



# Relations for quadratic Hodge integrals via stable maps

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*Abstract.* Following Faber–Pandharipande, we use the virtual localization formula for the moduli space of stable maps to  $\mathbb{P}^1$  to compute relations between Hodge integrals. We prove that certain generating series of these integrals are polynomials.

## 1 Introduction

Let  $\overline{\mathcal{M}}_{g,n}$  be the moduli space of  $n$ -pointed genus  $g$  stable curves. It is a proper smooth Deligne Mumford (DM) stack of dimension  $3g - 3 + n$ . We denote by  $\pi : \overline{\mathcal{C}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n}$  the universal curve and by  $\sigma_i : \overline{\mathcal{M}}_{g,n} \rightarrow \overline{\mathcal{C}}_{g,n}$  the sections associated with the marking  $i$  for all  $1 \leq i \leq n$ . We denote by  $\omega_{\overline{\mathcal{C}}_{g,n}/\overline{\mathcal{M}}_{g,n}}$  the relative dualizing sheaf of  $\pi$ . We will consider the following classes in  $A^*(\overline{\mathcal{M}}_{g,n})$ :

- For all  $0 \leq i \leq g$ ,  $\lambda_i$  stands for the  $i$ th Chern class of the Hodge bundle, i.e., the vector bundle  $\mathbb{E} = \pi_* \omega_{\overline{\mathcal{C}}_{g,n}/\overline{\mathcal{M}}_{g,n}}$ . For all  $\alpha \in \mathbb{C}$ , we denote  $\Lambda_g(\alpha) = \sum_{j=0}^g \alpha^{g-j} \lambda_j$ , and  $\Lambda_g^\vee(\alpha) = (-1)^g \Lambda_g(-\alpha)$ .<sup>1</sup>
- For all  $1 \leq i \leq n$ , we denote  $\psi_i$  the Chern class of the cotangent line at the  $i$ th marking  $\mathcal{L}_i = \sigma_i^*(\omega_{\overline{\mathcal{C}}_{g,n}/\overline{\mathcal{M}}_{g,n}})$ .

A *Hodge integral* is an intersection number of the form:

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \dots \psi_n^{k_n} \Lambda_g(t_1) \dots \Lambda_g(t_m),$$

where  $k_1, \dots, k_n$  are nonnegative integers and  $t_1, \dots, t_m$  are complex numbers. If  $m = 1, 2$ , or  $3$ , then the above integral is called a linear, double, or triple Hodge integrals, respectively. Relations between linear Hodge integrals were proved in [FP00a] using the Gromov–Witten theory of  $\mathbb{P}^1$  and the localization formula of [GP99]. This approach was also used in [FP00b] and [TZ03] to prove certain properties of triple Hodge integrals. Linear and triple Hodge integrals naturally appeared in the GW-theory of Calabi–Yau 3-folds, thus explaining a more abundant literature on the topic. However, double Hodge integrals have appeared recently in the quantization of Witten–Kontsevich generating series (see [Blo20]), in the theory of spin Hurwitz

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<sup>1</sup>Here, we use the convention of [FP00a] for  $\Lambda_g^\vee(\alpha)$  and  $\Lambda_g(\alpha)$ .



numbers (see [GKL21]), and in the GW theory of blow-ups of smooth surfaces (see [GKLS22]).

In the present note, we consider the following power series in  $\mathbb{C}[\alpha][[t]]$  defined using double Hodge integrals:

$$P_a(\alpha, t) = \sum_{g \geq 0} t^g \left( \int_{\overline{\mathcal{M}}_{g,n+1}} \frac{\Lambda_g^\vee(1) \Lambda_g^\vee(\alpha)}{1 - \psi_0} \prod_{i=1}^n (2a_i + 1)!! (-4\psi_i)^{a_i} \right) \exp\left(\frac{t}{24}\right),$$

where  $a = (a_1, \dots, a_n)$  is a vector of nonnegative integers. If  $n = 1$ , we use the convention:  $\int_{\overline{\mathcal{M}}_{0,2}} \psi_1^a \frac{\Lambda_1^\vee(1) \Lambda_1^\vee(\alpha)}{1 - \psi_2} = (-1)^a$ .

**Theorem 1.1**  $P_a(\alpha, t)$  is a monic polynomial in  $\mathbb{C}[\alpha][t]$  of degree  $|a|$  in  $t$ .

Here, we provide the first values of  $P_a(-\alpha - 1, t)$ . In the list below, we omit the variables  $-\alpha - 1$  and  $t$  in the notation:

$$P_{()} = 1.$$

$$P_{(1)} = t + 12.$$

$$P_{(2)} = t^2 - 10\alpha t + 240.$$

$$P_{(1,1)} = t^2 - 12t.$$

$$P_{(3)} = t^3 + (-77/3\alpha - 28)t^2 + 280t + 6720.$$

$$P_{(2,1)} = t^3 + (-10\alpha - 48)t^2 + (240\alpha + 240)t.$$

$$P_{(1,1,1)} = t^3 - 72t^2 + 432t.$$

$$P_{(4)} = t^4 + (-43\alpha - 72)t^3 + (126\alpha^2 + 756\alpha + 840)t^2 + 10080t + 241920.$$

$$P_{(3,1)} = t^4 + (-77/3\alpha - 100)t^3 + (1232\alpha + 1624)t^2.$$

$$P_{(2,2)} = t^4 + (20\alpha + 100)t^3 + (-100\alpha^2 - 1360\alpha - 1680)t^2.$$

$$P_{(2,1,1)} = t^4 + (-10\alpha - 132)t^3 + (840\alpha + 3120)t^2 + (-8640\alpha - 8640)t.$$

$$P_{(1,1,1,1)} = t^4 - 168t^3 + 5616t^2 - 20736t.$$

Considering these first values, we conjecture that  $P_a$  is a polynomial of total degree  $|a|$  in both variables  $t$  and  $\alpha$ .

## 2 Preliminaries

We denote by  $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, 1)$ , the moduli space of stable maps of degree 1 to  $\mathbb{P}^1$ . It is a proper DM stack of virtual dimension  $2g + n$ . Here, we can define in an analogous way the Hodge bundle  $\mathbb{E}$ , the cotangent line bundles  $\mathcal{L}_i$  and we denote again  $\lambda_i$  and  $\psi_i$  the respective Chern classes. We also have the forgetful and evaluation maps

$$\pi: \overline{\mathcal{M}}_{g,n+1}(\mathbb{P}^1, 1) \rightarrow \overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, 1), \text{ and } \epsilon\nu_i: \overline{\mathcal{M}}_{g,n+1}(\mathbb{P}^1, 1) \rightarrow \mathbb{P}^1.$$

Throughout this note, the enumeration of markings starts from 0. Furthermore,  $\pi$  is the morphism that forgets the marking  $p_0$  and  $ev_i$  is the evaluation of a stable map to the  $i$ th marked point. The vector bundle  $T := R^1\pi_*(ev_0^*\mathcal{O}_{\mathbb{P}^1}(-1))$  is of rank  $g$  and we denote by  $y$  its top Chern class. We will denote:

$$\langle \prod_{i=0}^{n-1} \tau_{a_i}(\omega) | y \rangle_{g,1}^{\mathbb{P}^1} := \int_{[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1,1)]^{\text{vir}}} \prod_{i=0}^{n-1} \psi_i^{a_i} ev_i^*(\omega) y,$$

where  $\omega$  denotes the class of a point in  $\mathbb{P}^1$ .

**Theorem 2.1** (Localization Formula [GP99, FP00a]) *Let  $g \in \mathbb{Z}_{\geq 0}$ , and let  $a \in \mathbb{Z}_{\geq 0}^n$  such that  $|a| \leq g$ . Then, for all complex numbers  $\alpha$ , and  $t \in \mathbb{C}^*$ , we have*

$$\begin{aligned} \langle \prod_{i=1}^n \tau_{a_i}(\omega) | y \rangle_{g,1}^{\mathbb{P}^1} &= \sum_{g_1+g_2=g} \int_{\overline{\mathcal{M}}_{g_1,n+1}} t^n \prod_{i=1}^n \psi_i^{a_i} \frac{\Lambda_{g_1}^\vee(t) \Lambda_{g_1}^\vee(\alpha)}{t(t-\psi_0)} \\ &\quad \times \int_{\overline{\mathcal{M}}_{g_2,1}} \frac{\Lambda_{g_2}^\vee(-t) \Lambda_{g_2}^\vee((\alpha+1)t)}{-t(-t-\psi_0)}. \end{aligned}$$

Here, we use the convention  $\int_{\overline{\mathcal{M}}_{0,1}} \psi_0^a = 1$ .

**Proposition 2.2** (Proposition 4.1 of [TZ03]) *For all complex numbers  $\alpha$ , we have*

$$F(\alpha, t) = 1 + \sum_{g>0} t^{2g} \int_{\overline{\mathcal{M}}_{g,1}} \frac{\Lambda_g^\vee(1) \Lambda_g^\vee(\alpha)}{1-\psi_0} = \exp\left(-\frac{t^2}{24}\right).$$

Besides, we have the String and Dilaton equation for Hodge integrals.

**Proposition 2.3** *Let  $g, n \in \mathbb{Z}_{\geq 0}$  such that  $2g - 2 + n > 0$ .*

(i) *[Dilaton equation for Hodge integrals] Let  $(a_1, \dots, a_n) \in \mathbb{Z}_{\geq 0}^n$  and assume that there exist  $i_0$  such that  $a_{i_0} = 1$ . Then*

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \frac{\psi_{i_0} \prod_{i \neq i_0} \psi_i^{a_i} \prod_{j=1}^g \lambda_k^{b_k}}{1-\psi_0} = (2g-2+n) \int_{\overline{\mathcal{M}}_{g,n}} \frac{\prod_{i=1}^{n-1} \psi_i^{a_i} \prod_{j=1}^g \lambda_k^{b_k}}{1-\psi_0}.$$

(ii) *[String equation for Hodge integrals] Let  $(a_1, \dots, a_n) \in \mathbb{Z}_{\geq 0}^n$  and assume that there exist  $i_0$  such that  $a_{i_0} = 0$ . Then we have*

$$\begin{aligned} \int_{\overline{\mathcal{M}}_{g,n+1}} \frac{\prod_{i=1}^n \psi_i^{a_i} \prod_{j=1}^g \lambda_k^{b_k}}{1-\psi_0} &= \int_{\overline{\mathcal{M}}_{g,n}} \frac{\prod_{i=1}^{n-1} \psi_i^{a_i} \prod_{j=1}^g \lambda_k^{b_k}}{1-\psi_0} \\ &\quad + \sum_{j=1}^n \int_{\overline{\mathcal{M}}_{g,n}} \frac{\psi_j^{a_j-1} \prod_{i \neq j} \psi_i^{a_i} \prod_{k=1}^g \lambda_k^{b_k}}{1-\psi_0}. \end{aligned}$$

### 3 The calculation

Note that the GW-invariant  $\langle \prod_{i=1}^n \tau_{a_i}(\omega) | y \rangle_{g,1}^{\mathbb{P}^1}$  is 0 unless  $|a| = g$  for dimensional reasons. Indeed,  $\dim_{\mathbb{C}}[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1,1)]^{\text{vir}} = 2g+n$  and the cycle we are integrating is in codimension  $g+|a|+n$ . Using the above localization formula, and Lemma 2.1 of

[TZ03] the intersection number  $\langle \prod_{i=1}^n \tau_{a_i}(\omega) | y \rangle_{g,1}^{\mathbb{P}^1}$  is expressed as

$$\begin{aligned} & \sum_{g_1+g_2=g} \int_{\overline{\mathcal{M}}_{g_1,n+1}} t^n \prod_{i=1}^n \psi_i^{a_i} \frac{\Lambda_{g_1}^\vee(t) \Lambda_{g_1}^\vee(\alpha t)}{t(t-\psi_0)} \cdot \int_{\overline{\mathcal{M}}_{g_2,1}} \frac{\Lambda_{g_2}^\vee(-t) \Lambda_{g_2}^\vee((\alpha+1)t)}{-t(-t-\psi_0)} \\ &= \sum_{g_1+g_2=g} t^{|a|-g_1} (-t)^{-g_2} \int_{\overline{\mathcal{M}}_{g_1,n+1}} \prod_{i=1}^n \psi_i^{a_i} \frac{\Lambda_{g_1}^\vee(1) \Lambda_{g_1}^\vee(\alpha)}{1-\psi_0} \times \int_{\overline{\mathcal{M}}_{g_2,1}} \frac{\Lambda_{g_2}^\vee(1) \Lambda_{g_2}^\vee(-(\alpha+1))}{1-\psi_0} \\ &= t^{|a|-g} \sum_{g_1+g_2=g} \int_{\overline{\mathcal{M}}_{g_1,n+1}} \prod_{i=1}^n \psi_i^{a_i} \frac{\Lambda_{g_1}^\vee(1) \Lambda_{g_1}^\vee(\alpha)}{1-\psi_0} \cdot \int_{\overline{\mathcal{M}}_{g_2,1}} \psi_0^{3g_2-2}. \end{aligned}$$

In the last equation, we used Proposition 2.2 in order to replace  $\int_{\overline{\mathcal{M}}_{g_2,1}} \frac{\Lambda_{g_2}^\vee(1) \Lambda_{g_2}^\vee(-(\alpha+1))}{1-\psi_0}$  with  $(-1)^{g_2} \int_{\overline{\mathcal{M}}_{g_2,1}} \psi_0^{3g_2-2}$ .

We define

$$A_{g,a}(\alpha) = \sum_{g_1+g_2=g} \int_{\overline{\mathcal{M}}_{g_1,n+1}} \prod_{i=1}^n \psi_i^{a_i} \frac{\Lambda_{g_1}^\vee(1) \Lambda_{g_1}^\vee(\alpha)}{1-\psi_0} \cdot \int_{\overline{\mathcal{M}}_{g_2,1}} \psi_0^{3g_2-2}.$$

Then, we have

$$A_{g,a}(\alpha) = \begin{cases} 0, & |a| < g, \\ \langle \prod_{i=1}^n \tau_{a_i}(\omega) | y \rangle_{g,1}^{\mathbb{P}^1}, & |a| = g. \end{cases}$$

By the definition of  $\Lambda_g^\vee(t)$ , we see that  $\Lambda_g^\vee(1) \Lambda_g^\vee(-(\alpha+1))$  is a polynomial in  $\alpha$  of degree  $g$ , which actually determines the degree of  $A_g(\alpha)$ .

We now present a proof for the main result.

**Proof (of Theorem 1.1)** We begin by stating the well-known fact

$$1 + \sum_{g>0} t^g \int_{\overline{\mathcal{M}}_{g,1}} \psi_0^{3g-2} = \exp\left(\frac{t}{24}\right)$$

proven in Section 3.1 of [FP00a]. Now, we consider the product of  $\exp\left(\frac{t}{24}\right)$  and

$$\sum_{g \geq 0} t^g \left( \int_{\overline{\mathcal{M}}_{g,n+1}} \frac{\Lambda_g^\vee(1) \Lambda_g^\vee(\alpha)}{1-\psi_0} \prod_{i=1}^n (2a_i + 1)!! (-4\psi_i)^{a_i} \right)$$

to obtain a new power series whose coefficients in degree  $g$  are given by

$$\sum_{g_1+g_2=g} \int_{\overline{\mathcal{M}}_{g_1,n+1}} \prod_{i=1}^n (2a_i + 1)!! (-4)^{a_i} \prod_{i=1}^n \psi_i^{a_i} \frac{\Lambda_{g_1}^\vee(1) \Lambda_{g_1}^\vee(\alpha)}{1-\psi_0} \cdot \int_{\overline{\mathcal{M}}_{g_2,1}} \psi_0^{3g_2-2}.$$

This is exactly  $A_{g,a}(\alpha) \cdot \prod_{i=1}^n (2a_i + 1)!! (-4)^{a_i}$ . Hence, we can rewrite the power series  $P_a(\alpha, t)$  in the form

$$P_a(\alpha, t) = \prod_{i=1}^n (2a_i + 1)!! (-4)^{a_i} \sum_{g \geq 0} t^g A_{g,a}(\alpha).$$

As it is computed in the start of Section 3, we have that the numbers  $A_{g,a}(\alpha)$  vanish when  $g > |a|$ . Hence, we get that all coefficients of the power series  $P_a(\alpha, t)$

vanish when  $g > |a|$ , i.e.  $P_a(\alpha, t)$  is a polynomial of degree  $|a|$ . Furthermore, the top coefficient of  $P_a(\alpha, t)$ , i.e., the coefficient of  $t^{|a|}$  is given by

$$\left\langle \prod_{i=1}^n (-4)^{a_i} (2a_i + 1)!! \tau_{a_i}(\omega) \middle| y \right\rangle_{|a|,1}^{\mathbb{P}^1}.$$

This value is computed in [KL11] and is actually equal to 1. In particular, the number  $\prod_{i=1}^n (-4)^{a_i} (2a_i + 1)!!$  is here to make the polynomial monic. ■

We now prove several other properties of the polynomials  $P_a$ .

**Proposition 3.1** *The constant term  $c_0$  of  $P_a(\alpha, t)$  is nonzero if and only if  $n = 1$ , where then  $c_0 = (-1)^a \prod_{i=1}^n (-4)^{a_i} (2a_i + 1)!!$  or if  $n > 1$  and  $\sum_{i=1}^n a_i \leq n - 2$  where then*

$$c_0 = \prod_{i=1}^n (-4)^{a_i} (2a_i + 1)!! \frac{(n - 2)!}{a_1! \dots (n - 2 - \sum a_i)!}.$$

**Proof** We only compute the integrals appearing in the constant term of this polynomial since then we only have to multiply with  $\prod_{i=1}^n (2a_i + 1)!! (-4)^{a_i}$ . The integral in the constant term of  $P_a(\alpha, t)$  is given by  $\int_{\overline{\mathcal{M}}_{0,n+1}} \frac{\prod_{i=1}^n \psi_i^{a_i}}{1 - \psi_0}$ . When  $n = 1$ , using the convention  $\int_{\overline{\mathcal{M}}_{0,2}} \frac{\psi_1^a}{1 - \psi_0} = (-1)^a$ , we get that

$$c_0 = (-1)^a \prod_{i=1}^n (-4)^{a_i} (2a_i + 1)!!.$$

When  $n > 1$ , if  $\sum_{i=1}^n a_i > n - 2$ , then  $c_0$  is zero for dimensional reasons. Otherwise, we have

$$\int_{\overline{\mathcal{M}}_{0,n+1}} \frac{\prod_{i=1}^n \psi_i^{a_i}}{1 - \psi_0} = \int_{\overline{\mathcal{M}}_{0,n+1}} \psi_0^{n-2-\sum a_i} \prod_{i=1}^n \psi_i^{a_i} = \frac{(n - 2)!}{a_1! \dots (n - 2 - \sum a_i)!}. \quad \blacksquare$$

**Proposition 3.2** *Let  $n \geq 3$ . Then we have the following rules:*

(i) [String equation]

$$P_{(a_1, \dots, a_{n-1}, 0)}(\alpha, t) = P_{(a_1, \dots, a_{n-1})}(\alpha, t) - \sum_{i=1}^n (8a_i + 4) P_{(a_1, \dots, a_i-1, \dots, a_{n-1})}(\alpha, t).$$

(ii) [Dilaton equation]

$$P_{(a_1, \dots, a_{n-1}, 1)}(\alpha, t) = (t - 12n + 24) P_{(a_1, \dots, a_{n-1})}(\alpha, t) - 24t P'_{(a_1, \dots, a_{n-1})}(\alpha, t).$$

**Proof** We define the power series

$$\tilde{P}_a(\alpha, t) = \sum_{g \geq 0} t^g \left( \int_{\overline{\mathcal{M}}_{g,n+1}} \prod_{i=1}^n \psi_i^{a_i} \frac{\Lambda_g^\vee(1) \Lambda_g^\vee(\alpha)}{1 - \psi_0} \right).$$

Note that the following equation holds:

$$P_a(\alpha, t) = \prod_{i=1}^n (2a_i + 1)!! (-4)^{a_i} \tilde{P}_a(\alpha, t) \exp\left(\frac{t}{24}\right).$$

We can rewrite the coefficients of  $\tilde{P}_a(\alpha, t)$  as

$$\sum_{k=0}^g \sum_{j=0}^g (-1)^{g+k} (a+1)^{g-j} \int_{\overline{\mathcal{M}}_{g,n+1}} \frac{\prod_{i=1}^n \psi_i^{a_i} \lambda_k \lambda_j}{1 - \psi_0}.$$

(i) Applying the String equation for Hodge integrals, we obtain the following formula:

$$\tilde{P}_{(a_1, \dots, a_{n-1}, 0)}(\alpha, t) = \tilde{P}_{(a_1, \dots, a_{n-1})}(\alpha, t) + \sum_{i=1}^n \tilde{P}_{(a_1, \dots, a_{i-1}, \dots, a_{n-1})}(\alpha, t).$$

Hence, multiplying with  $\prod_{i=1}^{n-1} (2a_i + 1)!! (-4)^{a_i} \exp\left(\frac{t}{24}\right)$ , we obtain the desired result after a straightforward calculation.

(ii) Applying Dilaton equation for Hodge integrals, we obtain the following formula:

$$\begin{aligned} \tilde{P}_{(a_1, \dots, a_{n-1}, 1)}(\alpha, t) &= 2 \sum_{g \geq 0} g t^g \int_{\overline{\mathcal{M}}_{g,n-1}} \prod_{i=1}^{n-1} \psi_i^{a_i} \frac{\Lambda_g^\vee(1) \Lambda_g^\vee(\alpha)}{1 - \psi_0} \\ &\quad + (n-2) \tilde{P}_{(a_1, \dots, a_{n-1})}(\alpha, t). \end{aligned}$$

Note that the first term of the sum is equal to  $2t \tilde{P}'_{(a_1, \dots, a_{n-1})}(\alpha, t)$ . Now, multiplying both sides of the equation above with

$$\prod_{i=1}^{n-1} (2a_i + 1) (-4)^{a_i} \exp\left(\frac{t}{24}\right),$$

we have

$$\begin{aligned} \frac{-1}{12} P_{(a_1, \dots, a_{n-1}, 1)}(\alpha, t) &= (n-2) P_{(a_1, \dots, a_{n-1})}(\alpha, t) \\ &\quad + 2t \left( \prod_{i=1}^{n-1} (-4)^{a_i} (2a_i + 1)!! \right) \tilde{P}'_{(a_1, \dots, a_{n-1})}(\alpha, t) e^{t/24} \\ &= (n-2) P_{(a_1, \dots, a_{n-1})}(\alpha, t) \\ &\quad + 2t (P'_{(a_1, \dots, a_{n-1})}(\alpha, t) - \frac{1}{24} P_{(a_1, \dots, a_{n-1})}(\alpha, t)). \end{aligned}$$

Finally, clearing the denominators, we obtain the desired result. ■

We recall Mumford's relation  $\Lambda_g^\vee(1) \cdot \Lambda_g^\vee(-1) = 1$  (see [Mum83]). In particular,  $P_a(-1, t)$  is defined by integrals of  $\psi$ -classes.

**Corollary 3.3** For any vector  $a \in \mathbb{Z}_{\geq 0}^n$ , the power series

$$P_a(-1, t) = \prod_{i=1}^n (2a_i + 1)!! (-4)^{a_i} \exp\left(\frac{t}{24}\right) \cdot \sum_{g \geq 0} (-t)^g \int_{\overline{\mathcal{M}}_{g,n+1}} \frac{\prod_{i=1}^n \psi_i^{a_i}}{1 - \psi_0}$$

is a polynomial of degree  $|a|$ .

In this case, the polynomiality as well as a closed expression were proved in [LX11].

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