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POLYNOMIALS ON BANACH SPACES WHOSE DUALS ARE ISOMORPHIC TO $\ell_1(\Gamma)$

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We prove that the dual of a Banach space E is isomorphic to an $\ell_1(\Gamma)$ space if and only if, for a fixed integer m, every m-homogeneous 1-dominated polynomial on E is nuclear. This extends a result for linear operators due to Lewis and Stegall. The same techniques used for this result allow us to prove that, if every m-homogeneous integral polynomial between two Banach spaces is nuclear, then every integral (linear) operator between the same spaces is nuclear.

The following result is proved in [**8**], for (a) $\Leftrightarrow$ (b), and in [**14**], for (a) $\Leftrightarrow$ (c):

THEOREM 1. *Given a Banach space E, the following assertions are equivalent:*

(a) *the dual of E is isomorphic to $\ell_1(\Gamma)$ for some set Γ;*

(b) *every absolutely summing operator on E is nuclear;*

(c) *every absolutely summing and compact operator on E is nuclear.*

In this paper we extend it to the polynomial case, proving that the dual of a Banach space E is an $\ell_1(\Gamma)$ space if and only if, for a fixed integer m, every m-homogeneous 1-dominated polynomial on E is nuclear, if and only if every m-homogeneous 1-dominated and compact polynomial on E is nuclear.

The same techniques allow us to prove that, for a fixed integer m and Banach spaces E, F, if every m-homogeneous integral polynomial from E into F is nuclear, then every integral operator from E into F is nuclear.

Throughout, E and F denote Banach spaces, E^* is the dual of E, and B_E stands for its closed unit ball. By $\mathbb{N}$ we represent the set of all natural numbers and by $\mathbb{K}$ the scalar field (real or complex). By an operator we always mean a linear bounded mapping between Banach spaces. Given $m \in \mathbb{N}$, we denote by $\mathcal{P}(^mE, F)$ the space of all m-homogeneous (continuous) polynomials from E into F, and by $\mathcal{L}(^mE, F)$ the space

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of all m-linear (continuous) mappings from $E \times \overset{(m)}{\cdots} \times E$ into F. Recall that to each $P \in \mathcal{P}(^mE, F)$ we can associate a unique symmetric $\widehat{P} \in \mathcal{L}(^mE, F)$ so that

$$P(x) = \widehat{P}(x, \overset{(m)}{\ldots}, x) \qquad (x \in E).$$

For the general theory of polynomials on Banach spaces, we refer to [**5**] and [**11**].

Given $1 \leqslant r < \infty$, a polynomial $P \in \mathcal{P}(^mE, F)$ is *r-dominated* (see, for example, [**9, 10**]) if there exists a constant $k > 0$ such that, for all $n \in \mathbb{N}$ and $(x_i)_{i=1}^n \subset E$, we have

$$\left(\sum_{i=1}^n \left\|P(x_i)\right\|^{r/m}\right)^{m/r} \leqslant k \sup_{x^* \in B_{E^*}} \left(\sum_{i=1}^n \left|x^*(x_i)\right|^r\right)^{m/r}.$$

For $m = 1$, we obtain the absolutely r-summing operators. We denote by $\mathcal{P}_{\mathrm{as}}(^mE, F)$ the space of all 1-dominated polynomials from E into F.

A polynomial $P \in \mathcal{P}(^mE, F)$ is *nuclear* [**5**, Definition 2.9] if it can be written in the form

$$P(x) = \sum_{i=1}^{\infty} \left[x_i^*(x)\right]^m y_i \qquad (x \in E)$$

where $(x_i^*) \subset E^*$ and $(y_i) \subset F$ are sequences such that

$$\sum_{i=1}^{\infty} \|x_i^*\|^m \|y_i\| < \infty.$$

We denote by $\mathcal{P}_{\mathrm{N}}(^mE, F)$ the space of all nuclear m-homogeneous polynomials from E into F.

The following definition of integral polynomial was given in [**2**] and extends the one given in [**12**] for multilinear functionals.

We say that a polynomial $P \in \mathcal{P}(^mE, F)$ is *integral* if there exists a constant $C \geqslant 0$ such that, for every $n \in \mathbb{N}$ and all families $(x_i)_{i=1}^n \subset E$ and $(f_i^*)_{i=1}^n \subset F^*$, we have

$$\left|\sum_{i=1}^n \langle P(x_i), f_i^* \rangle\right| \leqslant C \sup_{x^* \in B_{E^*}} \left\|\sum_{i=1}^n \left[x^*(x_i)\right]^m f_i^*\right\|_{F^*}.$$

By $\mathcal{P}_{\mathrm{I}}(^mE, F)$ we denote the space of all m-homogeneous integral polynomials from E into F. Easily, for $m = 1$, we obtain the (Grothendieck) integral operators [**4**, page 232]. A definition of integral polynomial, using an integral expression, has been given in [**15**]. This definition is equivalent to ours (see [**2**, Proposition 2.2] and [**15**, Proposition 2.6]).

We say that $P \in \mathcal{P}(^mE, F)$ is *compact* if $P(B_E)$ is relatively compact in F. We denote by $\mathcal{P}_{\mathrm{K}}(^mE, F)$ the space of all compact polynomials from E into F.

We use the notation $\overset{m}{\bigotimes} E := E \otimes \overset{(m)}{\cdots} \otimes E$ for the m-fold tensor product of E, $\overset{m}{\underset{\varepsilon}{\bigotimes}} E := E \underset{\varepsilon}{\bigotimes} \overset{(m)}{\cdots} \underset{\varepsilon}{\bigotimes} E$ for the m-fold injective tensor product of E, and $\overset{m}{\underset{\pi}{\bigotimes}} E$ for the

m-fold projective tensor product of E (see [4] for the theory of tensor products). By $\bigotimes_s^m E := E \bigotimes_s \overset{(m)}{\cdots} \bigotimes_s E$ we denote the m-fold symmetric tensor product of E, that is, the set of all elements $u \in \bigotimes^m E$ of the form

$$u = \sum_{j=1}^{n} \lambda_j x_j \otimes \overset{(m)}{\cdots} \otimes x_j \qquad (n \in \mathbb{N}, \lambda_j \in \mathbb{K}, x_j \in E, 1 \leqslant j \leqslant n).$$

By $\bigotimes_{\varepsilon,s}^m E$ we denote the closure of $\bigotimes_s^m E$ in $\bigotimes_\varepsilon^m E$. Analogously, $\bigotimes_{\pi,s}^m E$ is the closure of $\bigotimes_s^m E$ in $\bigotimes_\pi^m E$. For symmetric tensor products, we refer to [6]. For simplicity, we write $\bigotimes^m x := x \otimes \overset{(m)}{\cdots} \otimes x$.

We use the following notation for spaces of operators from E into F: $\mathcal{AS}(E,F)$ for the space of all absolutely summing operators, $\mathcal{I}(E,F)$ for the space of all integral operators, $\mathcal{N}(E,F)$ for the space of all nuclear operators, and $\mathcal{K}(E,F)$ for the space of all compact operators. The definitions may be seen in [3, 4].

We shall use the fact [4, page 232] that an operator $T : E \to F$ is integral if and only if the functional $\widetilde{T} : E \bigotimes_\varepsilon F^* \to \mathbb{K}$, given by $\widetilde{T}(x \otimes f^*) = \langle T(x), f^* \rangle$ for $x \in E$, $f^* \in F^*$, is well-defined and continuous.

For $P \in \mathcal{P}(^mE, F)$, let

$$\overline{P} : \bigotimes_s^m E \longrightarrow F$$

be the linearisation of P, given by

$$\overline{P}\left(\sum_{j=1}^{n} \lambda_j x_j \otimes \overset{(m)}{\cdots} \otimes x_j\right) = \sum_{j=1}^{n} \lambda_j P(x_j)$$

for all $\lambda_j \in \mathbb{K}$, $x_j \in E$ $(1 \leqslant j \leqslant n)$.

It is shown in [2] that P is integral if and only if $\overline{P} : \bigotimes_{\varepsilon,s}^m E \to F$ is well-defined and integral.

PROPOSITION 2. *Fix $m \in \mathbb{N}$ and Banach spaces E, F. Suppose that $\mathcal{P}_{\mathrm{as}}(^mE, F) \subseteq \mathcal{P}_{\mathrm{N}}(^mE, F)$. Then, $\mathcal{AS}(E,F) = \mathcal{N}(E,F)$.*

PROOF: We only have to prove that $\mathcal{AS}(E,F) \subseteq \mathcal{N}(E,F)$ since the other inclusion is always true.

Let $T \in \mathcal{AS}(E,F)$. For every index $i = 1, \ldots, m-1$, there are operators

$$j_i : \bigotimes_{\pi,s}^{i} E \longrightarrow \bigotimes_{\pi,s}^{i+1} E \qquad \text{and} \qquad \pi_i : \bigotimes_{\pi,s}^{i+1} E \longrightarrow \bigotimes_{\pi,s}^{i} E$$

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such that $\pi_i \circ j_i$ is the identity map on $\bigotimes_{\pi,s}^{i} E$ [**1**, p. 168].

Let $\delta_m : E \to \bigotimes_{\pi,s}^{m} E$ be the polynomial given by $\delta_m(x) = \bigotimes^{m} x$ $(x \in E)$. Consider the polynomial

$$P := T \circ \pi_1 \circ \pi_2 \circ \ldots \circ \pi_{m-1} \circ \delta_m : E \longrightarrow F$$

Using that T is absolutely summing, it is shown in [**2**, Proposition 3.1] that P is 1-dominated so, by our hypothesis, it is nuclear. It follows that there exist sequences $(x_n^*) \subset E^*$ and $(y_n) \subset F$ such that

$$P(x) = \sum_{n=1}^{\infty} \left[x_n^*(x)\right]^m y_n \qquad (x \in E)$$

with

$$\sum_{n=1}^{\infty} \|x_n^*\|^m \|y_n\| < \infty.$$

Now, for every n, we consider the m-homogeneous polynomial of finite type $P_n = (x_n^*)^m$. By the isomorphism $\mathcal{P}(^mE) \simeq \left(\bigotimes_{\pi,s}^{m} E\right)^*$, we associate to P_n a functional $\Phi_n \in \left(\bigotimes_{\pi,s}^{m} E\right)^*$ such that

$$\Phi_n\left(\sum_{j=1}^{l} \lambda_j \left(\bigotimes^{m} x_j\right)\right) = \sum_{j=1}^{l} \lambda_j \Phi_n\left(\bigotimes^{m} x_j\right) = \sum_{j=1}^{l} \lambda_j P_n(x_j)$$

for every $\sum_{j=1}^{l} \lambda_j \left(\bigotimes^{m} x_j\right) \in \bigotimes_{\pi,s}^{m} E$. So we have

$$\begin{aligned}
(T \circ \pi_1 \circ \ldots \circ \pi_{m-1})\left(\sum_{j=1}^{l} \lambda_j \left(\bigotimes^{m} x_j\right)\right) &= \sum_{j=1}^{l} \lambda_j (T \circ \pi_1 \circ \ldots \circ \pi_{m-1} \circ \delta_m)(x_j) \\
&= \sum_{j=1}^{l} \lambda_j P(x_j) \\
&= \sum_{j=1}^{l} \lambda_j \sum_{n=1}^{\infty} \left[x_n^*(x_j)\right]^m y_n \\
&= \sum_{n=1}^{\infty} \sum_{j=1}^{l} \lambda_j \left[x_n^*(x_j)\right]^m y_n \\
&= \sum_{n=1}^{\infty} \Phi_n\left(\sum_{j=1}^{l} \lambda_j \left(\bigotimes^{m} x_j\right)\right) y_n.
\end{aligned}$$

It follows that

$$(T \circ \pi_1 \circ \ldots \circ \pi_{m-1})(u) = \sum_{n=1}^{\infty} \Phi_n(u) y_n \qquad \text{for all } u \in \bigotimes_{\pi,s}^{m} E.$$

Moreover, there is $k > 0$ such that

$$\sum_{n=1}^{\infty} \|\Phi_n\| \, \|y_n\| \leqslant \sum_{n=1}^{\infty} k \, \|P_n\| \, \|y_n\| = k \sum_{n=1}^{\infty} \|x_n^*\|^m \, \|y_n\| < \infty.$$

So $T \circ \pi_1 \circ \ldots \circ \pi_{m-1}$ is nuclear and this implies that

$$T = T \circ \pi_1 \circ \ldots \circ \pi_{m-1} \circ j_{m-1} \circ \ldots \circ j_1$$

is nuclear. ∎

Obvious modifications in the proof of Proposition 2 yield:

Proposition 3. *Fix $m \in \mathbb{N}$ and Banach spaces E, F. Suppose that $\mathcal{P}_{as}(^mE, F) \cap \mathcal{P}_K(^mE, F) \subseteq \mathcal{P}_N(^mE, F)$. Then $\mathcal{AS}(E, F) \cap \mathcal{K}(E, F) = \mathcal{N}(E, F)$.*

Easily, $P \in \mathcal{P}(^mE, F)$ is nuclear if and only if there are $(\lambda_i) \in \ell_1$ and bounded sequences $(x_i^*) \subset E^*$, $(y_i) \subset F$ such that

$$P(x) = \sum_{i=1}^{\infty} \lambda_i \big[x_i^*(x)\big]^m y_i \qquad (x \in E).$$

The following simple result will be needed:

Proposition 4. *Let $T \in \mathcal{N}(E, F)$ and $Q \in \mathcal{P}(^mF, G)$. Then $P := Q \circ T \in \mathcal{P}(^mE, G)$ is nuclear.*

Proof: There are $(\lambda_i) \in \ell_1$ and bounded sequences $(x_i^*) \subset E^*$, $(y_i) \subset F$ such that

$$T(x) = \sum_{i=1}^{\infty} \lambda_i x_i^*(x) y_i \qquad (x \in E).$$

Hence, by the polarisation formula [**11**, Theorem 1.10],

$$\begin{aligned} P(x) = Q\Big(\sum_{i=1}^{\infty} \lambda_i x_i^*(x) y_i\Big) &= \sum_{i_1, \ldots, i_m = 1}^{\infty} \lambda_{i_1} \cdots \lambda_{i_m} x_{i_1}^*(x) \cdots x_{i_m}^*(x) \widehat{Q}(y_{i_1}, \ldots, y_{i_m}) \\ &= \frac{1}{2^m m!} \sum_{i_1, \ldots, i_m = 1}^{\infty} \lambda_{i_1} \cdots \lambda_{i_m} \sum_{\varepsilon_j = \pm 1} \varepsilon_1 \cdots \varepsilon_m \big[(\varepsilon_1 x_{i_1}^* + \cdots + \varepsilon_m x_{i_m}^*)(x)\big]^m \widehat{Q}(y_{i_1}, \ldots, y_{i_m}) \end{aligned}$$

from which the result follows. ∎

It is proved in [**9**, Proposition 3.1] that a polynomial $P \in \mathcal{P}(^mE, F)$ is r-dominated if and only if there are a constant $C \geqslant 0$ and a regular Borel probability measure μ on B_{E^*} (endowed with the weak-star topology) such that

$$\|P(x)\| \leqslant C \Big[\int_{B_{E^*}} |\varphi(x)|^r \, d\mu(\varphi)\Big]^{m/r} \qquad (x \in E). \tag{1}$$

The next result is stated in [**12**, Theorem 14] for the multilinear, scalar-valued case, and in [**13**, Proposition 3.6] for the vector-valued case. It will be needed in Theorem 6. Following the referee's suggestion, for the sake of completeness, we include the proof which is an easy modification of [**7**, 3.2.4].

THEOREM 5. *A polynomial $P \in \mathcal{P}(^mE, F)$ is r-dominated if and only if there are a Banach space G, an absolutely r-summing operator $T \in \mathcal{L}(E, G)$ and a polynomial $Q \in \mathcal{P}(^mG, F)$ such that $P = Q \circ T$.*

PROOF: Let $P \in \mathcal{P}(^mE, F)$ be r-dominated. Then there is a regular Borel probability measure μ on B_{E^*} such that the inequality (1) holds. Let $T_0 : E \to L_r(B_{E^*}, \mu)$ be given by $T_0(x)(\varphi) := \varphi(x)$ for all $x \in E$ and $\varphi \in B_{E^*}$. Clearly, T_0 is linear. Moreover,

$$\|T_0(x)\| = \left[\int_{B_{E^*}} |\varphi(x)|^r \, d\mu(\varphi)\right]^{1/r} \leqslant \|x\|.$$

Let G be the closure of $T_0(E)$ in $L_r(B_{E^*}, \mu)$. Let $T : E \to G$ be given by $T(x) := T_0(x)$. Then T is linear and, by [**3**, Theorem 2.12], absolutely r-summing. Define $Q_0 : T_0(E) \to F$ by $Q_0(T_0(x)) := P(x)$. Using the inequality (1), we have:

$$\|P(x)\| \leqslant C\left[\int_{B_{E^*}} |\varphi(x)|^r \, d\mu(\varphi)\right]^{m/r} = C\|T_0(x)\|^m,$$

so Q_0 is a continuous m-homogeneous polynomial. Let $Q : G \to F$ be its extension to G. Then, $P = Q \circ T$.

The converse is shown in [**10**, Theorem 10]. ∎

We can now give the polynomial characterisation of Banach spaces whose duals are isomorphic to $\ell_1(\Gamma)$.

THEOREM 6. *Given a Banach space E, the following assertions are equivalent:*

(a) *E^* is isomorphic to $\ell_1(\Gamma)$ for some set Γ;*

(b) *for all $m \in \mathbb{N}$ and every Banach space F, we have $\mathcal{P}_{\text{as}}(^mE, F) \subseteq \mathcal{P}_{\text{N}}(^mE, F)$;*

(c) *there is $m \in \mathbb{N}$ such that for every Banach space F we have $\mathcal{P}_{\text{as}}(^mE, F) \subseteq \mathcal{P}_{\text{N}}(^mE, F)$;*

(d) *there is $m \in \mathbb{N}$ such that for every Banach space F we have $\mathcal{P}_{\text{as}}(^mE, F) \cap \mathcal{P}_{\text{K}}(^mE, F) \subseteq \mathcal{P}_{\text{N}}(^mE, F)$.*

PROOF: (a) $\Rightarrow$ (b). Suppose $E^* \simeq \ell_1(\Gamma)$. Let $P \in \mathcal{P}_{\text{as}}(^mE, F)$. By Theorem 5, there are a Banach space G, an operator $T \in \mathcal{AS}(E, G)$, and a polynomial $Q \in \mathcal{P}(^mG, F)$ such that $P = Q \circ T$. By Theorem 1, T is nuclear. By Proposition 4, P is nuclear.

(b) $\Rightarrow$ (c) $\Rightarrow$ (d) are obvious.

(d) $\Rightarrow$ (a). It is enough to apply Proposition 3 and Theorem 1. ∎

The same techniques used above allow us to prove the following result:

Proposition 7. *Let E and F be Banach spaces and let $m \in \mathbb{N}$. Suppose that $\mathcal{P}_{\mathrm{I}}({}^mE, F) \subseteq \mathcal{P}_{\mathrm{N}}({}^mE, F)$. Then $\mathcal{I}(E,F) = \mathcal{N}(E,F)$.*

Proof: Let $T \in \mathcal{I}(E,F)$. With the operators

$$\pi_i : \bigotimes_{\pi,s}^{i+1} E \longrightarrow \bigotimes_{\pi,s}^{i} E$$

used in the proof of Proposition 2, we construct the polynomial

$$P := T \circ \pi_1 \circ \ldots \circ \pi_{m-1} \circ \delta_m : E \longrightarrow F,$$

where $\delta_m : E \to \bigotimes_{\pi,s}^{m} E$ is the canonical polynomial. We shall prove that P is integral, equivalently, that $\overline{P} : \bigotimes_{\varepsilon,s}^{m} E \to F$ is well-defined and integral. Easily, the operators π_i are also continuous when the spaces are endowed with the ε-norm. Since $\overline{P} = T \circ \pi_1 \circ \ldots \circ \pi_{m-1}$, we have that $\overline{P}$ is well-defined on $\bigotimes_{\varepsilon,s}^{m} E$. Since T is integral, $\overline{P}$ is integral as well.

By our hypothesis, P is nuclear and then, as in the last part of the proof of Proposition 2, T is nuclear too. So we are done. ∎

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