AVERAGING OPERATORS IN NON COMMUTATIVE L^p SPACES II

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(Received 26 November, 1982)

1. Introduction. This paper is the sequel to [**1**]. Briefly, the context in which we shall work is as follows. Let $\mathcal{A}$ be a finite von Neumann algebra acting on a Hilbert space $\mathcal{H}$. Let ϕ be a faithful normal finite trace on $\mathcal{A}$ with $\phi(I)=1$, where I is the identity of $\mathcal{A}$. For $1\leqslant p<\infty$, let $L^p(\mathcal{A})$ denote the non commutative Lebsegue spaces associated with $(\mathcal{A},\phi)$ [**9**]. We note that $L^p(\mathcal{A})$ is a linear space of (possibly unbounded) operators X, affiliated to $\mathcal{A}$, for which $\phi(|X|^p)<\infty$, where $|X|=(X^*X)^{1/2}$. $L^p(\mathcal{A})$ is a Banach space under the norm $\|X\|_p=\phi(|X|^p)^{1/p}$. The following facts will be used freely. For $1\leqslant p\leqslant\infty$, let q be defined by $1/p+1/q=1$ with $q=\infty$ if $p=1$. Then the dual of $L^p(\mathcal{A})$ is $L^q(\mathcal{A})$ under the bilinear form $\langle x,y\rangle=\phi(xy)$, where $x\in L^p(\mathcal{A})$ and $y\in L^q(\mathcal{A})$, with the convention that $L^\infty(\mathcal{A})\equiv\mathcal{A}$. If $1\leqslant r\leqslant s\leqslant\infty$, then $L^s(\mathcal{A})\subseteq L^r(\mathcal{A})\subseteq L^1(\mathcal{A})$. The L^1 norm may be defined as

$$\|x\|_1=\phi(|x|)=\sup_{B\in\mathcal{A}_1}|\phi(xB)|.$$

If $x\in L^p(\mathcal{A})$ then x^*, the (Hilbert space) adjoint of x, is in $L^p(\mathcal{A})$ too. For a fixed $p\in[1,\infty)$ we define an *average* to be a linear contraction A of $L^p(\mathcal{A})$ satisfying

$$A(x^*)=A(x)^* \qquad (x\in L^p(\mathcal{A})),$$
$$A(xA(y))=A(x)A(y) \qquad (x\in\mathcal{A},\ y\in L^p(\mathcal{A})).$$

In [**1**] it was shown that an average which preserves the identity of $\mathcal{A}$ is the conditional expectation onto its range, which has the form $L^p(\mathcal{B})$ for some von Neumann subalgebra $\mathcal{B}$ of $\mathcal{A}$. In this paper we shall characterise those averages that do not necessarily map I to I.

2. Characterisation of adjoint preserving averages. We begin by characterising those subspaces of $L^p(\mathcal{A})$, which are $L^p(\mathcal{B})$ for some von Neumann subalgebra $\mathcal{B}$ of $\mathcal{A}$.

THEOREM 2.1. *Let $p\in[1,\infty)$ be fixed and let M be a closed subspace of $L^p(\mathcal{A})$ which contains a $*$-subalgebra $\mathcal{B}^0$ of $\mathcal{A}$ with $I\in\mathcal{B}^0$ and such that $\mathcal{B}^0$ is $\|\cdot\|_p$-dense in M. Then $M=L^p(\mathcal{B},\phi)$, where $\mathcal{B}$ is the von Neumann algebra generated by $\mathcal{B}^0$.*

Proof. The idea is that the $\|\cdot\|_p$ closure picks up the strong operator closure too. The algebra $\mathcal{B}^0$ is dense in $\mathcal{B}$ in the strong operator topology (hereafter denoted τ_s). By Kaplansky's density theorem, the self adjoint part of $\mathcal{B}_1^0$, the unit ball of $\mathcal{B}^0$, is τ_s-dense in the self adjoint part of $\mathcal{B}_1$, the unit ball of $\mathcal{B}$. Given $x=x^*\in\mathcal{B}_1$ there is a net (x_α) of self-adjoint operators in $\mathcal{B}_1^0$ converging strongly to x. Let $0\neq n$ be a natural number and

https://doi.org/10.1017/S0017089500005504 Published online by Cambridge University Press

$y \in \mathscr{H}$; since $|x-x_\alpha|^2=(x-x_\alpha)^2$ we have

$$\| \, |x-x_\alpha|^{2n}(y)\| = \|(x-x_\alpha)^{2n}(y)\| \leqslant \|x-x_\alpha\|_\infty^{2n-1} \,.\, \|(x-x_\alpha)(y)\|$$
$$\leqslant 2^{2n}\|(x-x_\alpha)(y)\| \to 0 \quad \text{as} \quad \alpha \uparrow ,$$

where $\|\cdot\|_\infty$ denotes the operator norm. So $|x-x_\alpha|^{2n}\to 0-\tau_s$ as $\alpha\uparrow$. It follows that $\|x-x_\alpha\|_{2n}\to 0$ as $\alpha\uparrow$ because ϕ is weak operator continuous on bounded sets in $\mathscr{A}$. For $1\leqslant p<\infty$ we can choose n sufficiently large so that $p\leqslant 2n$. We then have

$$\|x-x_\alpha\|_p \leqslant \|x-x_\alpha\|_{2n} \qquad (n \geqslant p/2).$$

This relation follows from the corresponding relation for real valued functions from 2·4 of [**9**]. So $\|x-x_\alpha\|_p\to 0$ as $\alpha\uparrow$. Hence $\bar{\mathscr{B}}^{\|\cdot\|_p}=L^p(\mathscr{B})\subseteq M$. The reverse inclusion is obvious.

In connection with the next result, see [**6**].

THEOREM 2.2. *Let* $A: L^p(\mathscr{A})\to L^p(\mathscr{A})$ *be an average. Then* $A(x)=M_{\mathscr{B}}(ux)$, *where* $M_{\mathscr{B}}(\cdot)$ *is the conditional expectation with respect to a von Neumann subalgebra* $\mathscr{B}$ *of* $\mathscr{A}$ *and* $u=A^+(I)$. (A^+ *is the* L^p*-adjoint of* A.)

Proof. Let $\mathscr{B}^0=\{x\in\mathscr{A} : A(yx)=A(y)x,\, A(xy)=xA(y)\ \forall y\in\mathscr{A}\}$. Then $\mathscr{B}^0$ is a $*$-subalgebra of $\mathscr{A}$ containing I. By Theorem 2.1, $\bar{\mathscr{B}}^{0\|\cdot\|_p}=L^p(\mathscr{B})$, where $\mathscr{B}$ denotes the von Neumann algebra generated by $\mathscr{B}^0$. Hence for $y\in L^p(\mathscr{B})$ there is a sequence $(y_n)\subseteq\mathscr{B}^0$ such that $y_n\to y$ in $\|\cdot\|_p$. If $x\in\mathscr{A}$ we have

$$\|xy_n-xy\|_p \leqslant \|x\|_\infty\|y_n-y\|_p \to 0 \quad \text{as} \quad n\to\infty.$$

Hence $A(xy_n)\to A(xy)$ in $\|\cdot\|_p$. From [**1**], we know that $x\in\mathscr{A}\Rightarrow A(x)\in\mathscr{A}$. It follows for $x\in\mathscr{A}$ and $y\in L^p(\mathscr{B})$ that

$$A(xy)=\lim_n A(xy_n)=\lim_n A(x)y_n=A(x)y.$$

Now let $x\in\mathscr{A}$, $y\in\mathscr{B}$. Both $M_{\mathscr{B}}(A^+(I)x)$ and $A(x)$ are in $L^1(\mathscr{B})$; the latter because $A(x)\in\mathscr{B}^0$. Since

$$\phi(M_{\mathscr{B}}(A^+(I)x)y)=\phi(A^+(I)xy)=\phi(A(xy))=\phi(A(x)y)$$

it follows (from the definition of the $L^1(\mathscr{B})$ norm) that $A(x)=M_{\mathscr{B}}(A^+(I)x)$ for $x\in\mathscr{A}$. Now suppose that $x\in L^p(\mathscr{A})$. We can choose $x_n\in\mathscr{A}$ with $x=\lim_n x_n$ and $A(x)=\lim_n A(x_n)$ in $\|\cdot\|_p$. But $M_{\mathscr{B}}(A^+(I)\,\cdot\,):L^p(\mathscr{A})\to L^1(\mathscr{B})$ is continuous, and so $(M_{\mathscr{B}}(A^+(I)x_n))$ converges to $M_{\mathscr{B}}(A^+(I)x)$ in L^1 $(\mathscr{B})$ norm. But $\|\cdot\|_1\leqslant\|\cdot\|_p$ for a finite algebra so for $x\in L^p(\mathscr{A})$,

$$A(x)=\|\cdot\|_p-\lim_n A(x_n)=\|\cdot\|_1-\lim_n M_{\mathscr{B}}(A^+(I)x_n)=M_{\mathscr{B}}(A^+(I)x).$$

REMARKS

(i) An average is a translation followed by a conditional expectation.

(ii) Theorem 2.2 shows A to be a *left* translation followed by an expectation.

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Why not a right translation? In fact it makes no difference for if $y \in \mathcal{B}_1$, then

$$\phi(M_{\mathcal{B}}(A^+(I)x)y) = \phi(A^+(I)xy) = \phi(A(xy));$$

but $y \in \mathcal{B}$ so that

$$\begin{aligned}\phi(A(xy)) &= \phi(A(x)y) = \phi(yA(x))\\ &= \phi(A(yx)) = \phi(A^+(I)yx)\\ &= \phi(xA^+(I)y) = \phi(M_{\mathcal{B}}(xA^+(I))y).\end{aligned}$$

3. Fixed points of averages. Let $\mathcal{B}$ be a von Neumann subalgebra of $\mathcal{A}$ that does not (necessarily) contain I. We shall denote the $\|\cdot\|_p$ closure of $\mathcal{B}$ in $L^p(\mathcal{A})$ by $L^p(\mathcal{B})$.

The next result shows that there is a projection onto the fixed points of an average $A : L^p(\mathcal{A}) \to L^p(\mathcal{A})$ with some nice properties. In particular the fixed points of A are a closed subspace of $L^p(\mathcal{A})$ of the form $L^p(\mathcal{B})$, where $\mathcal{B}$ is a von Neumann subalgebra of $\mathcal{A}$.

THEOREM 3.1. *Let $A : L^p(\mathcal{A}) \to L^p(\mathcal{A})$ be an average. Then there is a projection F from $L^p(\mathcal{A})$ onto the fixed points of A with the following properties.*

(i) $F(x) = F(I)A(x) = A(x)F(I)$ $(x \in L^p(\mathcal{A}))$.
(ii) *$F(I)$ is a projection.*
(iii) *F is normal on $\mathcal{A}$.*
(iv) *$F(\mathcal{A})$ is a von Neumann subalgebra of $\mathcal{A}$.*
(v) *F maps the centre of $\mathcal{A}$ into the centre of $F(\mathcal{A})$.*
(vi) $F(L^p(\mathcal{A})) = L^p(F(\mathcal{A}))$.
(vii) *F is an average.*

Proof. We deal with the cases $1 < p < \infty$ first and then deduce the case of $p = 1$ from these.

(i) For $1 < p < \infty$, $L^p(\mathcal{A})$ is reflexive [**9**]. The ergodic averages $S_n(\cdot) = \frac{1}{n}\sum_{k=0}^{n-1} A^k(\cdot)$ are uniformly bounded as maps $L^p(\mathcal{A}) \to L^p(\mathcal{A})$ and, for $x \in L^p(\mathcal{A})$, $A^k(x)/k$ converges to 0 in $L^p(\mathcal{A})$ as $k \to \infty$. It follows from Corollaries 2 and 4 of VIII.5 of [**2**] that for $x \in L^p(\mathcal{A})$,

$$F(x) = \lim_n S_n(x) \quad \text{in} \quad \|\cdot\|_p$$

exists and the map $x \to F(x)$ is a $\|\ \|_p$ contractive projection onto the fixed points of A. The averaging property indicates that $A^k(x) = A^{k-1}(I)A(x)$; therefore,

$$\begin{aligned}S_n(x) &= \frac{1}{n}(x + A(x) + A(I)A(x) + A(I)^2A(x) + \ldots + A^{n-2}(I)A(x))\\ &= \frac{x}{n} + \frac{(I + A(I) + \ldots + A^{n-1}(I))A(x)}{n} - \frac{A^{n-1}(I)A(x)}{n}\\ &= \frac{x}{n} + S_n(I)A(x) - \frac{A^{n-1}(I)A(x)}{n}.\end{aligned}$$

Since $A(\mathscr{A}_1)\subseteq\mathscr{A}_1$ (2.2 of [**1**]) we have for $x\in\mathscr{A}$ that $S_n(I)A(x)\underset{n}{\rightarrow}F(I)A(x)$ in $\|\cdot\|_p$ and hence $F(x)=F(I)A(\mathbf{x})$ for $x\in\mathscr{A}$. Moreover $\|S_n(I)\|_\infty\leqslant 1$ for $n=1,2,3,\ldots$. We claim that $F(I)\in\mathscr{A}_1$ too. To see this, note that $\mathscr{A}_1$ is compact in the weak operator topology τ_ω and hence $(S_n(I))$ has a subnet $(S_{n_\alpha}(I))$ converging in τ_ω to some $T\in\mathscr{A}_1$. Since $(S_n(I))$ is convergent in $\|\cdot\|_p$ to $F(I)$ the subnet converges in $\|\cdot\|_p$—and hence weakly in $L^p(\mathscr{A})$ to $F(I)$ too. Now on $\mathscr{A}_1$ the weak and ultraweak topologies coincide, and so every weak operator continuous linear functional on $\mathscr{A}_1$ is given by an element of $L^1(\mathscr{A})$. But $L^q(\mathscr{A})\subseteq L^1(\mathscr{A})$; hence for every $X\in L^q(\mathscr{A})$, where $1/p+1/q=1$,

$$\phi(XT)=\lim_\alpha \phi(XS_n(I))=\phi(XF(I))$$

and so $F(I)=T\in\mathscr{A}_1$. It follows that $F(x)=F(I)A(x)$ for each $x\in L^p(\mathscr{A})$. To see that $F(x)=A(x)F(I)$ note that $A^k(x)=A(x)A^{k-1}(I)$, and proceed as above.

(ii) We now use (i) noting that $x=F(I)$ is a fixed point in $\mathscr{A}_1$. $F(I)$ is self adjoint because A preserves adjoints.

(iii) Using (i) and the averaging property with the fact that A is $*$-preserving we see that $F\restriction\mathscr{A}$ is a projection of norm one onto $F(\mathscr{A})$ which is a C*-algebra. Hence F is positive [**7**]. Let $x_\alpha\uparrow x$ in $\mathscr{A}$. By scaling if necessary we can take $0\leqslant x-x_\alpha\leqslant I$. Using the normality of ϕ we conclude that $x_\alpha\rightarrow x$ in $L^1(\mathscr{A})$. Now for $1<p<\infty$

$$0\leqslant|x-x_\alpha|^p\leqslant|x-x_\alpha|=x-x_\alpha\leqslant I.$$

So for $1<p<\infty$,

$$\phi(|x-x_\alpha|^p)\leqslant\phi(x-x_\alpha)=\|x-x_\alpha\|_1\rightarrow 0 \quad \text{as} \quad \alpha\uparrow.$$

That is, $\|x-x_\alpha\|_p\rightarrow 0$ as $n\rightarrow\infty$. By continuity $F(x_\alpha)\rightarrow F(x)$ in $\|\cdot\|_p$ and hence in $\|\cdot\|_1$, and as F is positive and $x\geqslant x_\alpha\ \forall\alpha$ we have, by §2 of [**9**],

$$S=\sup_\alpha F(x_\alpha)=\|\cdot\|_1-\lim F(x_\alpha)=F(x)=F\Big(\sup_\alpha x\Big).$$

So F is normal.

(iv) As we noted in (iii), $F(\mathscr{A})$ is a C*-algebra. Again by (iii), $F(\mathscr{A})$ is monotone closed. Finally, because it has a trace defined on it, it has sufficiently many positive linear functions. We use 3.16 of [**7**] to get the result.

(v) This follows directly from the averaging property.

(vi) Because $F(\mathscr{A})$ is dense in the range of F.

Now we consider the case $p=1$. Since, in this case A contracts $\|\cdot\|_1$ and also $\|\cdot\|_\infty$ by [**1**], it follows from [**5**] that it contracts $\|\cdot\|_p$, $1<p<\infty$. For $x\in L^p(\mathscr{A})$ (where p is fixed) the relation (i) holds and (ii)–(vi) follow. Since A is an L^1 contraction and $L^p(\mathscr{A})$ is dense in $L^1(\mathscr{A})$ we can extend the map $F(\cdot)$ from $L^p(\mathscr{A})$ to $L^1(\mathscr{A})$ by using (i). We note that (i) then holds (obviously) for $x\in L^1(\mathscr{A})$ and that $F(x)$ is a fixed point of A, so that F is an idempotent.

(vii) This follows from (i), (ii) and the fact that A is an average.

It would be useful to know what conditions a contraction of $L^p(\mathscr{A})$ should satisfy in order for it to be an average. Kelley [**4**] has shown that a positive idempotent operator on

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$C_\infty(X)$ is averaging if and only if its range is a subalgebra. The following result is along these lines.

LEMMA 3.2. *For a fixed $p \in [1, \infty)$ let $T : L^p(\mathcal{A}) \to L^p(\mathcal{A})$ be a positive linear mapping that preserves the identity operator. Then T preserves the trace.*

Proof. See 2.4 of [**1**]. The result is proved for an average, but the proof works just as well for the T of the hypothesis.

THEOREM 3.3. *Let $p \in [1, \infty)$ be fixed and $T : L^p(\mathcal{A}) \to L^p(\mathcal{A})$ be a positive contractive idempotent linear mapping with $T(I) = I$ and $T(\mathcal{A})$ an algebra. Then T is the conditional expectation onto $L^p(\mathcal{B})$, where $\mathcal{B}$ is a von Neumann subalgebra of $\mathcal{A}$.*

Proof. By the lemma, T preserves the trace and so for each projection $E \in \mathcal{A}$ we have $0 \leq T(E) \leq I$ and $\phi(T(E)) \leq \phi(E)$. These are the conditions of Proposition 1 of [**10**]. This shows that T extends to a map of $L^{p'}(\mathcal{A})$ into itself for $1 \leq p' \leq \infty$, and $T(\mathcal{A}) \subseteq \mathcal{A}$; moreover if $x = x^*$, then $\|T(x)\|_{p'} \leq \|x\|_{p'}$. We shall use the extension of T to $L^2(A)$ below.

So $T(\mathcal{A})$ is a $*$-subalgebra of $\mathcal{A}$ containing I. By Theorem 2.1 above, the range of T is $L^p(T(\mathcal{A})'')$, where $T(\mathcal{A})''$ denotes the von Neumann algebra generated by $T(\mathcal{A})$. Let M be the conditional expectation $L^p(\mathcal{A}) \xrightarrow{\text{onto}} L^p(T(\mathcal{A})'')$. Suppose that for each $y \in T(\mathcal{A})$, $z \in L^p(\mathcal{A})$,

$$\phi(M(z)y) = \phi(T(z)y). \tag{$*$}$$

Then by ultraweak continuity we have $(*)$ for $y \in T(\mathcal{A})''$. This shows that $M(z) = T(z)$; (consider the L^1 norm). So it remains to show that $(*)$ holds. What we shall show is that if $y \in \mathcal{A}$ is a fixed point of T then it is a fixed point of T^+ (the "L^q" adjoint of T). We then get, for $y \in T(\mathcal{A})$,

$$\phi(M(z)y) = \phi(zy) = \phi(zT^+(y)) = \phi(T(z)y),$$

which finishes the proof.

Consider $x \in \mathcal{A}$; we know that $T(x) \in L^2(\mathcal{A})$ but only that $T^+(x) \in L^q(\mathcal{A})$, where $1/p + 1/q = 1$. However it is clear that on $\mathcal{A}$ the L^q adjoint of T agrees with the L^2 adjoint of T. So for $z \in L^2(\mathcal{A})$ we have

$$\begin{aligned} \sup_{y \in \mathcal{A}_1} |\phi(T^+(x)zy)| &= \sup_{y \in \mathcal{A}_1} |\phi(xT(zy))| \qquad (zy \in L^2(\mathcal{A})) \\ &\leq \sup_{y \in \mathcal{A}_1} 2\{\|x\|_2 \|y\|_\infty \|z\|_2\}, \end{aligned}$$

so that $T^+(x)z \in L^1(\mathcal{A})$ and hence $T^+(x) \in L^2(\mathcal{A})$. Suppose now that $x = x^* \in \mathcal{A}$ and $T(x) = x$. We note first that, since T preserves positivity so does T^+, and hence they both preserve adjoints. Thus

$$\begin{aligned} 0 \leq \|T^+(x) - x\|_2^2 &= \phi((T^+(x) - x)(T^+(x) - x)) \\ &= \phi(T^+(x)T^+(x)) - \phi(T^+(x)x) - \phi(xT^+(x)) + \phi(xx) \\ &= \phi(T^+(x)T^+(x)) - \phi(xx). \end{aligned}$$

But by Proposition 1(iii) of [10] we can see that T^+ is $\|\cdot\|_2$ contractive on self adjoint elements in $\mathscr{A}$ just as T is. Hence $T^+(x) = x$. It follows that T and T^+ have the same fixed points in $\mathscr{A}$.

REFERENCES

1. C. Barnett. Averaging operators in non commutative L^p spaces I *Glasgow Math. J.* **24** (1983), 71–74.

2. N. Dunford and J. T. Schwartz, *Linear operators Part* 1 (Interscience, 1958).

3. N. Dunford and J. T. Schwartz, *Linear operators Part* 2 (Interscience, 1963).

4. J. L. Kelley. Averaging operators in $C_\infty(X)$, *Illinois J. Math.* **2** (1958), 214–223.

5. R. A. Kunze. L^p Fourier transforms on locally compact unimodular groups, *Trans. Amer. Math. Soc.* **89** (1958), 519–540.

6. J. Neveu. *Discrete parameter martingales.* (North Holland, 1975).

7. M. Takesaki. *Theory of operator algebras I.* (Springer-Verlag, 1979).

8. H. Umegaki. Conditional expectation in an operator algebra *Tôhoku Math. J.* **6** (1954), 177–181.

9. F. J. Yeadon. Non commutative L^p Spaces, *Math. Proc. Cambridge Philos. Soc.* **77** (1975), 91–102.

10. F. J. Yeadon. Ergodic theorems for semifinite von Neumann algebras I. *J. London Math. Soc.* (2) **16** (1977), 326–332.

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