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Relations for quadratic Hodge integrals via stable maps

Georgios Politopoulos

Abstract. Following Faber–Pandharipande, we use the virtual localization formula for the moduli space of stable maps to $\mathbb{P}^1$ to compute relations between Hodge integrals. We prove that certain generating series of these integrals are polynomials.

1 Introduction

Let $\overline{\mathcal{M}}_{g,n}$ be the moduli space of n-pointed genus g stable curves. It is a proper smooth Deligne Mumford (DM) stack of dimension $3g-3+n$. We denote by $\pi : \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ the universal curve and by $\sigma_i : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{C}}_{g,n}$ the sections associated with the marking i for all $1 \le i \le n$. We denote by $\omega_{\overline{\mathcal{C}}_{g,n}/\overline{\mathcal{M}}_{g,n}}$ the relative dualizing sheaf of π. We will consider the following classes in $A^*(\overline{\mathcal{M}}_{g,n})$:

- For all $0 \le i \le g$, λ_i stands for the ith Chern class of the Hodge bundle, i.e., the vector bundle $\mathbb{E} = \pi_* \omega_{\overline{\mathcal{C}}_{g,n}/\overline{\mathcal{M}}_{g,n}}$. For all $\alpha \in \mathbb{C}$, we denote $\Lambda_g(\alpha) = \sum_{j=0}^{g} \alpha^{g-j}\lambda_j$, and $\Lambda_g^\vee(\alpha) = (-1)^g \Lambda_g(-\alpha)$.[1]
- For all $1 \le i \le n$, we denote ψ_i the Chern class of the cotangent line at the ith marking $\mathcal{L}_i = \sigma_i^*(\omega_{\overline{\mathcal{C}}_{g,n}/\overline{\mathcal{M}}_{g,n}})$.

A *Hodge integral* is an intersection number of the form:

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \dots \psi_n^{k_n} \Lambda_g(t_1) \dots \Lambda_g(t_m),$$

where $k_1, \dots, k_n$ are nonnegative integers and $t_1, \dots, t_m$ are complex numbers. If $m = 1, 2,$ or 3, then the above integral is called a linear, double, or triple Hodge integrals, respectively. Relations between linear Hodge integrals where proved in [FP00a] using the Gromov–Witten theory of $\mathbb{P}^1$ and the localization formula of [GP99]. This approach was also used in [FP00b] and [TZ03] to prove certain properties of triple Hodge integrals. Linear and triple Hodge integrals naturally appeared in the GW-theory of Calabi–Yau 3-folds, thus explaining a more abundant literature on the topic. However, double Hodge integrals have appeared recently in the quantization of Witten–Kontsevich generating series (see [Blo20]), in the theory of spin Hurwitz

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[1] Here, we use the convention of [FP00a] for $\Lambda_g^\vee(\alpha)$ and $\Lambda_g(\alpha)$.

numbers (see [GKL21]), and in the GW theory of blow-ups of smooth surfaces (see [GKLS22]).

In the present note, we consider the following power series in $\mathbb{C}[\alpha][[t]]$ defined using double Hodge integrals:

$$P_a(\alpha,t) = \sum_{g\geq 0} t^g \left(\int_{\overline{\mathcal{M}}_{g,n+1}} \frac{\Lambda_g^\vee(1)\Lambda_g^\vee(\alpha)}{1-\psi_0} \prod_{i=1}^{n} (2a_i+1)!!(-4\psi_i)^{a_i} \right) \exp\left(\frac{t}{24}\right),$$

where $a = (a_1,\ldots,a_n)$ is a vector of nonnegative integers. If $n=1$, we use the convention: $\int_{\overline{\mathcal{M}}_{0,2}} \psi_1^a \frac{\Lambda_g^\vee(1)\Lambda_g^\vee(\alpha))}{1-\psi_2} = (-1)^a$.

Theorem 1.1 *$P_a(\alpha,t)$ is a monic polynomial in $\mathbb{C}[\alpha][t]$ of degree $|a|$ in t.*

Here, we provide the first values of $P_a(-\alpha-1,t)$. In the list below, we omit the variables $-\alpha-1$ and t in the notation:

$P_{()} = 1.$

$P_{(1)} = t + 12.$

$P_{(2)} = t^2 - 10\alpha t + 240.$
$P_{(1,1)} = t^2 - 12t.$

$P_{(3)} = t^3 + (-77/3\alpha - 28)t^2 + 280t + 6720.$
$P_{(2,1)} = t^3 + (-10\alpha - 48)t^2 + (240\alpha + 240)t.$
$P_{(1,1,1)} = t^3 - 72t^2 + 432t.$

$P_{(4)} = t^4 + (-43\alpha - 72)t^3 + (126\alpha^2 + 756\alpha + 840)t^2 + 10080t + 241920.$
$P_{(3,1)} = t^4 + (-77/3\alpha - 100)t^3 + (1232\alpha + 1624)t^2.$
$P_{(2,2)} = t^4 + (20\alpha + 100)t^3 + (-100\alpha^2 - 1360\alpha - 1680)t^2.$
$P_{(2,1,1)} = t^4 + (-10\alpha - 132)t^3 + (840\alpha + 3120)t^2 + (-8640\alpha - 8640)t.$
$P_{(1,1,1,1)} = t^4 - 168t^3 + 5616t^2 - 20736t.$

Considering these first values, we conjecture that P_a is a polynomial of total degree $|a|$ in both variables t and α.

2 Preliminaries

We denote by $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1,1)$, the moduli space of stable maps of degree 1 to $\mathbb{P}^1$. It is a proper DM stack of virtual dimension $2g+n$. Here, we can define in an analogous way the Hodge bundle $\mathbb{E}$, the cotangent line bundles $\mathcal{L}_i$ and we denote again λ_i and ψ_i the respective Chern classes. We also have the forgetful and evaluation maps

$$\pi\colon \overline{\mathcal{M}}_{g,n+1}(\mathbb{P}^1,1) \to \overline{\mathcal{M}}_{g,n}(\mathbb{P}^1,1), \text{ and } \; ev_i\colon \overline{\mathcal{M}}_{g,n+1}(\mathbb{P}^1,1) \to \mathbb{P}^1.$$

Throughout this note, the enumeration of markings starts from 0. Furthermore, π is the morphism that forgets the marking p_0 and ev_i is the evaluation of a stable map to the ith marked point. The vector bundle $T := R^1\pi_*(ev_0^*\mathcal{O}_{\mathbb{P}^1}(-1))$ is of rank g and we denote by y its top Chern class. We will denote:

$$\langle\prod_{i=0}^{n-1}\tau_{a_i}(\omega)|y\rangle_{g,1}^{\mathbb{P}^1} := \int_{[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1,1)]^{vir}}\prod_{i=0}^{n-1}\psi_i^{a_i}ev_i^*(\omega)y,$$

where ω denotes the class of a point in $\mathbb{P}^1$.

Theorem 2.1 (Localization Formula [GP99, FP00a]) *Let $g \in \mathbb{Z}_{\geq 0}$, and let $a \in \mathbb{Z}_{\geq 0}^n$ such that $|a| \leq g$. Then, for all complex numbers α, and $t \in \mathbb{C}^*$, we have*

$$\langle\prod_{i=1}^{n}\tau_{a_i}(\omega)|y\rangle_{g,1}^{\mathbb{P}^1} = \sum_{g_1+g_2=g}\int_{\overline{\mathcal{M}}_{g_1,n+1}} t^n\prod_{i=1}^{n}\psi_i^{a_i}\frac{\Lambda_{g_1}^{\vee}(t)\Lambda_{g_1}^{\vee}(\alpha t)}{t(t-\psi_0)}$$
$$\times\int_{\overline{\mathcal{M}}_{g_2,1}}\frac{\Lambda_{g_2}^{\vee}(-t)\Lambda_{g_2}^{\vee}((\alpha+1)t)}{-t(-t-\psi_0)}.$$

Here, we use the convention $\int_{\overline{\mathcal{M}}_{0,1}}\psi_0^a = 1$.

Proposition 2.2 (Proposition 4.1 of [TZ03]) *For all complex numbers α, we have*

$$F(\alpha,t) = 1 + \sum_{g>0}t^{2g}\int_{\overline{\mathcal{M}}_{g,1}}\frac{\Lambda_g^{\vee}(1)\Lambda_g^{\vee}(\alpha)}{1-\psi_0} = exp\left(-\frac{t^2}{24}\right).$$

Besides, we have the String and Dilaton equation for Hodge integrals.

Proposition 2.3 *Let $g,n \in \mathbb{Z}_{\geq 0}$ such that $2g-2+n>0$.*

(i) *[Dilaton equation for Hodge integrals] Let $(a_1,\ldots,a_n) \in \mathbb{Z}_{\geq 0}^n$ and assume that there exist i_0 such that $a_{i_0} = 1$. Then*

$$\int_{\overline{\mathcal{M}}_{g,n+1}}\frac{\psi_{i_0}\prod_{i\neq i_0}\psi_i^{a_i}\prod_{j=1}^{g}\lambda_k^{b_k}}{1-\psi_0} = (2g-2+n)\int_{\overline{\mathcal{M}}_{g,n}}\frac{\prod_{i=1}^{n-1}\psi_i^{a_i}\prod_{j=1}^{g}\lambda_k^{b_k}}{1-\psi_0}.$$

(ii) *[String equation for Hodge integrals] Let $(a_1,\ldots,a_n) \in \mathbb{Z}_{\geq 0}^n$ and assume that there exist i_0 such that $a_{i_0} = 0$. Then we have*

$$\int_{\overline{\mathcal{M}}_{g,n+1}}\frac{\prod_{i=1}^{n}\psi_i^{a_i}\prod_{j=1}^{g}\lambda_k^{b_k}}{1-\psi_0} = \int_{\overline{\mathcal{M}}_{g,n}}\frac{\prod_{i=1}^{n-1}\psi_i^{a_i}\prod_{j=1}^{g}\lambda_k^{b_k}}{1-\psi_0}$$
$$+\sum_{j=1}^{n}\int_{\overline{\mathcal{M}}_{g,n}}\frac{\psi_j^{a_j-1}\prod_{i\neq j}\psi_i^{a_i}\prod_{k=1}^{g}\lambda_k^{b_k}}{1-\psi_0}.$$

3 The calculation

Note that the GW-invariant $\langle\prod_{i=1}^{n}\tau_{a_i}(\omega)|y\rangle_{g,1}^{\mathbb{P}^1}$ is 0 unless $|a| = g$ for dimensional reasons. Indeed, $\dim_{\mathbb{C}}[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1,1)]^{\text{vir}} = 2g+n$ and the cycle we are integrating is in codimension $g+|a|+n$. Using the above localization formula, and Lemma 2.1 of

[TZ03] the intersection number $\langle \prod_{i=1}^{n} \tau_{a_i}(\omega)|y\rangle_{g,1}^{\mathbb{P}^1}$ is expressed as

$$\begin{aligned}
&\sum_{g_1+g_2=g} \int_{\overline{\mathcal{M}}_{g_1,n+1}} t^n \prod_{i=1}^{n} \psi_i^{a_i} \frac{\Lambda_{g_1}^{\vee}(t)\Lambda_{g_1}^{\vee}(\alpha t)}{t(t-\psi_0)} \cdot \int_{\overline{\mathcal{M}}_{g_2,1}} \frac{\Lambda_{g_2}^{\vee}(-t)\Lambda_{g_2}^{\vee}((\alpha+1)t)}{-t(-t-\psi_0)}\\
&= \sum_{g_1+g_2=g} t^{|a|-g_1}(-t)^{-g_2} \int_{\overline{\mathcal{M}}_{g_1,n+1}} \prod_{i=1}^{n} \psi^{a_i} \frac{\Lambda_{g_1}^{\vee}(1)\Lambda_{g_1}^{\vee}(\alpha)}{1-\psi_0} \times \int_{\overline{\mathcal{M}}_{g_2,1}} \frac{\Lambda_{g_2}^{\vee}(1)\Lambda_{g_2}^{\vee}(-(\alpha+1))}{1-\psi_0}\\
&= t^{|a|-g} \sum_{g_1+g_2=g} \int_{\overline{\mathcal{M}}_{g_1,n+1}} \prod_{i=1}^{n} \psi^{a_i} \frac{\Lambda_{g_1}^{\vee}(1)\Lambda_{g_1}^{\vee}(\alpha)}{1-\psi_0} \cdot \int_{\overline{\mathcal{M}}_{g_2,1}} \psi_0^{3g_2-2}.
\end{aligned}$$

In the last equation, we used Proposition 2.2 in order to replace $\int_{\overline{\mathcal{M}}_{g_2,1}} \frac{\Lambda_{g_2}^{\vee}(1)\Lambda_{g_2}^{\vee}(-(\alpha+1))}{1-\psi_0}$ with $(-1)^{g_2} \int_{\overline{\mathcal{M}}_{g_2,1}} \psi_0^{3g_2-2}$.

We define

$$A_{g,a}(\alpha) = \sum_{g_1+g_2=g} \int_{\overline{\mathcal{M}}_{g_1,n+1}} \prod_{i=1}^{n} \psi^{a_i} \frac{\Lambda_{g_1}^{\vee}(1)\Lambda_{g_1}^{\vee}(\alpha)}{1-\psi_0} \cdot \int_{\overline{\mathcal{M}}_{g_2,1}} \psi_0^{3g_2-2}.$$

Then, we have

$$A_{g,a}(\alpha) = \begin{cases} 0, & |a| < g, \\ \langle \prod_{i=1}^{n} \tau_{a_i}(\omega)|y\rangle_{g,1}^{\mathbb{P}^1}, & |a| = g. \end{cases}$$

By the definition of $\Lambda_g^{\vee}(t)$, we see that $\Lambda_g^{\vee}(1)\Lambda_g^{\vee}(-(\alpha+1))$ is a polynomial in α of degree g, which actually determines the degree of $A_g(\alpha)$.

We now present a proof for the main result.

Proof (of Theorem 1.1) We begin by stating the well-known fact

$$1 + \sum_{g>0} t^g \int_{\overline{\mathcal{M}}_{g,1}} \psi_0^{3g-2} = \exp\left(\frac{t}{24}\right)$$

proven in Section 3.1 of [FP00a]. Now, we consider the product of $\exp\left(\frac{t}{24}\right)$ and

$$\sum_{g\geq 0} t^g \left(\int_{\overline{\mathcal{M}}_{g,n+1}} \frac{\Lambda_g^{\vee}(1)\Lambda_g^{\vee}(\alpha)}{1-\psi_0} \prod_{i=1}^{n} (2a_i+1)!!(-4\psi_i)^{a_i} \right)$$

to obtain a new power series whose coefficients in degree g are given by

$$\sum_{g_1+g_2=g} \int_{\overline{\mathcal{M}}_{g_1,n+1}} \prod_{i=1}^{n} (2a_i+1)!!(-4)^{a_i} \prod_{i=1}^{n} \psi^{a_i} \frac{\Lambda_{g_1}^{\vee}(1)\Lambda_{g_1}^{\vee}(\alpha)}{1-\psi_0} \cdot \int_{\overline{\mathcal{M}}_{g_2,1}} \psi_0^{3g_2-2}.$$

This is exactly $A_{g,a}(\alpha) \cdot \prod_{i=1}^{n}(2a_i+1)!!(-4)^{a_i}$. Hence, we can rewrite the power series $P_a(\alpha,t)$ in the form

$$P_a(\alpha,t) = \prod_{i=1}^{n} (2a_i+1)!!(-4)^{a_i} \sum_{g\geq 0} t^g A_{g,a}(\alpha).$$

As it is computed in the start of Section 3, we have that the numbers $A_{g,a}(\alpha)$ vanish when $g > |a|$. Hence, we get that all coefficients of the power series $P_a(\alpha,t)$

vanish when $g > |a|$, i.e. $P_a(\alpha, t)$ is a polynomial of degree $|a|$. Furthermore, the top coefficient of $P_a(\alpha, t)$, i.e., the coefficient of $t^{|a|}$ is given by

$$\langle \prod_{i=1}^{n} (-4)^{a_i}(2a_i+1)!!\tau_{a_i}(\omega)|y\rangle^{\mathbb{P}^1}_{|a|,1}.$$

This value is computed in [KL11] and is actually equal to 1. In particular, the number $\prod_{i=1}^{n}(-4)^{a_i}(2a_i+1)!!$ is here to make the polynomial monic. ■

We now prove several other properties of the polynomials P_a.

Proposition 3.1 *The constant term c_0 of $P_a(\alpha, t)$ is nonzero if and only if $n = 1$, where then $c_0 = (-1)^a \prod_{i=1}^{n}(-4)^{a_i}(2a_i+1)!!$ or if $n > 1$ and $\sum_{i=1}^{n} a_i \leq n-2$ where then*

$$c_0 = \prod_{i=1}^{n}(-4)^{a_i}(2a_i+1)!! \frac{(n-2)!}{a_1! \dots (n-2-\sum a_i)!}.$$

Proof We only compute the integrals appearing in the constant term of this polynomial since then we only have to multiply with $\prod_{i=1}^{n}(2a_i+1)!!(-4)^{a_i}$. The integral in the constant term of $P_a(\alpha, t)$ is given by $\int_{\overline{\mathcal{M}}_{0,n+1}} \frac{\prod_{i=1}^{n}\psi_i^{a_i}}{1-\psi_0}$. When $n = 1$, using the convention $\int_{\overline{\mathcal{M}}_{0,2}} \frac{\psi_1^a}{1-\psi_0} = (-1)^a$, we get that

$$c_0 = (-1)^a \prod_{i=1}^{n}(-4)^{a_i}(2a_i+1)!!.$$

When $n > 1$, if $\sum_{i=1}^{n} a_i > n-2$, then c_0 is zero for dimensional reasons. Otherwise, we have

$$\int_{\overline{\mathcal{M}}_{0,n+1}} \frac{\prod_{i=1}^{n}\psi_i^{a_i}}{1-\psi_0} = \int_{\overline{\mathcal{M}}_{0,n+1}} \psi_0^{n-2-\sum a_i} \prod_{i=1}^{n}\psi_i^{a_i} = \frac{(n-2)!}{a_1! \dots (n-2-\sum a_i)!}.$$ ■

Proposition 3.2 *Let $n \geq 3$. Then we have the following rules:*

(i) *[String equation]*

$$P_{(a_1,\dots,a_{n-1},0)}(\alpha, t) = P_{(a_1,\dots,a_{n-1})}(\alpha, t) - \sum_{i=1}^{n}(8a_i+4)P_{(a_1,\dots,a_i-1,\dots,a_{n-1})}(\alpha, t).$$

(ii) *[Dilaton equation]*

$$P_{(a_1,\dots,a_{n-1},1)}(\alpha, t) = (t-12n+24)P_{(a_1,\dots,a_{n-1})}(\alpha, t) - 24tP'_{(a_1,\dots,a_{n-1})}(\alpha, t)).$$

Proof We define the power series

$$\widetilde{P}_a(\alpha, t) = \sum_{g\geq 0} t^g \left(\int_{\overline{\mathcal{M}}_{g,n+1}} \prod_{i=1}^{n} \psi^{a_i} \frac{\Lambda_g^\vee(1)\Lambda_g^\vee(\alpha)}{1-\psi_0} \right).$$

Note that the following equation holds:

$$P_a(\alpha, t) = \prod_{i=1}^{n}(2a_i+1)!!(-4)^{a_i}\widetilde{P}_a(\alpha, t)\exp\left(\frac{t}{24}\right).$$

We can rewrite the coefficients of $\widetilde{P}_a(\alpha, t)$ as

$$\sum_{k=0}^{g}\sum_{j=0}^{g}(-1)^{g+k}(a+1)^{g-j}\int_{\overline{\mathcal{M}}_{g,n+1}}\frac{\prod_{i=1}^{n}\psi_i^{a_i}\lambda_k\lambda_j}{1-\psi_0}.$$

(i) Applying the String equation for Hodge integrals, we obtain the following formula:

$$\widetilde{P}_{(a_1,\dots,a_{n-1},0)}(\alpha,t)=\widetilde{P}_{(a_1,\dots,a_{n-1})}(\alpha,t)+\sum_{i=1}^{n}\widetilde{P}_{(a_1,..,a_i-1,\dots,a_{n-1})}(\alpha,t).$$

Hence, multiplying with $\prod_{i=1}^{n-1}(2a_i+1)!!(-4)^{a_i}\exp\left(\frac{t}{24}\right)$, we obtain the desired result after a straightforward calculation.

(ii) Applying Dilaton equation for Hodge integrals, we obtain the following formula:

$$\begin{aligned}\widetilde{P}_{(a_1,\dots,a_{n-1},1)}(\alpha,t)=2\sum_{g\geq 0}gt^g\int_{\overline{\mathcal{M}}_{g,n-1}}\prod_{i=1}^{n-1}\psi_i^{a_i}\frac{\Lambda_g^\vee(1)\Lambda_g^\vee(\alpha)}{1-\psi_0}\\+(n-2)\widetilde{P}_{(a_1,\dots,a_{n-1})}(\alpha,t).\end{aligned}$$

Note that the first term of the sum is equal to $2t\widetilde{P}'_{(a_1,\dots,a_{n-1})}(\alpha,t)$. Now, multiplying both sides of the equation above with

$$\prod_{i=1}^{n-1}(2a_i+1)(-4)^{a_i}\exp\left(\frac{t}{24}\right),$$

we have

$$\begin{aligned}\frac{-1}{12}P_{(a_1,\dots,a_{n-1},1)}(\alpha,t)&=(n-2)P_{(a_1,\dots,a_{n-1})}(\alpha,t)\\&+2t\left(\prod_{i=1}^{n-1}(-4)^{a_i}(2a_i+1)!!\right)\widetilde{P}'_{(a_1,\dots,a_{n-1})}(\alpha,t)e^{t/24}\\&=(n-2)P_{(a_1,\dots,a_{n-1})}(\alpha,t)\\&+2t(P'_{(a_1,\dots,a_{n-1})}(\alpha,t)-\frac{1}{24}P_{(a_1,\dots,a_{n-1})}(\alpha,t)).\end{aligned}$$

Finally, clearing the denominators, we obtain the desired result. ■

We recall Mumford's relation $\Lambda_g^\vee(1)\cdot\Lambda_g^\vee(-1)=1$ (see [Mum83]). In particular, $P_a(-1,t)$ is defined by integrals of ψ-classes.

Corollary 3.3 *For any vector $a\in\mathbb{Z}_{\geq 0}^n$, the power series*

$$P_a(-1,t)=\prod_{i=1}^{n}(2a_i+1)!!(-4)^{a_i}exp\left(\frac{t}{24}\right)\cdot\sum_{g\geq 0}(-t)^g\int_{\overline{\mathcal{M}}_{g,n+1}}\frac{\prod_{i=1}^{n}\psi_i^{a_i}}{1-\psi_0}$$

is a polynomial of degree $|a|$.

In this case, the polynomiality as well as a closed expression were proved in [LX11].

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Laboratoire AGM, Cergy University, 2 Avenue Adolphe Chauvin, 95300 Cergy-Pontoise, France

Mathematical Institute, Leiden University, 2300 CA RA Leiden, Netherlands
e-mail: g.politopoulos@math.leidenuniv.nl