Can. J. Math., Vol. XLI, No. 2, 1989, pp. 274–284

A MIXED PARSEVAL'S EQUATION AND A GENERALIZED HANKEL TRANSFORMATION OF DISTRIBUTIONS

J. J. BETANCOR

1. Introduction. Let an integral transform $T\{f\}$ of a complex valued function $f(x)$ defined over the interval $(0, \infty)$ be defined as

$$\text{(1.a)} \quad g(y) = T\{f(x)\}(y) = \int_0^\infty f(x)K(x, y)dx.$$

One of the most usual procedures to extend the classical transform (1.a) to generalized functions consists in constructing a space A of testing functions over $(0, \infty)$ which is closed with respect to the classical transform (1.a) and then the corresponding transform of the generalized function f of the dual space of A is defined through

$$\text{(2.a)} \quad \langle Tf, \phi \rangle = \langle f, T\phi \rangle, \quad \text{for every } \phi \in A.$$

This approach has been followed by L. Schwartz [**13**] and A. H. Zemanian [**20**], amongst others.

Sometimes (for example in [**13**] and [**20**]) the above definition (2.a) appears to be a generalization of a suitable Parseval's equation, but, most of the time (see [**7**] and [**12**]) the needed formula does not occur. Recently, J. M. Mendez [**11**] introduced a modification in this method based on a mixed Parseval's equation involving two different integral transforms.

In a previous paper [**2**] we introduced the integral transform depending on three real parameters $(\alpha_0, \alpha_1, \alpha_2)$ defined by

$$\begin{aligned} F(y) &= F_{\alpha_0,\alpha_1,\alpha_2}\{f(x)\}(y) \\ &= y^{\alpha_0 - \alpha_2} \int_0^\infty J(xy;\, \alpha_2, \alpha_1, \alpha_0) f(x)dx \end{aligned}$$

where

$$J(z;\, \alpha_0, \alpha_1, \alpha_2) = z^{(1-\alpha_1-2\alpha_2)/2} J_\nu\left(\frac{2}{2+k} z^{(2+k)/2}\right)$$

where

Received January 5, 1988 and in revised form May 3, 1988.

https://doi.org/10.4153/CJM-1989-014-9 Published online by Cambridge University Press

$$v = \frac{\alpha_1 - 1}{2 + k} \geqq -\frac{1}{2}$$

and J_μ denotes the Bessel function of the first kind and order μ.

Note that the $F_{\alpha_0,\alpha_1,\alpha_2}$-transform reduces to certain Hankel type transformations for suitable values of the parameters (for example

$$F_{-2\mu-1,2\mu+1,0} = B_\mu \quad ([\mathbf{1}]),$$

$$F_{-\mu-1,2\mu+1,-\mu} = \mathscr{H}_\mu \quad ([\mathbf{18}]),$$

$$F_{-\mu-(1/2),2\mu+1,-\mu-(1/2)} = h_\mu \quad ([\mathbf{20}]) \quad \text{and}$$

$$F_{0,\mu+1,-\mu} = {}_c h_\mu \quad ([\mathbf{9}])).$$

In [**2**] the following statements concerning the $F_{\alpha_0,\alpha_1,\alpha_2}$-transformations were established.

THEOREM 1 (Inversion formula). *If f is of bounded variation in a vicinity of the point $x_0 > 0$, $(\alpha_1 - 1)/(2 + k) \geqq -1/2$ and the integral*

$$\int_0^\infty |f(x)| x^{((1-\alpha_1-2\alpha_0)/2)-((2+k)/2)} dx$$

exists, then

$$\lim_{\lambda\to\infty} x_0^{\alpha_0-\alpha_2} \int_0^\infty f(x) \int_0^\lambda J(ux;\, \alpha_2,\, \alpha_1,\, \alpha_0)$$

$$\times J(ux_0;\, \alpha_2,\, \alpha_1,\, \alpha_0) u^{\alpha_0-\alpha_2} du dx$$

$$= \frac{1}{2}\{f(x_0 + 0) + f(x_0 - 0)\}.$$

THEOREM 2 (Parseval's equation). *If $f(x)x^{-\alpha_0}$ and $G(y)y^{-\alpha_0}$ are in $L_1(0, \infty)$ and, in addition, $(\alpha_1 - 1)/(2 + k) \geqq -1/2$,*

$$F(y) = F_{\alpha_0,\alpha_1,\alpha_2}\{f(x)\}(y) \quad \textit{and}$$

$$g(x) = F_{\alpha_0,\alpha_1,\alpha_2}\{G(y)\}(x),$$

then

$$\int_0^\infty x^{\alpha_2-\alpha_0} f(x) g(x) dx = \int_0^\infty y^{\alpha_2-\alpha_0} F(y) G(y) dy.$$

By invoking the Fubini's theorem we also can prove

THEOREM 3. *If $f(x)x^{-\alpha_0}$ and $G(y)y^{-\alpha_2}$ are in $L_1(0, \infty)$,*

$$F(y) = F_{\alpha_0,\alpha_1,\alpha_2}\{f(x)\}(y), \quad \textit{and}$$

$$G(y) = F_{\alpha_2,\alpha_1,\alpha_0}\{g(x)\}(y),$$

https://doi.org/10.4153/CJM-1989-014-9 Published online by Cambridge University Press

then

$$(1) \qquad \int_0^\infty f(x)g(x)dx = \int_0^\infty F(y)G(y)dy$$

provided that $(\alpha_1 - 1)/(2 + k) \geqq -1/2$.

According to J. M. Mendez [**11**] the equality (1) can be called the mixed Parseval equation for the $F_{\alpha_0,\alpha_1,\alpha_2}$-transformations.

In this paper we extend the $F_{\alpha_0,\alpha_1,\alpha_2}$-transform to a space of generalized functions employing the same procedure as the one introduced by J. M. Mendez [**11**] to extend the Hankel transformation. For every α_2, $k \in R$ with $k > -2$, we construct a space $H_{\alpha_2,k}$ of testing functions such that the $F_{\alpha_0,\alpha_1,\alpha_2}$-transform is an automorphism on it provided that

$$\alpha_0 + \alpha_2 < 1, \left(\frac{\alpha_1 - 1}{2 + k}\right) \geqq -\frac{1}{2} \quad \text{and} \quad k = -\alpha_0 - \alpha_1 - \alpha_2.$$

The generalized transformation $F'_{\alpha_0,\alpha_1,\alpha_2}f$ of $f \in H'_{\alpha_0,k}$ (the dual space of $H_{\alpha_0,k}$) is defined on $H_{\alpha_0,k}$ as the adjoint of $F_{\alpha_2,\alpha_1,\alpha_0}$, so that

$$(2) \qquad \langle F'_{\alpha_0,\alpha_1,\alpha_2}f, \phi\rangle = \langle f, F_{\alpha_2,\alpha_1,\alpha_0}\phi\rangle, \text{ for every } \phi \in H_{\alpha_0,k}.$$

$F'_{\alpha_0,\alpha_1,\alpha_2}$ is an automorphism onto $H'_{\alpha_0,k}$, if $\alpha_0 + \alpha_2 < 1$, $(\alpha_1 - 1)/(2 + k) \geqq -1/2$.

We prove some operational formulas concerning the $F_{\alpha_0,\alpha_1,\alpha_2}$ and $F'_{\alpha_0,\alpha_1,\alpha_2}$ transforms, and the coincidence of both conventional and generalized transformations when they act on conventional functions is established.

Our results generalize the ones obtained by A. H. Zemanian [**20**] and J. M. Mendez [**11**], and improve the results due to W. Y. Lee [**7**] and A. Schuitman [**12**], as is shown in Remark 2.

Throughout this paper, I denotes the real interval $0 < x < \infty$. $D(I)$, $E(I)$, $D'(I)$ and $E'(I)$ denote well-known spaces of testing functions and distribution encountered in [**13**] and [**21**].

2. The space of testing functions $H_{\alpha_2,k}$ and its dual. Let α_2, k be real numbers, with $2 + k > 0$. In [**2**] we defined the space $H_{\alpha_2,k}$ of functions as the space of all complex valued smooth functions $\phi(x)$ on $0 < x < \infty$ for which all of the following quantities $\gamma^{m,n}_{\alpha_2,k}(\phi)$, $m, n \in N$, exist, where:

$$\gamma^{m,n}_{\alpha_2,k}(\phi) = \sup_{x \in I} |x^m(x^{-1-k}D)^n(x^{\alpha_2}\phi(x))|.$$

$H_{\alpha_2,k}$ is a linear space over the field of complex numbers. We assigned to it the topology generated by using $\gamma^{m,n}_{\alpha_2,k}$, $m, n \in N$, as seminorms. Hence $H_{\alpha_2,k}$ is a Hausdorff locally convex topological linear space that satisfies the first countability axiom.

We now list some properties of the space $H_{\alpha_2,k}$ and its dual.

PROPOSITION 1. *A function ϕ is in $H_{\alpha_2,k}$ if, and only if, ϕ satisfies all three conditions*:

(i) *ϕ is in $C^\infty(I)$,*

$$\text{(ii) } \phi(x) = x^{-\alpha_2}(b_0 + b_1x^{2+k} + b_2x^{2(2+k)} + \dots + b_nx^{n(2+k)} + B_n(x))$$

where

$$b_m = \frac{(2+k)^{-m}}{m!} \lim_{x\to 0^+} (x^{-1-k}D)^m(x^{\alpha_2}\phi(x)),$$

for every $m = 0, 1, 2, \dots, n$ and

$$\lim_{x\to 0^+} (x^{-1-k}D)^nB_n(x) = 0,$$

for each $n \in N$.

(iii) *$D^m\phi(x)$ is of rapid descent as $x \to \infty$, for every $m \in N$.*

PROPOSITION 2. *$H_{\alpha_2,k}$ is a Frechet space. Hence the dual space $H'_{\alpha_2,k}$ is complete endowed with both weak and strong topologies.*

Therefore $H_{\alpha_2,k}$ is a space of testing functions and $H'_{\alpha_2,k}$ is a space of generalized functions.

PROPOSITION 3. *$D(I) \subset H_{\alpha_2,k}$ and the topology of $D(I)$ is stronger than the one induced on it by $H_{\alpha_2,k}$. Hence, the restriction of any $f \in H'_{\alpha_2,k}$ to $D(I)$ is in $D'(I)$, and the convergence in $H'_{\alpha_2,k}$ implies the weak convergence in $D'(I)$.*

However $D(I)$ is not dense in $H_{\alpha_2,k}$.

PROPOSITION 4. *$H_{\alpha_2,k}$ is contained in $E(I)$ and the inclusion is continuous.*

Moreover $H_{\alpha_2,k}$ is dense in $E(I)$ because $D(I) \subset H_{\alpha_2,k}$ and $D(I)$ is dense in $E(I)$.

PROPOSITION 5. *If $p = (2 + k)n$, for $n \in N$, then*

$$H_{\alpha_2-p,k} \subset H_{\alpha_2,k}$$

and the inclusion is continuous.

Remark 1. Note that the space $H_{\alpha_2,k}$ of testing functions reduces to other spaces earlier studied related to certain variants of the Hankel transform. For example: $H_{-\mu-(1/2),0} = H_\mu$ **[20]**, $H_{0,0} = H$ **[1]**, $H_{-\mu,0} = \mathscr{H}_\mu$ **[11]**, $H_{-\mu,1} = {}_cH_\mu$ **[10]**, $H_{0,1} = S$ **[16]**, amongst others.

PROPOSITION 6. *If $\alpha_0 \in R$, and $\alpha_0 + \alpha_2 < 1$ then every member $f \in H_{\alpha_0,k}$ gives rise to a regular generalized function f in $H'_{\alpha_2,k}$, through*

$$\langle f, \phi\rangle = \int_0^\infty f(x)\phi(x)dx, \quad \text{for every } \phi \in H_{\alpha_2,k}.$$

https://doi.org/10.4153/CJM-1989-014-9 Published online by Cambridge University Press

Proof. Firstly, f is a linear operator on $H_{\alpha_2,k}$. Moreover, one has

$$|\langle f, \phi\rangle \leqq \int_0^\infty |x^{\alpha_2}\phi(x)|\,|x^{\alpha_0}f(x)|x^{-\alpha_0-\alpha_2}dx \leqq A\gamma^{0,0}_{\alpha_2,k}(\phi)$$

for a certain constant A, since $\alpha_0 + \alpha_2 < 1$, $f(x) = o(x^{-\alpha_0})$, as $x \to 0^+$, and $f(x)$ is of rapid descent as $x \to \infty$. Hence f is a continuous mapping.

It can be also proved that two members of $H_{\alpha_0,k}$ which generate the same regular distribution of $H'_{\alpha_2,k}$, are identical. Therefore, $H_{\alpha_0,k}$ can be identified with a subspace of $H'_{\alpha_2,k}$ and the inclusion $H_{\alpha_0,k} \subset H'_{\alpha_2,k}$ is justified provided that $\alpha_0 + \alpha_2 < 1$.

We now introduce the differential operators

$$(3)\qquad \begin{aligned} N_{\alpha_0,\alpha_1,\alpha_2} &= x^{\alpha_1+\alpha_2}Dx^{\alpha_2} \\ M_{\alpha_0} &= x^{\alpha_0}Dx^{-\alpha_0} \\ N^*_{\alpha_0,\alpha_1,\alpha_2} &= -x^{\alpha_2}Dx^{\alpha_1+\alpha_0} \\ M^*_{\alpha_0} &= -x^{-\alpha_0}Dx^{\alpha_0} \end{aligned}$$

α_0, α_1 and α_2 being real numbers. For the moment, we interpret the differentiations in the conventional sense. Next, we study these operators on the space $H_{\alpha_2,k}$.

PROPOSITION 7. *If* $k = -\alpha_0 - \alpha_1 - \alpha_2$, *then*

(a) $N_{\alpha_0,\alpha_1,\alpha_2}$ *is an isomorphism between* $H_{\alpha_2,k}$ *and* $H_{\alpha_2-1,k}$.

(b) *The operator* M_{α_0} *is a continuous linear mapping from* $H_{\alpha_2-1,k}$ *into* $H_{\alpha_2,k}$.

(c) $M^*_{\alpha_0}$ *is an isomorphism between* $H_{\alpha_0,k}$ *and* $H_{\alpha_0-1-k,k}$.

(d) *The operator* $N^*_{\alpha_0,\alpha_1,\alpha_2}$ *is a continuous linear mapping from* $H_{\alpha_0-1-k,k}$ *into* $H_{\alpha_0,k}$.

In the dual space we follow the notational convention. Now if A denotes some of the operators in (3) then we define the dual operator A' as follows

$$\langle A'f, \phi\rangle = \langle f, A\phi\rangle$$

where f and ϕ are suitable generalized and testing functions, respectively. According to well-known results about dual operators (see [**21**]), we have

PROPOSITION 8. *If* $k = -\alpha_0 - \alpha_1 - \alpha_2$, *then*

(a) $N_{\alpha_0,\alpha_1,\alpha_2}$ *is a continuous linear mapping from* $H'_{\alpha_0,k}$ *into* $H'_{\alpha_0-1-k,k}$.

(b) M_{α_0} *is an isomorphism of* $H'_{\alpha_0-1-k,k}$ *into* $H'_{\alpha_0,k}$.

(c) $M^*_{\alpha_0}$ *is a continuous linear mapping from* $H'_{\alpha_2,k}$ *into* $H'_{\alpha_2-1,k}$.

(d) $N^*_{\alpha_0,\alpha_1,\alpha_2}$ *is an isomorphism from* $H'_{\alpha_2-1,k}$ *into* $H'_{\alpha_2,k}$.

Propositions 7 and 8 imply that the Bessel type operator

$$B_{\alpha_0,\alpha_1,\alpha_2} = M_{\alpha_0}N_{\alpha_0,\alpha_1,\alpha_2} = x^{\alpha_0}Dx^{\alpha_1}Dx^{\alpha_2}$$

is a continuous mapping of the spaces $H_{\alpha_2,k}$ and $H'_{\alpha_0,k}$ into themselves. This operator, and some of its special cases, has been extensively studied because it arises in many problems of mathematical physics. Significant studies were made by, amongst others, I. Dimovski [**3**], A. McBride [**8**], A. H. Zemanian [**21**] and J. M. Mendez [**10**].

3. A generalized $F_{\alpha_0,\alpha_1,\alpha_2}$-transformation. Throughout this section we restrict the parameters α_0, α_1 and α_2 to $\alpha_0 + \alpha_1 < 1$ and $(\alpha_1 - 1)/(2 + k) \geqq -1/2$. Here, as before, $k = -\alpha_0 - \alpha_1 - \alpha_2 > -2$.

According to the behaviour of the function $J(z; \alpha_0, \alpha_1, \alpha_2)$ at the origin and at infinity and in view of Proposition 1, we can establish, after some straightforward manipulations, the following operational formulas, involving the $F_{\alpha_0,\alpha_1,\alpha_2}$-transformation defined in (1).

PROPOSITION 9. *If ϕ is in $H_{\alpha_2,k}$, then*

(a) $$F_{\alpha_0-1-k,\alpha_1+2+k,\alpha_2-1}\{N_{\alpha_0,\alpha_1,\alpha_2}\phi\}(y) = -yF_{\alpha_0,\alpha_1,\alpha_2}\{\phi\}(y)$$

(b) $$F_{\alpha_0,\alpha_1,\alpha_2}\{B_{\alpha_0,\alpha_1,\alpha_2}\phi\}(y) = -y^{2+k}F_{\alpha_0,\alpha_1,\alpha_2}\{\phi\}(y)$$

(c) $$N_{\alpha_0,\alpha_1,\alpha_2}F_{\alpha_0,\alpha_1,\alpha_2}\{\phi\}(y) = F_{\alpha_0-1-k,\alpha_1+2+k,\alpha_2-1}\{-x\phi\}(y)$$

(d) $$B_{\alpha_0,\alpha_1,\alpha_2}F_{\alpha_0,\alpha_1,\alpha_2}\{\phi\}(y) = F_{\alpha_0,\alpha_1,\alpha_2}\{-x^{2+k}\phi\}(y).$$

If ϕ is in $H_{\alpha_2-1,k}$, then

(e) $$F_{\alpha_0,\alpha_1,\alpha_2}\{M_{\alpha_0}\phi\}(y) = y^{1+k}F_{\alpha_0-1-k,\alpha_1+2+k,\alpha_2-1}\{\phi\}(y)$$

(f) $$M_{\alpha_0}F_{\alpha_0-1-k,\alpha_1+2+k,\alpha_2-1}\{\phi\}(y) = F_{\alpha_0,\alpha_1,\alpha_2}\{x^{1+k}\phi\}(y).$$

We now prove a fundamental theorem which describes the behaviour of the $F_{\alpha_0,\alpha_1,\alpha_2}$-transformation on the spaces $H_{\alpha_2,k}$.

THEOREM 4. *$F_{\alpha_0,\alpha_1,\alpha_2}$ is an automorphism on $H_{\alpha_2,k}$.*

Proof. Let ϕ be in $H_{\alpha_2,k}$ and let n and m be any pair of nonnegative integers. Denote

$$\psi(y) = F_{\alpha_0,\alpha_1,\alpha_2}\{\phi\}(y).$$

We must analyze the expression

$$\sup_{y\in I} |y^{m(2+k)/2}(y^{-1-k}D)^n(y^{\alpha_2}\psi(y))|.$$

According to the operational rules (a) and (c) in Proposition 9, we get

$$(-y)^m N_{\alpha_0-n+1,\alpha_1+2(n-1),\alpha_2-n+1} \ldots N_{\alpha_0,\alpha_1,\alpha_2} F_{\alpha_0,\alpha_1,\alpha_2}\{\phi\}(y)$$
$$= F_{\alpha_0-(n+m)(1+k),\alpha_1+(2+k)(n+m),\alpha_2-(n+m)}\{(-x)^n$$
$$\cdot N_{\alpha_0-m+1,\alpha_1+2(m-1),\alpha_2-m+1} \ldots N_{\alpha_0-1,\alpha_1+2,\alpha_0-1} N_{\alpha_0,\alpha_1,\alpha_2}\phi(x)\}(y).$$

Therefore

$$y^{n-\alpha_2}(-1)^{m+n} y^n (y^{-1-k}D)^n (y^{\alpha_2}\psi(y))$$
$$= y^{\alpha_0-\alpha_2-k(n+m)} \int_0^\infty J(xy;\, \alpha_2 - (n+m),\, \alpha_1 + (n+m)(2+k),$$
$$\alpha_0 - (n+m)(1+k))x^{n+m-\alpha_2}(x^{-1-k}D)^m (x^{\alpha_2}\phi(x))dx$$

which can be written

$$(-1)^{n+m} y^{m(2+k)/2} (y^{-1-k}D)^n (y^{\alpha_2}\psi(y))$$
$$= \int_0^\infty J(xy;\, \alpha_2 - n - m,\, \alpha_1 + (2+k)(n+m),$$
$$\alpha_0 - (n+m)(1+k))(xy)^{\alpha_0 - n(k+1) - k(m/2)}$$
$$\times x^{-\alpha_0-\alpha_2} x^{(2+k)(2n+m)/2} (x^{-1-k}D)^m (x^{\alpha_2}\phi(x))dx.$$

Hence, since the function

$$J(z;\, \alpha_2 - n - m,\, \alpha_1 + (2+k)(n+m),$$
$$\alpha_0 - (n+m)(1+k))z^{\alpha_0-(1/2)(n+m)+(1/2)(2+k)}$$

is bounded on $0 < z < \infty$, one has

$$\gamma_{\alpha_2,k}^{m,n}(\psi) \leqq A \sum_{r=0}^{m} \gamma_{\alpha_2,k}^{m(r),n(r)}(\phi)$$

provided that $\alpha_0 + \alpha_1 < 1$, where A is a positive real number and $m(r)$ and $n(r)$ are suitable nonnegative integers.

The last inequality proves that ψ is also in $H_{\alpha_2,k}$ and that the linear mapping $F_{\alpha_0,\alpha_1,\alpha_2}$ is also continuous from $H_{\alpha_2,k}$ into themselves.

Furthermore, the classical inversion theorem (Theorem 1) applies in this case; if ϕ is in $H_{\alpha_2,k}$ then it satisfies the conditions in Theorem 1. Also

$$F_{\alpha_0,\alpha_1,\alpha_2}^{-1} = F_{\alpha_0,\alpha_1,\alpha_2}.$$

Hence $F_{\alpha_0,\alpha_1,\alpha_2}$ is a one to one mapping of $H_{\alpha_2,k}$ onto $H_{\alpha_2,k}$ and indeed an automorphism.

On the other hand, proceeding as in Proposition 9, we can establish the following operational formulas involving the operators $N^*_{\alpha_0,\alpha_1,\alpha_2}$, $M^*_{\alpha_0}$ and $B^*_{\alpha_0,\alpha_1,\alpha_2} = N^*_{\alpha_0,\alpha_1,\alpha_2} M^*_{\alpha_0}$.

https://doi.org/10.4153/CJM-1989-014-9 Published online by Cambridge University Press

PROPOSITION 10. *If* ϕ *is in* $H_{\alpha_0,k}$ *then*

(a) $F_{\alpha_2-1,\alpha_1+2+k,\alpha_0-1-k}\{M^*_{\alpha_0}\phi\}(y) = y^{1+k}F_{\alpha_2,\alpha_1,\alpha_0}\{\phi\}(y)$

(b) $M^*_{\alpha_0}F_{\alpha_2,\alpha_1,\alpha_0}\{\phi\}(y) = F_{\alpha_2-1,\alpha_1+2+k,\alpha_0-1-k}\{x^{1+k}\phi\}(y)$

(c) $F_{\alpha_2,\alpha_1,\alpha_0}\{B^*_{\alpha_0,\alpha_1,\alpha_2}\phi\}(y) = -y^{2+k}F_{\alpha_2,\alpha_1,\alpha_0}\{\phi\}(y)$

(d) $B^*_{\alpha_0,\alpha_1,\alpha_2}F_{\alpha_2,\alpha_1,\alpha_0}\{\phi\}(y) = F_{\alpha_2,\alpha_1,\alpha_0}\{-x^{2+k}\phi\}(y).$

If ϕ *is in* $H_{\alpha_0-1-k,k}$, *then*

(e) $F_{\alpha_2,\alpha_1,\alpha_0}\{N^*_{\alpha_0,\alpha_1,\alpha_2}\phi\}(y) = -yF_{\alpha_2-1,\alpha_1+2+k,\alpha_0-1-k}\{\phi\}(y)$

(f) $N^*_{\alpha_0,\alpha_1,\alpha_2}F_{\alpha_2-1,\alpha_1+2+k,\alpha_0-1-k}\{\phi\}(y) = F_{\alpha_2,\alpha_1,\alpha_0}\{-x\phi\}(y).$

We now define the generalized transformation $F'_{\alpha_0,\alpha_1,\alpha_2}$ on $H'_{\alpha_0,k}$ as the adjoint of the $F_{\alpha_2,\alpha_1,\alpha_0}$ on $H_{\alpha_0,k}$. More specifically, for arbitrary $\phi \in H_{\alpha_0,k}$ and for every $f \in H'_{\alpha_0,k}$ we define $F'_{\alpha_0,\alpha_1,\alpha_2}f$ by

(4) $\langle F'_{\alpha_0,\alpha_1,\alpha_2}f, \phi\rangle = \langle f, F_{\alpha_2,\alpha_1,\alpha_0}\phi\rangle.$

Applying the theory of adjoint operators to Theorem 4 we derive

THEOREM 5. *The generalized transformation* $F'_{\alpha_0,\alpha_1,\alpha_2}$ *is an automorphism on* $H'_{\alpha_0,k}$.

Note that by setting $\psi = F_{\alpha_2,\alpha_1,\alpha_0}\phi$ and by invoking the inversion theorem, (4) can be written as

(5) $\langle F'_{\alpha_0,\alpha_1,\alpha_2}f, F_{\alpha_2,\alpha_1,\alpha_0}\phi\rangle = \langle f, \phi\rangle, f \in H'_{\alpha_0,k}, \phi \in H_{\alpha_0,k}$

and in this form it appears as a generalization of Parseval's equation (1).

The conventional $F_{\alpha_0,\alpha_1,\alpha_2}$-transformation when acting on a function $f \in H_{\alpha_2,k}$ (recall that $H_{\alpha_2,k} \subset H'_{\alpha_0,k}$) is a special case of our generalized transformation. Indeed, set

$$F(y) = F_{\alpha_0,\alpha_1,\alpha_2}\{f(x)\}(y) = y^{\alpha_0-\alpha_2}\int_0^\infty J(xy;\ \alpha_2,\ \alpha_1,\ \alpha_0)f(x)dx.$$

Since $z^{\alpha_0}J(z;\ \alpha_2,\ \alpha_1,\ \alpha_0)$ is bounded on $0 < z < \infty$ and $\alpha_0 + \alpha_2 < 1$, $F(y)$ generates a regular generalized function in $H'_{\alpha_0,k}$. Furthermore, by our definition (5) we have

$$\langle F'_{\alpha_0,\alpha_1,\alpha_2}f, F_{\alpha_2,\alpha_1,\alpha_0}\phi\rangle = \langle f, \phi\rangle = \int_0^\infty f(x)\phi(x)dx$$

for every $\phi \in H_{\alpha_0,k}$. Since

$$x^{-\alpha_2}F_{\alpha_2,\alpha_1,\alpha_0}\phi \in L_1(0, \infty)$$

when $\alpha_0 + \alpha_2 < 1$ and we may invoke Parseval's equation (1) to write

$$\int_0^\infty f(x)\phi(x)dx = \int_0^\infty F(y)F_{\alpha_2,\alpha_1,\alpha_0}\{\phi(x)\}dy.$$

Therefore

(6) $\langle F'_{\alpha_0,\alpha_1,\alpha_2}f, F_{\alpha_2,\alpha_1,\alpha_0}\phi\rangle = \langle F(y), F_{\alpha_2,\alpha_1,\alpha_0}\phi\rangle.$

Thus, our generalized transformation $F'_{\alpha_0,\alpha_1,\alpha_2}f$ is the regular generalized function corresponding to the conventional $F_{\alpha_0,\alpha_1,\alpha_2}$ transform

$$F(y) = F_{\alpha_0,\alpha_1,\alpha_2}\{f\}(y).$$

According to definition (3) and Proposition 10 we can prove the following operational formulas for the generalized $F'_{\alpha_0,\alpha_1,\alpha_2}$-transformation.

PROPOSITION 11. *If ϕ is in $H'_{\alpha_0,k}$, then*

(a) $F'_{\alpha_0-1-k,\alpha_1+2+k,\alpha_2-1}\{N_{\alpha_0,\alpha_1,\alpha_2}\phi\}(y) = -yF'_{\alpha_0,\alpha_1,\alpha_2}\{\phi\}(y)$

(b) $F'_{\alpha_0,\alpha_1,\alpha_2}\{B_{\alpha_0,\alpha_1,\alpha_2}\phi\}(y) = -y^{2+k}F'_{\alpha_0,\alpha_1,\alpha_2}\{\phi\}(y)$

(c) $N_{\alpha_0,\alpha_1,\alpha_2}F'_{\alpha_0,\alpha_1,\alpha_2}\{\phi\}(y) = F'_{\alpha_0-1-k,\alpha_1+2+k,\alpha_2-1}\{-x\phi\}(y)$

(d) $B_{\alpha_0,\alpha_1,\alpha_2}F'_{\alpha_0,\alpha_1,\alpha_2}\{\phi\}(y) = F'_{\alpha_0,\alpha_1,\alpha_2}\{-x^{2+k}\phi\}(y).$

If ϕ is in $H'_{\alpha_0-1-k,k}$, then

(e) $F'_{\alpha_0,\alpha_1,\alpha_2}\{M_{\alpha_0}\phi\}(y) = y^{1+k}F'_{\alpha_0-1-k,\alpha_1+2+k,\alpha_2-1}\{\phi\}(y)$

(f) $M_{\alpha_0}F'_{\alpha_0-1-k,\alpha_1+2+k,\alpha_2-1}\{\phi\}(y) = F'_{\alpha_0,\alpha_1,\alpha_2}\{x^{1+k}\phi\}(y).$

Since $H_{\alpha_2,k} \subset H'_{\alpha_0,k}$ and $H_{\alpha_2-1,k} \subset H'_{\alpha_0-1-k,k}$ provided that $\alpha_0 + \alpha_2 < 1$, the results in Proposition 11 can be seen as an extension of those in Proposition 9.

Remark 2. As we said above, the $F_{\alpha_0,\alpha_1,\alpha_2}$-transformation reduces to other Hankel type transformations setting for the parameters α_0, α_1 and α_2 suitable values. Our results extend those due to A. H. Zemanian [**20**], and J. M. Mendez [**11**], which can be obtained as special cases of ours. Moreover, when $\alpha_0 = -\alpha_1 = -2\mu - 1$ and $\alpha_2 = 0$, the $F_{\alpha_0,\alpha_1,\alpha_2}$-transform reduces to the Hankel-Schwartz transform B_μ introduced by A. L. Schwartz [**14**]. In this paper we improve the work of A. Schuitman [**12**] on a generalized Hankel-Schwartz transformation. This author proved that, for $\mu \geqq -1/2$, the B_μ-transform is an automorphism onto a space H of testing functions. He gave the following definition of the generalized B'_μ-transformation for every $f \in H'$:

$$\langle B'_\mu f, \phi\rangle = \langle f, B_\mu\phi\rangle, \quad \forall\phi \in H.$$

For both conventional and generalized B_μ-transformations the following equality holds:

$$B'_\mu f = y^{2\mu+1} B_\mu \{x^{-2\mu-1} f\}, \quad \forall f \in H$$

instead of the natural equality (6). Moreover, the operational formulas corresponding to the generalized Hankel-Schwartz transformation B'_μ [**1**, pp. 211] do not coincide with the respective classic results [**1**, pp. 207]. Also, the paper of W. Y. Lee, [**7**] is improved in the same sense.

If (4) is replaced by the more usual definition

$$\langle F'_{\alpha_0,\alpha_1,\alpha_2} f, \phi \rangle = \langle f, F_{\alpha_2,\alpha_1,\alpha_2} \phi \rangle$$

the same problems that we have just noted in [**12**] and [**7**] arise again.

References

1. G. Altenburg, *Bessel transformationen in Raumen von Grundfunktionen uber dem intervall* $\Omega = (0, \infty)$ *und derem dual raumen*, Math. Nachr. *108* (1982), 197-218.

2. J. J. Betancor, *La transformacion integral* $F_{\alpha_0,\alpha_1,\alpha_2}$, Doctoral Thesis, Depart. Math. Anal., Univ. of La Laguna, Tenerife, Spain (1987).

3. I. Dimovski, *Operational calculus for a class of differential operators*, Compt. Rend. l'Academie Bulgare Sci. *19* (1966), 1111-1114.

4. I. M. Gelfand and G. E. Shilov, *Generalized functions*, *1* and *2* (Academic Press, New York, 1964, 1968).

5. A. Gray, G. B. Mathews and T. M. McRobert, *A treatise on Bessel functions and their applications to physics* (McMillan and Co. Ltd., London, 1952).

6. E. L. Koh and A. H. Zemanian, *The complex Hankel and I-transformations of generalized functions*. SIAM J. Appl. Math. *16* (1968), 945-957.

7. W. Y. Lee, *On Schwartz's Hankel transformation of certain spaces of distributions*, SIAM J. Math. Anal., *6* (2) (1975), 427-432.

8. A. McBride, *The Hankel transformation of some classes of generalized functions and connections with fractional integration*, Proc. Roy. Soc. Edinburgh *81* A (1978), 95-117.

9. J. M. Mendez, *La transformacion integral de Hankel-Clifford*, Secretariado de Publicaciones de la Universidad de La Laguna, Coleccion Monografias, *8* (Tenerife, Spain, 1981).

10. ——— *On the Bessel transform of arbitrary order*, Math. Nachr. *136* (1988), 233-239.

11. ——— *A mixed Parseval equation and the generalized Hankel transformation*, Proc. Amer. Math. Soc. *102* (1988), 619-624.

12. A. Schuitman, *On a certain test function space for the Schwartz's Hankel transformation*, Delft Progr. Report *2* (1977), 193-206.

13. L. Schwartz, *Theorie des distributions* (Hermann, Paris, 1966).

14. A. L. Schwartz, *An inversion theorem for Hankel transforms*, Proc. Amer. Math. Soc. *22* (1969), 713-717.

15. I. N. Sneddon, *The use of integral transforms* (McGraw Hill Book Co., New York, 1972).

16. M. M. Socas, *La transformacion integral de Hankel-Clifford en ciertos espacios de funciones generalizadas*, Doctoral Thesis, Depart. Math. Anal., Univ. of La Laguna, Tenerife, Spain, 1986.

17. E. C. Titchmarsh, *Introduction to the theory of Fourier integrals* (Oxford University Press, Oxford, 1948).

18. G. N. Watson, *A treatise on the theory of Bessel functions* (Cambridge University Press, Cambridge, 1958).

19. A. H. Zemanian, *Distribution theory and transform analysis* (McGraw Hill Book Co., New York, 1965).

20. ——— *A distributional Hankel transformation*, J. SIAM Appl. Math. *14* (1966), 561-576.

21. ——— *Generalized integral transformation* (Interscience Publishers, New York, 1968).

University of La Laguna,
La Laguna, Canary Islands