NOTE ON PRIMARY IDEAL DECOMPOSITIONS

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Let R be a ring with a unity element. An ideal Q of R is called (right) *primary* if for ideals A and B of R, $AB \subset Q$ and $A \not\subset Q$ imply that $B^n \subset Q$ for some positive integer n. If R satisfies the ascending chain condition for ideals (ACC), then R is said to have a *Noetherian ideal theory* if every ideal of R is an intersection of a finite number of primary ideals. If R is a commutative ring that satisfies the ACC, then R has a Noetherian ideal theory. However, it is known that in general R may satisfy the ACC without having a Noetherian ideal theory (an example of such a ring is given in **(2)**). Thus there is some interest in conditions that imply that a ring R satisfying the ACC will have a Noetherian ideal theory.

If A and B are ideals of R we set

$$AB^{-1} = \{c; c \in R \text{ and } cB \subset A\}.$$

Then AB^{-1} is an ideal of R such that $(AB^{-1})B \subset A$. If C is also an ideal of R, then $(AB^{-1})C^{-1} = A(CB)^{-1}$. We write AB^{-n} in place of $A(B^n)^{-1}$. Furthermore, if $B \subset C$, then $AC^{-1} \subset AB^{-1}$. In this note we shall prove the following:

THEOREM. *If R satisfies the* ACC, *then the following statements are equivalent:*

(1) *R has a Noetherian ideal theory.*

(2) *If A and B are ideals of R, then for all large positive integers n we have $A \cap B^n \subset AB$.*

(3) *If A and B are ideals of R, then for all large positive integers n we have $A = (A + B^n) \cap (AB^{-n})$.*

The equivalence of (1) and (2) was noted by Dilworth **(3)** and again, recently, by Riley **(5)**. We include (2) because we shall use it in showing that (1) implies (3). Barnes and Cunnea **(1)** proved that (3) holds in a Noetherian commutative ring. Their proof made direct use of the fact that in such a ring every ideal is finitely generated, and did not require prior knowledge of the fact that a Noetherian commutative ring has a Noetherian ideal theory. Then they used (3) to obtain the existence of a Noetherian ideal theory without introducing the intermediate notion of irreducible ideal.

Suppose that (1) holds for R and let A and B be ideals of R. We have $AB = Q_1 \cap \ldots \cap Q_r$ where each Q_i is primary. For each i, $AB \subset Q_i$ and so either $A \subset Q_i$ or $B^{k_i} \subset Q_i$ for some positive integer k_i. If $A \subset Q_i$ we set $n_i = 1$; otherwise we set $n_i = k_i$. Then

$$A \cap B^n \subseteq Q_1 \cap \ldots \cap Q_r = AB \qquad \text{for all } n \geqslant \max n_i.$$

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This is the proof given by Dilworth in **(3)**. We note that it does not make use of the fact that R satisfies the ACC.

Now assume that (2) holds for R and let A and B be ideals of R. Since R satisfies the ACC, there is a positive integer k such that

$$AB^{-k} = AB^{-(k+1)} = \ldots .$$

Furthermore, by (2), there is a positive integer s such that

$$(AB^{-k}) \cap B^{ks} \subset (AB^{-k})B^{k} \subset A.$$

Hence, for all $n \geqslant ks$, $(AB^{-k}) \cap B^{n} \subset A$. But $AB^{-k} = AB^{-n}$, so

$$(AB^{-n}) \cap B^{n} \subset A.$$

Then, by the modular law, we have for all large positive integers n,

$$(A + B^{n}) \cap (AB^{-n}) = A + ((AB^{-n}) \cap B^{n}) \subset A.$$

Since the reverse inclusion holds for all n, the equality of (3) holds for all large positive integers n.

An ideal A or R is called *irreducible* if whenever $A = B \cap C$, where B and C are ideals of R, then either $B = A$ or $C = A$. It is a consequence of the fact that R satisfies the ACC that every ideal of R is an intersection of a finite number of irreducible ideals.

Assume that (3) holds for R. To prove that R has a Noetherian ideal theory, it is sufficient to show that every irreducible ideal of R is primary. Suppose that the ideal A of R is not primary. Then there are ideals B and C of R such that $BC \subset A$, $B \not\subset A$, and $C^{n} \not\subset A$ for all positive integers n. For all large positive integers n we have $A = (A + C^{n}) \cap (AC^{-n})$. For all n, $A \neq A + C^{n}$. Also,

$$(A + B)C^{n} = AC^{n} + BC^{n} \subset A,$$

so that $A + B \subseteq AC^{-n}$. Since $A \neq A + B$, we have $A \neq AC^{-n}$ for all n. Therefore, A is not irreducible, and all is proved.

COROLLARY. *If R satisfies the* ACC *and has a Noetherian ideal theory, and if A is an ideal of R, then for all large positive integers n, $A^{n} \cap 0A^{-n} = 0$.*

We can extend another result of Barnes and Cunnea **(1**, p. 180**)** to the non-commutative case with the following:

THEOREM. *Suppose that R satisfies the* ACC *and has a Noetherian ideal theory. Let A be an ideal of R and let $P_1, \ldots, P_k$ be the minimal prime divisors of A. Then, for all large positive integers n,*

$$A = (A + P_1^{n}) \cap \ldots \cap (A + P_k^{n}).$$

By a result of Murdoch **(4**, Theorem 10**)**, there is an ordered listing of the minimal prime divisors of A, say $P_1, P_2, \ldots, P_r$, with repetitions allowed, such that $P_r \ldots P_1 \subset A$. Here the numbering of the P's may not be the same

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as in the statement of the theorem. Since (3) of the first theorem holds for R, there are positive integers $n_1, \ldots, n_r$ such that

$$A = (A + P_1^{n_1}) \cap AP_1^{-n_1}$$
$$= (A + P_1^{n_1}) \cap (AP_1^{-n_1} + P_2^{n_2}) \cap A(P_2^{n_2}P_1^{n_1})^{-1},$$

and so on, until we finally have

$$A = (A + P_1^{n_1}) \cap (AP_1^{-n_1} + P_2^{n_2}) \cap \ldots$$
$$\cap (A(P_{r-1}^{n_{r-1}} \ldots P_1^{n_1})^{-1} + P_r^{n_r}) \cap A(P_r^{n_r} \ldots P_1^{n_1})^{-1}.$$

But $P_r^{n_r} \ldots P_1^{n_1} \subset P_r \ldots P_1 \subseteq A$, so that the last term is equal to R and consequently may be dropped. Let $n \geqslant \max n_i$. For $i = 1, \ldots, r - 1$, we have

$$A \subset A + P_{i+1}^n \subset A(P_i^{n_i} \ldots P_1^{n_1})^{-1} + P_{i+1}^{n_{i+1}}$$

and so, on forming the intersection of A and the various $A + P_{i+1}^n$, we have

$$A = (A + P_1^n) \cap \ldots \cap (A + P_r^n)$$

for all large positive integers n. If we drop repetitions from this intersection we obtain the result of the theorem.

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