (respectively $\mathrm{Cht}'^{\leqslant\mu,\,\nu}_{M,N,I,W}$) is also the inverse image of $\mathrm{Bun}_M^{\leqslant\mu}$ (respectively $\mathrm{Bun}_M^{\leqslant\mu,\,\nu}$) by $\mathrm{Cht}'_{M,N,I,W} \to \mathrm{Bun}_M$.

Definition 3.4.7. Just as in § 2.6, we construct a morphism $\epsilon'^{\Xi}_{M,d} : \mathrm{Cht}'_{M,N,I,W}/\Xi \to [\overline{M}_{I,d}\backslash \mathrm{Gr}_{M,I,W}]$ and we define $\mathcal{F}'^{\,\Xi}_{M,N,I,W}$ to be the inverse image of $\mathcal{S}^{\mathrm{ad},d}_{M,I,W}$. We define $\mathcal{H}'^{\leqslant\mu,\,\nu}_{M,N,I,W} := R(\mathfrak{p}_M)_!(\mathcal{F}'^{\,\Xi}_{M,I,N,W}\big|_{\mathrm{Cht}'^{\leqslant\mu,\,\nu}_{M,N,I,W}/\Xi})$, $\mathcal{H}'^{\,j,\,\leqslant\mu,\,\nu}_{M,N,I,W} := H^j(\mathcal{H}'^{\leqslant\mu,\,\nu}_{M,N,I,W})$ and $H'^{\,j,\,\leqslant\mu,\,\nu}_{M,N,I,W} := \mathcal{H}'^{\,j,\,\leqslant\mu,\,\nu}_{M,N,I,W}\big|_{\overline{\eta^I}}$.

3.4.8 Just as in 2.6.5, if $\nu \notin \widehat{\Lambda}^{\mu}_{Z_M/Z_G}$ (defined in 1.5.20), then $\mathrm{Cht}'^{\leqslant\mu,\,\nu}_{M,N,I,W}$ is empty and $H'^{\,j,\,\leqslant\mu,\,\nu}_{M,N,I,W} = 0$.

Definition 3.4.9. Just as in Definitions 2.6.6 and 2.6.8, we define

$$H'^{\,j,\,\leqslant\mu}_{M,N,I,W} := \prod_{\nu\in\widehat{\Lambda}^{\mu}_{Z_M/Z_G}} H'^{\,j,\,\leqslant\mu,\,\nu}_{M,N,I,W}; \quad H'^{\,j}_{M,N,I,W} := \varinjlim_{\mu} H'^{\,j,\,\leqslant\mu}_{M,N,I,W}.$$

Definition 3.4.10. For any $\nu \in \widehat{\Lambda}^{\mathbb{Q}}_{Z_M/Z_G}$, we define $H'^{\,j,\,\nu}_{M,N,I,W} := \varinjlim_{\mu} H'^{\,j,\,\leqslant\mu,\,\nu}_{M,N,I,W}$.

3.5 Constant term morphism for cohomology groups

3.5.1 Morphisms (3.24) induce morphisms over $(X\smallsetminus N)^I$.

$$\begin{tikzcd} \mathrm{Cht}_{G,N,I,W}/\Xi \arrow[d, "\epsilon^{\Xi}_{G,N,I,d}"] & \mathrm{Cht}'_{P,N,I,W}/\Xi \arrow[l, "i'"'] \arrow[r, "\pi'"] \arrow[d, "\epsilon'^{\Xi}_{P,N,I,d}"] & \mathrm{Cht}'_{M,N,I,W}/\Xi \arrow[d, "\epsilon'^{\Xi}_{M,N,I,d}"] \\ {[G^{\mathrm{ad}}_{I,d}\backslash \mathrm{Gr}_{G,I,W}]} & {[\overline{P}_{I,d}\backslash \mathrm{Gr}_{P,I,W}]} \arrow[l, "i^0_d"'] \arrow[r, "\pi^0_d"] & {[\overline{M}_{I,d}\backslash \mathrm{Gr}_{M,I,W}]} \end{tikzcd} \tag{3.28}$$

3.5.2 For any $\mu \in \widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}}$ and any $\nu \in \widehat{\Lambda}^{\mathbb{Q}}_{Z_M/Z_G}$, the first line of morphisms (3.28) induces morphisms over $(X\smallsetminus N)^I$:

$$\mathrm{Cht}^{\leqslant\mu}_{G,N,I,W}/\Xi \xleftarrow{i'} \mathrm{Cht}'^{\leqslant\mu,\nu}_{P,N,I,W}/\Xi \xrightarrow{\pi'} \mathrm{Cht}'^{\leqslant\mu,\nu}_{M,N,I,W}/\Xi. \tag{3.29}$$

The proof of [Var04, Proposition 5.7] in fact proves the following.

PROPOSITION 3.5.3 [Var04, Proposition 5.7]. *For any $\mu \in \widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}}$ and any $\nu \in \widehat{\Lambda}^{\mu}_{Z_M/Z_G}$ (defined in 1.5.20), there exists an open dense subscheme $\Omega^{\leqslant\mu,\nu}$ of $(X \smallsetminus N)^I$ such that the restriction of the morphism i' on $\mathrm{Cht}'^{\leqslant\mu,\nu}_{P,N,I,W}/\Xi\big|_{\Omega^{\leqslant\mu,\nu}}$ is proper. In particular, the restriction of the morphism i' on $\mathrm{Cht}'^{\leqslant\mu,\nu}_{P,N,I,W}/\Xi\big|_{\eta^I}$ is proper.*

Remark 3.5.4. In [Var04, Proposition 5.7], the level is denoted by D, the paws are indexed by n, the index d is related to our ν, the index $\overline{k}$ is related to our W, and the index $[g]$ is in $G(\mathcal{O}_N)/P(\mathcal{O}_N)$. The open subscheme $\Omega^{\leqslant\mu,\nu}$ is of the form

$$\Omega(m) = \{(x_i)_{i\in I} \in (X \smallsetminus N)^I,\ x_i \neq {}^{\tau^r}x_j \text{ for all } i,j \text{ and } r = 1,2,\ldots,m\},$$

where ${}^{\tau^r}x$ is the image of x by $\mathrm{Frob}^r : X \to X$ and m is some positive integer.

In the proof of [Var04, Proposition 5.7], $\mathrm{Bun}_G^{\leqslant\mu}$ is denoted by V and $\Omega(m)$ is denoted by U. Varshavsky shows that for fixed μ and ν, there exists a level D large enough and an integer m large enough (both depending on μ and ν), such that over $\mathrm{Bun}_G^{\leqslant\mu} \times \Omega(m) \subset \mathrm{Bun}_G \times (X \smallsetminus N)^I$, the morphism However,

$$\mathrm{Cht}'^{\leqslant\mu,\nu}_{P,D,I,W}\big|_{\Omega(m)} \to \mathrm{Cht}^{\leqslant\mu}_{G,D,I,W}\big|_{\Omega(m)}$$

is a closed embedding. In particular, it is proper. Then we descend to level N.

Note that i' is schematic (i.e. representable). This is implied by the well-known fact that $\mathrm{Bun}_P \to \mathrm{Bun}_G$ is schematic (a P-structure of a G-bundle $\mathcal{G}$ over $X \times S$ is a section of the fibration $\mathcal{G}/P \to X \times S$).

3.5.5 Now consider the following commutative diagram.

$$\begin{tikzcd}
 & \mathrm{Cht}'^{\leqslant\mu,\nu}_{P,N,I,W}/\Xi\big|_{\Omega^{\leqslant\mu,\nu}} \arrow[dl, "i'"'] \arrow[dd, "\mathfrak{p}_P"] \arrow[dr, "\pi'"] & \\
\mathrm{Cht}^{\leqslant\mu}_{G,N,I,W}/\Xi\big|_{\Omega^{\leqslant\mu,\nu}} \arrow[dr, "\mathfrak{p}_G"'] & & \mathrm{Cht}'^{\leqslant\mu,\nu}_{M,N,I,W}/\Xi\big|_{\Omega^{\leqslant\mu,\nu}} \arrow[dl, "\mathfrak{p}_M"] \\
 & \Omega^{\leqslant\mu,\nu} &
\end{tikzcd} \tag{3.30}$$

To simplify the notations, we denote by $\mathcal{F}^{\Xi}_{G,N,\Omega^{\leqslant\mu,\nu},W}$ the restriction of $\mathcal{F}^{\Xi}_{G,I,N,W}$ to $\mathrm{Cht}^{\leqslant\mu}_{G,N,I,W}/\Xi\big|_{\Omega^{\leqslant\mu,\nu}}$ and by $\mathcal{F}'^{\Xi,\nu}_{M,N,\Omega^{\leqslant\mu,\nu},W}$ the restriction of $\mathcal{F}'^{\Xi}_{M,I,N,W}$ to $\mathrm{Cht}'^{\leqslant\mu,\nu}_{M,N,I,W}/\Xi\big|_{\Omega^{\leqslant\mu,\nu}}$.

The commutative diagram (3.28) is compatible with the Harder–Narasimhan stratification. Just as in Construction 3.3.4, we construct a canonical morphism of complexes

$$(\pi')_!(i')^*\mathcal{F}^{\Xi}_{G,N,\Omega^{\leqslant\mu,\nu},W} \to \mathcal{F}'^{\Xi,\nu}_{M,N,\Omega^{\leqslant\mu,\nu},W} \tag{3.31}$$

in $D^b_c(\mathrm{Cht}'^{\leqslant\mu,\nu}_{M,N,I,W}/\Xi\big|_{\Omega^{\leqslant\mu,\nu}}, \mathbb{Q}_\ell)$.

3.5.6 Thanks to Proposition 3.5.3, we can apply [SGA5, III 3] to diagram (3.30) and the cohomological correspondence (3.31).

Concretely, first we have morphisms of functors from $D^b_c(\mathrm{Cht}^{\leqslant\mu}_{G,N,I,W}/\Xi\big|_{\Omega^{\leqslant\mu,\nu}}, \mathbb{Q}_\ell)$ to $D^b_c(\Omega^{\leqslant\mu,\nu}, \mathbb{Q}_\ell)$ (all functors are considered as derived functors):

$$(\mathfrak{p}_G)_! \overset{(a)}{\to} (\mathfrak{p}_G)_!(i')_*(i')^* \overset{(b)}{\simeq} (\mathfrak{p}_G)_!(i')_!(i')^* \overset{(c)}{\simeq} (\mathfrak{p}_M)_!(\pi')_!(i')^*, \tag{3.32}$$

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where (a) is the adjunction morphism, (b) is induced by $i'_! \xrightarrow{\sim} i'_*$ which is because that i' is schematic and proper (Proposition 3.5.3), and (c) is induced by the commutativity of diagram (3.30).

Second we combine (3.32) with (3.31). We obtain a composition of morphisms of complexes in $D_c^b(\Omega^{\leqslant \mu,\, \nu}, \mathbb{Q}_\ell)$.

$$(\mathfrak{p}_G)_! \mathcal{F}^{\Xi}_{G,N,\Omega^{\leqslant \mu,\, \nu},W} \xrightarrow{(3.32)} (\mathfrak{p}_M)_! (\pi')_! (i')^* \mathcal{F}^{\Xi}_{G,N,\Omega^{\leqslant \mu,\, \nu},W} \xrightarrow{(3.31)} (\mathfrak{p}_M)_! \mathcal{F}'^{\,\Xi,\, \nu}_{M,N,\Omega^{\leqslant \mu,\, \nu},W}. \tag{3.33}$$

By Definition 2.5.1 and Definition 3.4.7, (3.33) is also written as

$$\mathcal{C}_G^{P, \leqslant \mu,\, \nu} : \mathcal{H}^{\leqslant \mu}_{G,N,I,W}\Big|_{\Omega^{\leqslant \mu,\, \nu}} \to \mathcal{H}'^{\,\leqslant \mu,\, \nu}_{M,N,I,W}\Big|_{\Omega^{\leqslant \mu,\, \nu}}. \tag{3.34}$$

3.5.7 From now on, we restrict everything to the geometric generic point $\overline{\eta^I}$ of X^I fixed in 2.5.4. Recall that we have defined $H^{j, \leqslant \mu}_{G,N,I,W} = \mathcal{H}^{j, \leqslant \mu}_{G,N,I,W}\Big|_{\overline{\eta^I}}$ in Definition 2.5.5 and $H'^{\,j, \leqslant \mu,\, \nu}_{M,N,I,W} = \mathcal{H}'^{\,j, \leqslant \mu,\, \nu}_{M,N,I,W}\Big|_{\overline{\eta^I}}$ in Definition 3.4.7.

For any $j \in \mathbb{Z}$, morphism (3.34) induces a morphism of cohomology groups

$$C_G^{P, j, \leqslant \mu,\, \nu} : H^{j, \leqslant \mu}_{G,N,I,W} \to H'^{\,j, \leqslant \mu,\, \nu}_{M,N,I,W}. \tag{3.35}$$

By 3.4.8, for $\nu \notin \widehat{\Lambda}^{\mu}_{Z_M/Z_G}$, the morphism $C_G^{P, j, \leqslant \mu,\, \nu}$ is the zero morphism.

3.5.8 We define a morphism:

$$C_G^{P, j, \leqslant \mu} = \prod_{\nu \in \widehat{\Lambda}^{\mu}_{Z_M/Z_G}} C_G^{P, j, \leqslant \mu,\, \nu} : H^{j, \leqslant \mu}_{G,N,I,W} \to H'^{\,j, \leqslant \mu}_{M,N,I,W} \tag{3.36}$$

where $H'^{\,j, \leqslant \mu}_{M,N,I,W}$ is defined in Definition 3.4.9.

3.5.9 Let $\mu_1, \mu_2 \in \widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}}$ with $\mu_1 \leqslant \mu_2$. By Lemma A.0.8, the commutative diagram of stacks

$$\begin{tikzcd}
\mathrm{Cht}^{\leqslant \mu_2}_{G,N,I,W}/\Xi & \mathrm{Cht}'^{\,\leqslant \mu_2}_{P,N,I,W}/\Xi \arrow[l, "i'"'] \arrow[r, "\pi'"] & \mathrm{Cht}'^{\,\leqslant \mu_2}_{M,N,I,W}/\Xi \\
\mathrm{Cht}^{\leqslant \mu_1}_{G,N,I,W}/\Xi \arrow[u, hook] & \mathrm{Cht}'^{\,\leqslant \mu_1}_{P,N,I,W}/\Xi \arrow[l, "i'"'] \arrow[r, "\pi'"] \arrow[u, hook] & \mathrm{Cht}'^{\,\leqslant \mu_1}_{M,N,I,W}/\Xi \arrow[u, hook]
\end{tikzcd} \tag{3.37}$$

induces a commutative diagram of cohomology groups.

$$\begin{tikzcd}
H^{j, \leqslant \mu_1}_{G,N,I,W} \arrow[r] \arrow[d, "C_G^{P, j, \leqslant \mu_1}"] & H^{j, \leqslant \mu_2}_{G,N,I,W} \arrow[d, "C_G^{P, j, \leqslant \mu_2}"] \\
H'^{\,j, \leqslant \mu_1}_{M,N,I,W} \arrow[r] & H'^{\,j, \leqslant \mu_2}_{M,N,I,W}
\end{tikzcd} \tag{3.38}$$

We have defined $H^{j}_{G,N,I,W} = \varinjlim_{\mu} H^{j, \leqslant \mu}_{G,N,I,W}$ in Definition 2.5.5 and $H'^{\,j}_{M,N,I,W} = \varinjlim_{\mu} H'^{\,j, \leqslant \mu}_{M,N,I,W}$ in Definition 3.4.9. The commutative diagram (3.38) induces a morphism between inductive limits.

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Definition 3.5.10. For all parabolic subgroups P, for all degrees $j \in \mathbb{Z}$, we define the constant term morphism of cohomology groups:

$$C_{G,N}^{P,j} : H_{G,N,I,W}^{j} \to H'^{\,j}_{M,N,I,W}. \tag{3.39}$$

Remark 3.5.11. The morphisms $H'^{\,j,\leqslant\mu,\,\nu}_{M,N,I,W} \to \varinjlim_{\mu'} H'^{\,j,\leqslant\mu',\,\nu}_{M,N,I,W}$ for each $\nu \in \widehat{\Lambda}^{\mathbb{Q}}_{Z_M/Z_G}$ induce a morphism

$$\varinjlim_{\mu} \prod_{\nu \in \widehat{\Lambda}^{\mathbb{Q}}_{Z_M/Z_G}} H'^{\,j,\leqslant\mu,\,\nu}_{M,N,I,W} \to \prod_{\nu \in \widehat{\Lambda}^{\mathbb{Q}}_{Z_M/Z_G}} \varinjlim_{\mu'} H'^{\,j,\leqslant\mu',\,\nu}_{M,N,I,W}. \tag{3.40}$$

With the notations in Definitions 3.4.9 and 3.4.10, morphism (3.40) is the natural map

$$H'^{\,j}_{M,N,I,W} \to \prod_{\nu \in \widehat{\Lambda}^{\mathbb{Q}}_{Z_M/Z_G}} H'^{\,j,\,\nu}_{M,N,I,W}. \tag{3.41}$$

For each $\nu \in \widehat{\Lambda}^{\mathbb{Q}}_{Z_M/Z_G}$, taking inductive limit over μ of (3.35), we define $C_{G,N}^{P,j,\nu} : H^{j}_{G,N,I,W} \to H'^{\,j,\,\nu}_{M,N,I,W}$. We form a morphism

$$\prod_{\nu \in \widehat{\Lambda}^{\mathbb{Q}}_{Z_M/Z_G}} C_{G,N}^{P,j,\nu} : H^{j}_{G,N,I,W} \to \prod_{\nu \in \widehat{\Lambda}^{\mathbb{Q}}_{Z_M/Z_G}} H'^{\,j,\,\nu}_{M,N,I,W}. \tag{3.42}$$

It is equal to the composition of (3.39) and (3.41).

In Lemma 5.3.4 below, we will prove that, for μ large enough,

$$H'^{\,j,\leqslant\mu}_{M,N,I,W} \to \prod_{\nu \in \widehat{\Lambda}^{\mathbb{Q}}_{Z_M/Z_G}} H'^{\,j,\,\nu}_{M,N,I,W}$$

is injective. This implies that (3.41) is injective. Thus the kernel of (3.42) is the same as the kernel of (3.39).

Remark 3.5.12. Now consider all parabolic subgroups (not only the standard ones). If P_1 and P_2 are conjugated, then the conjugation induces an isomorphism $M_1 \simeq M_2$. This induces for any j an isomorphism $H'^{\,j}_{M_1,N,I,W} \simeq H'^{\,j}_{M_2,N,I,W}$. The following diagram commutes

$$\begin{tikzcd} H^{j}_{G,N,I,W} \arrow[r, "C_{G,N}^{P_1,j}"] \arrow[dr, "C_{G,N}^{P_2,j}"'] & H'^{\,j}_{M_1,N,I,W} \arrow[d, "\simeq"'] \\ & H'^{\,j}_{M_2,N,I,W} \end{tikzcd}$$

and thus we have $\operatorname{Ker} C_{G,N}^{P_1,j} = \operatorname{Ker} C_{G,N}^{P_2,j}$ in $H^{j}_{G,N,I,W}$.

However, we do not know how to compare the constant term morphism along different parabolic subgroups which have a common Levi subgroup. It is perhaps possible to do that, but quite difficult because it would be a generalization of the functional equation for Eisenstein series.

Definition 3.5.13. For any degree $j \in \mathbb{Z}$, we define the cuspidal cohomology group:

$$H^{j,\,\mathrm{cusp}}_{G,N,I,W} := \bigcap_{P \subsetneq G} \operatorname{Ker} C_{G,N}^{P,j}. \tag{3.43}$$

This is a $\mathbb{Q}_\ell$-vector subspace of $H^{j}_{G,N,I,W}$.

Remark 3.5.14. For

$$\begin{tikzcd}[column sep=small]
& & P_1 \arrow[dl] \arrow[dr] & & \\
& P_2 \arrow[dl] \arrow[dr] & & P_{1,2} \arrow[dl] \arrow[dr] & \\
G & & M_2 & & M_1
\end{tikzcd}$$

we have $C_{M_2,N}^{P_{1,2},\,j} \circ C_{G,N}^{P_2,\,j} = C_{G,N}^{P_1,\,j}$. Thus we have an equivalent definition:

$$H_{G,N,I,W}^{j,\,\mathrm{cusp}} = \bigcap_{P \text{ maximal parabolic}} \operatorname{Ker} C_{G,N}^{P,\,j}.$$

Example 3.5.15 (Shtukas without paws). When $I = \emptyset$ and $W = \mathbf{1}$, we have $\mathrm{Cht}_{G,N,\emptyset,\mathbf{1}} = G(F)\backslash G(\mathbb{A})/K_{G,N}$. (Note that G is split. See [Laf18, (0.5) and Remarque 8.21] for more details.) Moreover, let $K_{P,N} := K_{G,N} \cap P(\mathbb{O})$, $K_{U,N} := K_{G,N} \cap U(\mathbb{O})$ and $K_{M,N} := K_{P,N}/K_{U,N}$. We write $\overset{\text{set}}{=}$ for equalities of sets which are not equalities of groupoids. We have

$$\begin{aligned}
\mathrm{Cht}'_{P,N,\emptyset,\mathbf{1}} &= (P(F)\backslash P(\mathbb{A})/K_{P,N}) \overset{P(\mathcal{O}_N)}{\times} G(\mathcal{O}_N) = P(F)\backslash (P(\mathbb{A}) \overset{P(\mathbb{O})}{\times} G(\mathbb{O}))/K_{G,N} \\
&= P(F)\backslash G(\mathbb{A})/K_{G,N}, \\
\mathrm{Cht}'_{M,N,\emptyset,\mathbf{1}} &= (M(F)\backslash M(\mathbb{A})/K_{M,N}) \overset{P(\mathcal{O}_N)}{\times} G(\mathcal{O}_N) \\
&= M(F)\backslash (M(\mathbb{A}) \overset{P(\mathbb{O})/K_{U,N}}{\times} G(\mathbb{O})/K_{G,N}) \overset{\text{set}}{=} M(F)\backslash (M(\mathbb{A}) \overset{P(\mathbb{O})}{\times} G(\mathbb{O}))/K_{G,N} \\
&= M(F)U(\mathbb{A})\backslash (P(\mathbb{A}) \overset{P(\mathbb{O})}{\times} G(\mathbb{O}))/K_{G,N} = M(F)U(\mathbb{A})\backslash G(\mathbb{A})/K_{G,N}.
\end{aligned}$$

In this case, $\mathrm{Gr}_{P,\emptyset,\mathbf{1}} = \mathrm{Gr}_{M,\emptyset,\mathbf{1}} = \operatorname{Spec} \overline{\mathbb{F}_q}$. We can choose $d = 0$ in (3.9). Thus $\widetilde{\mathrm{Cht}}'_{M,N,\emptyset,1} = \mathrm{Cht}'_{M,N,\emptyset,1}$. The constant term morphism $C_{G,N}^{P,\,j}$ in Definition 3.5.10 coincides (up to constants depending on $\nu \in \widehat{\Lambda}^{\mathbb{Q}}_{Z_M/Z_G}$ component by component) with the classical constant term morphism:

$$\begin{aligned}
C_c(G(F)\backslash G(\mathbb{A})/K_{G,N}\Xi, \mathbb{Q}_\ell) &\to C(U(\mathbb{A})M(F)\backslash G(\mathbb{A})/K_{G,N}\Xi, \mathbb{Q}_\ell) \\
f \qquad &\mapsto \quad f^P : g \mapsto \int_{U(F)\backslash U(\mathbb{A})} f(ug)\, du.
\end{aligned} \tag{3.44}$$

Therefore $H_{G,N,\emptyset,\mathbf{1}}^{0,\,\mathrm{cusp}} = C_c^{\mathrm{cusp}}(G(F)\backslash G(\mathbb{A})/K_{G,N}\Xi, \mathbb{Q}_\ell)$.

Remark 3.5.16. When $I = \emptyset$, $W = \mathbf{1}$ and $N = \emptyset$ (without level), for any $\mu \in \widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}}$, $H'^{\,0,\leqslant\mu}_{M,N,I,W}$ is included in the subspace of $C(U(\mathbb{A})M(F)\backslash G(\mathbb{A})/G(\mathbb{O})\Xi, \mathbb{Q}_\ell)$ of functions supported on the components of $U(\mathbb{A})M(F)\backslash G(\mathbb{A})/G(\mathbb{O})\Xi$ indexed by a translated cone $\widehat{\Lambda}^{\mu}_{Z_M/Z_G}$ in $\widehat{\Lambda}^{\mathbb{Q}}_{Z_M/Z_G}$. The image of the constant term morphism is included in $H'^{\,0}_{M,N,I,W} = \varinjlim_\mu H'^{\,0,\leqslant\mu}_{M,N,I,W}$. This space is already defined independently by Wang in [Wan18, § 5.1] and is denoted by $\mathcal{C}_{P,-}$ in [Wan18].

4. Contractibility of deep enough horospheres

In this section, let P be a parabolic subgroup of G and M its Levi quotient. The goal is to prove Proposition 4.6.4, which will be a consequence of Theorems 4.2.1 and 4.2.4.

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4.1 More on Harder–Narasimhan stratification

To state Theorems 4.2.1 and 4.2.4, we need to introduce some locally closed substacks of $\mathrm{Cht}_{G,N,I,W}$.

Definition 4.1.1. Let $\mu \in \widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}}$. We define a set

$$\begin{aligned} S_M(\mu) &:= \{\lambda \in \widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}} \mid \lambda \leqslant^{\overline{M}} \mu\} \\ &= \{\lambda \in \widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}} \mid \lambda \leqslant \mu\} \cap \{\lambda \in \widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}} \mid \mathrm{pr}^{\mathrm{ad}}_P(\lambda) = \mathrm{pr}^{\mathrm{ad}}_P(\mu)\}, \end{aligned}$$

where the second equality follows from 1.5.16 (taking into account Notation 1.7.1). The set $S_M(\mu)$ is bounded.

Remark 4.1.2. The set $S_M(\mu)$ is the same as the one (modulo $\widehat{\Lambda}^{\mathbb{Q}}_{Z_G}$) used in [DG15, §§ 8 and 9].

Definition 4.1.3. We define

$$\mathrm{Bun}_G^{=\mu} := \bigcup_{\lambda \in \widehat{\Lambda}^{+,\mathbb{Q}}_G,\ \Upsilon_G(\lambda)=\mu} \mathrm{Bun}_G^{(\lambda)}, \quad \mathrm{Bun}_M^{=\mu} := \bigcup_{\lambda \in \widehat{\Lambda}^{+,\mathbb{Q}}_M,\ \Upsilon_G(\lambda)=\mu} \mathrm{Bun}_M^{(\lambda)},$$

where $\mathrm{Bun}_G^{(\lambda)}$ (respectively $\mathrm{Bun}_M^{(\lambda)}$) is defined in Definition 1.4.9 (respectively Definition 1.5.3).

Definition 4.1.4. We define

$$\mathrm{Bun}_G^{S_M(\mu)} := \bigcup_{\lambda \in S_M(\mu)} \mathrm{Bun}_G^{=\lambda}; \quad \mathrm{Bun}_M^{S_M(\mu)} := \bigcup_{\lambda \in S_M(\mu)} \mathrm{Bun}_M^{=\lambda}.$$

4.1.5 If $\lambda \in S_M(\mu)$, $\lambda' \in \widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}}$ and $\lambda \leqslant \lambda' \leqslant \mu$, then $\mathrm{pr}^{\mathrm{ad}}_P(\lambda) = \mathrm{pr}^{\mathrm{ad}}_P(\lambda') = \mathrm{pr}^{\mathrm{ad}}_P(\mu)$. This implies that $\lambda' \in S_M(\mu)$. Using [DG15, Corollary 7.4.11], we deduce the following.

Lemma 4.1.6. *The substack* $\mathrm{Bun}_G^{S_M(\mu)}$ *is closed in* $\mathrm{Bun}_G^{\leqslant\mu}$.

4.1.7 We deduce from the definition of $S_M(\mu)$ and 1.5.12 that

$$\mathrm{Bun}_M^{S_M(\mu)} = \mathrm{Bun}_M^{\leqslant\mu,\ \mathrm{pr}^{\mathrm{ad}}_P(\mu)}. \tag{4.1}$$

Recall that $\mathrm{Bun}_M^{\leqslant\mu}$ is open in Bun_M (see Lemma 1.5.5) and $\mathrm{Bun}_M^{\mathrm{pr}^{\mathrm{ad}}_P(\mu)}$ is open and closed in Bun_M (see 1.5.7).

Lemma 4.1.8 [DG15, Corollary 7.4.11, Lemma 8.2.6]. *The substack* $\mathrm{Bun}_M^{S_M(\mu)}$ *is open and closed in* $\mathrm{Bun}_M^{\leqslant\mu}$*, and is open in* $\mathrm{Bun}_M^{\mathrm{pr}^{\mathrm{ad}}_P(\mu)}$ *and in* Bun_M.

We define $\mathrm{Bun}_P^{S_M(\mu)} := \mathrm{Bun}_P^{\leqslant\mu} \cap \pi^{-1}(\mathrm{Bun}_M^{S_M(\mu)})$. By Lemma 4.1.8, it is open and closed in $\mathrm{Bun}_P^{\leqslant\mu}$, and is open in Bun_P. So it is reduced.

Lemma 4.1.9. *Morphisms (1.25) induce morphisms*

$$\mathrm{Bun}_G^{S_M(\mu)} \leftarrow \mathrm{Bun}_P^{S_M(\mu)} \to \mathrm{Bun}_M^{S_M(\mu)}. \tag{4.2}$$

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Proof. We need to verify that the image of $\mathrm{Bun}_P^{S_M(\mu)} \to \mathrm{Bun}_G^{\leqslant\mu}$ is in the closed substack $\mathrm{Bun}_G^{S_M(\mu)}$. Since $\mathrm{Bun}_P^{S_M(\mu)}$ is reduced, it is enough to consider geometric points. Let $\mathcal{P} \in \mathrm{Bun}_P^{S_M(\mu)}$ be a geometric point. Let $\mathcal{M}$ be its image in $\mathrm{Bun}_M^{S_M(\mu)}$. By definition of $\mathrm{Bun}_M^{S_M(\mu)}$, there exists $\lambda \in S_M(\mu)$ such that $\mathcal{M} \in \mathrm{Bun}_M^{=\lambda}$.

Let $\mathcal{G}$ be the image of $\mathcal{P}$ in $\mathrm{Bun}_G^{\leqslant\mu}$. By 1.4.14, there exists $\lambda' \leqslant \mu$ such that $\mathcal{G} \in \mathrm{Bun}_G^{=\lambda'}$. Taking into account that $\mathrm{Bun}_G^{=\lambda'} \subset \mathrm{Bun}_G^{\leqslant\lambda'}$, by Lemma 1.6.2, we deduce that $\mathcal{M} \in \mathrm{Bun}_M^{\leqslant\lambda'}$. Hence $\lambda \leqslant \lambda'$. By 4.1.5, this implies that $\lambda' \in S_M(\mu)$. Thus $\mathcal{G} \in \mathrm{Bun}_G^{S_M(\mu)}$. □

Definition 4.1.10. We define $\mathrm{Cht}_{G,N,I,W}^{=\mu}$ (respectively $\mathrm{Cht}_{G,N,I,W}^{S_M(\mu)}$) as the inverse image of $\mathrm{Bun}_G^{=\mu}$ (respectively $\mathrm{Bun}_G^{S_M(\mu)}$) by the morphism

$$\mathrm{Cht}_{G,N,I,W} \to \mathrm{Bun}_G, \quad ((x_i)_{i\in I}, (\mathcal{G}, \psi) \xrightarrow{\phi} ({}^\tau\mathcal{G}, {}^\tau\psi)) \mapsto \mathcal{G}.$$

Similarly, we define $\mathrm{Cht}_{P,N,I,W}^{S_M(\mu)}$, $\mathrm{Cht}_{M,N,I,W}^{=\mu}$ and $\mathrm{Cht}_{M,N,I,W}^{S_M(\mu)}$.

4.1.11 We deduce from Lemma 4.1.6 that $\mathrm{Cht}_{G,N,I,W}^{S_M(\mu)}$ is closed in $\mathrm{Cht}_{G,N,I,W}^{\leqslant\mu}$. We deduce from Lemma 4.1.8 that $\mathrm{Cht}_{M,N,I,W}^{S_M(\mu)}$ is open and closed in $\mathrm{Cht}_{M,N,I,W}^{\leqslant\mu}$, and is open in $\mathrm{Cht}_{M,N,I,W}^{\mathrm{pr}_P^{\mathrm{ad}}(\mu)}$ and in $\mathrm{Cht}_{M,N,I,W}$.

4.1.12 The commutativity of diagram (1.28) and Lemma 4.1.9 imply that $\mathrm{Cht}_{P,N,I,W}^{S_M(\mu)} = \mathrm{Cht}_{P,N,I,W}^{\leqslant\mu} \cap \pi^{-1}(\mathrm{Cht}_{M,N,I,W}^{S_M(\mu)})$. Morphisms (1.29) induce morphisms:

$$\mathrm{Cht}_{G,N,I,W}^{S_M(\mu)} \xleftarrow{i^{S_M(\mu)}} \mathrm{Cht}_{P,N,I,W}^{S_M(\mu)} \xrightarrow{\pi^{S_M(\mu)}} \mathrm{Cht}_{M,N,I,W}^{S_M(\mu)}.$$

4.1.13 As in Definition 3.4.2, we define

$$\mathrm{Cht}'^{S_M(\mu)}_{P,N,I,W} := \mathrm{Cht}_{P,N,I,W}^{S_M(\mu)} \overset{P(\mathcal{O}_N)}{\times} G(\mathcal{O}_N), \quad \mathrm{Cht}'^{S_M(\mu)}_{M,N,I,W} := \mathrm{Cht}_{M,N,I,W}^{S_M(\mu)} \overset{P(\mathcal{O}_N)}{\times} G(\mathcal{O}_N).$$

Morphisms (3.26) induce morphisms

$$\mathrm{Cht}_{G,N,I,W}^{S_M(\mu)} \xleftarrow{i'^{S_M(\mu)}} \mathrm{Cht}'^{S_M(\mu)}_{P,N,I,W} \xrightarrow{\pi'^{S_M(\mu)}} \mathrm{Cht}'^{S_M(\mu)}_{M,N,I,W}. \tag{4.3}$$

4.2 Geometric statements

First consider the morphism $i'^{S_M(\mu)}$.

Theorem 4.2.1 ([Var04, Theorem 2.25 and Proposition 5.7], [DG15, Proposition 9.2.2]). *There exists a constant $C'(G,X,N,W)$, such that if $\mu \in \widehat{\Lambda}_{G^{\mathrm{ad}}}^{+,\mathbb{Q}}$ and $\langle\mu,\alpha\rangle > C'(G,X,N,W)$ for all $\alpha \in \Gamma_G - \Gamma_M$, then the morphism $i'^{S_M(\mu)}$ is a schematic finite universal homeomorphism.*

Proof. (1) Schematic and finite follows from [Var04, Proposition 5.7] (recalled in Proposition 3.5.3 and Remark 3.5.4).

(2) Surjectivity is implied by [Var04, Theorem 2.25].

(3) Universally injectivity is implied by the fact that $\mathrm{Bun}_P^{S_M(\mu)} \to \mathrm{Bun}_G^{S_M(\mu)}$ is an isomorphism for μ satisfying the assumption of Theorem 4.2.1 (see [DG15, Proposition 9.2.2]) and the

well-known fact that $\mathrm{Gr}_{P,I,W} \to \mathrm{Gr}_{G,I,W}$ is bijective. (More concretely, it is enough to prove that for any algebraically closed field k containing $\overline{\mathbb{F}_q}$, the map $\mathrm{Cht}_{P,I,W}^{S_M(\mu)}(k) \to \mathrm{Cht}_{G,I,W}^{S_M(\mu)}(k)$ is injective. Let $((x_i), \mathcal{G} \xrightarrow{\phi_G} {}^\tau\mathcal{G}) \in \mathrm{Cht}_{G,I,W}^{S_M(\mu)}(k)$. By (3), there exists $((x_i), \mathcal{P} \xrightarrow{\phi_P} {}^\tau\mathcal{P}) \in \mathrm{Cht}_{P,I,W}^{S_M(\mu)}(k)$ such that $\mathcal{P} \overset{P}{\times} G \simeq \mathcal{G}$ and $\phi_P \overset{P}{\times} G \simeq \phi_G$. Since $\mathrm{Bun}_P^{S_M(\mu)}(k) \to \mathrm{Bun}_G^{S_M(\mu)}(k)$ is injective, $\mathcal{P}$ is unique. Choosing a trivialization of $\mathcal{P}$ over $\Gamma_{\sum \infty x_i}$, we deduce from the injectivity of $\mathrm{Gr}_{P,I,W}(k) \to \mathrm{Gr}_{G,I,W}(k)$ that ϕ_P is unique.) $\square$

4.2.2 Now we consider the morphism $\pi'^{S_M(\mu)}$. For all d large enough, similar to diagram (3.9), we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Cht}'^{S_M(\mu)}_{P,N,I,W} \arrow[dr, dashed, "\pi'^{S_M(\mu)}_d"'] \arrow[drr, "\pi'^{S_M(\mu)}"] \arrow[ddr] & & \\
& \widetilde{\mathrm{Cht}}'^{S_M(\mu)}_{M,N,I,W} \arrow[r] \arrow[d] & \mathrm{Cht}'^{S_M(\mu)}_{M,N,I,W} \arrow[d] \\
& {[P_{I,d}\backslash \mathrm{Gr}_{P,I,W}]} \arrow[r] & {[M_{I,d}\backslash \mathrm{Gr}_{M,I,W}]}
\end{tikzcd} \tag{4.4}$$

where $\widetilde{\mathrm{Cht}}'^{S_M(\mu)}_{M,N,I,W}$ is the fiber product, which depends on d. By 4.1.11, $\mathrm{Cht}'^{S_M(\mu)}_{M,N,I,W}$ is open in $\mathrm{Cht}'_{M,N,I,W}$ and $\mathrm{Cht}'^{S_M(\mu)}_{P,N,I,W}$ is open in $\mathrm{Cht}'_{P,N,I,W}$. By Lemma 3.1.8, the morphism $\pi'^{S_M(\mu)}_d$ is smooth of relative dimension $\dim_{X^I} U_{I,d}$.

We now introduce a notion of unipotent group scheme (which should rather be called 'elementary unipotent group scheme').

DEFINITION 4.2.3. (a) Let H be a group scheme of finite dimension over a scheme S. We say that H is a unipotent group scheme if H admits a filtration $H = H^{(0)} \supset H^{(1)} \supset \cdots \supset H^{(m)} \supset H^{(m+1)} = 0$ such that for every j, the quotient $H^{(j)}/H^{(j+1)}$ is an additive group scheme (i.e. isomorphic to $\mathbb{G}_{a,S}^n$ for some n locally for the étale topology) over S.

(b) A morphism of algebraic stacks $f : \mathcal{X} \to \mathcal{Y}$ is called unipotent if for any scheme S and any morphism $S \to \mathcal{Y}$, the fiber product $S \underset{\mathcal{Y}}{\times} \mathcal{X}$ is locally for the smooth topology on S isomorphic to a quotient stack $[H_1/H_2]$, where H_1 and H_2 are unipotent group schemes over S and H_2 acts on H_1 as a group scheme over S acting on a scheme over S.

THEOREM 4.2.4. *There exists a constant $C(G, X, N, I, d)$, such that if $\mu \in \widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}}$ and $\langle \mu, \alpha \rangle > C(G, X, N, I, d)$ for all $\alpha \in \Gamma_G - \Gamma_M$, then the morphism $\pi'^{S_M(\mu)}_d$ is unipotent in the sense of Definition 4.2.3.*

The proof will be given in §§ 4.3–4.5.

Remark 4.2.5. Theorem 4.2.4 will be used to prove Proposition 4.6.4, where only the statement for the geometric fibers of $\pi'^{S_M(\mu)}_d$ is needed. Since the proof is the same for a geometric fiber or a fiber over a general base, we prove it over a general base.

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4.3 Proof of Theorem 4.2.4: step 1

4.3.1 We have a similar diagram as (4.4) without index $'$. The morphism

$$\pi_d^{S_M(\mu)} : \mathrm{Cht}_{P,N,I,W}^{S_M(\mu)} \to \widetilde{\mathrm{Cht}}_{M,N,I,W}^{S_M(\mu)}$$

is $P(\mathcal{O}_N)$-equivariant and the morphism $\pi_d'^{S_M(\mu)}$ is induced by $\pi_d^{S_M(\mu)} \overset{P(\mathcal{O}_N)}{\times} G(\mathcal{O}_N)$. So to prove Theorem 4.2.4, it is enough to prove the statement for $\pi_d^{S_M(\mu)}$ instead of $\pi_d'^{S_M(\mu)}$.

The problem is local for the smooth topology. So it is enough to prove the statement for the base change by $\mathrm{Gr}_{P,I,W} \to [P_{I,d} \backslash \mathrm{Gr}_{P,I,W}]$:

$$\pi_d^{\times, S_M(\mu)} : \mathrm{Cht}_{P,N,I,W}^{S_M(\mu)} \underset{[P_{I,d}\backslash \mathrm{Gr}_{P,I,W}]}{\times} \mathrm{Gr}_{P,I,W} \to \widetilde{\mathrm{Cht}}_{M,N,I,W}^{S_M(\mu)} \underset{[P_{I,d}\backslash \mathrm{Gr}_{P,I,W}]}{\times} \mathrm{Gr}_{P,I,W}.$$

4.3.2 Note that $\widetilde{\mathrm{Cht}}_{M,N,I,W}^{S_M(\mu)} \underset{[P_{I,d}\backslash \mathrm{Gr}_{P,I,W}]}{\times} \mathrm{Gr}_{P,I,W} \simeq \mathrm{Cht}_{M,N,I,W}^{S_M(\mu)} \underset{[M_{I,d}\backslash \mathrm{Gr}_{M,I,W}]}{\times} \mathrm{Gr}_{P,I,W}$. We have the following commutative diagram, where the front and back Cartesian squares are defined in the proof of [Laf18, Proposition 2.8] (replace G by P and M, respectively). We have already used these Cartesian squares in (3.11) and (3.12).

$$\begin{tikzcd}
\mathrm{Cht}_{P,N,I,W}^{S_M(\mu)} \underset{[P_{I,d}\backslash \mathrm{Gr}_{P,I,W}]}{\times} \mathrm{Gr}_{P,I,W} \arrow[rr] \arrow[dr, "\pi_d^{\times, S_M(\mu)}"] \arrow[dd] & & \mathrm{Bun}_{P,N,I,d}^{S_M(\mu)} \underset{(X \smallsetminus N)^I}{\times} \mathrm{Gr}_{P,I,W} \arrow[dr, "(\pi^{\mathrm{Bun}}, \mathrm{Id})"] \arrow[dd, dotted] & \\
& \widetilde{\mathrm{Cht}}_{M,N,I,W}^{S_M(\mu)} \underset{[P_{I,d}\backslash \mathrm{Gr}_{P,I,W}]}{\times} \mathrm{Gr}_{P,I,W} \arrow[rr, "(b_1^P, b_2^P)"] \arrow[dd] & & \mathrm{Bun}_{M,N,I,d}^{S_M(\mu)} \underset{(X \smallsetminus N)^I}{\times} \mathrm{Gr}_{P,I,W} \arrow[dd, "(b_1^M, b_2^M)"] \\
\mathrm{Bun}_{P,N}^{S_M(\mu)} \arrow[rr, dotted, "(\mathrm{Id},\mathrm{Id})"] \arrow[dr, "\pi^{\mathrm{Bun}}"] & & \mathrm{Bun}_{P,N}^{S_M(\mu)} \times \mathrm{Bun}_{P,N} \arrow[dr, dotted, "(\pi^{\mathrm{Bun}}, \pi^{\mathrm{Bun}})"] & \\
& \mathrm{Bun}_{M,N}^{S_M(\mu)} \arrow[rr, "(\mathrm{Id},\mathrm{Id})"] & & \mathrm{Bun}_{M,N}^{S_M(\mu)} \times \mathrm{Bun}_{M,N}
\end{tikzcd}$$

4.3.3 Now let S be an affine scheme over $\overline{\mathbb{F}_q}$ and let

$$((x_i), (\mathcal{M}, \psi) \xrightarrow{\phi} ({}^{\tau}\mathcal{M}, {}^{\tau}\psi), s) : S \to \widetilde{\mathrm{Cht}}_{M,N,I,W}^{S_M(\mu)} \underset{[P_{I,d}\backslash \mathrm{Gr}_{P,I,W}]}{\times} \mathrm{Gr}_{P,I,W}$$

be an S-point. Consider

$$S \to \widetilde{\mathrm{Cht}}_{M,N,I,W}^{S_M(\mu)} \underset{[P_{I,d}\backslash \mathrm{Gr}_{P,I,W}]}{\times} \mathrm{Gr}_{P,I,W} \to \mathrm{Bun}_{M,N,I,d}^{S_M(\mu)} \underset{(X \smallsetminus N)^I}{\times} \mathrm{Gr}_{P,I,W}$$

and

$$S \to \widetilde{\mathrm{Cht}}_{M,N,I,W}^{S_M(\mu)} \underset{[P_{I,d}\backslash \mathrm{Gr}_{P,I,W}]}{\times} \mathrm{Gr}_{P,I,W} \to \mathrm{Bun}_{M,N}^{S_M(\mu)}.$$

We define $\mathscr{Z}$, $\mathscr{Y}_{N,d}$ and $\mathscr{Y}_N$ to be the following fiber products.

$$\begin{tikzcd} \mathscr{Z} \arrow[r] \arrow[d] & \mathrm{Cht}^{S_M(\mu)}_{P,N,I,W} \underset{[P_{I,d}\backslash \mathrm{Gr}_{P,I,W}]}{\times} \mathrm{Gr}_{P,I,W} \arrow[d, "\pi_d^{\times,\, S_M(\mu)}"] \\ S \arrow[r] & \widetilde{\mathrm{Cht}}^{S_M(\mu)}_{M,N,I,W} \underset{[P_{I,d}\backslash \mathrm{Gr}_{P,I,W}]}{\times} \mathrm{Gr}_{P,I,W} \end{tikzcd} \quad \begin{tikzcd} \mathscr{Y}_{N,d} \arrow[r] \arrow[d] & \mathrm{Bun}^{S_M(\mu)}_{P,N,I,d} \underset{(X\smallsetminus N)^I}{\times} \mathrm{Gr}_{P,I,W} \arrow[d, "(\pi^{\mathrm{Bun}},\mathrm{Id})"] \\ S \arrow[r] & \mathrm{Bun}^{S_M(\mu)}_{M,N,I,d} \underset{(X\smallsetminus N)^I}{\times} \mathrm{Gr}_{P,I,W} \end{tikzcd} \quad \begin{tikzcd} \mathscr{Y}_N \arrow[r] \arrow[d] & \mathrm{Bun}^{S_M(\mu)}_{P,N} \arrow[d, "\pi^{\mathrm{Bun}}"] \\ S \arrow[r] & \mathrm{Bun}^{S_M(\mu)}_{M,N} \end{tikzcd}$$

Applying Lemma B.0.1 to the diagram in 4.3.2, we deduce a Cartesian square

$$\begin{tikzcd} \mathscr{Z} \arrow[r] \arrow[d] & \mathscr{Y}_{N,d} \arrow[d, "(b_1,b_2)"] \\ \mathscr{Y}_N \arrow[r, "(\mathrm{Id},\mathrm{Id})"] & \mathscr{Y}_N \underset{S}{\times} \mathscr{Y}_N \end{tikzcd} \tag{4.5}$$

where b_1 (respectively b_2) is induced by b_1^P (respectively b_2^P).

Remark 4.3.4. By the proof of [Laf18, Proposition 2.8], b_1^P (respectively b_1^M) is the forgetful morphism of the level structure on I (thus smooth) and b_2^P (respectively b_2^M) is the composition of the Frobenius morphism with some other morphism. We deduce that b_1 is smooth and b_2 has zero differential. Moreover, the morphism $\mathrm{Bun}_{P,N} \to \mathrm{Bun}_{M,N}$ is smooth, and thus $\mathscr{Y}_N$ is smooth over S. Similarly $\mathscr{Y}_{N,d}$ is smooth over S. We deduce that $\mathscr{Z}$ is smooth over S. Note that the same argument without $S_M(\mu)$ would give another proof of Lemma 3.1.8.

4.4 Proof of Theorem 4.2.4: step 2

The goal of this subsection is to describe $\mathscr{Y}_N$ and $\mathscr{Y}_{N,d}$.

(1) First we describe the fiber of $\mathrm{Bun}_P^{S_M(\mu)} \to \mathrm{Bun}_M^{S_M(\mu)}$ in Proposition 4.4.4.

4.4.1 We fix a maximal torus $T \subset B$. This allows us to view the Levi quotient M of a standard parabolic subgroup P as a subgroup $M \subset P$ (the unique splitting that contains T). Then $P = M \ltimes U$, where M acts on U via the embedding $M \subset P$ and P acts on U by the adjoint action.

4.4.2 Let $S \to \mathrm{Bun}_M^{S_M(\mu)}$ be a morphism and $\mathcal{M}$ the corresponding M-bundle over $X \times S$. We define the fiber space $U_{\mathcal{M}} := (U \times \mathcal{M})/M$. It is easy to check that it is a group scheme over $X \times S$ (see [Xue17, C.2] for more details).

Definition 4.4.3. Let S be an affine scheme over $\overline{\mathbb{F}_q}$. Let A be a sheaf of groups on $X \times S$. We denote by $\mathrm{pr}_S : X \times S \to S$ the second projection.

(a) We define $R^0(\mathrm{pr}_S)_* A$ as the sheaf of groups on S:

$$(S' \to S) \mapsto \mathrm{Hom}_{X\times S}(X \times S', A'),$$

where A' is the inverse image of A by $X \times S' \to X \times S$.

(b) [Gir71, V.2.1] We define $R^1(\mathrm{pr}_S)_* A$ as the sheaf of sets on S associated to the presheaf:

$$(S' \to S) \mapsto H^1(X \times S', A').$$

Indeed $R^1(\mathrm{pr}_S)_* A$ is a sheaf of pointed sets with a canonical section which corresponds to the trivial A-torsor.

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Proposition 4.4.4. *There exists a constant $C(G,X) \in \mathbb{Q}_{\geqslant 0}$, such that if $\langle \mu, \alpha \rangle > C(G,X)$ for all $\alpha \in \Gamma_G - \Gamma_M$, then $R^0(\mathrm{pr}_S)_* U_{\mathcal{M}}$ is a unipotent group scheme over S and the fiber of $\mathrm{Bun}_P^{S_M(\mu)} \to \mathrm{Bun}_M^{S_M(\mu)}$ over S is the classifying stack $[S/R^0(\mathrm{pr}_S)_* U_{\mathcal{M}}]$.*

Proof. We denote by $\mathscr{Y}$ the fiber of $\mathrm{Bun}_P^{S_M(\mu)} \to \mathrm{Bun}_M^{S_M(\mu)}$ over S. For any scheme $S' \to S$, the groupoid $\mathscr{Y}(S')$ classifies the $U_{\mathcal{M}}\big|_{X\times S'}$-bundle over $X \times S'$ (see [Xue17, Lemme C.3.2] for more details).

By Lemma 4.4.5(b) below, all $U_{\mathcal{M}}$-bundles are trivial. Taking into account that $R^0(\mathrm{pr}_S)_* U_{\mathcal{M}}(S')$ is the group of automorphisms of the trivial $U_{\mathcal{M}}\big|_{X\times S'}$-bundle on $X \times S'$ and Lemma 4.4.5(a), we deduce the proposition. □

Lemma 4.4.5. *There exists a constant $C(G,X) \in \mathbb{Q}_{\geqslant 0}$, such that if $\langle \mu, \alpha \rangle > C(G,X)$ for all $\alpha \in \Gamma_G - \Gamma_M$, then we have the following.*

(a) *The sheaf of groups $R^0(\mathrm{pr}_S)_* U_{\mathcal{M}}$ is a unipotent group scheme.*

(b) *The sheaf of pointed sets $R^1(\mathrm{pr}_S)_* U_{\mathcal{M}}$ is trivial.*

Remark 4.4.6. If U is commutative, then $U_{\mathcal{M}}$ is an additive group scheme over $X \times S$ (in the sense of Definition 4.2.3). Part (a) of Lemma 4.4.5 is automatic and part (b) follows directly from [DG15, Proposition 10.4.5].

The difficulty is that in general, U is not commutative. To prove Lemma 4.4.5, we will need to use a filtration of U where the graded are commutative groups.

4.4.7 We have a canonical filtration of U (see the proof of [DG15, Proposition 11.1.4(c)] for more details):

$$U = U^{(0)} \supset U^{(1)} \supset \cdots \supset U^{(m)} \supset U^{(m+1)} = 0, \tag{4.6}$$

where $U^{(j)}$ is the subgroup generated by the root subgroups corresponding to the positive roots α of G, such that

$$\sum_{\beta \in \Gamma_G - \Gamma_M} \mathrm{coeff}_\beta(\alpha) \geqslant j+1.$$

(Here $\mathrm{coeff}_\beta(\alpha)$ denotes the coefficient of α in simple root β.) For each j, the subgroup $U^{(j+1)}$ of $U^{(j)}$ is normal and the quotient is equipped with an isomorphism $\vartheta^{(j)} : \mathbb{G}_a^{n_j} \xrightarrow{\sim} U^{(j)}/U^{(j+1)}$ for some $n_j \in \mathbb{N}$.

4.4.8 The filtration (4.6) induces for every $j \in \{1, \ldots, m+1\}$ an exact sequence of groups:

$$0 \to U^{(j-1)}/U^{(j)} \to U/U^{(j)} \to U/U^{(j-1)} \to 0. \tag{4.7}$$

For every j, the subgroup $U^{(j)}$ of P is normal. Then P acts on $U^{(j)}$ by the adjoint action and M acts on $U^{(j)}$ via $M \hookrightarrow P$. We deduce that M acts on $U^{(j)}/U^{(j+1)}$ and $U/U^{(j)}$.

We define the fiber spaces $(U^{(j)}/U^{(j+1)})_{\mathcal{M}} := (\mathcal{M} \times U^{(j)}/U^{(j+1)})/M$, it is an additive group scheme over $X \times S$. We define the fiber space $(U/U^{(j)})_{\mathcal{M}} := (\mathcal{M} \times U/U^{(j)})/M$, it is a group scheme over $X \times S$ (see [Xue17, C.2] for more details).

Proposition 4.4.9. *There exists a constant $C(G,X)$ such that for $\mu \in \widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}}$, if $\langle \mu, \alpha \rangle > C(G,X)$ for all $\alpha \in \Gamma_G - \Gamma_M$, then for any $\mathcal{M} \in \mathrm{Bun}_M^{S_M(\mu)}(S)$ and any j, the sheaf $R^1(\mathrm{pr}_S)_*((U^{(j)}/U^{(j+1)})_{\mathcal{M}})$ is trivial.*

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Proof. This is [DG15, Proposition 10.4.5(a)]. We take $C(G,X) := \max_i\{c_i'\}$, where c_i' are the constants in [DG15, Proposition 10.4.5(a)]. □

LEMMA 4.4.10. *Let $0 \to A \to B \to C \to 0$ be an exact sequence of sheaves of groups on $X \times S$.*

(a) *If the sheaf of pointed sets $R^1(\mathrm{pr}_S)_*A$ is trivial, then we have an exact sequence of sheaves of groups:*
$$0 \to R^0(\mathrm{pr}_S)_*A \to R^0(\mathrm{pr}_S)_*B \to R^0(\mathrm{pr}_S)_*C \to 0.$$

(b) *If moreover the sheaf of pointed sets $R^1(\mathrm{pr}_S)_*C$ is also trivial, then the sheaf of pointed sets $R^1(\mathrm{pr}_S)_*B$ is trivial.*

Proof. By [Gir71, V Proposition 2.3], the exact sequence $0 \to A \to B \to C \to 0$ induces an exact sequence of sheaves of pointed sets on S:
$$0 \to R^0(\mathrm{pr}_S)_*A \to R^0(\mathrm{pr}_S)_*B \to R^0(\mathrm{pr}_S)_*C \to R^1(\mathrm{pr}_S)_*A \to R^1(\mathrm{pr}_S)_*B \to R^1(\mathrm{pr}_S)_*C.$$
We deduce the lemma. □

Proof of Lemma 4.4.5. For each j, the exact sequence (4.7) induces an exact sequence of group schemes over $X \times S$:
$$0 \to (U^{(j-1)}/U^{(j)})_{\mathcal{M}} \to (U/U^{(j)})_{\mathcal{M}} \to (U/U^{(j-1)})_{\mathcal{M}} \to 0. \tag{4.8}$$
We apply Lemma 4.4.10 to (4.8) successively for $j = 1, j = 2, \ldots,$ until $j = m+1$. Taking into account the fact that $R^1(\mathrm{pr}_S)_*((U^{(j)}/U^{(j+1)})_{\mathcal{M}})$ is trivial (by Proposition 4.4.9) and $R^0(\mathrm{pr}_S)_*(U^{(j)}/U^{(j+1)})_{\mathcal{M}}$ is additive in the sense of Definition 4.2.3 (because $U^{(j)}/U^{(j+1)} \xrightarrow{\sim} \mathbb{G}_a^{n_j}$), we deduce Lemma 4.4.5. □

(2) Now we add level structure on $N \times S + \Gamma_{\sum dx_i}$ to the argument in (1), i.e. we describe the fiber of $\mathrm{Bun}_{P,N,I,d}^{S_M(\mu)} \to \mathrm{Bun}_{M,N,I,d}^{S_M(\mu)}$ in Proposition 4.4.13.

4.4.11 Let V be a group scheme on $X \times S$. For any divisor $i_D : D \hookrightarrow X \times S$, we denote by $V\big|_D$ the fiber product $D \underset{X\times S}{\times} V$. We denote by $\mathcal{V}$ and $\mathcal{V}\big|_D$ the associated sheaves of groups. We define the sheaf of groups $\mathcal{K}er_{V,D}$ on $X \times S$ as the kernel of the morphism $\mathcal{V} \to (i_D)_*(\mathcal{V}\big|_D)$. If V is smooth, the morphism $\mathcal{V} \to (i_D)_*(\mathcal{V}\big|_D)$ is surjective.

4.4.12 Let S be an affine scheme over $\overline{\mathbb{F}_q}$. Let $((x_i)_{i\in I}, \mathcal{M}, \psi_M)$ be an S-point of $\mathrm{Bun}_{M,N,I,d}^{S_M(\mu)}$. Let $D := N \times S + \Gamma_{\sum dx_i}$. Applying 4.4.11 to the group scheme $U_{\mathcal{M}}$ on $X \times S$, we obtain an exact sequence of sheaves of groups:
$$0 \to \mathcal{K}er_{U_{\mathcal{M}},D} \to \mathcal{U}_{\mathcal{M}} \to (i_D)_*\mathcal{U}_{\mathcal{M}}\big|_D \to 0. \tag{4.9}$$

PROPOSITION 4.4.13. *There exists a constant $C(G,X,N,I,d) \in \mathbb{Q}_{\geqslant 0}$, such that if $\langle \mu, \alpha\rangle > C(G, X, N, I, d)$ for all $\alpha \in \Gamma_G - \Gamma_M$, then $R^0(\mathrm{pr}_S)_*\mathcal{K}er_{U_{\mathcal{M}},D}$ is a unipotent group scheme over S and the fiber of $\mathrm{Bun}_{P,N,I,d}^{S_M(\mu)} \to \mathrm{Bun}_{M,N,I,d}^{S_M(\mu)}$ over S is the classifying stack $[S/R^0(\mathrm{pr}_S)_*\mathcal{K}er_{U_{\mathcal{M}},D}]$.*

Proof. We recall that $\mathscr{Y}_{N,d}$ denotes the fiber of $\mathrm{Bun}_{P,N,I,d}^{S_M(\mu)} \to \mathrm{Bun}_{M,N,I,d}^{S_M(\mu)}$ over S. For any scheme $S' \to S$, the groupoid $\mathscr{Y}_{N,d}(S')$ classifies the data of $(\mathcal{F}, \beta)$, where $\mathcal{F}$ is a $\mathcal{U}_{\mathcal{M}}$-bundle on $X \times S'$ and β is an isomorphism of $\mathcal{U}_{\mathcal{M}}$-bundles $\mathcal{F}\big|_D \xrightarrow{\sim} \mathcal{U}_{\mathcal{M}}\big|_D$. By (4.9), this groupoid is equivalent to the groupoid of $\mathcal{K}er_{U_{\mathcal{M}},D}$-bundles on $X \times S'$.

Similarly to the case without level, Proposition 4.4.13 follows from Lemma 4.4.14 below. □

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Lemma 4.4.14. *There exists a constant $C(G,X,N,I,d)\in\mathbb{Q}_{\geqslant 0}$, such that if $\langle\mu,\alpha\rangle > C(G,X,N,I,d)$ for all $\alpha\in\Gamma_G-\Gamma_M$, then we have the following.*

(a) *The sheaf of groups $R^0(\mathrm{pr}_S)_*\mathcal{K}er_{U_{\mathcal{M}},D}$ is a unipotent group scheme.*

(b) *The sheaf of pointed sets $R^1(\mathrm{pr}_S)_*\mathcal{K}er_{U_{\mathcal{M}},D}$ is trivial.*

Proof. The proof is the same as Lemma 4.4.5, except that we replace $(U^{(j-1)}/U^{(j)})_{\mathcal{M}}$ by $\mathcal{K}er_{(U^{(j-1)}/U^{(j)})_{\mathcal{M}},D}$, and that we use Lemma 4.4.15 below instead of Proposition 4.4.9. $\square$

Lemma 4.4.15. *There exists a constant $C(G,X,N,I,d)\in\mathbb{Q}_{\geqslant 0}$ such that for $\mu\in\widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}}$, if $\langle\mu,\alpha\rangle > C(G,X,N,I,d)$, for all $\alpha\in\Gamma_G-\Gamma_M$, then for any $((x_i),\mathcal{M},\psi)\in\mathrm{Bun}^{S_M(\mu)}_{M,N,I,d}(S)$ and any j, the sheaf $R^1(\mathrm{pr}_S)_*((U^{(j)}/U^{(j+1)})_{\mathcal{M}}(-N\times S-\Gamma_{\sum dx_i}))$ is trivial.*

Proof. Let $C(G,X,N,I,d):=C(G,X)+\deg N+|I|\cdot d$, where $C(G,X)$ is the constant in Proposition 4.4.9. We repeat the argument in [DG15, Proposition 10.4.5], except that in [DG15, Remark 10.3.5] we replace the reductive group $\widetilde{G}$ by $\widetilde{G}\times\mathbb{G}_m$ and the $\widetilde{G}$-bundle $\mathcal{F}_{\widetilde{G}}$ by the $\widetilde{G}\times\mathbb{G}_m$-bundle $\mathcal{F}_{\widetilde{G}}\times\mathcal{O}(-N\times S-\Gamma_{\sum dx_i})$. $\square$

4.5 Proof of Theorem 4.2.4: step 3

4.5.1 Let S be a scheme over $\mathbb{F}_q$. Let H_S and H'_S be two group schemes over S. Let $f: H'_S\to H_S$ be a morphism of group schemes over S. We denote by $[S/H'_S]$ the classifying stack of H'_S on S. Similarly for $[S/H_S]$. Then f induces a morphism of stacks: $\overline{f}:[S/H'_S]\to[S/H_S]$.

Lemma 4.5.2. *Let $f,g: H'_S\to H_S$ be two morphisms of connected group schemes. Let $[H_S/H'_S]$ be the quotient stack where H'_S acts on H_S by $h'\cdot h=f(h')hg(h')^{-1}$. Then the following diagram is Cartesian*

$$\begin{tikzcd}
{[H_S/H'_S]} \arrow[r] \arrow[d] & {[S/H'_S]} \arrow[d, "(\overline{f},\overline{g})"] \\
{[S/H_S]} \arrow[r, "(\mathrm{id},\mathrm{id})"] & {[S/H_S]\underset{S}{\times}[S/H_S]}
\end{tikzcd} \tag{4.10}$$

where the morphism $[H_S/H'_S]\to[S/H_S]$ is induced by $H_S\to S$ and $H'_S\xrightarrow{g}H_S$.

Proof. The fiber product is $[H_S\times_S H_S/H'_S\times_S H_S]$, where H'_S acts on $H_S\times_S H_S$ by (f,g) (from the left) and H_S acts on $H_S\times_S H_S$ by diagonal action (from the right). The morphism $\alpha:[H_S\times_S H_S/H'_S\times_S H_S]\to[S/H_S]$ (respectively $\beta:[H_S\times_S H_S/H'_S\times_S H_S]\to[S/H'_S]$) is given by $H_S\times_S H_S\to S$ and the second projection $H'_S\times_S H_S\to H_S$ (respectively the first projection $H'_S\times_S H_S\to H'_S$).

The morphism of group schemes over S

$$H_S\times_S H_S\to H_S\times_S H_S,\quad (x,y)\mapsto(xy^{-1},y) \tag{4.11}$$

is an isomorphism. Moreover, it is $H'_S\times_S H_S$-equivariant for the action of $H'_S\times_S H_S$ on the left-hand side as above and the action of $H'_S\times_S H_S$ on the right-hand side given by (h',h) $(z,t)=(f(h')zg(h')^{-1},g(h')th^{-1})$. The isomorphism (4.11) induces an isomorphism of quotient stacks

$$[H_S\times_S H_S/H'_S\times_S H_S]\underset{(4.11)}{\xrightarrow{\sim}}[H_S\times_S H_S/H'_S\times_S H_S]\simeq[H_S/H'_S], \tag{4.12}$$

where H'_S acts on H_S by $h'\cdot x=f(h')xg(h')^{-1}$. The morphism $[H_S/H'_S]\to[S/H_S]$ is the composition of the inverse of (4.12) and α. $\square$

LEMMA 4.5.3. *Let S be an affine scheme. Let H_1 and H_2 be two unipotent group schemes over S. Let $\varphi : [S/H_1] \to [S/H_2]$ be a morphism of stacks. Then there exists $f : H_1 \to H_2$ a morphism of group schemes over S such that $\varphi = \overline{f}$.*

Proof. Since S is affine and H_2 is unipotent, all H_2-torsors on S are trivial. The morphism φ is given by a H_2-torsor $\mathscr{H}$ on S which is H_1-equivariant. We trivialize $\mathscr{H}$ as a H_2-torsor. Then the action of H_1 on $\mathscr{H}$ gives the morphism f. □

End of the proof of Theorem 4.2.4. Let μ satisfy the hypothesis in Proposition 4.4.13; then $\mathscr{Y}_{N,d} = [S/H_{N,d}]$ (respectively $\mathscr{Y}_N = [S/H_N]$), where $H_{N,d} := R^0(\mathrm{pr}_S)_*\mathcal{K}er_{U_{\mathcal{M}},N\times S+\Gamma_{\sum dx_i}}$ (respectively $H_N := R^0(\mathrm{pr}_S)_*\mathcal{K}er_{U_{\mathcal{M}},N\times S}$) is a unipotent group scheme over S.

By Lemma 4.5.3, the two morphisms b_1 and b_2 in diagram (4.5) are induced by two morphisms of group schemes $f_1, f_2 : H_{N,d} \to H_N$. By Lemma 4.5.2, $\mathscr{Z}$ is isomorphic to $[H_N/H_{N,d}]$, where $H_{N,d}$ acts on H_N by $h' \cdot h = f_1(h')hf_2(h')^{-1}$. □

4.6 Cohomological statements

DEFINITION 4.6.1. Let d_W be the smallest integer in Proposition 2.2.1 such that the action of $G_{I,\infty}$ on $\mathrm{Gr}_{G,I,W}$ factors through G_{I,d_W}. We have defined the constants $C'(G,X,N,W)$ and $C(G,X,N,I,d_W)$ in Theorems 4.2.1 and 4.2.4 respectively. We take

$$\widetilde{C}(G,X,N,W) := \mathrm{Max}\{C'(G,X,N,W),\ C(G,X,N,I,d_W)\}.$$

DEFINITION 4.6.2. Let $\mu \in \widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}}$. For any $j \in \mathbb{Z}$, we define degree j cohomology sheaves

$$\mathcal{H}^{j,\,S_M(\mu)}_{G,N,I,W} = R^j(\mathfrak{p}_G)_!(\mathcal{F}^{\Xi}_{G,N,I,W}\big|_{\mathrm{Cht}^{S_M(\mu)}_{G,N,I,W}/\Xi}),$$
$$\mathcal{H}'^{\,j,\,S_M(\mu)}_{M,N,I,W} = R^j(\mathfrak{p}'_M)_!(\mathcal{F}'^{\,\Xi}_{M,N,I,W}\Big|_{\mathrm{Cht}'^{S_M(\mu)}_{M,N,I,W}/\Xi}).$$

4.6.3 If $\langle\mu,\alpha\rangle > \widetilde{C}(G,X,N,W)$ for all $\alpha \in \Gamma_G - \Gamma_M$, then by Theorem 4.2.1, the morphism $i'^{\,S_M(\mu)} : \mathrm{Cht}'^{S_M(\mu)}_{P,N,I,W} \to \mathrm{Cht}^{S_M(\mu)}_{G,N,I,W}$ is proper and schematic. Applying the construction in § 3 to the truncation $S_M(\mu)$, we obtain a constant term morphism (in $D^b_c((X\smallsetminus N)^I, \mathbb{Q}_\ell)$):

$$\mathcal{C}^{P,\,j,\,S_M(\mu)}_{G,N} : \mathcal{H}^{j,\,S_M(\mu)}_{G,N,I,W} \to \mathcal{H}'^{\,j,\,S_M(\mu)}_{M,N,I,W}. \tag{4.13}$$

Here is the main result of § 4.

PROPOSITION 4.6.4. *Let P be a parabolic subgroup of G and M its Levi quotient. For $\mu \in \widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}}$, if $\langle\mu,\alpha\rangle > \widetilde{C}(G,X,N,W)$ for all $\alpha \in \Gamma_G - \Gamma_M$, then for any j, morphism (4.13) is an isomorphism.*

Proof. By (3.33), $\mathcal{C}^{P,\,j,\,S_M(\mu)}_{G,N}$ is the composition of two morphisms:

$$R^j(\mathfrak{p}_G)_!(\mathcal{F}^{\Xi}_{G,N,I,W}\big|_{\mathrm{Cht}^{S_M(\mu)}_{G,N,I,W}}) \xrightarrow{(1)} R^j(\mathfrak{p}'_M)_!(\pi'^{\,S_M(\mu)})_!(i'^{\,S_M(\mu)})^*(\mathcal{F}^{\Xi}_{G,N,I,W}\big|_{\mathrm{Cht}^{S_M(\mu)}_{G,N,I,W}})$$
$$\xrightarrow{(2)} R^j(\mathfrak{p}'_M)_!(\mathcal{F}'^{\,\Xi}_{M,N,I,W}\big|_{\mathrm{Cht}'^{S_M(\mu)}_{M,N,I,W}}).$$

The morphism (1) is induced by the composition of functors

$$R^j(\mathfrak{p}_G)_! \to R^j(\mathfrak{p}_G)_!(i'^{\,S_M(\mu)})_*(i'^{\,S_M(\mu)})^* \simeq R^j(\mathfrak{p}_G)_!(i'^{\,S_M(\mu)})_!(i'^{\,S_M(\mu)})^*$$
$$\xrightarrow{\sim} R^j(\mathfrak{p}'_M)_!(\pi'^{\,S_M(\mu)})_!(i'^{\,S_M(\mu)})^*$$

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defined in (3.32). By Theorem 4.2.1 and Lemma 4.6.5 below applied to $i'^{S_M(\mu)}$, the morphism (1) is an isomorphism.

The morphism (2) is induced by the morphism

$$(\pi'^{S_M(\mu)})_!(i'^{S_M(\mu)})^*\mathcal{F}^{\Xi}_{G,N,I,W} \to \mathcal{F}'^{\Xi}_{M,N,I,W}$$

defined in (3.23), which is a composition of the counit map

$$\mathrm{Co} : (\pi'^{S_M(\mu)}_d)_!(\pi'^{S_M(\mu)}_d)^! \to \mathrm{Id}$$

and some isomorphisms. By Theorem 4.2.4 and Lemma 4.6.6 below applied to $\pi'^{S_M(\mu)}_d$, the morphism (2) is an isomorphism. □

Lemma 4.6.5. *Let $f : \mathscr{X} \to \mathscr{Y}$ be a schematic finite universal homeomorphism of algebraic stacks; then the unit map* $\mathrm{Id} \to f_*f^*$ *is an isomorphism.*

Lemma 4.6.6. *Let $f : \mathscr{X} \to \mathscr{Y}$ be an unipotent morphism of algebraic stacks (see Definition 4.2.3); then the counit map $f_!f^! \to \mathrm{Id}$ is an isomorphism.*

Proof. The proof consists of four steps.

(i) Using proper base change and the fact that f is smooth, we reduce to the case when $\mathscr{Y} = \operatorname{Spec} k$ is a point, and thus $\mathscr{X} = U_1/U_2$ is a quotient of unipotent group schemes U_1 and U_2 over k.

Indeed, to prove the lemma, it is enough to prove that for any geometric point $i_y : y \to \mathscr{Y}$, the morphism $(i_y)^*f_!f^! \to (i_y)^*$ is an isomorphism. Form the following Cartesian square.

$$\begin{tikzcd} f^{-1}(y) \arrow[r, "\widetilde{i_y}"] \arrow[d, "\widetilde{f}"] & \mathscr{X} \arrow[d, "f"] \\ y \arrow[r, "i_y"] & \mathscr{Y} \end{tikzcd} \tag{4.14}$$

Since f is smooth, we have $f^! \simeq f^*[2n](n)$ and $(\widetilde{f})^! \simeq (\widetilde{f})^*[2n](n)$, where n is the dimension of f. We deduce that

$$(i_y)^*f_!f^! \simeq (\widetilde{f})_!(\widetilde{i_y})^*f^! \simeq (\widetilde{f})_!(\widetilde{i_y})^*(f)^*[2n](n) \simeq (\widetilde{f})_!(\widetilde{f})^*(i_y)^*[2n](n) \simeq (\widetilde{f})_!(\widetilde{f})^!(i_y)^*, \tag{4.15}$$

where the first isomorphism is the proper base change [LO08, § 12]. Thus it is enough to prove that $(\widetilde{f})_!(\widetilde{f})^!(i_y)^* \to (i_y)^*$ is an isomorphism.

(ii) We denote by BU_2 the classifying stack of U_2 over k. Let $f_1 : U_1/U_2 \to BU_2$ and $f_2 : BU_2 \to \operatorname{Spec} k$ be the canonical morphisms. Then $f = f_2 \circ f_1$. We have a commutative diagram of functors.

$$\begin{tikzcd} f_!f^! = (f_2)_!(f_1)_!(f_1)^!(f_2)^! \arrow[r] \arrow[d] & \mathrm{Id} \\ (f_2)_!(f_2)^! \arrow[ur] & \end{tikzcd} \tag{4.16}$$

Thus it is enough to prove that the counit maps $(f_1)_!(f_1)^! \to \mathrm{Id}$ and $(f_2)_!(f_2)^! \to \mathrm{Id}$ are isomorphisms.

(iii) Note that f_1 is a U_1-torsor over BU_2. By Definition 4.2.3, we reduce to the case of $\mathbb{A}^1$-torsor. Using (i) again, we reduce to the case when f_1 is the map $\mathbb{A}^1 \to \operatorname{Spec} k$, where it is clear that $(f_1)_!(f_1)^! \to \mathrm{Id}$ is an isomorphism.

(iv) Let $g_2 : \operatorname{Spec} k \to BU_2$ be the canonical morphism. Then $f_2 \circ g_2 \simeq \mathrm{Id}$. We have a commutative diagram of functors.

$$\begin{tikzcd} (f_2)_!(g_2)_!(g_2)^!(f_2)^! \arrow[r, "\simeq"] \arrow[d] & \mathrm{Id} \\ (f_2)_!(f_2)^! \arrow[ur] & \end{tikzcd} \tag{4.17}$$

We deduce that to prove that $(f_2)_!(f_2)^! \to \mathrm{Id}$ is an isomorphism, it is enough to prove that $(g_2)_!(g_2)^! \to \mathrm{Id}$ is an isomorphism. Note that g_2 is a U_2-torsor over BU_2. Just like in (iii), we prove that $(g_2)_!(g_2)^! \to \mathrm{Id}$ is an isomorphism. □

Remark 4.6.7. In fact, to prove that the morphism (2) in Proposition 4.6.4 is an isomorphism, it is enough to write $\pi_d'^{S_M(\mu)}$ as the tower

$$\mathrm{Cht}_P^{S_M(\mu)} \xrightarrow{\pi_{d,m}} \cdots \to \widetilde{\mathrm{Cht}}_{P/U^{(j+1)}} \xrightarrow{\pi_{d,j}} \widetilde{\mathrm{Cht}}_{P/U^{(j)}} \to \cdots \xrightarrow{\pi_{d,0}} \widetilde{\mathrm{Cht}}_M$$

and prove that for each j, the morphism $\mathrm{Co} : (\pi_{d,j})_!(\pi_{d,j})^! \to \mathrm{Id}$ is an isomorphism. For this, we only need the statement of Theorem 4.2.4 for each $\pi_{d,j}$ (and replace unipotent group scheme by additive group scheme). The proof of such a statement still uses the three steps, but in step 2 Remark 4.4.6 we only need to consider the case of commutative groups.

5. Finiteness of the cuspidal cohomology

The goal of this section is to prove the following.

Theorem 5.0.1. *The $\mathbb{Q}_\ell$-vector space $H^{j,\,\mathrm{cusp}}_{G,N,I,W}$ (defined in Definition 3.5.13) has finite dimension.*

Theorem 5.0.1 will be a direct consequence of the following proposition.

Proposition 5.0.2. *Let G, X, N, I, W as before. There exists $\mu_0 \in \widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}}$ (depending on G, X, N, W and j) such that*

$$H^{j,\,\mathrm{cusp}}_{G,N,I,W} \subset \mathrm{Im}(H^{j,\,\leqslant\mu_0}_{G,N,I,W} \to H^{j}_{G,N,I,W}).$$

The proof of this proposition is essentially based on Proposition 4.6.4 and an induction argument on the semisimple rank of the group G. We will present our strategy in § 5.1 and give the proof in §§ 5.2–5.4.

Notation 5.0.3. In the remaining part of this section, to simplify the notations, we will omit the indices N, I, W.

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5.1 Strategy of the proof

5.1.1 We denote by $\widehat{R}_{G^{\mathrm{ad}}}$ the coroot lattice of G^{ad}. We have $\widehat{R}_{G^{\mathrm{ad}}} \subset \widehat{\Lambda}_{G^{\mathrm{ad}}}$. Let $\widehat{R}^{+}_{G^{\mathrm{ad}}} := \widehat{\Lambda}^{+}_{G^{\mathrm{ad}}} \cap \widehat{R}_{G^{\mathrm{ad}}}$. For any $r \in \mathbb{N}$, we have $(1/r)\widehat{R}^{+}_{G^{\mathrm{ad}}} \subset \widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}}$ and

$$\varinjlim_{\mu \in \widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}}} H_G^{j, \leqslant \mu} = \varinjlim_{\mu \in (1/r)\widehat{R}^{+}_{G^{\mathrm{ad}}}} H_G^{j, \leqslant \mu}.$$

Let $\iota : \widehat{\Lambda}^{\mathbb{Q}}_{Z_M/Z_G} \subset \widehat{\Lambda}^{\mathbb{Q}}_{G^{\mathrm{ad}}}$ be the inclusion. We fix r such that $\bigcup_{P \subsetneq G} \iota \circ \mathrm{pr}_P^{\mathrm{ad}}(\widehat{\Lambda}^{+}_{G^{\mathrm{ad}}}) \subset (1/r)\widehat{R}^{+}_{G^{\mathrm{ad}}}$, where $\mathrm{pr}_P^{\mathrm{ad}} : \widehat{\Lambda}^{\mathbb{Q}}_{G^{\mathrm{ad}}} \to \widehat{\Lambda}^{\mathbb{Q}}_{Z_M/Z_G}$ is defined in (1.18).

5.1.2 For any $\alpha \in \Gamma_G$, we denote by $\check{\alpha} \in \widehat{\Gamma}_G$ the corresponding coroot, and vice versa. Let P_α be the maximal parabolic subgroup with Levi quotient M_α such that $\Gamma_G - \Gamma_{M_\alpha} = \{\alpha\}$.

In this section, for $\mu \in \widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}}$, we will write $\mu - (1/r)\check{\alpha}$ instead of $\mu - (1/r)\Upsilon_G(\check{\alpha})$, where $\Upsilon_G : \widehat{\Lambda}^{\mathbb{Q}}_{G} \to \widehat{\Lambda}^{\mathbb{Q}}_{G^{\mathrm{ad}}}$ is defined in 1.15.

5.1.3 We have defined the inductive limits H_G^{j} in Definition 2.5.5 and $H'^{\,j}_{M_\alpha}$ in Definition 3.4.9. For any $\lambda \in (1/r)\widehat{R}^{+}_{G^{\mathrm{ad}}}$, let $\mathcal{I}_\lambda : H_G^{j, \leqslant \lambda} \to H_G^{j}$ be the morphism to the inductive limit. Let $H_G^{j, \leqslant \lambda} \to H'^{\,j}_{M_\alpha}$ be the composition of morphisms $H_G^{j, \leqslant \lambda} \xrightarrow{\mathcal{I}_\lambda} H_G^{j} \xrightarrow{C_G^{P_\alpha, j}} H'^{\,j}_{M_\alpha}$, where the second morphism is defined in Definition 3.5.10.

5.1.4 Since for every $c \in H_G^{j}$, there exists $\lambda \in \widehat{R}^{+}_{G^{\mathrm{ad}}}$ large enough such that $c \in \mathrm{Im}(H_G^{j, \leqslant \lambda} \to H_G^{j})$, Proposition 5.0.2 will be a direct consequence of part (b) in the following proposition.

Proposition 5.1.5. *Let G be a connected split reductive group. There exists a constant $C_G^0 \in \mathbb{Q}^{\geqslant 0}$ (depending on G, X, N, W, j), such that the following properties hold.*

(a) *Let $\mu \in (1/r)\widehat{R}^{+}_{G^{\mathrm{ad}}}$ such that $\langle \mu, \gamma \rangle \geqslant C_G^0$ for all $\gamma \in \Gamma_G$. Then for any $\alpha \in \Gamma_G$ such that $\mu - (1/r)\check{\alpha} \in (1/r)\widehat{R}^{+}_{G^{\mathrm{ad}}}$ (which is automatic if $C_G^0 > 2/r$), the morphism*

$$\mathrm{Ker}(H_G^{j, \leqslant \mu - (1/r)\check{\alpha}} \to H'^{\,j}_{M_\alpha}) \to \mathrm{Ker}(H_G^{j, \leqslant \mu} \to H'^{\,j}_{M_\alpha})$$

is surjective.

(b) *There exists $\mu_0 \in (1/r)\widehat{R}^{+}_{G^{\mathrm{ad}}}$ (depending on C_G^0), such that for any $\lambda \in (1/r)\widehat{R}^{+}_{G^{\mathrm{ad}}}$ satisfying $\lambda \geqslant \mu_0$ and $\langle \lambda, \gamma \rangle \geqslant C_G^0$ for all $\gamma \in \Gamma_G$, the morphism*

$$\mathrm{Ker}\Big(H_G^{j, \leqslant \mu_0} \to \prod_{P \subsetneq G} H'^{\,j}_{M}\Big) \to \mathrm{Ker}\Big(H_G^{j, \leqslant \lambda} \to \prod_{P \subsetneq G} H'^{\,j}_{M}\Big)$$

is surjective.

(c) *There exists a constant $C_G \geqslant C_G^0$, such that for any $\lambda \in (1/r)\widehat{R}^{+}_{G^{\mathrm{ad}}}$ satisfying $\langle \lambda, \gamma \rangle \geqslant C_G$ for all $\gamma \in \Gamma_G$, the morphism $\mathcal{I}_\lambda : H_G^{j, \leqslant \lambda} \to H_G^{j}$ is injective.*

5.1.6 The proof of Proposition 5.1.5 uses an induction argument on the semisimple rank of the group G: first we prove the statements (a), (b) and (c) for every Levi subgroup of G of rank 0. Second we prove the key step: for $n \geqslant 1$, if (c) is true for all Levi subgroups of rank $n-1$, then (a) is true for all Levi subgroups of rank n. Then we deduce easily (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c) for all Levi subgroups of rank n.

5.1.7 As in 4.4.1, we fix a maximal torus $T \subset B$ and view the Levi quotient M of a (standard) parabolic P as a subgroup $M \subset P$.

Recall that we have fixed $\Xi = \Xi_G \subset Z_G(\mathbb{A})$ in 0.0.4. Applying 0.0.4 to each Levi subgroup M of G, we fix $\Xi_M \subset Z_M(\mathbb{A})$. Moreover, we choose Ξ_M for different Levi subgroups in a compatible way: if M_2 is a Levi subgroup of M_1, then we have $\Xi_G \subset \Xi_{M_1} \subset \Xi_{M_2} \subset T(\mathbb{A})$.

5.2 Beginning of the induction: semisimple rank 0

5.2.1 The only Levi subgroup of semisimple rank 0 is the maximal torus T. Then T^{ad} is trivial and $\widehat{\Lambda}^{+}_{T^{\mathrm{ad}}} = \widehat{\Lambda}_{T^{\mathrm{ad}}}$ has only one element: 0.

The algebraic stack Cht_T/Ξ_T is of finite type. There is only one term in the inductive limit H^j_T, which is of finite dimension.

There is no constant term morphism for T. So we have $H^{j,\,\mathrm{cusp}}_T = H^j_T$.

Lemma 5.2.2. *Take $C^0_T = C_T = 0$ and $\mu_0 = 0$. Proposition 5.1.5 is true for T.*

5.3 From semisimple rank $n-1$ to n

Lemma 5.3.1. *Let G be a connected split reductive group of semisimple rank n. Suppose that Proposition 5.1.5(c) is true for every Levi quotient M of G of semisimple rank $n-1$, with a constant C_M. We take*

$$C^0_G := \mathrm{Max}\{\{C_M \mid M \text{ Levi quotient of semisimple rank } n-1 \text{ of } G\},\ \widetilde{C}(G,X,N,W)\},$$

where $\widetilde{C}(G,X,N,W)$ is the constant defined in Definition 4.6.1. Then for this constant C^0_G Proposition 5.1.5(a) is true for G.

We need some preparations before the proof of Lemma 5.3.1.

5.3.2 Let $\mu \in (1/r)\widehat{R}^{+}_{G^{\mathrm{ad}}}$ such that $\langle \mu, \gamma\rangle \geqslant C^0_G$ for all $\gamma \in \Gamma_G$. Let $\alpha \in \Gamma_G$ such that $\mu - (1/r)\check{\alpha} \in (1/r)\widehat{R}^{+}_{G^{\mathrm{ad}}}$. Let $P := P_\alpha$ and $M := M_\alpha$ as in 5.1.2. Note that $\Gamma_G - \Gamma_M = \{\alpha\}$.

Lemma 5.3.3. *Let $S_1 = \{\lambda \in (1/r)\widehat{R}^{+}_{G^{\mathrm{ad}}} | \lambda \leqslant \mu - (1/r)\check{\alpha}\}$ and $S_2 = \{\lambda \in (1/r)\widehat{R}^{+}_{G^{\mathrm{ad}}} | \lambda \leqslant \mu\}$. Then*

$$S_2 - S_1 = S_M(\mu) \cap \left(\frac{1}{r}\widehat{R}^{+}_{G^{\mathrm{ad}}}\right), \tag{5.1}$$

where $S_M(\mu)$ is defined in Definition 4.1.1.

Proof. For any $\lambda \in S_2$, we have $\mu - \lambda = \sum_{\check{\gamma}\in\widehat{\Gamma}_G} (c_\gamma/r)\check{\gamma}$ for some $c_\gamma \in \mathbb{Z}_{\geqslant 0}$. Thus

$$\left(\mu - \frac{1}{r}\check{\alpha}\right) - \lambda = \left(\frac{c_\alpha}{r} - \frac{1}{r}\right)\check{\alpha} + \sum_{\check{\gamma}\in\widehat{\Gamma}_G,\, \check{\gamma}\neq\check{\alpha}} \frac{c_\gamma}{r}\check{\gamma}, \quad c_\gamma \in \mathbb{Z}_{\geqslant 0}. \tag{5.2}$$

If moreover $\lambda \notin S_1$, then in (5.2), there should be at least one coefficient strictly negative. So we must have $c_\alpha - 1 < 0$. Since $c_\alpha \in \mathbb{Z}_{\geqslant 0}$, we must have $c_\alpha = 0$. We deduce that

$$\mu - \lambda = \sum_{\check{\gamma}\in\widehat{\Gamma}_G,\, \check{\gamma}\neq\check{\alpha}} \frac{c_\gamma}{r}\check{\gamma} = \sum_{\check{\gamma}\in\widehat{\Gamma}_M} \frac{c_\gamma}{r}\check{\gamma}, \quad c_\gamma \in \mathbb{Z}_{\geqslant 0}.$$

By Definition 4.1.1, we have $\lambda \in S_M(\mu)$. □

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LEMMA 5.3.4. *Let μ and M as in 5.3.2. Suppose that Proposition 5.1.5(c) is true for M. Then for any $j \in \mathbb{Z}$, the morphism $H'^{j,\leqslant\mu}_M \to H'^{j}_M$ is injective.*

The point of the proof of this lemma is to replace the quotient by Ξ_M in (5.3) by the quotient by Ξ_G in (5.5).

Proof. By Proposition 5.1.5(c) for M, for any $\lambda \in (1/r)\widehat{R}^{+}_{M^{\mathrm{ad}}}$ satisfying $\langle\lambda,\gamma\rangle > C_M$ for all $\gamma \in \Gamma_M$, the morphism

$$H^j_c(\mathrm{Cht}^{\leqslant^{M^{\mathrm{ad}}}\lambda}_{M,\overline{\eta^I}}/\Xi_M, \mathcal{F}_M) \to H^j_c(\mathrm{Cht}_{M,\overline{\eta^I}}/\Xi_M, \mathcal{F}_M) \tag{5.3}$$

is injective, where everything is defined as in § 2.5 by replacing G by M.

We can assume that Ξ_M in 5.1.7 is small enough (containing Ξ_G). Then for any $\nu \in A_M$ (defined in 1.5.7), the composition of morphisms

$$\mathrm{Cht}^{\nu}_M/\Xi_G \to \mathrm{Cht}_M/\Xi_G \to \mathrm{Cht}_M/\Xi_M \tag{5.4}$$

is an open and closed immersion.

(For the following discussion, see [Xue17, Illustration 7.4.4] for an example for $G = \mathrm{GL}_3$.) Let $\nu \leqslant \mathrm{pr}^{\mathrm{ad}}_P(\mu)$. We use a special case of 1.5.17. By 1.5.13, we have $\mathrm{pr}^{\mathrm{ad}}_P \circ \Upsilon_G(\check{\alpha}) > 0$. Let $c_\alpha \in \mathbb{Q}_{\geqslant 0}$ such that $\mathrm{pr}^{\mathrm{ad}}_P(\mu) - c_\alpha\,\mathrm{pr}^{\mathrm{ad}}_P \circ \Upsilon_G(\check{\alpha}) = \nu$. Let $\mu_\nu := \mu - c_\alpha\check{\alpha}$. For any $\lambda \in (1/r)\widehat{R}_{G^{\mathrm{ad}}}$, the condition $\lambda \leqslant \mu$ and $\mathrm{pr}^{\mathrm{ad}}_P(\lambda) = \nu$ is equivalent to $\lambda \leqslant^{\overline{M}} \mu_\nu$. We deduce that $\mathrm{Cht}^{\leqslant\mu,\,\nu}_M = \mathrm{Cht}^{\leqslant^{\overline{M}}\mu_\nu}_M$.

Let $\Psi : \overline{M} \twoheadrightarrow M^{\mathrm{ad}}$. If $\mu_1 \leqslant \mu_2$, then $\mu_{1,\nu} \leqslant^{\overline{M}} \mu_{2,\nu}$ and $\Psi(\mu_{1,\nu}) \leqslant^{M^{\mathrm{ad}}} \Psi(\mu_{2,\nu})$. For all $\gamma \in \Gamma_M$, since $\langle\check{\alpha},\gamma\rangle \leqslant 0$, we have $\langle\mu_\nu,\gamma\rangle \geqslant \langle\mu,\gamma\rangle$. By hypothesis $\langle\mu,\gamma\rangle \geqslant C^0_G \geqslant C_M$, so $\langle\mu_\nu,\gamma\rangle \geqslant C_M$. Then the injectivity of (5.3) with $\lambda = \Psi(\mu_\nu)$ implies that the morphism

$$H^j_c(\mathrm{Cht}^{\leqslant\mu,\,\nu}_{M,\overline{\eta^I}}/\Xi_G, \mathcal{F}_M) \to H^j_c(\mathrm{Cht}^{\nu}_{M,\overline{\eta^I}}/\Xi_G, \mathcal{F}_M) \tag{5.5}$$

is injective. Note that we have defined $H^{j,\leqslant\mu,\,\nu}_M = H^j_c(\mathrm{Cht}^{\leqslant\mu,\,\nu}_{M,\overline{\eta^I}}/\Xi_G, \mathcal{F}_M)$ in Definition 2.6.6 and $H^{j,\,\nu}_M = H^j_c(\mathrm{Cht}^{\nu}_{M,\overline{\eta^I}}/\Xi_G, \mathcal{F}_M)$ in Definition 2.6.9.

Moreover, since $\mathrm{Cht}'_M = \mathrm{Cht}_M \overset{P(\mathcal{O}_N)}{\times} G(\mathcal{O}_N)$ is a disjoint union of copies of Cht_M, we deduce that the morphism $H'^{j,\leqslant\mu,\,\nu}_M \to H'^{j,\,\nu}_M$ is also injective, where $H'^{j,\leqslant\mu,\,\nu}_M$ is defined in Definition 3.4.7 and $H'^{j,\,\nu}_M$ is defined in Definition 3.4.10.

Note that by Lemma 1.5.14, for $\nu \notin A_M$ or $\nu \not\leqslant \mathrm{pr}^{\mathrm{ad}}_P(\mu)$, the cohomology group $H'^{j,\,\leqslant\mu,\,\nu}_M = 0$. By Remark 3.5.11, we have a commutative diagram,

$$\begin{tikzcd}
\varinjlim_\mu H'^{j,\leqslant\mu}_M \arrow[r, "f"] & \prod_{\nu\in\widehat{\Lambda}^{\mathbb{Q}}_{Z_M/Z_G}} H'^{j,\,\nu}_M \\
H'^{j,\leqslant\mu}_M = \prod_{\nu\in\widehat{\Lambda}^{\mu}_{Z_M/Z_G}} H'^{j,\leqslant\mu,\,\nu}_M \arrow[u, "g"] \arrow[ur, "h"] &
\end{tikzcd}$$

where f is (3.41) and h is induced component by component by $H'^{j,\leqslant\mu,\,\nu}_M \to H'^{j,\,\nu}_M$. By the above discussion, h is injective. We deduce that the morphism g is injective. $\square$

Proof of Lemma 5.3.1. The proof consists of four steps.

(1) Let S_1 and S_2 as in Lemma 5.3.3. We define $\mathrm{Cht}_G^{S_2}$ and $\mathrm{Cht}_G^{S_1}$ as in A.0.2 (taking into account A.0.1). We deduce from Lemma 5.3.3 that $\mathrm{Cht}_G^{S_2} - \mathrm{Cht}_G^{S_1} = \mathrm{Cht}_G^{S_M(\mu)}$ and $\mathrm{Cht}'^{S_2}_M - \mathrm{Cht}'^{S_1}_M = \mathrm{Cht}'^{S_M(\mu)}_M$.

We deduce from 1.4.10 that

$$\mathrm{Bun}_G^{=\lambda} \neq \emptyset \Rightarrow \Upsilon_G(\lambda) \in \bigcup_{P \subsetneq G} \iota \circ \mathrm{pr}_P^{\mathrm{ad}}(\widehat{\Lambda}^+_{G^{\mathrm{ad}}}) \subset \frac{1}{r}\widehat{R}^+_{G^{\mathrm{ad}}},$$

where the last inclusion follows from the choice of r in 5.1.1. We deduce that $\mathrm{Cht}_G^{=\lambda} = \emptyset$ if $\lambda \notin (1/r)\widehat{R}^+_{G^{\mathrm{ad}}}$. Thus $\mathrm{Cht}_G^{S_2} = \mathrm{Cht}_G^{\leqslant\mu}$, $\mathrm{Cht}_G^{S_1} = \mathrm{Cht}_G^{\leqslant\mu-(1/r)\check{\alpha}}$, $\mathrm{Cht}'^{S_2}_M = \mathrm{Cht}'^{\leqslant\mu}_M$ and $\mathrm{Cht}'^{S_1}_M = \mathrm{Cht}'^{\leqslant\mu-(1/r)\check{\alpha}}_M$.

Applying Lemma A.0.8 to S_1 and S_2, we obtain a commutative diagram of cohomology groups, where the upper and lower lines are part of the long exact sequences in (A.2).

$$\begin{tikzcd}
H_G^{j,\leqslant\mu-(1/r)\check{\alpha}} \arrow[r] \arrow[d, "C_G^{P,j,\leqslant\mu-(1/r)\check{\alpha}}"] & H_G^{j,\leqslant\mu} \arrow[r] \arrow[d, "C_G^{P,j,\leqslant\mu}"] & H_G^{j,S_M(\mu)} \arrow[d, "C_G^{P,j,S_M(\mu)}"] \\
H'^{j,\leqslant\mu-(1/r)\check{\alpha}}_M \arrow[r] & H'^{j,\leqslant\mu}_M \arrow[r] & H'^{j,S_M(\mu)}_M
\end{tikzcd} \tag{5.6}$$

Note that if $\mathrm{Cht}_G^{S_M(\mu)} = \emptyset$, then the proof is finished.

(2) By the hypothesis of Lemma 5.3.1, $\langle\mu,\alpha\rangle \geqslant C_G^0 \geqslant \widetilde{C}(G,X,N,W)$. By Proposition 4.6.4, for any j, the morphism $C_G^{P,j,S_M(\mu)} : H_G^{j,S_M(\mu)} \to H'^{j,S_M(\mu)}_M$ is an isomorphism.

(3) We deduce from (3.38) a commutative diagram.

$$\begin{tikzcd}
H_G^{j,\leqslant\mu} \arrow[r, "\mathcal{I}_G"] \arrow[d, "C_G^{P,j,\leqslant\mu}"'] & H_G^{j} \arrow[d, "C_G^{P,j}"] \\
H'^{j,\leqslant\mu}_M \arrow[r, "\mathcal{I}_M"] & H'^{j}_M
\end{tikzcd} \tag{5.7}$$

By Lemma 5.3.4, the morphism $\mathcal{I}_M$ in (5.7) is injective.

(4) Let $a \in \mathrm{Ker}(H_G^{j,\leqslant\mu} \to H'^{j}_M)$. By the commutativity of (5.7), $\mathcal{I}_M \circ C_G^{P,j,\leqslant\mu}(a) = C_G^{P,j} \circ \mathcal{I}_G(a) = 0$. By step (3), $\mathcal{I}_M$ is injective. So $C_G^{P,j,\leqslant\mu}(a) = 0$.

By the commutativity of (5.6) and the isomorphism in step (2), we deduce that the image of a in $H_G^{j,S_M(\mu)}$ is zero. So there exists $a' \in H_G^{j,\leqslant\mu-(1/r)\check{\alpha}}$ whose image in $H_G^{j,\leqslant\mu}$ is a. $\square$

Remark 5.3.5. In fact, we have

$$H'^{j,\leqslant\mu}_M = \Big(\prod_{\nu < \mathrm{pr}_P^{\mathrm{ad}}(\mu)} H'^{j,\leqslant\mu,\nu}_M\Big) \oplus H'^{j,\leqslant\mu,\mathrm{pr}_P^{\mathrm{ad}}(\mu)}_M = H'^{j,\leqslant\mu-(1/r)\check{\alpha}}_M \oplus H'^{j,S_M(\mu)}_M.$$

Thus the bottom line of (5.6) was canonically split.

LEMMA 5.3.6. *If the property (a) of Proposition 5.1.5 is true for G, then the property (b) of Proposition 5.1.5 is true for G.*

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Proof. Let $\nabla(C_G^0)$ be the set of $\mu \in (1/r)\widehat{R}^+_{G^{\mathrm{ad}}}$ such that $\langle \mu, \gamma\rangle > C_G^0$ for all $\gamma \in \Gamma_G$. Let $\Omega(C_G^0)$ be the set of $\mu \in \nabla(C_G^0)$ such that $\mu - (1/r)\check{\alpha} \notin \nabla(C_G^0)$ for all $\check{\alpha} \in \widehat{\Gamma}_G$. The set $\Omega(C_G^0)$ is bounded, and thus is finite. Let $\mu_0 \in (1/r)\widehat{R}^+_{G^{\mathrm{ad}}}$ such that $\mu_0 > \mu$ for all $\mu \in \Omega(C_G^0)$.

For any $\lambda \in \nabla(C_G^0)$, there exists a (zigzag) chain $\lambda = \lambda^{(0)} > \lambda^{(1)} > \cdots > \lambda^{(m-1)} > \lambda^{(m)}$ in $(1/r)\widehat{R}^+_{G^{\mathrm{ad}}}$ for some $m \in \mathbb{Z}_{\geqslant 0}$ such that:

(i) for any j, we have $\lambda^{(j)} \in \nabla(C_G^0)$;

(ii) for any j, we have $\lambda^{(j)} - \lambda^{(j+1)} = (1/r)\check{\alpha}$ for some simple coroot $\check{\alpha} \in \widehat{\Gamma}_G$;

(iii) $\lambda^{(m)} \in \Omega(C_G^0)$.

(Indeed, $\lambda^{(0)}$ satisfies (i). Suppose that we have already constructed a chain until $\lambda^{(j)}$ which satisfies (i) and (ii). If $\lambda^{(j)}$ satisfies (iii), we are done. If not, then there exists some $\check{\alpha} \in \widehat{\Gamma}_G$ such that $\lambda^{(j)} - (1/r)\check{\alpha} \in \nabla(C_G^0)$. We define $\lambda^{(j+1)} := \lambda^{(j)} - (1/r)\check{\alpha}$ and continue the process.)

Applying successively the property (a) of Proposition 5.1.5 to $\lambda^{(0)}, \lambda^{(1)}, \ldots,$ until $\lambda^{(m)}$, we deduce that the morphism

$$\mathrm{Ker}\Big(H_G^{j, \leqslant \lambda^{(m)}} \to \prod_{P \subsetneq G} H'^{j}_M\Big) \to \mathrm{Ker}\Big(H_G^{j, \leqslant \lambda} \to \prod_{P \subsetneq G} H'^{j}_M\Big)$$

is surjective. Assume in addition that $\lambda \geqslant \mu_0$; then the morphism $H_G^{j, \leqslant \lambda^{(m)}} \to H_G^{j, \leqslant \lambda}$ factors through $H_G^{j, \leqslant \mu_0}$. We deduce the lemma. $\square$

5.4 Injectivity

Lemma 5.4.1. *If the property (b) of Proposition 5.1.5 is true for G, then the property (c) of Proposition 5.1.5 is true for G.*

We need some preparations before the proof of Lemma 5.4.1.

5.4.2 For $\mu \in (1/r)\widehat{R}^+_{G^{\mathrm{ad}}}$, let $\mathcal{I}_\mu : H_G^{j, \leqslant \mu} \to H_G^j$ be the morphism to the inductive limit as in 5.1.3. For $\lambda \in (1/r)\widehat{R}^+_{G^{\mathrm{ad}}}$ such that $\lambda \geqslant \mu$, we denote by $\mathcal{I}_\mu^\lambda : H_G^{j, \leqslant \mu} \to H_G^{j, \leqslant \lambda}$ the morphism defined in 2.5.2. We have $\mathrm{Ker}(\mathcal{I}_\mu^\lambda) \subset \mathrm{Ker}(\mathcal{I}_\mu) \subset H_G^{j, \leqslant \mu}$.

For $\lambda_2 \geqslant \lambda_1 \geqslant \mu$, we have $\mathrm{Ker}(\mathcal{I}_\mu^{\lambda_1}) \subset \mathrm{Ker}(\mathcal{I}_\mu^{\lambda_2})$.

Lemma 5.4.3. *Let $\mu \in (1/r)\widehat{R}^+_{G^{\mathrm{ad}}}$. There exists $\mu^\sharp \in \widehat{R}^+_{G^{\mathrm{ad}}}$ such that $\mu^\sharp \geqslant \mu$ and $\mathrm{Ker}(\mathcal{I}_\mu^{\mu^\sharp}) = \mathrm{Ker}(\mathcal{I}_\mu)$.*

Proof. We have the filtered system $\{\mathrm{Ker}(\mathcal{I}_\mu^\lambda) \mid \lambda \in (1/r)\widehat{R}^+_{G^{\mathrm{ad}}}, \lambda \geqslant \mu\}$ in $\mathrm{Ker}(\mathcal{I}_\mu)$ and $\mathrm{Ker}(\mathcal{I}_\mu) = \varinjlim_\lambda \mathrm{Ker}(\mathcal{I}_\mu^\lambda)$. Since $\mathrm{Ker}(\mathcal{I}_\mu)$ is of finite dimension, the result is clear. $\square$

Construction 5.4.4. Let μ_0 be the one in the property (b) of Proposition 5.1.5. Choose $\mu_0^\sharp \in (1/r)\widehat{R}^+_{G^{\mathrm{ad}}}$ which satisfies Lemma 5.4.3 for μ_0. Let $C_G = \max\{C_G^0, \max_{\gamma \in \Gamma_G}\{\langle \mu_0^\sharp, \gamma\rangle\}\}$.

Proof of Lemma 5.4.1. Let $\lambda \in (1/r)\widehat{R}^+_{G^{\mathrm{ad}}}$ such that $\langle \lambda, \gamma\rangle \geqslant C_G$ for all $\gamma \in \Gamma_G$. By Construction 5.4.4, $\langle \lambda - \mu_0^\sharp, \gamma\rangle = \langle \lambda, \gamma\rangle - \langle \mu_0^\sharp, \gamma\rangle \geqslant C_G - \langle \mu_0^\sharp, \gamma\rangle \geqslant 0$ for all $\gamma \in \Gamma_G$. Thus $\mu_0^\sharp \leqslant \lambda$. Consider the morphisms

$$H_G^{j, \leqslant \mu_0} \to H_G^{j, \leqslant \mu_0^\sharp} \to H_G^{j, \leqslant \lambda} \to H_G^j.$$

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We have $\mathrm{Ker}(\mathcal{I}_{\mu_0}^{\mu_0^\sharp}) \subset \mathrm{Ker}(\mathcal{I}_{\mu_0}^{\lambda}) \subset \mathrm{Ker}(\mathcal{I}_{\mu_0})$. By Lemma 5.4.3, $\mathrm{Ker}(\mathcal{I}_{\mu_0}^{\mu_0^\sharp}) = \mathrm{Ker}(\mathcal{I}_{\mu_0})$, and hence $\mathrm{Ker}(\mathcal{I}_{\mu_0}^{\lambda}) = \mathrm{Ker}(\mathcal{I}_{\mu_0})$.

For any element $b \in \mathrm{Ker}(H_G^{j,\leqslant\lambda} \to H_G^j)$, we have $b \in \mathrm{Ker}(H_G^{j,\leqslant\lambda} \to \prod H'^{j}_M)$. By the property (b) of Proposition 5.1.5, b is the image of an element $b_0 \in \mathrm{Ker}(H_G^{j,\leqslant\mu_0} \to \prod H'^{j}_M)$. We have $b_0 \in \mathrm{Ker}(\mathcal{I}_{\mu_0}) = \mathrm{Ker}(\mathcal{I}_{\mu_0}^{\lambda})$, so its image b in $H_G^{j,\leqslant\lambda}$ is zero. This implies that the morphism $H_G^{j,\leqslant\lambda} \to H_G^j$ is injective. □

6. Rational Hecke-finite cohomology

In this section, we will define a subspace $H_{G,N,I,W}^{j,\,\mathrm{Hf\text{-}rat}}$ of $H_{G,N,I,W}^{j}$ and prove the following.

Proposition 6.0.1. *The two $\mathbb{Q}_\ell$-vector subspaces $H_{G,N,I,W}^{j,\,\mathrm{cusp}}$ and $H_{G,N,I,W}^{j,\,\mathrm{Hf\text{-}rat}}$ of $H_{G,N,I,W}^{j}$ are equal.*

In § 6.1 we give some preparations. In § 6.2 we show that the constant term morphisms commute with the action of the Hecke algebra. Using this, in § 6.3 we prove Proposition 6.0.1.

In § 6, all the stacks are restricted to $\overline{\eta^I}$.

6.1 Compatibility of constant term morphisms and level change

6.1.1 Let K be a compact open subgroup of $G(\mathbb{O})$. Let N be a level such that $K_N \subset K$. We define

$$\mathrm{Cht}_{G,K,I,W} := \mathrm{Cht}_{G,N,I,W} / (K/K_N).$$

It is independent of the choice of N.

Let $d \in \mathbb{N}$ be large enough as in Proposition 2.2.1; we define $\mathcal{F}_{G,K,I,W}$ to be the inverse image of $\mathcal{S}_{G,I,W}^d$ by $\epsilon_{K,d} : \mathrm{Cht}_{G,K,I,W} \to [G_{I,d} \backslash \mathrm{Gr}_{G,I,W}]$. Just as in Remark 2.4.6, $\mathcal{F}_{G,K,I,W}$ is independent of d. Similarly we define $\mathcal{F}_{G,K,I,W}^{\Xi}$ over $\mathrm{Cht}_{G,K,I,W} / \Xi$. We define $H_{G,K,I,W}^j := \varinjlim_\mu H_c^j(\mathrm{Cht}_{G,K,I,W}^{\leqslant\mu} / \Xi, \mathcal{F}_{G,K,I,W}^{\Xi})$.

6.1.2 Let $K' \subset K$ be two compact open subgroups of $G(\mathbb{O})$. The inclusion $K'/K_N \hookrightarrow K/K_N$ induces a morphism $\mathrm{pr}_{K',K}^G : \mathrm{Cht}_{G,K',I,W} \to \mathrm{Cht}_{G,K,I,W}$. Note that all the stacks are restricted to $\overline{\eta^I}$. Morphism $\mathrm{pr}_{K',K}^G$ is finite étale of degree the cardinality of K/K'. The following diagram is commutative.

$$\begin{tikzcd}
\mathrm{Cht}_{G,K',I,W} \arrow[rr, "\mathrm{pr}_{K',K}^G"] \arrow[dr, "\epsilon_{K',d}"'] & & \mathrm{Cht}_{G,K,I,W} \arrow[dl, "\epsilon_{K,d}"] \\
& {[G_{I,d} \backslash \mathrm{Gr}_{G,I,W}]} &
\end{tikzcd}$$

Note that $(\mathrm{pr}_{K',K}^G)_* = (\mathrm{pr}_{K',K}^G)_!$ and

$$(\mathrm{pr}_{K',K}^G)^* \mathcal{F}_{G,K,I,W} = (\mathrm{pr}_{K',K}^G)^* (\epsilon_{K,d})^* \mathcal{S}_{G,I,W}^d = (\epsilon_{K',d})^* \mathcal{S}_{G,I,W}^d = \mathcal{F}_{G,K',I,W}.$$

The adjunction morphism $\mathrm{adj}(\mathrm{pr}_{K',K}^G) : \mathrm{Id} \to (\mathrm{pr}_{K',K}^G)_* (\mathrm{pr}_{K',K}^G)^*$ induces an (injective) morphism of cohomology groups, which we still denote by

$$\mathrm{adj}(\mathrm{pr}_{K',K}^G) : H_{G,K,I,W}^j \to H_{G,K',I,W}^j.$$

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Note that $(\mathrm{pr}^G_{K',K})^! = (\mathrm{pr}^G_{K',K})^*$. The counit morphism (in this case equal to the trace map) $\mathrm{Co}(\mathrm{pr}^G_{K',K}) : (\mathrm{pr}^G_{K',K})_!(\mathrm{pr}^G_{K',K})^! \to \mathrm{Id}$ induces a (surjective) morphism of cohomology groups, which we still denote by

$$\mathrm{Co}(\mathrm{pr}^G_{K',K}) : H^j_{G,K',I,W} \to H^j_{G,K,I,W}.$$

6.1.3 Let v be a place in X. Let $N = N^v + nv$. Taking projective limit over n, we define

$$\varprojlim_n \mathrm{Cht}_{G,N^v+nv,I,W}.$$

Let $g \in G(F_v)$. The right action of g (by left multiplication by g^{-1}) induces an isomorphism

$$\varprojlim_n \mathrm{Cht}_{G,N^v+nv,I,W} \xrightarrow{\sim} \varprojlim_n \mathrm{Cht}_{G,N^v+nv,I,W} \quad (\mathcal{G} \to {}^\tau\mathcal{G}, \psi^v, \psi_v) \mapsto (\mathcal{G}' \to {}^\tau\mathcal{G}', \psi^v, \psi'_v)$$

where ψ^v (respectively ψ_v) is the level structure outside v (respectively on v). The G-bundle $\mathcal{G}'$ is defined by gluing $G\big|_{\Gamma_{\infty v}}$ and $\mathcal{G}\big|_{X-v}$ by $G\big|_{\Gamma_{\infty v}-v} \xrightarrow{g} G\big|_{\Gamma_{\infty v}-v} \underset{\sim}{\xleftarrow{\psi_v}} \mathcal{G}\big|_{\Gamma_{\infty v}-v}$. We have $\psi'_v = g^{-1} \circ \psi_v$.

Let

$$\mathrm{Cht}_{G,\infty,I,W} := \varprojlim_N \mathrm{Cht}_{G,N,I,W}.$$

Similarly, $\mathrm{Cht}_{G,\infty,I,W}$ is equipped with an action of $G(\mathbb{A})$.

6.1.4 Let P be a parabolic subgroup of G and M its Levi quotient. We define

$$\mathrm{Cht}_{P,\infty,I,W} := \varprojlim_N \mathrm{Cht}_{P,N,I,W}.$$

Just as in 6.1.3, $\mathrm{Cht}_{P,\infty,I,W}$ is equipped with an action of $P(\mathbb{A})$. For any compact open subgroup $K \subset G(\mathbb{O})$, we define

$$\mathrm{Cht}'_{P,K,I,W} := \mathrm{Cht}_{P,\infty,I,W} \overset{P(\mathbb{O})}{\times} G(\mathbb{O})/K. \tag{6.1}$$

We have a morphism

$$\mathrm{Cht}_{P,\infty,I,W} \overset{P(\mathbb{O})}{\times} G(\mathbb{O}) \to \mathrm{Cht}_{G,\infty,I,W} \tag{6.2}$$

by sending $((\mathcal{P}, \psi_P) \to ({}^\tau\mathcal{P}, {}^\tau\psi_P), g \in G(\mathbb{O}))$ to $((\mathcal{G}, g^{-1} \circ \psi_G) \to ({}^\tau\mathcal{G}, g^{-1} \circ {}^\tau\psi_G))$, where $\mathcal{G} = \mathcal{P} \overset{P}{\times} G$ and $\psi_G = \psi_P \overset{P}{\times} G$. It induces a morphism

$$\mathrm{Cht}_{P,\infty,I,W} \overset{P(\mathbb{O})}{\times} G(\mathbb{O}) \to \mathrm{Cht}_{P,I,W} \underset{\mathrm{Cht}_{G,I,W}}{\times} \mathrm{Cht}_{G,\infty,I,W}. \tag{6.3}$$

This is a $G(\mathbb{O})$-equivariant morphism of $G(\mathbb{O})$-torsors over $\mathrm{Cht}_{P,I,W}$, where $G(\mathbb{O})$ acts on the left-hand side of (6.3) by right action (right multiplication) on $G(\mathbb{O})$ and acts on the right-hand side of (6.3) by the right action on $\mathrm{Cht}_{G,\infty,I,W}$ defined in 6.1.3. Thus (6.3) is an isomorphism. We have

$$\mathrm{Cht}_{P,\infty,I,W} \overset{P(\mathbb{O})}{\times} G(\mathbb{O})/K \xrightarrow{\sim} \mathrm{Cht}_{P,I,W} \underset{\mathrm{Cht}_{G,I,W}}{\times} \mathrm{Cht}_{G,\infty,I,W}/K,$$

i.e.

$$\mathrm{Cht}'_{P,K,I,W} = \mathrm{Cht}_{P,I,W} \underset{\mathrm{Cht}_{G,I,W}}{\times} \mathrm{Cht}_{G,K,I,W}. \tag{6.4}$$

When $K = K_N$ for some level N, we have $\mathrm{Cht}_{P,N,I,W} = \mathrm{Cht}_{P,\infty,I,W}/K_{P,N}$, where $K_{P,N} := K_N \cap P(\mathbb{O})$. We deduce that $\mathrm{Cht}'_{P,K_N,I,W}$ defined in (6.1) coincides with $\mathrm{Cht}'_{P,N,I,W}$ defined in Definition 3.4.2.

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6.1.5 We define

$$\mathrm{Cht}_{M,\infty,I,W} := \varprojlim_{N} \mathrm{Cht}_{M,N,I,W}.$$

Just as in 6.1.3, $\mathrm{Cht}_{M,\infty,I,W}$ is equipped with an action of $M(\mathbb{A})$. Recall that for any level N, in Definition 3.4.2, we defined $\mathrm{Cht}'_{M,N,I,W} = \mathrm{Cht}_{M,N,I,W} \overset{P(\mathcal{O}_N)}{\times} G(\mathcal{O}_N)$. Let $K_{U,N} := K_N \cap U(\mathbb{O})$ and $K_{M,N} := K_{P,N}/K_{U,N}$. Taking into account that $\mathrm{Cht}_{M,N,I,W} = \mathrm{Cht}_{M,\infty,I,W}/K_{M,N}$, we deduce

$$\mathrm{Cht}'_{M,N,I,W} = \mathrm{Cht}_{M,\infty,I,W} \overset{P(\mathbb{O})/K_{U,N}}{\times} G(\mathbb{O})/K_N. \tag{6.5}$$

When we consider the action of the Hecke algebras in 6.2.4 in the next section, we will need some functoriality on K_N. For this reason, we rewrite (6.5) in the following way. Note that K_N is normal in $G(\mathbb{O})$. The stabilizer of any $P(\mathbb{O})$-orbit in $G(\mathbb{O})/K_N$ is $K_{P,N}$. We deduce from (6.5) that

$$\begin{aligned}\mathrm{Cht}'_{M,N,I,W} &= \bigsqcup_{P(\mathbb{O})\text{-orbits in } G(\mathbb{O})/K_N} \mathrm{Cht}_{M,\infty,I,W}/(K_{P,N}/K_{U,N}) \\ &= \bigsqcup_{P(\mathbb{A})\text{-orbits in } G(\mathbb{A})/K_N} \mathrm{Cht}_{M,\infty,I,W}/(K_{P,N}/K_{U,N}).\end{aligned} \tag{6.6}$$

The second equation is because that $P(\mathbb{O})\backslash G(\mathbb{O}) = P(\mathbb{A})\backslash G(\mathbb{A})$, and that in each $P(\mathbb{A})$-orbit in $G(\mathbb{A})/K_N$, we can choose a representative in $G(\mathbb{O})/K_N$.

In the following, we want to generalize (6.6) for any compact open subgroup $K \subset G(\mathbb{O})$ (which may not be normal in $G(\mathbb{O})$).

6.1.6 Let $\mathcal{D}$ be the category of discrete sets S equipped with a continuous action of $P(\mathbb{A})$ with finitely many orbits such that the stabilizer of any point is conjugated to some open subgroup of finite index in $P(\mathbb{O})$. In particular, for any compact open subgroup $K \subset G(\mathbb{O})$, the set $S = G(\mathbb{A})/K$ is an object in $\mathcal{D}$.

For any $S \in \mathcal{D}$, we define functorially the cohomology group $H'_{M,S,I,W}$ in the following way.

When S has only one orbit, choose a point $s \in S$, and let H be the stabilizer of s. Then H is a subgroup of $P(\mathbb{A})$ conjugated to some open subgroup of finite index in $P(\mathbb{O})$. We have $S = P(\mathbb{A})/H$. Let R be a subgroup of finite index in $H \cap U(\mathbb{A})$ and normal in H. By 6.1.5, $\mathrm{Cht}_{M,\infty,I,W}$ is equipped with an action of $M(\mathbb{A})$, thus an action of $P(\mathbb{A})$ by the projection $P(\mathbb{A}) \twoheadrightarrow M(\mathbb{A})$. Note that $R \subset U(\mathbb{A})$ acts trivially on $\mathrm{Cht}_{M,\infty,I,W}$. We define a Deligne–Mumford stack

$$\mathrm{Cht}_{M,\infty,I,W}/(H/R).$$

We define the cohomology group $H'_{M,S,R,I,W}$ as in Definition 2.6.8 for $\mathrm{Cht}_{M,\infty,I,W}/(H/R)$ (instead of $\mathrm{Cht}_{M,N,I,W}$). Concretely, we have a morphism $\mathrm{Cht}_{M,\infty,I,W}/(H/R)\Xi \to \mathrm{Cht}_{M,I,W}/\Xi$, where $\mathrm{Cht}_{M,I,W}$ is the stack of M-shtukas without level structure. Let $\mathcal{F}^{\Xi}_{M,\infty,I,W}$ be the inverse image of $\mathcal{F}^{\Xi}_{M,I,W}$. We define

$$H'^{\,j}_{M,S,R,I,W} := \varinjlim_{\mu} \prod_{\nu} H^j_c(\mathrm{Cht}^{\leqslant\mu,\nu}_{M,\infty,I,W}/(H/R)\Xi, \mathcal{F}^{\Xi}_{M,\infty,I,W}).$$

Let $R_1 \subset R_2$ be two subgroups of finite index in $H \cap U(\mathbb{A})$ and normal in H. The projection $H/R_1 \twoheadrightarrow H/R_2$ induces a morphism

$$\mathfrak{q}_{R_1,R_2} : \mathrm{Cht}_{M,\infty,I,W}/(H/R_1) \to \mathrm{Cht}_{M,\infty,I,W}/(H/R_2).$$

It is a gerbe for the finite q-group R_2/R_1. The counit morphism (which is equal to the trace map because $\mathfrak{q}_{R_1,R_2}$ is smooth of dimension 0) $\mathrm{Co}(\mathfrak{q}_{R_1,R_2}) : (\mathfrak{q}_{R_1,R_2})_!(\mathfrak{q}_{R_1,R_2})^! \to \mathrm{Id}$ is an isomorphism. Indeed, just as in the proof (i) of Lemma 4.6.6, by proper base change and the fact that $\mathfrak{q}_{R_1,R_2}$ is smooth, we reduce to the case of Lemma 6.1.7 below with $\Gamma = R_2/R_1$. The morphism $\mathrm{Co}(\mathfrak{q}_{R_1,R_2})$ induces an isomorphism of cohomology groups

$$H'_{M,S,R_1,I,W} \xrightarrow{\sim} H'_{M,S,R_2,I,W}. \tag{6.7}$$

We define $H'_{M,S,I,W}$ to be any $H'_{M,S,R,I,W}$, where we identify $H'_{M,S,R_1,I,W}$ and $H'_{M,S,R_2,I,W}$ by (6.7).

Recall that S has only one orbit. $H'_{M,S,I,W}$ is independent of the choice of the point s in S. In fact, let s_1, s_2 be two points of S, and let H_1 (respectively H_2) be the stabilizer of s_1 (respectively s_2); then $H_2 = p^{-1}H_1p$ for some $p \in P(\mathbb{A})$. The action of p induces an isomorphism $\mathrm{Cht}_{M,\infty,I,W} /(H_1/R) \xrightarrow{\sim} \mathrm{Cht}_{M,\infty,I,W} /(p^{-1}H_1p/p^{-1}Rp)$. We deduce an isomorphism of cohomology groups by the adjunction morphism.

In general, $S = \bigsqcup_{\alpha\in A} \alpha$ is a finite union of orbits, and we define

$$H'_{M,S,I,W} := \bigoplus_{\alpha\in A} H'_{M,\alpha,I,W}.$$

When $S = G(\mathbb{A})/K$ for some compact open subgroup K in $G(\mathbb{O})$, we write

$$H'_{M,K,I,W} := H'_{M,S,I,W}. \tag{6.8}$$

Lemma 6.1.7. *Let Γ be a finite group over an algebraically closed field k over $\mathbb{F}_q$. We denote by $B\Gamma$ the classifying stack of Γ over k. Let $\mathfrak{q} : B\Gamma \to \mathrm{Spec}\, k$ be the structure morphism. Then the counit morphism (equal to the trace map) $\mathrm{Co}(\mathfrak{q}) : \mathfrak{q}_!\mathfrak{q}^! \to \mathrm{Id}$ of functors on $D_c(\mathrm{Spec}\, k, \mathbb{Q}_\ell)$ is an isomorphism.*

Proof. The counit morphism $\mathrm{Co}(\mathfrak{q})$ is the dual of the adjunction morphism $\mathrm{adj}(\mathfrak{q}) : \mathrm{Id} \to \mathfrak{q}_*\mathfrak{q}^*$. For any $\mathcal{F} \in D_c(\mathrm{Spec}\, k, \mathbb{Q}_\ell)$, $\mathfrak{q}^*\mathcal{F}$ is a complex F of Γ-modules with trivial action of Γ. Since $H^j(B\Gamma, \mathfrak{q}^*\mathcal{F}) = H^j(\Gamma, F)$ (group cohomology), we have $H^0(B\Gamma, \mathfrak{q}^*\mathcal{F}) = F^\Gamma = F$ and $H^j(B\Gamma, \mathfrak{q}^*\mathcal{F}) = 0$ for $j > 0$. So $\mathrm{adj}(\mathfrak{q})$ is an isomorphism. By duality, we deduce the lemma. □

6.1.8 Let $S \in \mathcal{D}$. We define

$$\mathrm{Cht}'_{P,S,I,W} := \mathrm{Cht}_{P,\infty,I,W} \overset{P(\mathbb{A})}{\times} S. \tag{6.9}$$

For each orbit α in S, choose a representative, and let H^α be the stabilizer (well defined up to conjugation). Then

$$\mathrm{Cht}'_{P,S,I,W} = \bigsqcup_{\alpha\in\{P(\mathbb{A})\text{-orbits in } S\}} \mathrm{Cht}_{P,\infty,I,W} /H^\alpha.$$

For each α, let R^α be a subgroup of finite index in $H^\alpha \cap U(\mathbb{A})$ and normal in H^α. Let $R = (R^\alpha)_{\alpha\in\{P(\mathbb{A})\text{-orbits in } S\}}$. We define

$$\mathrm{Cht}'_{M,S,R,I,W} := \bigsqcup_{\alpha\in\{P(\mathbb{A})\text{-orbits in } S\}} \mathrm{Cht}_{M,\infty,I,W} /(H^\alpha/R^\alpha). \tag{6.10}$$

For each α, we have morphisms of prestacks

$$\mathrm{Cht}_{P,\infty,I,W}/H^\alpha \to \mathrm{Cht}_{M,\infty,I,W}/H^\alpha \to \mathrm{Cht}_{M,\infty,I,W}/(H^\alpha/R^\alpha), \tag{6.11}$$

where the first and third prestacks are Deligne–Mumford stacks, while the second is only a prestack. Taking union over all the orbits, we deduce from (6.11) a morphism

$$\pi_{S,R} : \mathrm{Cht}'_{P,S,I,W} \to \mathrm{Cht}'_{M,S,R,I,W}. \tag{6.12}$$

In particular, when $S = G(\mathbb{A})/K_N$, the stack $\mathrm{Cht}'_{P,S,I,W}$ coincides with $\mathrm{Cht}'_{P,N,I,W}$. For every orbit α, we can choose a representative in $G(\mathbb{O})/K_N$ (so that $H^\alpha = K_{P,N}$) and choose $R^\alpha = K_{U,N}$. Then $\mathrm{Cht}'_{M,S,R,I,W}$ coincides with $\mathrm{Cht}'_{M,N,I,W}$, $H'_{M,S,R,I,W}$ coincides with $H'_{M,N,I,W}$ defined in Definition 3.4.9, and (6.12) coincides with π' defined in (3.26).

6.1.9 For any compact open subgroup $K \subset G(\mathbb{O})$, let $S = G(\mathbb{A})/K$. Note that in this case we have $\mathrm{Cht}'_{P,K,I,W} = \mathrm{Cht}'_{P,S,I,W}$. For any R as in 6.1.8, we have the following morphisms.

$$\begin{tikzcd} & \mathrm{Cht}'_{P,S,I,W} \arrow[dl, "i_K"'] \arrow[d, "\mathfrak{p}_P"] \arrow[dr, "\pi_{S,R}"] & \\ \mathrm{Cht}_{G,K,I,W} \arrow[r, "\mathfrak{p}_G"'] & \overline{\eta^I} & \mathrm{Cht}'_{M,S,R,I,W} \arrow[l, "\mathfrak{p}_M"] \end{tikzcd} \tag{6.13}$$

Just as in Proposition 3.5.3 and Remark 3.5.4, the morphism i_K is schematic and proper. Apply the construction in § 3 to (6.13). Similarly to (3.9), we have

$$\begin{tikzcd} \mathrm{Cht}'_{P,S,I,W} \arrow[dr, "\pi_{S,R,d}"] \arrow[drr, "\pi_{S,R}"] \arrow[ddr, "\epsilon_{P,d}"'] & & \\ & \widetilde{\mathrm{Cht}}'_{M,S,R,I,W} \arrow[r, "\widetilde{\pi_d^0}"'] \arrow[d, "\widetilde{\epsilon_{M,d}}"] & \mathrm{Cht}'_{M,S,R,I,W} \arrow[d, "\epsilon_{M,d}"] \\ & {[P_{I,d}\backslash \mathrm{Gr}_{P,I,W}]} \arrow[r, "\pi_d^0"'] & {[M_{I,d}\backslash \mathrm{Gr}_{M,I,W}]} \end{tikzcd} \tag{6.14}$$

where $\pi_{S,R,d}$ is smooth. Let $\mathcal{F}_G$ be the canonical Satake sheaf on $\mathrm{Cht}_{G,K,I,W}$ and $\mathcal{F}_M$ be the canonical Satake sheaf on $\mathrm{Cht}'_{M,S,R,I,W}$. We construct a morphism $c^P_{G,K} : (\pi_{S,R})_!(i_K)^*\mathcal{F}_G \to \mathcal{F}_M$ similar to (3.22) and (3.23). Namely, $c^P_{G,K}$ is the composition of some isomorphisms and the counit morphism $(\pi_{S,R,d})_!(\pi_{S,R,d})^! \to \mathrm{Id}$. Note that since $\pi_{S,R,d}$ is smooth, the composition $(\pi_{S,R,d})_!(\pi_{S,R,d})^*[2m](m) \xrightarrow{\sim} (\pi_d)_!(\pi_d)^! \to \mathrm{Id}$ is the trace map in [SGA4, XVIII 2], where m is the dimension of $\pi_{S,R,d}$.

Similar to (3.33), we have a composition of morphisms of functors in $D^b_c(\overline{\eta^I}, \mathbb{Q}_\ell)$:

$$(\mathfrak{p}_G)_!\mathcal{F}_G \xrightarrow{\mathrm{adj}(i_K)} (\mathfrak{p}_G)_!(i_K)_*(i_K)^*\mathcal{F}_G \simeq (\mathfrak{p}_M)_!(\pi_{S,R})_!(i_K)^*\mathcal{F}_G \xrightarrow{c^P_{G,K}} (\mathfrak{p}_M)_!\mathcal{F}_M.$$

We define

$$H'_{P,K,I,W} := H'_{P,S,I,W} := (\mathfrak{p}_P)_!(i_K)^*\mathcal{F}_G.$$

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The morphism $(\mathfrak{p}_G)_!\mathcal{F}_G \xrightarrow{\mathrm{adj}(i_K)} (\mathfrak{p}_G)_!(i_K)_*(i_K)^*\mathcal{F}_G$ induces a morphism

$$H^j_{G,K,I,W} \to H'_{P,S,I,W}. \tag{6.15}$$

The morphism $(\mathfrak{p}_M)_!(\pi_{S,R})_!(i_K)^*\mathcal{F}_G \xrightarrow{c^P_{G,K}} (\mathfrak{p}_M)_!\mathcal{F}_M$ induces a morphism

$$H'_{P,S,I,W} \to H'^{\,j}_{M,S,R,I,W}. \tag{6.16}$$

We define the constant term morphism to be the composition of (6.15) and (6.16)

$$C^{P,\,j}_{G,S,R} : H^j_{G,K,I,W} \to H'^{\,j}_{M,S,R,I,W}. \tag{6.17}$$

For $R_1 \subset R_2$ as in 6.1.6, the following diagram is commutative

$$\begin{tikzcd} H^j_{G,K,I,W} \arrow[r, "C^{P,\,j}_{G,S,R_1}"] \arrow[dr, "C^{P,\,j}_{G,S,R_2}"'] & H'^{\,j}_{M,S,R_1,I,W} \arrow[d, "\simeq"', "(6.7)"] \\ & H'^{\,j}_{M,S,R_2,I,W} \end{tikzcd} \tag{6.18}$$

because $C^{P,\,j}_{G,S,R_1}$, $C^{P,\,j}_{G,S,R_2}$ and (6.7) are defined by counit morphisms (which in these cases are equal to trace maps), and by [SGA4, XVIII Théorème 2.9], the trace morphism is compatible with composition.

In 6.1.6 we defined $H'^{\,j}_{M,K,I,W}$. We deduce from (6.18) a morphism

$$C^{P,\,j}_{G,K} : H^j_{G,K,I,W} \to H'^{\,j}_{M,K,I,W}, \tag{6.19}$$

which is the composition $H^j_{G,K,I,W} \to H'^{\,j}_{P,K,I,W} \to H'^{\,j}_{M,K,I,W}$.

6.1.10 Let $S_1, S_2 \in \mathcal{D}$ and $f : S_1 \to S_2$ be a morphism in $\mathcal{D}$. Note that f is $P(\mathbb{A})$-equivariant and it sends orbit to orbit. For each $P(\mathbb{A})$-orbit β in S_2, choose a representative $s^\beta \in \beta$ with stabilizer H^β. If $f^{-1}(\beta)$ is empty, take any R^β subgroup of finite index in $H_2^\beta \cap U(\mathbb{A})$ and normal in H_2^β. If $f^{-1}(\beta)$ is non-empty, for every $P(\mathbb{A})$-orbit $\alpha \in f^{-1}(\beta)$, choose a representative $s^\alpha \in \alpha$ such that $f(s^\alpha) = s^\beta$. Let H_1^α be the stabilizer of s^α. Then $H_1^\alpha \subset H_2^\alpha$. Let R^β be a subgroup of finite index in $(\bigcap_{\alpha \in f^{-1}(\beta)} H_1^\alpha) \cap U(\mathbb{A}) \subset H_2^\beta \cap U(\mathbb{A})$ and normal in H_2^β.

The morphism $H_1^\alpha / R^\beta \hookrightarrow H_2^\beta / R^\beta$ for $\beta = f(\alpha)$ induces a morphism

$$\mathfrak{q}^M_\alpha : \mathrm{Cht}_{M,\infty,I,W} / (H_1^\alpha / R^\beta) \to \mathrm{Cht}_{M,\infty,I,W} / (H_2^\beta / R^\beta). \tag{6.20}$$

Let $R = ((R^\beta)_{\beta \in \{P(\mathbb{A})\text{-orbits in } S_2\}})$. Similarly to (6.10), we define $\mathrm{Cht}'_{M,S_1,R,I,W}$ and $\mathrm{Cht}'_{M,S_2,R,I,W}$. Then (6.20) for every orbit α induces a morphism

$$\mathfrak{q}^M_f : \mathrm{Cht}'_{M,S_1,R,I,W} \to \mathrm{Cht}'_{M,S_2,R,I,W}. \tag{6.21}$$

Similarly to 6.1.2, the adjunction morphism $\mathrm{Id} \to (\mathfrak{q}^M_f)_*(\mathfrak{q}^M_f)^*$ induces a morphism

$$\mathrm{adj}(\mathfrak{q}^M_f) : H'_{M,S_2,I,W} \to H'_{M,S_1,I,W}. \tag{6.22}$$

The counit morphism $(\mathfrak{q}^M_f)_!(\mathfrak{q}^M_f)^! \to \mathrm{Id}$ induces a morphism

$$\mathrm{Co}(\mathfrak{q}^M_f) : H'_{M,S_1,I,W} \to H'_{M,S_2,I,W}. \tag{6.23}$$

In the following, we will apply the functoriality to the cases:

- $K' \subset K$, $S_1 = G(\mathbb{A})/K'$, $S_2 = G(\mathbb{A})/K$ and f is the projection $G(\mathbb{A})/K' \twoheadrightarrow G(\mathbb{A})/K$;
- $S_1 = G(\mathbb{A})/\widetilde{K}$, $S_2 = G(\mathbb{A})/g^{-1}\widetilde{K}g$ and f is the isomorphism induced by the right multiplication by g: $G(\mathbb{A})/\widetilde{K} \xrightarrow{\sim} G(\mathbb{A})/g^{-1}\widetilde{K}g$.

Remark 6.1.11. In 6.1.10, we can also first define morphisms of cohomology groups for each orbit α: the adjunction morphism $\mathrm{Id} \to (\mathfrak{q}_\alpha^M)_*(\mathfrak{q}_\alpha^M)^*$ induces a morphism

$$\mathrm{adj}(\mathfrak{q}_\alpha^M) : H'_{M,f(\alpha),I,W} \to H'_{M,\alpha,I,W}, \tag{6.24}$$

where the orbit α (respectively $f(\alpha)$) is considered as subset of S_1 (respectively S_2). The counit morphism $(\mathfrak{q}_\alpha^M)_!(\mathfrak{q}_\alpha^M)^! \to \mathrm{Id}$ induces a morphism

$$\mathrm{Co}(\mathfrak{q}_\alpha^M) : H'_{M,\alpha,I,W} \to H'_{M,f(\alpha),I,W}. \tag{6.25}$$

Then taking sum over all the orbits, we obtain (6.22) and (6.23).

Similarly, in 6.1.12 below, we can first prove the statement for cohomology groups orbit by orbit, and then take the sum over all the orbits. But the notations would be more complicated.

6.1.12 Any $S_1, S_2 \in \mathcal{D}$ and $f : S_1 \to S_2$ morphism in $\mathcal{D}$ induce a morphism

$$\mathfrak{q}_f^P : \mathrm{Cht}'_{P,S_1,I,W} \to \mathrm{Cht}'_{P,S_2,I,W}.$$

The adjunction morphism $\mathrm{Id} \to (\mathfrak{q}_f^P)_*(\mathfrak{q}_f^P)^*$ induces a morphism

$$\mathrm{adj}(\mathfrak{q}_f^P) : H'_{P,S_2,I,W} \to H'_{P,S_1,I,W}.$$

The counit morphism $(\mathfrak{q}_f^P)_!(\mathfrak{q}_f^P)^! \to \mathrm{Id}$ induces a morphism

$$\mathrm{Co}(\mathfrak{q}_f^P) : H'_{P,S_1,I,W} \to H'_{P,S_2,I,W}.$$

For each orbit α in S_1 with $\beta = f(\alpha)$, let H_1^α, H_2^β and R^β as in 6.1.10. We have a Cartesian square.

$$\begin{tikzcd}
\mathrm{Cht}_{P,\infty,I,W}/H_1^\alpha \arrow[r, "\mathfrak{q}_\alpha^P"] \arrow[d, "(6.11)"'] & \mathrm{Cht}_{P,\infty,I,W}/H_2^\beta \arrow[d, "(6.11)"] \\
\mathrm{Cht}_{M,\infty,I,W}/(H_1^\alpha/R^\beta) \arrow[r, "\mathfrak{q}_\alpha^M"] & \mathrm{Cht}_{M,\infty,I,W}/(H_2^\beta/R^\beta)
\end{tikzcd} \tag{6.26}$$

Taking union over all the orbits, with the notations in 6.1.8 and 6.1.10, we deduce a Cartesian square.

$$\begin{tikzcd}
\mathrm{Cht}'_{P,S_1,I,W} \arrow[r, "\mathfrak{q}_f^P"] \arrow[d, "\pi_{S_1,R}"'] & \mathrm{Cht}'_{P,S_2,I,W} \arrow[d, "\pi_{S_2,R}"] \\
\mathrm{Cht}'_{M,S_1,R,I,W} \arrow[r, "\mathfrak{q}_f^M"] & \mathrm{Cht}'_{M,S_2,R,I,W}
\end{tikzcd} \tag{6.27}$$

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Diagram (6.27) induces a commutative diagram of cohomology groups

$$\begin{tikzcd} H'_{P,S_2,I,W} \arrow[r, "\mathrm{adj}(\mathfrak{q}_f^P)"] \arrow[d, "(6.16)"'] & H'_{P,S_1,I,W} \arrow[d, "(6.16)"] \\ H'_{M,S_2,I,W} \arrow[r, "\mathrm{adj}(\mathfrak{q}_f^M)"] & H'_{M,S_1,I,W} \end{tikzcd} \tag{6.28}$$

because (6.27) is Cartesian, (6.16) is given by a counit morphism (equal to the trace morphism), and by [SGA4, XVIII Théorème 2.9], the trace morphism commutes with base change.

Diagram (6.27) induces a commutative diagram of cohomology groups

$$\begin{tikzcd} H'_{P,S_1,I,W} \arrow[r, "\mathrm{Co}(\mathfrak{q}_f^P)"] \arrow[d, "(6.16)"'] & H'_{P,S_2,I,W} \arrow[d, "(6.16)"] \\ H'_{M,S_1,I,W} \arrow[r, "\mathrm{Co}(\mathfrak{q}_f^M)"] & H'_{M,S_2,I,W} \end{tikzcd} \tag{6.29}$$

because by [SGA4, XVIII Théorème 2.9], the trace morphism is compatible with composition.

Remark 6.1.13. When $S_1 = G(\mathbb{A})/K_{N_1}$ and $S_2 = G(\mathbb{A})/K_{N_2}$ with $N_1 \supset N_2$, we have the projection $f : G(\mathbb{A})/K_{N_1} \twoheadrightarrow G(\mathbb{A})/K_{N_2}$. We have $\mathrm{Cht}'_{M,N_1,I,W} = \mathrm{Cht}'_{M,S_1,R_1,I,W}$ with $R_1^\alpha = K_{U,N_1}$ for each $P(\mathbb{A})$-orbit α in S_1 and $\mathrm{Cht}'_{M,N_2,I,W} = \mathrm{Cht}'_{M,S_2,R_2,I,W}$ with $R_2^\beta = K_{U,N_2}$ for each $P(\mathbb{A})$-orbit β in S_2. Note that $R_1^\alpha \neq R_2^{f(\alpha)}$, and thus the commutative diagram

$$\begin{tikzcd} \mathrm{Cht}'_{P,N_1,I,W} \arrow[r] \arrow[d, "\pi'"'] & \mathrm{Cht}'_{P,N_2,I,W} \arrow[d, "\pi'"] \\ \mathrm{Cht}'_{M,N_1,I,W} \arrow[r] & \mathrm{Cht}'_{M,N_2,I,W} \end{tikzcd} \tag{6.30}$$

does NOT coincide with diagram (6.27). In particular, diagram (6.30) is not Cartesian (the morphism from $\mathrm{Cht}'_{P,N_1,I,W}$ to the fiber product is finite étale of degree $\sharp(K_{U,N_2}/K_{U,N_1})$ which is a power of q).

6.1.14 Let $K' \subset K$ be two compact open subgroups of $G(\mathbb{O})$. Applying 6.1.10 to $S_1 = G(\mathbb{A})/K'$, $S_2 = G(\mathbb{A})/K$ and the projection $f : G(\mathbb{A})/K' \twoheadrightarrow G(\mathbb{A})/K$, we deduce a finite étale morphism (denoted by $\mathfrak{q}_f^M$ in 6.1.10)

$$\mathrm{pr}^M_{K',K} : \mathrm{Cht}'_{M,S_1,R,I,W} \to \mathrm{Cht}'_{M,S_2,R,I,W},$$

where R is defined in 6.1.10. The adjunction morphism $\mathrm{adj}(\mathrm{pr}^M_{K',K}) : \mathrm{Id} \to (\mathrm{pr}^M_{K',K})_*(\mathrm{pr}^M_{K',K})^*$ induces

$$\mathrm{adj}(\mathrm{pr}^M_{K',K}) : H'^{j}_{M,K,I,W} \to H'^{j}_{M,K',I,W}.$$

The counit morphism $\mathrm{Co}(\mathrm{pr}^M_{K',K}) : (\mathrm{pr}^M_{K',K})_!(\mathrm{pr}^M_{K',K})^! \to \mathrm{Id}$ induces

$$\mathrm{Co}(\mathrm{pr}^M_{K',K}) : H'^{j}_{M,K',I,W} \to H'^{j}_{M,K,I,W}.$$

Lemma 6.1.15. *For $K' \subset K$ as in 6.1.14, the following diagram of cohomology groups commutes.*

$$\begin{tikzcd} H^j_{G,K,I,W} \arrow[r, "\mathrm{adj}(\mathrm{pr}^G_{K',K})"] \arrow[d, "C^{P,j}_{G,K}"'] & H^j_{G,K',I,W} \arrow[d, "C^{P,j}_{G,K'}"] \\ H'^j_{M,K,I,W} \arrow[r, "\mathrm{adj}(\mathrm{pr}^M_{K',K})"] & H'^j_{M,K',I,W} \end{tikzcd} \tag{6.31}$$

Proof. (1) By (6.4), we have a Cartesian square.

$$\begin{tikzcd} \mathrm{Cht}_{G,K,I,W} & \mathrm{Cht}_{G,K',I,W} \arrow[l, "\mathrm{pr}^G_{K',K}"'] \\ \mathrm{Cht}'_{P,K,I,W} \arrow[u, "i_K"] & \mathrm{Cht}'_{P,K',I,W} \arrow[l, "\mathrm{pr}^P_{K',K}"] \arrow[u, "i_{K'}"'] \end{tikzcd} \tag{6.32}$$

Since adjunction morphism is compatible with composition, we deduce that the following diagram is commutative.

$$\begin{tikzcd} H^j_{G,K,I,W} \arrow[r, "\mathrm{adj}(\mathrm{pr}^G_{K',K})"] \arrow[d, "\mathrm{adj}(i_K)"'] & H^j_{G,K',I,W} \arrow[d, "\mathrm{adj}(i_{K'})"] \\ H'^j_{P,K,I,W} \arrow[r, "\mathrm{adj}(\mathrm{pr}^P_{K',K})"] & H'^j_{P,K',I,W} \end{tikzcd}$$

(2) Applying 6.1.12 to $S_1 = G(\mathbb{A})/K'$, $S_2 = G(\mathbb{A})/K$ and the projection $f : G(\mathbb{A})/K' \twoheadrightarrow G(\mathbb{A})/K$, we deduce from (6.28) that the following diagram is commutative.

$$\begin{tikzcd} H'^j_{P,K,I,W} \arrow[r, "\mathrm{adj}(\mathrm{pr}^P_{K',K})"] \arrow[d, "(6.16)"'] & H'^j_{P,K',I,W} \arrow[d, "(6.16)"] \\ H'^j_{M,K,I,W} \arrow[r, "\mathrm{adj}(\mathrm{pr}^M_{K',K})"] & H'^j_{M,K',I,W} \end{tikzcd}$$

□

Lemma 6.1.16. *For $K' \subset K$ as in 6.1.14, the following diagram of cohomology groups commutes.*

$$\begin{tikzcd} H^j_{G,K',I,W} \arrow[r, "\mathrm{Co}(\mathrm{pr}^G_{K',K})"] \arrow[d, "C^{P,j}_{G,K'}"'] & H^j_{G,K,I,W} \arrow[d, "C^{P,j}_{G,K}"] \\ H'^j_{M,K',I,W} \arrow[r, "\mathrm{Co}(\mathrm{pr}^M_{K',K})"] & H'^j_{M,K,I,W} \end{tikzcd}$$

Proof. (1) By [SGA4, XVIII Théorème 2.9], the trace morphism commutes with base change. Since (6.32) is Cartesian, we deduce that the following diagram is commutative.

$$\begin{tikzcd} H^j_{G,K',I,W} \arrow[r, "\mathrm{Co}(\mathrm{pr}^G_{K',K})"] \arrow[d, "\mathrm{adj}(i_{K'})"'] & H^j_{G,K,I,W} \arrow[d, "\mathrm{adj}(i_K)"] \\ H'^j_{P,K',I,W} \arrow[r, "\mathrm{Co}(\mathrm{pr}^P_{K',K})"] & H'^j_{P,K,I,W} \end{tikzcd}$$

(2) Applying 6.1.12 to $S_1 = G(\mathbb{A})/K'$, $S_2 = G(\mathbb{A})/K$ and the projection $f : G(\mathbb{A})/K' \twoheadrightarrow G(\mathbb{A})/K$, we deduce from (6.29) that the following diagram is commutative.

$$\begin{tikzcd} H'^{j}_{P,K',I,W} \arrow[r, "\mathrm{Co}(\mathrm{pr}^{P}_{K',K})"] \arrow[d, "(6.16)"'] & H'^{j}_{P,K,I,W} \arrow[d, "(6.16)"] \\ H'^{j}_{M,K',I,W} \arrow[r, "\mathrm{Co}(\mathrm{pr}^{M}_{K',K})"] & H'^{j}_{M,K,I,W} \end{tikzcd}$$

□

6.2 Compatibility of constant term morphisms and actions of Hecke algebras

We first recall the action of the local Hecke algebras. The goal of this subsection is Lemmas 6.2.6 and 6.2.12.

6.2.1 Let v be a place in X. Let $g \in G(F_v)$. By 6.1.3, the right action of g induces an isomorphism

$$\mathrm{Cht}_{G,\infty,I,W} \xrightarrow{\sim} \mathrm{Cht}_{G,\infty,I,W}. \tag{6.33}$$

Let $\widetilde{K} \subset G(\mathbb{O})$ be a compact open subgroup such that $g^{-1}\widetilde{K}g \subset G(\mathbb{O})$. The isomorphism (6.33) is $\widetilde{K}$-equivariant, where $k \in \widetilde{K}$ acts on the second stack by $g^{-1}kg$. It induces an isomorphism

$$\mathrm{Cht}_{G,\infty,I,W}/\widetilde{K} \xrightarrow{\sim} \mathrm{Cht}_{G,\infty,I,W}/g^{-1}\widetilde{K}g,$$

i.e. $\mathrm{Cht}_{G,\widetilde{K},I,W} \xrightarrow{\sim} \mathrm{Cht}_{G,g^{-1}\widetilde{K}g,I,W}$. It induces (by adjunction) an isomorphism of cohomology groups

$$\mathrm{adj}(g) : H^{j}_{G,g^{-1}\widetilde{K}g,I,W} \xrightarrow{\sim} H^{j}_{G,\widetilde{K},I,W}. \tag{6.34}$$

6.2.2 We denote by $\mathbb{O}^v$ the ring of integral adèles outside v. Let $K = K^vK_v \subset G(\mathbb{O}^v)G(\mathcal{O}_v) = G(\mathbb{O})$ be an open compact subgroup. Let $h = \mathbf{1}_{K_vgK_v} \in C_c(K_v\backslash G(F_v)/K_v, \mathbb{Q}_\ell)$ be the characteristic function of K_vgK_v for some $g \in G(F_v)$. The action of h on $H^j_{G,K,I,W}$ is given by the following composition of morphisms

$$T(h) : H^{j}_{G,K,I,W} \xrightarrow{\mathrm{adj}} H^{j}_{G,K\cap g^{-1}Kg,I,W} \xrightarrow[\sim]{\mathrm{adj}(g)} H^{j}_{G,gKg^{-1}\cap K,I,W} \xrightarrow{\mathrm{Co}} H^{j}_{G,K,I,W}, \tag{6.35}$$

where $\mathrm{adj} = \mathrm{adj}(\mathrm{pr}^{G}_{K\cap g^{-1}Kg,K})$ and $\mathrm{Co} = \mathrm{Co}(\mathrm{pr}^{G}_{gKg^{-1}\cap K,K})$, the isomorphism $\mathrm{adj}(g)$ is induced by (6.34) applied to $\widetilde{K} = gKg^{-1} \cap K$. Note that (6.35) depends only on the class K_vgK_v of g in $G(F_v)$. The action of $T(h)$ is equivalent to the one constructed by Hecke correspondence (see [Laf18, 2.20 and 4.4]).

6.2.3 Let $\widetilde{K}$ and g as in 6.2.1. The right action of g (by right multiplication by g) on $G(\mathbb{A})$ induces an isomorphism

$$G(\mathbb{A})/\widetilde{K} \xrightarrow{\sim} G(\mathbb{A})/g^{-1}\widetilde{K}g. \tag{6.36}$$

Applying 6.1.10 to $S_1 = G(\mathbb{A})/\widetilde{K}$, $S_2 = G(\mathbb{A})/g^{-1}\widetilde{K}g$ and the isomorphism (6.36), we deduce an isomorphism of cohomology groups

$$\mathrm{adj}(g) : H'^{j}_{M,g^{-1}\widetilde{K}g,I,W} \xrightarrow{\sim} H'^{j}_{M,\widetilde{K},I,W}. \tag{6.37}$$

6.2.4 Let K and h as in 6.2.2. The action of h on $H'^{j}_{M,K,I,W}$ is given by the following composition of morphisms

$$T(h) : H'^{j}_{M,K,I,W} \xrightarrow{\mathrm{adj}} H'^{j}_{M,K\cap g^{-1}Kg,I,W} \xrightarrow[\sim]{\mathrm{adj}(g)} H'^{j}_{M,gKg^{-1}\cap K,I,W} \xrightarrow{\mathrm{Co}} H'^{j}_{M,K,I,W}, \tag{6.38}$$

where $\mathrm{adj} = \mathrm{adj}(\mathrm{pr}^{M}_{K\cap g^{-1}Kg,K})$ and $\mathrm{Co} = \mathrm{Co}(\mathrm{pr}^{M}_{gKg^{-1}\cap K,K})$, the isomorphism $\mathrm{adj}(g)$ is induced by (6.37) applied to $\widetilde{K} = gKg^{-1} \cap K$. Note that $\widetilde{K}$ may not be normal in $G(\mathbb{O})$. Note that (6.38) depends only on the class $K_v g K_v$ of g in $G(F_v)$.

Lemma 6.2.5. *Let $\widetilde{K}$ and g as in 6.2.1. The following diagram of cohomology groups commutes.*

$$\begin{tikzcd} H^{j}_{G,g^{-1}\widetilde{K}g,I,W} \arrow[r, "\mathrm{adj}(g)", "\sim"'] \arrow[d, "C^{P,j}_{G,g^{-1}\widetilde{K}g}"'] & H^{j}_{G,\widetilde{K},I,W} \arrow[d, "C^{P,j}_{G,\widetilde{K}}"] \\ H'^{j}_{M,g^{-1}\widetilde{K}g,I,W} \arrow[r, "\mathrm{adj}(g)", "\sim"'] & H'^{j}_{M,\widetilde{K},I,W} \end{tikzcd} \tag{6.39}$$

Proof. (1) Since the isomorphism (6.3) is $G(\mathbb{O})$-equivariant, we deduce a Cartesian square.

$$\begin{tikzcd} \mathrm{Cht}_{P,I,W} \underset{\mathrm{Cht}_{G,I,W}}{\times} \mathrm{Cht}_{G,\infty,I,W} / \widetilde{K} \arrow[r, "g", "\simeq"'] & \mathrm{Cht}_{P,I,W} \underset{\mathrm{Cht}_{G,I,W}}{\times} \mathrm{Cht}_{G,\infty,I,W} / g^{-1}\widetilde{K}g \\ \mathrm{Cht}_{P,\infty,I,W} \overset{P(\mathbb{A})}{\times} G(\mathbb{A})/\widetilde{K} \arrow[u, "\simeq"] \arrow[r, "g", "\simeq"'] & \mathrm{Cht}_{P,\infty,I,W} \overset{P(\mathbb{A})}{\times} G(\mathbb{A})/g^{-1}\widetilde{K}g \arrow[u, "\simeq"'] \end{tikzcd}$$

We deduce a Cartesian square.

$$\begin{tikzcd} \mathrm{Cht}_{G,\widetilde{K},I,W} \arrow[r, "g", "\simeq"'] & \mathrm{Cht}_{G,g^{-1}\widetilde{K}g,I,W} \\ \mathrm{Cht}'_{P,\widetilde{K},I,W} \arrow[u, "i'_{\widetilde{K}}"] \arrow[r, "g", "\simeq"'] & \mathrm{Cht}'_{P,g^{-1}\widetilde{K}g,I,W} \arrow[u, "i'_{g^{-1}\widetilde{K}g}"'] \end{tikzcd}$$

It induces a commutative diagram.

$$\begin{tikzcd} H^{j}_{G,g^{-1}\widetilde{K}g,I,W} \arrow[r, "\mathrm{adj}(g)", "\sim"'] \arrow[d, "\mathrm{adj}(i'_{g^{-1}\widetilde{K}g})"'] & H^{j}_{G,\widetilde{K},I,W} \arrow[d, "\mathrm{adj}(i'_{\widetilde{K}})"] \\ H'^{j}_{P,g^{-1}\widetilde{K}g,I,W} \arrow[r, "\mathrm{adj}(g)", "\sim"'] & H'^{j}_{P,\widetilde{K},I,W} \end{tikzcd}$$

(2) Applying 6.1.12 to $S_1 = G(\mathbb{A})/\widetilde{K}$, $S_2 = G(\mathbb{A})/g^{-1}\widetilde{K}g$ and f the isomorphism (6.36), we deduce from (6.29) a commutative diagram.

$$\begin{tikzcd} H'^{j}_{P,g^{-1}\widetilde{K}g,I,W} \arrow[r, "\mathrm{adj}(g)", "\sim"'] \arrow[d, "(6.16)"'] & H'^{j}_{P,\widetilde{K},I,W} \arrow[d, "(6.16)"] \\ H'^{j}_{M,g^{-1}\widetilde{K}g,I,W} \arrow[r, "\mathrm{adj}(g)", "\sim"'] & H'^{j}_{M,\widetilde{K},I,W} \end{tikzcd}$$

□

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Lemma 6.2.6. *For any place v of X, any K and $h \in C_c(K_v \backslash G(F_v)/K_v, \mathbb{Q}_\ell)$ as in 6.2.2, the following diagram of cohomology groups commutes*

$$\begin{tikzcd} H^j_{G,K,I,W} \arrow[r, "T(h)"] \arrow[d, "C^{P,j}_{G,K}"] & H^j_{G,K,I,W} \arrow[d, "C^{P,j}_{G,K}"] \\ H'^j_{M,K,I,W} \arrow[r, "T(h)"] & H'^j_{M,K,I,W} \end{tikzcd} \tag{6.40}$$

where the horizontal morphisms are defined in 6.2.2 and 6.2.4, the vertical morphisms are the constant term morphism defined in (6.19).

Proof. By Lemma 6.1.15, Lemma 6.2.5 and Lemma 6.1.16. □

6.2.7 From now on let $N \subset X$ be a closed subscheme and v be a place in $X \smallsetminus N$. We have the (unnormalized) Satake transform:

$$\begin{aligned} C_c(G(\mathcal{O}_v)\backslash G(F_v)/G(\mathcal{O}_v), \mathbb{Q}_\ell) &\hookrightarrow C_c(M(\mathcal{O}_v)\backslash M(F_v)/M(\mathcal{O}_v), \mathbb{Q}_\ell) \\ h \quad &\mapsto \quad h^M : m \mapsto \sum_{u \in U(F_v)/U(\mathcal{O}_v)} h(mu). \end{aligned} \tag{6.41}$$

6.2.8 We have $K_{M,N} = K^v_{M,N} K_{M,N,v} \subset M(\mathbb{O}^v) M(\mathcal{O}_v)$. For any $K_{M,v} \subset M(\mathcal{O}_v)$ open compact subgroup, we have $K^v_{M,N} K_{M,v} \subset M(\mathbb{O}^v) M(\mathcal{O}_v)$. We define $H^j_{M,K^v_{M,N}K_{M,v},I,W}$ as in Definition 2.6.8 (replacing $\mathrm{Cht}_{M,N,I,W}$ by $\mathrm{Cht}_{M,K^v_{M,N}K_{M,v},I,W}$). We define

$$\varinjlim_{K_{M,v}} H^j_{M,K^v_{M,N}K_{M,v},I,W}.$$

As in 6.2.1 (by replacing G by M), for any $m \in M(F_v)$ and $K_{M,v}$ such that $m^{-1}K_{M,v}m \subset M(\mathcal{O}_v)$, we have an isomorphism $H^j_{M,m^{-1}K^v_{M,N}K_{M,v}m,I,W} \xrightarrow{\sim} H^j_{M,K^v_{M,N}K_{M,v},I,W}$. Taking limit on $K_{M,v}$, we deduce an action of $M(F_v)$ on $\varinjlim_{K_{M,v}} H^j_{M,K^v_{M,N}K_{M,v},I,W}$.

We have $K_N = K^v_N K_{N,v} \subset G(\mathbb{O}^v) G(\mathcal{O}_v)$. For any $K_v \subset G(\mathcal{O}_v)$ open compact subgroup, we have $K^v_N K_v \subset G(\mathbb{O}^v) G(\mathcal{O}_v)$. Applying 6.1.6 to $S = G(\mathbb{A})/K^v_N K_v$, we define $H'^j_{M,K^v_N K_v,I,W}$. We define

$$\varinjlim_{K_v} H'^j_{M,K^v_N K_v,I,W}.$$

Note that v is a place in $X \smallsetminus N$, so $K_{N,v} = G(\mathcal{O}_v)$ and $K_{M,N,v} = M(\mathcal{O}_v)$. We have

$$H'^j_{M,K^v_N G(\mathcal{O}_v),I,W} = H'^j_{M,N,I,W} = H^j_{M,N,I,W} \overset{P(\mathcal{O}_N)}{\times} G(\mathcal{O}_N) = H^j_{M,K^v_{M,N}M(\mathcal{O}_v),I,W} \overset{P(\mathcal{O}_N)}{\times} G(\mathcal{O}_N),$$

where $H'^j_{M,N,I,W}$ is defined in Definition 3.4.9. We deduce

$$\varinjlim_{K_v} H'^j_{M,K^v_N K_v,I,W} = \mathrm{Ind}^{G(F_v)}_{P(F_v)} \Big(\Big(\varinjlim_{K_{M,v}} H^j_{M,K^v_{M,N}K_{M,v},I,W} \Big) \overset{P(\mathcal{O}_N)}{\times} G(\mathcal{O}_N) \Big), \tag{6.42}$$

where $\mathrm{Ind}^{G(F_v)}_{P(F_v)}$ is the (unnormalized) parabolic induction.

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6.2.9 Let V be a $\mathbb{Q}_\ell$-vector space equipped with a continuous action of $M(F_v)$, denoted by $\sigma : M(F_v) \to \mathrm{GL}(V)$. We recall that

$$\mathrm{Ind}_{P(F_v)}^{G(F_v)} V = \{f : G(F_v) \to V \text{ continuous}, f(pg) = \sigma(p)f(g), p \in P(F_v), g \in G(F_v)\}.$$

We have a morphism

$$(\mathrm{Ind}_{P(F_v)}^{G(F_v)} V)^{G(\mathcal{O}_v)} \to V^{M(\mathcal{O}_v)} : \quad f \mapsto f(1). \tag{6.43}$$

LEMMA 6.2.10. *Morphism (6.43) is an isomorphism. Moreover, for $h \in C_c(G(\mathcal{O}_v)\backslash G(F_v)/G(\mathcal{O}_v), \mathbb{Q}_\ell)$, the action of $T(h)$ on $(\mathrm{Ind}_{P(F_v)}^{G(F_v)} V)^{G(\mathcal{O}_v)}$ coincides with the action of $T(h^M)$ on $V^{M(\mathcal{O}_v)}$.*

Proof. Morphism (6.43) admits an inverse $f(1) \mapsto f$ given by

$$f(x) = f(x_P x_K) = \sigma(x_P) f(x_K) = \sigma(x_P) f(1),$$

where $x = x_P x_K \in G(F_v) = P(F_v)G(\mathcal{O}_v)$. Thus $(\mathrm{Ind}_{P(F_v)}^{G(F_v)} V)^{G(\mathcal{O}_v)} = V^{M(\mathcal{O}_v)}$.

Moreover, for $g = g_P g_K \in G(F_v)$ and $f \in (\mathrm{Ind}_{P(F_v)}^{G(F_v)} V)^{G(\mathcal{O}_v)}$, we have

$$gf(1) = f(g_P g_K) = \sigma(g_P) f(g_K) = \sigma(g_P) f(1). \tag{6.44}$$

Note that $G(F_v) = M(F_v)U(F_v)G(\mathcal{O}_v)$. Denote by dg (respectively dm, du, dk) the Haar measure on $G(F_v)$ (respectively $M(F_v), U(F_v), G(\mathcal{O}_v)$) such that the volume of $G(O_v)$ (respectively $M(\mathcal{O}_v), U(\mathcal{O}_v), G(\mathcal{O}_v)$) is 1. We have $dg = dm\, du\, dk$. Taking the integral over $G(F_v)$ of the product by $h(g)$ of (6.44), we deduce that the action of $T(h)$ on $(\mathrm{Ind}_{P(F_v)}^{G(F_v)} V)^{G(\mathcal{O}_v)}$ coincides with the action of $T(h^M)$ on $V^{M(\mathcal{O}_v)}$. □

6.2.11 Let $V = (\varinjlim_{K_{M,v}} H^j_{M,K^v_{M,N}K_{M,v},I,W}) \overset{P(\mathcal{O}_N)}{\times} G(\mathcal{O}_N)$. We have:

$$V^{M(\mathcal{O}_v)} = H^j_{M,K^v_{M,N}M(\mathcal{O}_v),I,W} \overset{P(\mathcal{O}_N)}{\times} G(\mathcal{O}_N) = H'^j_{M,N,I,W};$$

$$(\mathrm{Ind}_{P(F_v)}^{G(F_v)} V)^{G(\mathcal{O}_v)} = \left(\varinjlim_{K_v} H'^j_{M,K^v_N K_v,I,W}\right)^{G(\mathcal{O}_v)} = H'^j_{M,K^v_N G(\mathcal{O}_v),I,W} = H'^j_{M,N,I,W}.$$

By Lemma 6.2.10, the action of $T(h)$ on $H'^j_{M,N,I,W}$ (defined in (6.38)) coincides with the action of $T(h^M)$ on $H'^j_{M,N,I,W}$ (induced by the action of $T(h^M)$ on $H^j_{M,N,I,W}$). Combining this fact and Lemma 6.2.6, we deduce the following.

LEMMA 6.2.12. *For any place v of $X \smallsetminus N$ and any $h \in C_c(G(\mathcal{O}_v)\backslash G(F_v)/G(\mathcal{O}_v), \mathbb{Q}_\ell)$, the following diagram of cohomology groups is commutative*

$$\begin{tikzcd} H^j_{G,N,I,W} \arrow[r, "T(h)"] \arrow[d, "C^{P,j}_{G,N}"] & H^j_{G,N,I,W} \arrow[d, "C^{P,j}_{G,N}"] \\ H'^j_{M,N,I,W} \arrow[r, "T(h^M)"] & H'^j_{M,N,I,W} \end{tikzcd} \tag{6.45}$$

where the vertical morphisms are the constant term morphism defined in Definition 3.5.10.

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Remark 6.2.13. For a direct proof of Lemma 6.2.12, see [Xue17, Lemme 8.1.1].

Remark 6.2.14. We could normalize the constant term morphism $C_{G,N}^{P,j}$ and the Satake transform (6.41) by $\delta^{1/2}$ as usual, where δ is the modular function of $P(F_v)$. But we do not need this normalization in this paper.

Remark 6.2.15. When $I = \emptyset$ and $W = \mathbf{1}$, $S = G(\mathbb{A})/K$, we have $H'^{0}_{M,S,R,I,W}$ included in $C(M(F)U(\mathbb{A})\backslash G(\mathbb{A})/K\Xi, \mathbb{Q}_\ell)$. In (6.17), we defined $C^{P,0}_{G,S,R,I,W}$. Commutative diagram (6.18) implies that for a given Haar measure du on $U(\mathbb{A})$, $(\int_R du) \cdot C^{P,0}_{G,S,R,I,W}$ is independent on R. This identifies $C^{P,0}_{G,S,I,W}$ with the classical constant term morphism (3.44) associated to du.

6.3 Cuspidal cohomology and rational Hecke-finite cohomology

Definition 6.3.1. We define

$$H^{j,\,\text{Hf-rat}}_{G,N,I,W} := \{c \in H^{j}_{G,N,I,W},\ \dim_{\mathbb{Q}_\ell} C_c(K_N\backslash G(\mathbb{A})/K_N, \mathbb{Q}_\ell)\cdot c < +\infty\}.$$

Proposition 6.0.1 will follow from Lemmas 6.3.2 and 6.3.3 below.

Lemma 6.3.2. *We have an inclusion*

$$H^{j,\,\text{cusp}}_{G,N,I,W} \subset H^{j,\,\text{Hf-rat}}_{G,N,I,W}. \tag{6.46}$$

Proof. By Theorem 5.0.1, the $\mathbb{Q}_\ell$-vector space $H^{j,\,\text{cusp}}_{G,N,I,W}$ has finite dimension. By Lemma 6.2.6, it is stable under the action of the Hecke algebra $C_c(K_N\backslash G(\mathbb{A})/K_N, \mathbb{Q}_\ell)$. We complete the proof by Definition 6.3.1. □

Lemma 6.3.3. *We have an inclusion*

$$H^{j,\,\text{cusp}}_{G,N,I,W} \supset H^{j,\,\text{Hf-rat}}_{G,N,I,W}. \tag{6.47}$$

The proof of Lemma 6.3.3 will use the fact that any non-zero image of a constant term morphism $C_P^{G,j}$ is supported on the components $H'^{j,\nu}_{M}$ indexed by ν in a translated cone in $\widehat{\Lambda}^{+,\mathbb{Q}}_{Z_G/Z_M}$. The proof will also need the following lemma, which is for example a consequence of the Satake isomorphism.

Lemma 6.3.4. *Under the Satake transformation (6.41), the algebra $C_c(M(O_v)\backslash M(F_v)/M(O_v), \mathbb{Q}_\ell)$ is finite over $C_c(G(O_v)\backslash G(F_v)/G(O_v), \mathbb{Q}_\ell)$.*

Proof of Lemma 6.3.3. Let $a \in H^{\text{Hf-rat}}_{G,N,I,W}$. We argue by contradiction. Suppose that $a \notin H^{j,\,\text{cusp}}_{G,N,I,W}$. Then there exists a maximal parabolic subgroup P such that $C_G^{P,\,j}(a) \neq 0$. We denote by M the Levi quotient of P. Let v be a place in $X \smallsetminus N$.

(1) On the one hand, by Definition 6.3.1, the $\mathbb{Q}_\ell$-vector subspace $C_c(G(O_v)\backslash G(F_v)/G(O_v), \mathbb{Q}_\ell)\cdot a$ has finite dimension. Then Lemma 6.2.12 applied to $K = K_N$ and Lemma 6.3.4 imply that the $\mathbb{Q}_\ell$-vector space $C_c(M(O_v)\backslash M(F_v)/M(O_v), \mathbb{Q}_\ell)\cdot C_G^{P,\,j}(a)$ has finite dimension.

(2) On the other hand, since $a \in H^{j}_{G,N,I,W}$, there exists $\mu \in \widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\text{ad}}}$ such that $a \in \operatorname{Im}(H^{j,\leqslant\mu}_{G,N,I,W} \to H^{j}_{G,N,I,W})$. We deduce from (3.38) that $C_G^{P,\,j}(a)$ is the image of an element

$a^0 \in H'^{j, \leqslant\mu}_{M,N,I,W}$ in $H'^{j}_{M,N,I,W}$. By 3.4.8, a^0 is supported on the components $H'^{j,\nu}_{M,N,I,W}$ of $H'^{j}_{M,N,I,W}$ indexed by ν in the translated cone $\widehat{\Lambda}^{\mu}_{Z_M/Z_G} \subset \widehat{\Lambda}^{\mathbb{Q}}_{Z_M/Z_G}$. So is $C_G^{P,j}(a)$.

Let $g \in Z_M(F_v)$ such that $g \notin Z_M(O_v)Z_G(F_v)$. We denote by $\xi(g)$ the image of g by the composition of morphisms

$$Z_M(F_v) \to Z_M(\mathbb{A}) \to \mathrm{Bun}_M(\mathbb{F}_q) \to \mathrm{Bun}_M \xrightarrow{(1.16)} \widehat{\Lambda}^{\mathbb{Q}}_{Z_M/Z_G}.$$

The choice of g implies that $\xi(g) \neq 0$. Note that P is maximal, so $\widehat{\Lambda}^{\mathbb{Q}}_{Z_M/Z_G} \simeq \mathbb{Q}$. For all $\nu \in \widehat{\Lambda}^{\mathbb{Q}}_{Z_M/Z_G}$, the action of g on Cht_M/Ξ induces an isomorphism $g : \mathrm{Cht}_M^{\nu}/\Xi \xrightarrow{\sim} \mathrm{Cht}_M^{\nu+\xi(g)}/\Xi$ (the inverse is induced by g^{-1}). We denote by $T(g) \in C_c(M(O_v)\backslash M(F_v)/M(\mathcal{O}_v), \mathbb{Q}_\ell)$ the Hecke operator associated to g. Then $T(g)$ induces an isomorphism $H'^{j,\nu}_{M,N,I,W} \xrightarrow{\sim} H'^{j,\nu+\xi(g)}_{M,N,I,W}$.

Suppose that $\xi(g) > 0$ (if not, we take g^{-1} in place of g). Since $C_G^{P,j}(a) \neq 0$, there exists $m \in \mathbb{Z}_{>0}$ such that $T(g)^m \cdot C_G^{P,j}(a)$ is supported on the cone $\widehat{\Lambda}^{\mu+m\xi(g)}_{Z_M/Z_G} \supset \widehat{\Lambda}^{\mu}_{Z_M/Z_G}$, but not supported on $\widehat{\Lambda}^{\mu}_{Z_M/Z_G}$. Therefore $T(g)^{2m} \cdot C_G^{P,j}(a)$ is supported on the cone $\widehat{\Lambda}^{\mu+2m\xi(g)}_{Z_M/Z_G}$, but not supported on $\widehat{\Lambda}^{\mu+m\xi(g)}_{Z_M/Z_G}$, etc. We deduce that

$$C_G^{P,j}(a), T(g)^m \cdot C_G^{P,j}(a), T(g)^{2m} \cdot C_G^{P,j}(a), T(g)^{3m} \cdot C_G^{P,j}(a), \ldots$$

are linearly independent. So the $\mathbb{Q}_\ell$-vector space generated by $T(g)^{\mathbb{Z}} \cdot C_G^{P,j}(a)$ has infinite dimension. Hence $C_c(M(O_v)\backslash M(F_v)/M(O_v), \mathbb{Q}_\ell) \cdot C_G^{P,j}(a)$ has infinite dimension.

(3) We deduce from (1) and (2) a contradiction. So $a \in H^{j,\,\mathrm{cusp}}_{G,N,I,W}$. □

Definition 6.3.5 [Laf18, Définition 8.19]. We define

$$H^{j,\,\mathrm{Hf}}_{G,N,I,W} := \{c \in H^{j}_{G,N,I,W},\ C_c(K_N\backslash G(\mathbb{A})/K_N, \mathbb{Z}_\ell) \cdot c \text{ is a finitely generated } \mathbb{Z}_\ell\text{-submodule}\}.$$

By definition, $H^{j,\,\mathrm{Hf}}_{G,N,I,W} \subset H^{j,\,\mathrm{Hf\text{-}rat}}_{G,N,I,W}$. Thus Proposition 6.0.1 has the following corollary.

Corollary 6.3.6.

$$H^{j,\,\mathrm{Hf}}_{G,N,I,W} \subset H^{j,\,\mathrm{Hf\text{-}rat}}_{G,N,I,W} = H^{j,\,\mathrm{cusp}}_{G,N,I,W}.$$

In particular, $H^{j,\,\mathrm{Hf}}_{G,N,I,W}$ *has finite dimension.*

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Appendix A. Exact sequences associated to an open and a closed substack of the stack of shtukas

For simplicity of the notation, we do not write the indices N, I and W.

A.0.1 In the following, we use $\widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}}$. But everything remains true if we replace it by $(1/r)\widehat{R}^{+}_{G^{\mathrm{ad}}}$.

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A.0.2 As in [DG15, 7.4.10], we equip the set $\widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}}$ with the order topology, i.e. the one where a base of open subsets is formed by subsets of the form $\{\lambda \in \widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}} | \lambda \leqslant \lambda_0\}$ for $\lambda_0 \in \widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}}$. Let S be a subset of $\widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}}$. We define

$$\mathrm{Bun}_G^S := \bigcup_{\lambda \in S} \mathrm{Bun}_G^{=\lambda}, \quad \mathrm{Cht}_G^S := \bigcup_{\lambda \in S} \mathrm{Cht}_G^{=\lambda}, \quad \mathrm{Cht}'^{S}_M := \bigcup_{\lambda \in S} \mathrm{Cht}'^{=\lambda}_M,$$

where $\mathrm{Cht}_G^{=\lambda}$ and $\mathrm{Cht}'^{=\lambda}_M$ are defined in Definition 4.1.10. If the subset S is open (respectively closed) in $\widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}}$, then Bun_G^S is open (respectively closed) in Bun_G. So Cht_G^S is open (respectively closed) in Cht_G and Cht'^{S}_M is open (respectively closed) in Cht'_M.

If S is a bounded locally closed subset of $\widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}}$, then Cht_G^S and Cht'^{S}_M are Deligne–Mumford stacks of finite type.

A.0.3 Let $\mu \in \widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}}$. Let $S_2 = \{\lambda \in \widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}} \mid \lambda \leqslant \mu\}$. By definition it is an open subset of $\widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}}$ for the order topology of G^{ad}. It is also open in $\widehat{\Lambda}^{+,\mathbb{Q}}_{\overline{M}}$ for the order topology of $\overline{M} = M/Z_G$ (because $\lambda \leqslant^{\overline{M}} \mu$ implies $\lambda \leqslant \mu$).

Let S_1 be an open subset of S_2 for the order topology of G^{ad}. Thus the morphism of stacks $\mathrm{Cht}_G^{S_1} \xrightarrow{j_G} \mathrm{Cht}_G^{S_2}$ (respectively $\mathrm{Cht}'^{S_1}_M \xrightarrow{j_M} \mathrm{Cht}'^{S_2}_M$) is an open immersion. By definition, $\mathrm{Cht}_G^{S_2 - S_1}$ (respectively $\mathrm{Cht}'^{S_2-S_1}_M$) is the closed substack in $\mathrm{Cht}_G^{S_2}$ (respectively $\mathrm{Cht}'^{S_2}_M$) which is the complement of $\mathrm{Cht}_G^{S_1}$ (respectively $\mathrm{Cht}'^{S_1}_M$).

We define $\mathrm{Cht}'^{S_2}_P$ (respectively $\mathrm{Cht}'^{S_1}_P$) to be the inverse image of $\mathrm{Cht}_G^{S_2}$ (respectively $\mathrm{Cht}_G^{S_1}$) in Cht'_P. Just as in Lemma 1.7.4, we have $\pi_2 : \mathrm{Cht}'^{S_2}_P \to \mathrm{Cht}'^{S_2}_M$ (respectively $\pi_1 : \mathrm{Cht}'^{S_1}_P \to \mathrm{Cht}'^{S_1}_M$). We have $\mathrm{Cht}'^{S_1}_P \xrightarrow{j_P} \mathrm{Cht}'^{S_2}_P$, which is an open immersion. We define $\mathrm{Cht}'^{S_2-S_1}_P := \mathrm{Cht}'^{S_2}_P \cap \pi^{-1}(\mathrm{Cht}'^{S_2-S_1}_M)$. It is a closed substack in the complement of $\mathrm{Cht}'^{S_1}_P$ in $\mathrm{Cht}'^{S_2}_P$, but may not be equal to it.

Lemma A.0.4. *The following diagram of algebraic stacks is commutative.*

$$\begin{tikzcd}
\mathrm{Cht}_G^{S_2-S_1} \arrow[d, hook, "i_G"'] & \mathrm{Cht}'^{S_2-S_1}_P \arrow[l, "i_{12}"'] \arrow[r, "\pi_{12}"] \arrow[d, hook, "i_P"'] & \mathrm{Cht}'^{S_2-S_1}_M \arrow[d, hook, "i_M"] \\
\mathrm{Cht}_G^{S_2} & \mathrm{Cht}'^{S_2}_P \arrow[l, "i_2"'] \arrow[r, "\pi_2"] & \mathrm{Cht}'^{S_2}_M \\
\mathrm{Cht}_G^{S_1} \arrow[u, hook, "j_G"] & \mathrm{Cht}'^{S_1}_P \arrow[l, "i_1"'] \arrow[r, "\pi_1"] \arrow[u, hook, "j_P"] & \mathrm{Cht}'^{S_1}_M \arrow[u, hook, "j_M"']
\end{tikzcd} \tag{A.1}$$

Moreover, the left bottom square and the right top square are Cartesian.

A.0.5 For any j, any $\nu \in \widehat{\Lambda}^{\mathbb{Q}}_{Z_M/Z_G}$ and any bounded locally closed subset $S \subset \widehat{\Lambda}^{+,\mathbb{Q}}_{G^{\mathrm{ad}}}$, we define

$$H_G^{j,\,S} := H_c^j(\mathrm{Cht}^S_{G,\overline{\eta^I}}/\Xi_G, \mathcal{F}_G); \quad H'^{j,\,S,\,\nu}_M := H_c^j(\mathrm{Cht}'^{S,\nu}_{M,\overline{\eta^I}}/\Xi_G, \mathcal{F}'_M).$$

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A.0.6 By Proposition 3.5.3, the restriction of morphism i_1 (respectively i_2) to $\overline{\eta^I}$ is proper. The restriction of morphism i_{12} to $\overline{\eta^I}$ is also proper because $\mathrm{Cht}'^{S_2-S_1}_P \to \mathrm{Cht}^{S_2-S_1}_G \underset{\mathrm{Cht}^{S_2}_G}{\times} \mathrm{Cht}'^{S_2}_P$ is a closed immersion. Moreover i_1, i_2 and i_{12} are schematic. Applying the construction in § 3 to each line in diagram (A.1), respectively, we obtain the constant term morphism $C_G^{P,\,j,\,S_1} : H_G^{j,\,S_1} \to H'^{j,\,S_1}_M$, $C_G^{P,\,j,\,S_2}$ and $C_G^{P,\,j,\,S_2-S_1}$ (note that the morphism $\pi_{12,d} : \mathrm{Cht}'^{S_2-S_1}_P \to \widetilde{\mathrm{Cht}}'^{S_2-S_1}_M$ is smooth because the right top square of diagram (A.1) is Cartesian).

A.0.7 Diagram (A.1) induces a diagram of cohomology groups with compact support for which we will study the commutativity.

$$\begin{tikzcd}
\cdots \arrow[r] & H_G^{j-1,\,S_2-S_1} \arrow[r] \arrow[d, "C_G^{P,\,j-1,\,S_2-S_1}"] & H_G^{j,\,S_1} \arrow[r] \arrow[d, "C_G^{P,\,j,\,S_1}"] & H_G^{j,\,S_2} \arrow[r] \arrow[d, "C_G^{P,\,j,\,S_2}"] & H_G^{j,\,S_2-S_1} \arrow[r] \arrow[d, "C_G^{P,\,j,\,S_2-S_1}"] & \cdots \\
\cdots \arrow[r] & H'^{j-1,\,S_2-S_1}_M \arrow[r] & H'^{j,\,S_1}_M \arrow[r] & H'^{j,\,S_2}_M \arrow[r] & H'^{j,\,S_2-S_1}_M \arrow[r] & \cdots
\end{tikzcd} \tag{A.2}$$

The horizontal maps are the long exact sequences associated to an open substack and the complementary closed substack. The vertical maps are the constant term morphisms.

Lemma A.0.8. *For any j, the following diagram is commutative.*

$$\begin{tikzcd}
H_G^{j,\,S_1} \arrow[r] \arrow[d, "C_G^{P,\,j,\,S_1}"] & H_G^{j,\,S_2} \arrow[r] \arrow[d, "C_G^{P,\,j,\,S_2}"] & H_G^{j,\,S_2-S_1} \arrow[d, "C_G^{P,\,j,\,S_2-S_1}"] \\
H'^{j,\,S_1}_M \arrow[r] & H'^{j,\,S_2}_M \arrow[r] & H'^{j,\,S_2-S_1}_M
\end{tikzcd}$$

Proof. We denote the morphisms of paws by $p_G : \mathrm{Cht}^{S_2}_G \to \eta^I$ and $p_M : \mathrm{Cht}'^{S_2}_M \to \eta^I$. For $S = S_1$ or S_2 or $S_2 - S_1$, denote $\mathcal{F}^S_G := \mathcal{F}_G\big|_{\mathrm{Cht}^S_G}$ and $\mathcal{F}^S_M := \mathcal{F}'_M\big|_{\mathrm{Cht}'^S_M}$. Note that $\mathcal{F}^{S_1}_G = (j_G)^*\mathcal{F}^{S_2}_G$ and $\mathcal{F}^{S_2-S_1}_G = (i_G)^*\mathcal{F}^{S_2}_G$. Similarly $\mathcal{F}^{S_1}_M = (j_M)^*\mathcal{F}^{S_2}_M$ and $\mathcal{F}^{S_2-S_1}_M = (i_M)^*\mathcal{F}^{S_2}_M$. Lemma A.0.8 will follow from the commutativity of the following diagram of complexes in $D^b_c(\eta^I, \mathbb{Q}_\ell)$.

$$\begin{tikzcd}
(p_G)_!(j_G)_!(j_G)^*\mathcal{F}^{S_2}_G \arrow[r] \arrow[d, "c_G^{P,\,S_1}"] & (p_G)_!\mathcal{F}^{S_2}_G \arrow[r] \arrow[d, "c_G^{P,\,S_2}"] & (p_G)_!(i_G)_!(i_G)^*\mathcal{F}^{S_2}_G \arrow[d, "c_G^{P,\,S_2-S_1}"] \\
(p_M)_!(j_M)_!(j_M)^*\mathcal{F}^{S_2}_M \arrow[r] & (p_M)_!\mathcal{F}^{S_2}_M \arrow[r] & (p_M)_!(i_M)_!(i_M)^*\mathcal{F}^{S_2}_M
\end{tikzcd} \tag{A.3}$$

The commutativity of the left square is induced by (1) and (2) below. The commutativity of the right square is induced by (3) and (4) below.

We consider the left square of (A.3)

$$\begin{tikzcd}
(p_G)_!(j_G)_!(j_G)^*\mathcal{F}^{S_2}_G \arrow[r, "\mathrm{Tr}_{j_G}"] \arrow[d] \arrow[dr, phantom, "(1)"] & (p_G)_!\mathcal{F}^{S_2}_G \arrow[d] \\
(p_M)_!(j_M)_!(\pi_1)_!(i_1)^*(j_G)^*\mathcal{F}^{S_2}_G \arrow[r, "\mathrm{Tr}_{j_P}"] \arrow[d, "f_1"] \arrow[dr, phantom, "(2)"] & (p_M)_!(\pi_2)_!(i_2)^*\mathcal{F}^{S_2}_G \arrow[d, "f_2"] \\
(p_M)_!(j_M)_!(j_M)^*\mathcal{F}^{S_2}_M \arrow[r, "\mathrm{Tr}_{j_M}"'] & (p_M)_!\mathcal{F}^{S_2}_M
\end{tikzcd} \tag{A.4}$$

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where (1) and (2) are detailed below.

(1) The following diagram of functors is commutative

$$\begin{tikzcd}
(p_G)_!(j_G)_!(j_G)^* \arrow[rr, "\mathrm{Tr}_{j_G}"] \arrow[d, "\mathrm{adj}_{i_1}"] \arrow[dr, "\mathrm{adj}_{i_2}"] & & (p_G)_! \arrow[d, "\mathrm{adj}_{i_2}"] \\
(p_G)_!(j_G)_!(i_1)_!(i_1)^*(j_G)^* \arrow[r, "\simeq", "(*)"'] \arrow[d, "\simeq"] & (p_G)_!(i_2)_!(i_2)^*(j_G)_!(j_G)^* \arrow[r, "\mathrm{Tr}_{j_G}"] & (p_G)_!(i_2)_!(i_2)^* \arrow[d, "\simeq"] \\
(p_M)_!(j_M)_!(\pi_1)_!(i_1)^*(j_G)^* \arrow[r, "\simeq"] & (p_M)_!(\pi_2)_!(j_P)_!(j_P)^*(i_2)^* \arrow[r, "\mathrm{Tr}_{j_P}"] & (p_M)_!(\pi_2)_!(i_2)^*
\end{tikzcd}$$

where $(*)$ is given by $(j_G)_!(i_1)_!(i_1)^* \simeq (i_2)_!(j_P)_!(i_1)^* \simeq (i_2)_!(i_2)^*(j_G)_!$, the last isomorphism is the proper base change for the left bottom square of diagram (A.1). The commutativity of (1) follows from the fact that the adjunction morphism commutes with base change and the trace morphism commutes with base change [SGA4, XVIII Théorème 2.9].

(2) Taking (3.9) into account, we have a commutative diagram, where π_2 (respectively π_1) is the composition $\widetilde{\pi^0_{2,\underline{d}}} \circ \pi_{2,\underline{d}}$ (respectively $\widetilde{\pi^0_{1,\underline{d}}} \circ \pi_{1,\underline{d}}$) for some d large enough as in Proposition 2.2.1.

$$\begin{tikzcd}
\mathrm{Cht}'^{S_2}_P \arrow[r, "\pi_{2,d}"] \arrow[dr, phantom, "(b)"] & \widetilde{\mathrm{Cht}}'^{S_2}_M \arrow[r, "\widetilde{\pi^0_{2,d}}"] \arrow[dr, phantom, "(c)"] & \mathrm{Cht}'^{S_2}_M \\
\mathrm{Cht}'^{S_1}_P \arrow[r, "\pi_{1,d}"] \arrow[u, hook, "j_P"] & \widetilde{\mathrm{Cht}}'^{S_1}_M \arrow[r, "\widetilde{\pi^0_{1,d}}"] \arrow[u, hook, "\widetilde{j_M}"] & \mathrm{Cht}'^{S_1}_M \arrow[u, hook, "j_M"]
\end{tikzcd} \tag{A.5}$$

The square (c) is Cartesian. The square (b) may not be Cartesian. As in Lemma 3.1.8, $\pi_{1,d}$ and $\pi_{2,d}$ are smooth. We have $\dim(\pi_{1,d}) = \dim(\pi_{2,d}) = d \cdot |I| \dim U$. We denote this dimension by m.

By (3.22) and (3.23), the morphism f_1 (respectively f_2) defined in diagram (A.4) is the composition of $\mathrm{Tr}_{\pi_{1,d}} : (\pi_{1,d})_!(\pi_{1,d})^* \to \mathrm{Id}[-2m](-m)$ (respectively $\mathrm{Tr}_{\pi_{2,d}} : (\pi_{2,d})_!(\pi_{2,d})^* \to \mathrm{Id}[-2m](-m)$) with some isomorphisms. By [SGA4, XVIII Théorème 2.9], the trace morphism is compatible with composition, and thus

$$\mathrm{Tr}_{\pi_{2,d}} \circ \mathrm{Tr}_{j_P} \simeq \mathrm{Tr}_{\pi_{2,d} \circ j_P} \simeq \mathrm{Tr}_{\widetilde{j_M} \circ \pi_{1,d}} \simeq \mathrm{Tr}_{\widetilde{j_M}} \circ \mathrm{Tr}_{\pi_{1,d}},$$

where the middle isomorphism is due to the commutativity of square (b). Moreover, by [SGA4] the trace morphism is compatible with base change, and thus

$$\mathrm{Tr}_{\widetilde{j_M}} = (\widetilde{\pi^0_{1,d}})^* \, \mathrm{Tr}_{j_M}.$$

We deduce that (2) is commutative.

Now we consider the right square of (A.3)

$$\begin{tikzcd}
(p_G)_!\mathcal{F}^{S_2}_G \arrow[r, "\mathrm{adj}_{i_G}"] \arrow[d] \arrow[dr, phantom, "(3)"] & (p_G)_!(i_G)_!(i_G)^*\mathcal{F}^{S_2}_G \arrow[d] \\
(p_M)_!(\pi_2)_!(i_2)^*\mathcal{F}^{S_2}_G \arrow[r, "\mathrm{adj}_{i_P}"] \arrow[d, "f_2"] \arrow[dr, phantom, "(4)"] & (p_M)_!(i_M)_!(\pi_{12})_!(i_{12})^*(i_G)^*\mathcal{F}^{S_2}_G \arrow[d, "f_{12}"] \\
(p_M)_!\mathcal{F}^{S_2}_M \arrow[r, "\mathrm{adj}_{i_M}"] & (p_M)_!(i_M)_!(i_M)^*\mathcal{F}^{S_2}_M
\end{tikzcd} \tag{A.6}$$

where (3) and (4) are detailed below.

(3) The following diagram of functors is commutative.

$$\begin{tikzcd}
(p_G)_! \arrow[rr, "\mathrm{adj}_{i_G}"] \arrow[d, "\mathrm{adj}_{i_2}"] & & (p_G)_!(i_G)_!(i_G)^* \arrow[d, "\mathrm{adj}_{i_{12}}"] \\
(p_G)_!(i_2)_!(i_2)^* \arrow[r, "\mathrm{adj}_{i_P}"] \arrow[d, "\simeq"] & (p_G)_!(i_2)_!(i_P)_!(i_P)^*(i_2)^* \arrow[r, "\simeq"] \arrow[d, "\simeq"] & (p_G)_!(i_G)_!(i_{12})_!(i_{12})^*(i_G)^* \arrow[d, "\simeq"] \\
(p_M)_!(\pi_2)_!(i_2)^* \arrow[r, "\mathrm{adj}_{i_P}"] & (p_M)_!(\pi_2)_!(i_P)_!(i_P)^*(i_2)^* \arrow[r, "\simeq"] & (p_M)_!(i_M)_!(\pi_{12})_!(i_{12})^*(i_G)^*
\end{tikzcd}$$

(4) Taking (3.9) into account, we have a commutative diagram, where π_{12} is the composition $\widetilde{\pi^0_{12,\underline{d}}} \circ \pi_{12,\underline{d}}$.

$$\begin{tikzcd}
\mathrm{Cht}'^{S_2-S_1}_P \arrow[r, "\pi_{12,d}"] \arrow[d, hook, "i_P"'] \arrow[dr, phantom, "(e)"] & \widetilde{\mathrm{Cht}}'^{S_2-S_1}_M \arrow[r, "\widetilde{\pi^0_{12,d}}"] \arrow[d, hook, "\widetilde{i_M}"'] \arrow[dr, phantom, "(f)"] & \mathrm{Cht}'^{S_2-S_1}_M \arrow[d, hook, "i_M"] \\
\mathrm{Cht}'^{S_2}_P \arrow[r, "\pi_{2,d}"] & \widetilde{\mathrm{Cht}}'^{S_2}_M \arrow[r, "\widetilde{\pi^0_{2,d}}"] & \mathrm{Cht}'^{S_2}_M
\end{tikzcd}$$

The squares (e) and (f) are Cartesian.

By (3.22) and (3.23), f_{12} defined in diagram (A.6) is the composition of $\mathrm{Tr}_{\pi_{12,d}} : (\pi_{12,d})_!(\pi_{12,d})^* \to \mathrm{Id}[-2(\dim \pi_{12,d})](-\dim\pi_{12,d})$ with some isomorphisms. By [SGA4, XVIII Théorème 2.9], the trace morphism is compatible with base change, and thus

$$\mathrm{Tr}_{\pi_{12,d}} = (\widetilde{i_M})^* \, \mathrm{Tr}_{\pi_{2,d}}.$$

We deduce that (4) is commutative. □

Remark A.0.9. We do not know if the complete diagram (A.2) is commutative.

Appendix B. Lemma of the cubic commutative diagram

LEMMA B.0.1. *Let $\mathscr{X}, \mathscr{Y}, \mathscr{Z}, \mathscr{W}, \mathscr{X}', \mathscr{Y}', \mathscr{Z}', \mathscr{W}'$ be algebraic stacks. Suppose that we have two Cartesian squares.*

$$\begin{tikzcd}
\mathscr{Z} \arrow[r] \arrow[d] & \mathscr{Y} \arrow[d, "h"] \\
\mathscr{X} \arrow[r, "g"] & \mathscr{W}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathscr{Z}' \arrow[r] \arrow[d] & \mathscr{Y}' \arrow[d, "h'"] \\
\mathscr{X}' \arrow[r, "g'"] & \mathscr{W}'
\end{tikzcd}$$

If these two squares are the front and back faces of a commutative diagram

$$\begin{tikzcd}[row sep=small, column sep=small]
\mathscr{Z}' \arrow[rr] \arrow[dr, "f_{\mathscr{Z}}"] \arrow[dd] & & \mathscr{Y}' \arrow[dr, "f_{\mathscr{Y}}"] \arrow[dd, dotted] & \\
& \mathscr{Z} \arrow[rr] \arrow[dd] & & \mathscr{Y} \arrow[dd] \\
\mathscr{X}' \arrow[rr, dotted] \arrow[dr, "f_{\mathscr{X}}"] & & \mathscr{W}' \arrow[dr, dotted, "f_{\mathscr{W}}"] & \\
& \mathscr{X} \arrow[rr] & & \mathscr{W}
\end{tikzcd} \tag{B.1}$$

then the fibers $f_{\mathscr{Z}}$, $f_{\mathscr{X}}$, $f_{\mathscr{Y}}$ and $f_{\mathscr{W}}$ form a Cartesian square.

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Concretely, let T be a scheme. For any morphism $T \to \mathscr{Z}$, we have the compositions of morphisms $T \to \mathscr{Z} \to \mathscr{X}$, $T \to \mathscr{Z} \to \mathscr{Y}$ and $T \to \mathscr{Z} \to \mathscr{W}$. We denote by $\mathscr{Z}_T$ (respectively $\mathscr{X}_T$, $\mathscr{Y}_T$, $\mathscr{W}_T$) the fiber of $f_{\mathscr{Z}}$ (respectively $f_{\mathscr{X}}$, $f_{\mathscr{Y}}$, $f_{\mathscr{W}}$) over T. The lemma says that $\mathscr{Z}_T$ is equivalent to $\mathscr{X}_T \underset{\mathscr{W}_T}{\times} \mathscr{Y}_T$.

Proof. We will prove a more general statement. Suppose that we have another Cartesian square

$$\begin{tikzcd} \mathscr{Z}'' \arrow[r] \arrow[d] & \mathscr{Y}'' \arrow[d, "h''"] \\ \mathscr{X}'' \arrow[r, "g''"] & \mathscr{W}'' \end{tikzcd}$$

and a commutative diagram.

$$\begin{tikzcd} \mathscr{Z}'' \arrow[rr] \arrow[dr] \arrow[dd] & & \mathscr{Y}'' \arrow[dr] \arrow[dd, dotted] & \\ & \mathscr{Z} \arrow[rr] \arrow[dd] & & \mathscr{Y} \arrow[dd] \\ \mathscr{X}'' \arrow[rr, dotted] \arrow[dr] & & \mathscr{W}'' \arrow[dr, dotted] & \\ & \mathscr{X} \arrow[rr] & & \mathscr{W} \end{tikzcd} \tag{B.2}$$

Then we have a canonical isomorphism:

$$\mathscr{Z}' \times_{\mathscr{Z}} \mathscr{Z}'' \xrightarrow{\sim} (\mathscr{X}' \times_{\mathscr{X}} \mathscr{X}'') \underset{\mathscr{W}' \times_{\mathscr{W}} \mathscr{W}''}{\times} (\mathscr{Y}' \times_{\mathscr{Y}} \mathscr{Y}''). \tag{B.3}$$

In fact, by definition, we have

$$\mathscr{Z}' \times_{\mathscr{Z}} \mathscr{Z}'' \simeq (\mathscr{X}' \times_{\mathscr{W}'} \mathscr{Y}') \underset{\mathscr{X} \times_{\mathscr{W}} \mathscr{Y}}{\times} (\mathscr{X}'' \times_{\mathscr{W}''} \mathscr{Y}'').$$

For any scheme S, the S-points of both sides of (B.3) classify the data of S-points x' in $\mathscr{X}'$, x'' in $\mathscr{X}''$, y' in $\mathscr{Y}'$, y'' in $\mathscr{Y}''$, an isomorphism between the images of x' and x'' in $\mathscr{X}$, an isomorphism between the images of y' and y'' in $\mathscr{Y}$, an isomorphism between the images of x' and y' in $\mathscr{W}'$, an isomorphism between the images of x'' and y'' in $\mathscr{W}''$, such that the diagram deduced from these four isomorphisms between the images of x', x'', y', y'' in $\mathscr{W}$ is commutative. We deduce (B.3).

The lemma is the special case when $\mathscr{X}'' = \mathscr{Y}'' = \mathscr{W}'' = \mathscr{Z}'' = T$. □

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