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ON THE BOUNDARY BEHAVIOR OF HOLOMORPHIC FUNCTIONS IN THE UNIT DISC

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Seidel (1959) established various boundary properties of holomorphic functions with spiral asymptotic paths. Especially, Theorem 4 which is the fundamental result of the paper; was generalized by Faust (1962), using essentially the same method that was employed by Seidel.

In this paper, by tracing the proof of Theorem 4 in Seidel (1959) and the proof of Theorem in Faust (1962) more minutely in some parts, we shall prove a theorem (Theorem 1) concerning the boundary behavior of holomorphic functions in the unit disc. Further, as applications of this theorem, we shall prove some results about spiral functions and annular functions.

1. Definitions and Notations

In the following, we denote the unit disc $\{z; |z| < 1\}$ by D, the unit circle $\{z; |z| = 1\}$ by C and the finite w-plane by W.

Let τ be a point of C. We denote by $L(\tau; \alpha)$ the segment terminating at τ and making an angle α $(0 < \alpha < \pi)$ with the positive tangent of C at τ. We denote by $\Delta(\tau; \alpha, \beta, \delta)$ a Stolz angle having the vertex at τ, bounded by two segments $L(\tau; \alpha)$, $L(\tau; \beta)$ $(0 < \alpha < \beta < \pi)$ and lying in the set $\{z; |z| > \delta\}$, where δ, $0 < \delta < 1$, is sufficiently near to 1. In some cases, we use for $\Delta(\tau; \alpha,\beta,\delta)$ the short notation $\Delta(\tau; \alpha, \beta)$ without specifying δ, or the shorter notation $\Delta(\tau)$ without specifying α, β, δ.

Let $f(z)$ be a function defined in D and assuming values in W. We denote by $C_{\Delta(\tau;\alpha,\beta,\delta)}(f)$ the cluster set of $f(z)$ in $\Delta(\tau; \alpha, \beta, \delta)$, i.e., the set of points w of W such that there exists a sequence $\{z_n\}$, $z_n \to \tau$, $z_n \in \Delta(\tau; \alpha, \beta, \delta)$, satisfying $f(z_n) \to w$. We denote by $E_{\Delta\Delta}(f)$ the set of points $\tau \in C$ such that

$$C_{\Delta(\tau)}(f) \neq C_{\Delta'(\tau)}(f)$$

for some pair of two Stolz angles $\Delta(\tau)$ and $\Delta'(\tau)$.

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Let $f(z)$ be holomorphic in D. The set of all values $w \in W$ such that the equation $f(z) = w$ has infinitely many solutions in a Stolz angle $\Delta(\tau)$ having the vertex at $\tau \in C$ is called the range of $f(z)$ in $\Delta(\tau)$, and is denoted by $R_{\Delta(\tau)}(f)$. The angular range $\Lambda(f, \tau)$ of $f(z)$ at $\tau \in C$ is defined to be

$$\Lambda(f,\tau) = \bigcap_{\Delta(\tau)} R_{\Delta(\tau)}(f),$$

where the intersection is taken over all Stolz angles $\Delta(\tau)$. In case the complement of $\Lambda(f, \tau)$ with respect to W consists of at most one point, τ is said to be an *angular Picard point* of $f(z)$.

Let $f(z)$ be holomorphic in D. We shall call a point $\tau \in C$ a *Fatou point* of $f(z)$ with a Fatou value ∞, if $f(z)$ tends uniformly to ∞ in every Stolz angle having the vertex at τ.

Suppose a set $P \subset C$ and a point $\tau = e^{i\theta} \in C$ are given. For a number $\omega > 0$, we denote an arc $\{e^{i\theta'};\ \theta - \omega < \theta' < \theta + \omega\}$ by $\Gamma(\omega, \tau)$. Let $\gamma(\tau, \omega, P)$ be the largest of the lengths of arcs contained in $\Gamma(\omega, \tau)$ and not intersecting with P. The set P is *of porosity at* τ, if

$$\varlimsup_{\omega\to 0} \frac{1}{\omega}\gamma(\tau,\omega,P) > 0.$$

The set P is *of porosity on* C if it is so at each $\tau \in P$. A set which is a countable sum of sets of porosity on C is said to be of *σ-porosity on* C (for this definition, see) Dolzhenko (1967)). A set of σ-porosity on C is of the first Baire category on C. A set of σ-porosity on C has no points of density with respect to outer measure (i.e., no points of outer density), hence is of measure 0 [see Saks (1964; page 129, Theorem 10.2)]. But there exists a set, which is of measure 0 and not of σ-porosity on C [see Collingwood and Lohwater (1966; page 75)].

Let z', z'' be two points in D. We shall denote by $\rho(z', z'')$ the non-Euclidean (hyperbolic) distance between these two points:

$$\rho(z',z'') = \frac{1}{2}\log\frac{1+u}{1-u}, \quad u = \left|\frac{z'-z''}{1-\overline{z'}\cdot z''}\right|.$$

For a number ε, $0 < \varepsilon < 1$, and a point z', $z' \in D$, we shall denote by $D(z', \varepsilon)$ the open circular disc $\{z;\ |z-z'| < \varepsilon(1-|z'|)\}$ and shall denote by $D^*(z', \varepsilon)$ the open non-Euclidean circular disc with non-Euclidean center z' and non-Euclidean radius $\dfrac{1}{2}\log\dfrac{1+\varepsilon}{1-\varepsilon}$.

Let $f(z)$ be holomorphic in D. For a sequence $\{z_n\}$ of points in D, we shall denote the function

$$f\left(\frac{z+z_n}{1+\overline{z_n}\cdot z}\right)$$

by $f(z; z_n)$.

2. The main result

In this section, we shall prove some lemmas required for the proof of Theorem 1 and finally prove Theorem 1 which is the main result in this paper.

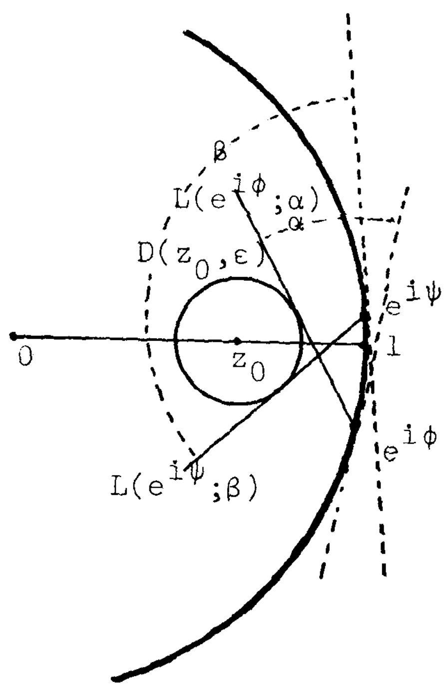
LEMMA 1. [See Dolzhenko (1967; Theorem 1)] *Let $w = f(z)$ be an arbitrary function defined in D and assuming values in W. Then the set $E_{\Delta\Delta}(f)$ is of σ-porosity on C.*

LEMMA 2. *Let P be a subset of C and P^* be the set of points $\tau \in C$ at which P is not of porosity. Suppose that three positive constants α, β and δ' satisfying $0 < \alpha < \beta < \pi$ and $0 < \delta' < 1$ are given.*

Let τ be a point of P^*. If we choose suitable positive numbers ε and δ, then for any z_0, $|z_0| > \delta$, on the radius terminating at τ, there exists a point $\tau' \in P$ depending on z_0 such that

$$\Delta(\tau'; \alpha, \beta, \delta') \supset D(z_0, \varepsilon).$$

PROOF. Without loss of generality, we may assume that $\tau = 1$.



Fix a number ε, $0 < \varepsilon < 1$. Let z_0 be a point on the radius terminating at 1. We denote by $e^{i\phi} = e^{i\phi(z_0)}$ (or $e^{i\psi} = e^{i\psi(z_0)}$) the point of C at which the segment $L(e^{i\phi}; \alpha)$ (or $L(e^{i\psi}; \beta)$), tangent to the disc $D(z_0, \varepsilon)$ from right (or left), terminates. If we choose a number ε, $0 < \varepsilon < 1$, satisfying

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$$(1)\qquad \begin{cases} \varepsilon < \cos(\alpha) & \left(\text{in the case } \alpha < \beta \leqq \dfrac{\pi}{2}\right) \\ \varepsilon < \cos(\alpha),\ \varepsilon < -\cos(\beta) & \left(\text{in the case } \alpha < \dfrac{\pi}{2} < \beta\right) \\ \varepsilon < -\cos(\beta) & \left(\text{in the case } \dfrac{\pi}{2} \leqq \alpha < \beta\right), \end{cases}$$

then we have from easy estimations,

$$(2)\qquad \begin{cases} \phi(z_0) = \left(\dfrac{\varepsilon - \cos(\alpha)}{\sin(\alpha)} + o(1)\right)(1-|z_0|) \quad (|z_0| \to 1) \\ \psi(z_0) = \left(\dfrac{-\varepsilon - \cos(\beta)}{\sin(\beta)} + o(1)\right)(1-|z_0|) \quad (|z_0| \to 1), \end{cases}$$

and hence we have

$$(3)\qquad \psi(z_0) - \phi(z_0) = \left(\frac{\sin(\beta-\alpha) - \varepsilon(\sin(\alpha)+\sin(\beta))}{\sin(\alpha)\cdot\sin(\beta)} + o(1)\right)(1-|z_0|) \quad (|z_0| \to 1).$$

Here, if we choose a number ε, $0 < \varepsilon < 1$, satisfying (1) and

$$\sin(\beta-\alpha) > \varepsilon(\sin(\alpha)+\sin(\beta)),$$

and we choose a number δ'', $0 < \delta'' < 1$, sufficiently near to 1, then we obtain from (3)

$$(4)\qquad \begin{cases} \psi(z_0) - \phi(z_0) > K(1-|z_0|) \\ \text{for any } z_0 \text{ satisfying } |z_0| > \delta'', \text{ on the radius} \\ \text{terminating at 1, where } K \text{ is a positive} \\ \text{constant independent of } z_0. \end{cases}$$

Further, we choose a number δ, $\delta'' < \delta < 1$, such that

$$D(z_0, \varepsilon) \subset \{z; |z| > \delta'\}, \text{ i.e., } |z_0|(1+\varepsilon) - \varepsilon > \delta'$$

for any z_0 satisfying $|z_0| > \delta$, on the radius terminating at 1.

Now, we suppose that Lemma 2 were false for ε and δ chosen above. Then, there exists a sequence $\{z_n\}$, $|z_n| > \delta$ $(n = 1, 2, 3, ---)$, $z_n \to 1$ $(n \to \infty)$, on the radius terminating at 1, such that

(5) the arc $\{e^{i\theta}; \phi(z_n) < \theta < \psi(z_n)\}$ contains no point of P.

For this sequence $\{z_n\}$, we have from (4)

$$(6)\qquad \psi(z_n) - \phi(z_n) > K(1 - |Z_n|).$$

Here, if we set

$$\omega_n = \max(|\psi(z_n)|, |\phi(z_n)|),$$

we have from (2)

$$\omega_n \leqq \left(\frac{2}{\min(\sin(\alpha), \sin(\beta))} + o(1)\right)(1 - |z_n|) \qquad (n \to \infty) \tag{7}$$

and from (5), (6)

$$\gamma(1, \omega_n, P) \geqq \psi(z_n) - \phi(z_n) > K(1 - |z_n|). \tag{8}$$

Thus, we have by (7) and (8)

$$\overline{\lim_{\omega \to 0}} \frac{\gamma(1, \omega, P)}{\omega} \geqq \overline{\lim_{n \to \infty}} \frac{\gamma(1, \omega_n, P)}{\omega_n} \geqq \frac{K}{2/\min(\sin(\alpha), \sin(\beta))} > 0.$$

This shows that the set P is of porosity at $\tau = 1$, and this contradicts the assumption $1 \in P^*$.

Lemma 3. [See Dragosh (1972; Lemma 3.)] *A family $\{g_n(z)\}$ of meromorphic functions in D is not normal at $z = 0$ if and only if, for each sequence*

$$\{\varepsilon_p\}, \ 0 < \varepsilon_p < 1, \ \lim_{p \to \infty} \varepsilon_p = 0,$$

$\{g_n(z)\}$ contains a subsequence $\{g_{n_p}(z)\}$ such that the image of $|z| < \varepsilon_p$ under $g_{n_p}(z)$ covers the Riemann sphere Ω with the possible exception of two sets each having spherical diameter less than ε_p.

Lemma 4. *Let z' be a point in D and ε be a number satisfying $0 < \varepsilon < \frac{1}{3}$. Then, we have*

$$D^*(z', \varepsilon) \subset D(z', \sqrt{3\varepsilon}).$$

Proof. Consider the linear transformation $z = (t + z')/(1 + \overline{z'} \cdot t)$ from $|t| < 1$ to $|z| < 1$. Then, since $D^*(z', \varepsilon)$ is the image of the set $\{t; |t| < \varepsilon\}$ by $z = (t + z')/(1 + \overline{z'} \cdot t)$, any z, $z \in D^*(z', \varepsilon)$, satisfies the inequality

$$|z - z'| = \frac{1 - |z'|^2}{|1 + \bar{z}' \cdot t|} |t| \leqq (1 - |z'|) \frac{2|t|}{1 - |t|} \leqq \sqrt{3|t|} \cdot (1 - |z'|).$$

This fact proves Lemma 4.

Lemma 5. *Let $f(z)$ be holomorphic in D. Let $\{z_n\}$ be a sequence of points in D satisfying $\lim_{n \to \infty} |z_n| = 1$. Suppose that the family $\{f(z; z_n)\}$ for this sequence $\{z_n\}$ is not normal at $z = 0$. Then, for each sequence $\{\varepsilon_p\}$, $0 < \varepsilon_p < \frac{1}{3}$, $\lim_{p \to \infty} \varepsilon_p = 0$, there exists a subsequence $\{z_{n_p}\}$ in the sequence $\{z_n\}$, such that the image of $D(z_{n_p}, \sqrt{3 \cdot \varepsilon_p})$ under $f(z)$ covers the Riemann sphere Ω with the possible exception of two sets each having spherical diameter less than ε_p.*

REMARK 1. Since $f(z)$ is holomorphic in D, there is always an exceptional set containing ∞ and having spherical diameter less than ε_p.

PROOF. From Lemma 3, we can see that for each sequence $\{\varepsilon_p\}$, $0 < \varepsilon_p < \frac{1}{3}$, $\lim_{p\to\infty} \varepsilon_p = 0$, there exists a subsequence $\{z_{n_p}\}$ in the sequence $\{z_n\}$ such that the image of $D^*(z_{n_p}, \varepsilon_p)$ under $f(z)$ covers the Riemann sphere Ω with the possible exception of two sets each having spherical diameter less than ε_p.

On the other hand, from Lemma 4, we have

$$D^*(z_{n_p}, \varepsilon_p) \subset D(z_{n_p}, \sqrt{3 \cdot \varepsilon_p})$$

for each pair (z_{n_p}, ε_p), and hence the image of $D^*(z_{n_p}, \varepsilon_p)$ under $f(z)$ is contained in the image of $D(z_{n_p}, \sqrt{3 \cdot \varepsilon_p})$ under $f(z)$. Thus, the image of $D(z_{n_p}, \sqrt{3 \cdot \varepsilon_p})$ under $f(z)$ covers the Riemann sphere Ω with the possible exception of two sets each having spherical diameter less than ε_p. This fact proves Lemma 5.

Now, we can prove Theorem 1.

THEOREM 1. *Let $f(z)$ be holomorphic in D. Let $E(f)$ be the set of points τ on C such that for each point $\tau \in E(f)$, there exists a sequence $\{z_n\}$, $\lim_{n\to\infty} |z_n| = 1$, on the radius terminating at τ, for which*

$$\rho(z_n, z_{n+1}) < M \qquad (n = 1, 2, 3, \cdots)$$

where $M = M(\tau)$ is a positive constant which may depend on τ, and

$$\lim_{n\to\infty} f(z_n) = \infty.$$

Then, except for a set of σ-porosity on C, every point of $E(f)$ is either a Fatou point of $f(z)$ with a Fatou value ∞ or an angular Picard point of $f(z)$.

PROOF. Let τ be any point of $E(f)$. We take a sequence $\{z_n\}$, $\lim_{n\to\infty} |z_n| = 1$, on the radius terminating at τ, for which

$$\rho(z_n, z_{n+1}) < M \qquad (n = 1, 2, 3, \cdots)$$

and

$$\lim_{n\to\infty} f(z_n) = \infty.$$

Considering the family $\{f(z; z_n)\}$ for this sequence $\{z_n\}$, the following three mutually exclusive cases can be considered:

1. The family $\{f(z; z_n)\}$ is normal in D;
2. The family $\{f(z; z_n)\}$ is not normal in D, but is normal at $z = 0$;
3. The family $\{f(z; z_n)\}$ is not normal at $z = 0$.

Faust (1962; pages 99–100) showed that

$$\lim_{r\to 1} f(r\tau) = \infty,$$

for each τ at which the case 2 happens. And hence, for each τ at which the case 2 happens, we can choose a sequence $\{z_n^1\}$, $\lim_{n\to\infty}|z_n^1| = 1$, on the radius terminating at τ, for which

$$\lim_{n\to\infty} \rho(z_n^1, z_{n+1}^1) = 0$$

and

$$\lim_{n\to\infty} f(z_n^1) = \infty.$$

For this sequence $\{z_n^1\}$, we have the following three mutually exclusive cases:

1′. The family $\{f(z; z_n^1)\}$ is normal in D;
2′. The family $\{f(z; z_n^1)\}$ is not normal in D, but is normal at $z = 0$;
3′. The family $\{f(z; z_n^1)\}$ is not normal at $z = 0$.

Thus, for each point $\tau \in E(f)$, one of the following three cases happens:

1″. There exists a sequence $\{z_n^2\}$, $\lim_{n\to\infty}|z_n^2| = 1$, on the radius terminating at τ, for which

$$\rho(z_n^2, z_{n+1}^2) < M'$$

where $M' = M'(\tau)$ is a positive constant which may depend on τ, and

$$\lim_{n\to\infty} f(z_n^2) = \infty,$$

such that the family $\{f(z; z_n^2)\}$ is normal in D. The set of these points τ of $E(f)$ will be denoted by $G(f)$.

2″. There exists a sequence $\{z_n^3\}$, $\lim_{n\to\infty}|z_n^3| = 1$, on the radius terminating at τ, for which

$$\lim_{n\to\infty} \rho(z_n^3, z_{n+1}^3) = 0,$$

and

$$\lim_{n\to\infty} f(z_n^3) = \infty,$$

such that the family $\{f(z; z_n^3)\}$ is not normal in D, but is normal at $z = 0$. The set of these points τ of $E(f)$ will be denoted by $H(f)$.

3″. There exists a sequence $\{z_n^4\}$, $\lim_{n\to\infty}|z_n^4| = 1$, on the radius terminating at τ, such that the family $\{f(z; z_n^4)\}$ is not normal at $z = 0$. The set of these points τ of $E(f)$ will be denoted by $J(f)$.

First, according to Seidel (1959; pages 167–168) and Faust (1962; page 99],

(9) every point $\tau \in G(f)$ is a Fatou point of $f(z)$ with a Fatou value ∞.

Next, according to Seidel (1959; page 169), at each point $\tau \in H(f)$, there exists a positive number c_τ, $0 < c_\tau < \pi/2$, such that $f(z)$ tends uniformly to ∞ as $z \to \tau$ in every Stolz angle

$$\Delta\left(\tau, \frac{\pi}{2} - c_\tau + \varepsilon, \frac{\pi}{2} + c_\tau - \varepsilon\right), \text{ with } 0 < \varepsilon < c_\tau$$

and the complement of

$$R_{\Delta(\tau, \pi/2 - c_\tau - \varepsilon, \pi/2 + c_\tau + \varepsilon, \delta)}(f)$$

with respect to W consists of at most one point for every ε, $0 < \varepsilon < \pi/2 - c_\tau$, and every δ, $0 < \delta < 1$. Thus, we see that $H(f)$ is contained in $E_{\Delta\Delta}(f)$, and hence

(10) $H(f)$ is of σ-porosity on C.

Further, we shall prove that except for a set of σ-porosity on C, every point of the set $J(f)$ is an angular Picard point of $f(z)$.

Let $\{\alpha_i\}$ (or $\{\beta_j\}$) be a sequence of all rational numbers satisfying

$$0 < \alpha_i < \pi \quad (\text{or } 0 < \beta_j < \pi),$$

and $\{\delta_k\}$ be a sequence of all rational numbers satisfying $0 < \delta_k < 1$. Let $\{Q_p\}_{p=1}^{\infty}$ denote a basis, which consists of closed discs on W.

We denote by $P(f)$ the set of points $\tau \in C$ which are not angular Picard points of $f(z)$. Then, at each $\tau \in P(f)$, there exist two disjoint closed discs Q_1, Q_2 on W and two Stolz angles $\Delta^1(\tau)$, $\Delta^2(\tau)$ such that $f(z)$ omits a value from Q_ν in $\Delta^\nu(\tau)$ $(\nu = 1, 2)$. Here, for positive integers $i_1, i_2, j_1, j_2, p_1, p_2, k_1$ and k_2, we denote by

$$P(i_1, i_2, j_1, j_2, p_1, p_2, k_1, k_2)(f)$$

the set of points $\tau \in P(f)$ such that $f(z)$ omits a value from Q_{p_ν} in the Stolz angle $\Delta(\tau; \alpha_{i_\nu}, \beta_{j_\nu}, \delta_{k_\nu})$ $(\nu = 1, 2)$. Then, we have

$$P(f) = \bigcup_{i_1, i_2, j_1, j_2, p_1, p_2, k_1, k_2} P(i_1, i_2, j_1, j_2, p_1, p_2, k_1, k_2)(f).$$

We denote by

$$P^*(i_1, i_2, j_1, j_2, p_1, p_2, k_1, k_2)(f)$$

the set of points $\tau \in C$ at which

$$P(i_1, i_2, j_1, j_2, p_1, p_2, k_1, k_2)(f)$$

is not of porosity. Then, at each point of the set

$$P(i_1, i_2, j_1, j_2, p_1, p_2, k_1, k_2)(f) - P^*(i_1, i_2, j_1, j_2, p_1, p_2, k_1, k_2)(f),$$

the set

$$P(i_1, i_2, j_1, j_2, p_1, p_2, k_1, k_2)(f)$$

is of porosity, and hence the set

$$P(i_1, i_2, j_1, j_2, p_1, p_2, k_1, k_2)(f) - P^*(i_1, i_2, j_1, j_2, p_1, p_2, k_1, k_2)(f)$$

is of porosity, too. Thus, we see that the set

$$(11) \qquad P(i_1, i_2, j_1, j_2, p_1, p_2, k_1, k_2)(f) - P^*(i_1, i_2, j_1, j_2, p_1, p_2, k_1, k_2)(f)$$

is a set of porosity on C.

Now, we shall show presently that

$$(12) \qquad P^*(i_1, i_2, j_1, j_2, p_1, p_2, k_1, k_2)(f) \cap J(f) = \phi$$

for any combination $(i_1, i_2, j_1, j_2, p_1, p_2, k_1, k_2)$. If (12) is proved, then we shall have

$$\begin{aligned} P(f) \cap J(f) &= \Bigg(\bigcup_{i_1, i_2, j_1, j_2, p_1, p_2, k_1, k_2} (P(i_1, i_2, j_1, j_2, p_1, p_2, k_1, k_2)(f) \\ &\qquad - P^*(i_1, i_2, j_1, j_2, p_1, p_2, k_1, k_2)(f)) \Bigg) \cap J(f) \\ &\subset \bigcup_{i_1, i_2, j_1, j_2, p_1, p_2, k_1, k_2} (P(i_1, i_2, j_1, j_2, p_1, p_2, k_1, k_2)(f) \\ &\qquad - P^*(i_1, i_2, j_1, j_2, p_1, p_2, k_1, k_2)(f)). \end{aligned}$$

With (11), this fact will show that $P(f) \cap J(f)$ is a set of σ-porosity on C, and hence

(13) except for a set of σ-porosity on C, every point of the set $J(f)$ is an angular Picard point of $f(z)$.

Thus, since $E(f) = G(f) \cup H(f) \cup J(f)$, we obtain the conclusion of Theorem 1 from (9), (10) and (13).

Now, it remains to show (12). Suppose that

$$P^*(i_1, i_2, j_1, j_2, p_1, p_2, k_1, k_2)(f) \cap J(f) \neq \phi$$

and let τ be a point of the set

$$P^*(i_1, i_2, j_1, j_2, p_1, p_2, k_1, k_2)(f) \cap J(f).$$

For each $\nu = 1, 2$, we put $\alpha = \alpha_{i_\nu}$, $\beta = \beta_{j_\nu}$, $\delta = \delta_{k_\nu}$ in Lemma 2. Then, from Lemma 2, by the fact

$$\tau \in P^*(i_1, i_2, j_1, j_2, p_1, p_2, k_1, k_2)(f),$$

we can choose suitable positive numbers ε_ν, δ_ν ($\nu = 1, 2$), such that for any z_0, $|z_0| > \delta_\nu$, on the radius terminating at τ, there exists a point

$$\tau' = \tau'(z_0) \in P(i_1, i_2, j_1, j_2, p_1, p_2, k_1, k_2)(f)$$

satisfying

$$\Delta(\tau'; \alpha_{i_\nu}, \beta_{j_\nu}, \delta_{k_\nu}) \supset D(z_0, \varepsilon_\nu) \qquad (\nu = 1, 2).$$

Thus, if we set $\varepsilon_3 = \min(\varepsilon_1, \varepsilon_2)$ and $\delta_3 = \max(\delta_1, \delta_2)$, $f(z)$ omits two values $w_1(z_0)$, $w_2(z_0)$, $w_\nu(z_0) \in Q_{p_\nu}$ ($\nu = 1, 2$) in $D(z_0, \varepsilon_3)$ with each z_0, $|z_0| > \delta_3$, on the radius terminating at τ. And there is a positive constant η, independent of $w_1(z_0)$, $w_2(z_0)$, ∞, such that

$$(14) \qquad \chi(w_1(z_0), \infty) > \eta, \ \chi(w_2(z_0), \infty) > \eta, \ \chi(w_1(z_0), w_2(z_0)) > \eta,$$

where $\chi(w_1, w_2)$ denotes the spherical metric between w_1 and w_2. Thus, $f(z)$ omits three values $w_1(z_0)$, $w_2(z_0)$, ∞ satisfying (14), in $D(z_0, \varepsilon_3)$ with each z_0, $|z_0| > \delta_3$, on the radius terminating at τ. But, since $\tau \in J(f)$, there exists a sequence $\{z_n^4\}$, $\lim_{n\to\infty}|z_n^4| = 1$, on the radius terminating at τ, such that the family $\{f(z; z_n^4)\}$ is not normal at $z = 0$. This contradicts the fact obtained from Lemma 5. Hence, we get (12), i.e.,

$$P^*(i_1, i_2, j_1, j_2, p_1, p_2, k_1, k_2)(f) \cap J(f) = \phi.$$

3. Some applications

Let $\zeta(t)$ be a continuous, complex-valued function for $0 \leqq t < \infty$ with the properties:

$$0 < |\zeta(t)| < 1, \ \lim_{t\to\infty} |\zeta(t)| = 1, \ \lim_{t\to\infty} \arg(\zeta(t)) = \infty.$$

A simple curve in D defined by $z = \zeta(t)$ is called a *spiral* and is denoted by S. Here, for any value of t, starting with the point $\zeta(t)$ on a spiral S, describe the curve $z = \zeta(t)$ in the sense of increasing t and let t' denote the first value of t for which

$$\arg(\zeta(t')) = \arg(\zeta(t)) + 2\pi.$$

We shall introduce the following measure for the tightness of a spiral S [see Seidel (1959; page 160)]:

$$\bar{\mu}(S) = \lim_{t\to\infty} \rho(\zeta(t), \zeta(t')).$$

A function $f(z)$ holomorphic in D is called a *spiral function* in D relative to a spiral S: $z = \zeta(t)$, $0 \leqq t < \infty$, if

$$\lim_{t\to\infty} f(\zeta(t)) = \infty.$$

A sequence $\{J_n\}$ of Jordan curves J_n in D which satisfy:

1. J_n is contained in the interior of J_{n+1},
2. $\min_{z\in J_n} |z| \to 1$ as $n \to \infty$,

is called an *annular sequence* in D. Here, let τ be any point of C and τ_n be the first intersection point of the radius terminating at τ with J_n. We shall introduce the following measure for the tightness of an annular sequence $\{J_n\}$:

$$\bar{\mu}(\{J_n\}) = \sup_{\tau\in C} \overline{\lim_{n\to\infty}} \rho(\tau_n, \tau_{n+1}),$$

where $\sup_{\tau \in C}$ means the supremum for all $\tau \in C$. A function $f(z)$ holomorphic in D is called an ***annular function*** in D relative to $\{J_n\}$ [see Bagemihl and Erdös (1964)]; if there exists an annular sequence $\{J_n\}$ in D satisfying

$$\operatorname*{Min}_{z \in J_n} |f(z)| \to \infty \text{ as } n \to \infty.$$

THEOREM 2. *Let $f(z)$ be a spiral function in D relative to a spiral S satisfying*

$$\bar{\mu}(S) < \infty.$$

Then, except for a set of σ-porosity on C, every point of C is either a Fatou point with a Fatou value ∞ or an angular Picard point of $f(z)$.

REMARK 2. This is an improvement of Seidel (1959; Theorem 5).

PROOF. From the assumption $\bar{\mu}(S) < \infty$, for each point τ of C, we can find a sequence $\{z_n\}$, $\lim_{n \to \infty} |z_n| = 1$, on the radius terminating at τ, for which

$$\rho(z_n, z_{n+1}) < M \qquad (n = 1, 2, 3, \cdots)$$

where M is a positive constant, and

$$\lim_{n \to \infty} f(z_n) = \infty.$$

Hence, Theorem 2 is easily obtained from Theorem 1.

COROLLARY 1. *Let $f(z)$ be a spiral function in D relative to a spiral S satisfying*

$$\bar{\mu}(S) < \infty.$$

Then, almost all points of C are angular Picard points of $f(z)$.

PROOF. By Lusin-Privaloff's theorem [see Tsuji (1959; Theorem VIII.28)], the set of Fatou points having a Fatou value ∞ is of measure 0. A set of σ-porosity on C is also of measure 0. Hence, Corollary 1 is easily obtained from Theorem 2.

THEOREM 3. *Let $f(z)$ be an annular function in D relative to an annular sequence $\{J_n\}$ satisfying*

$$\bar{\mu}(\{J_n\}) < \infty.$$

Then, except for a set of σ-porosity on C, every point of C is either a Fatou point having a Fatou value ∞ or an angular Picard point of $f(z)$.

PROOF. Theorem 3 follows from Theorem 1 by the same reason as Theorem 2.

REMARK 3. Since $f(z)$ is holomorphic in D, there exists an asymptotic path along which $f(z)$ tends to ∞. Hence, we can deduce Theorem 3 also from Theorem

2, if we make a spiral S satisfying $\bar{\mu}(S) < \infty$ from this asymptotic path and $\{J_n\}$ satisfying $\bar{\mu}(\{J_n\}) < \infty$.

COROLLARY 2. *Let $f(z)$ be an annular function in D relative to an annular sequence $\{J_n\}$ satisfying*

$$\bar{\mu}(\{J_n\}) < \infty.$$

Then, almost all points of C are angular Picard points of $f(z)$.

PROOF. Corollary 2 follows from Theorem 3 by the same reason as Corollary 1.

REMARK 4. At an angular Picard point τ of C, the complement of $\Lambda(f, \tau)$ with respect to W may not be empty. In fact, there exists an annular function $w(z)$ in D relative to an annular sequence $\{J_n\}$ satisfying

$$\bar{\mu}(\{J_n\}) < \infty,$$

such that

$$\Lambda(f, \tau) \not\ni 0 \text{ at every point } \tau \text{ of } C.$$

See the example in Barth and Schneider (1969), where a sequence $\{s_n\}$ satisfying

$$\frac{1}{2} \cdot \log \frac{1 - s_n}{1 - s_{n+1}} < M \qquad (M \text{ is a constant})$$

has to been chosen.

Further, by the same reason as Remark 3, this example give the analogous example concerning Corollary 1.

THEOREM 4. *There exists a spiral function (or an annular function) $\Psi(z)$ in D, whose maximum modulus tends to ∞ as slowly as one wishes, with the property that almost all points of C are angular Picard points of $\Psi(z)$.*

REMARK 5. This is an improvement of Seidel (1959; Theorem 7).

PROOF. It is evident that the function $\Psi(z)$ in Seidel (1959; pages 165–166) satisfies our condition.

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