A GENERALIZATION OF A THEOREM OF HILTON

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Let $f: A \times B \to X$ be a map. Let $J(f): \Sigma(A \wedge B) \to \Sigma X$ be the map obtained from f by means of the Hopf construction. Let $P(f)$ denote the space obtained from ΣX by attaching a cone on $\Sigma(A \wedge B)$ by means of $J(f)$. Let $\ell: \Sigma X \to P(f)$ be the inclusion and $\tau(\ell): X \to \Omega P(f)$ the adjoint of ℓ. Let $h_1: A_1 \to A$, $h_2: B_1 \to B$ be maps. Let $c: \Omega P(f) \times \Omega P(f) \to \Omega P(f)$ be the basic commutator. Then we prove that there exists a map $\Sigma A_1 \times \Sigma B_1 \to P(f)$ with axes $\ell\Sigma(fi_1h_1)$, $\ell\Sigma(fi_2h_2)$ if and only if $c(\tau(\ell)fi_1h_1 \times \tau(\ell)fi_2h_2) \simeq *$, where $i_1: A \to A \times B$ and $i_2: B \to A \times B$ are the inclusions. This generalizes a result of Hilton. Also, by letting f be an H-space multiplication and h_1 and h_2 the identity maps, we obtain a well known criterion of Stasheff for an H-space to be homotopy-commutative. Finally, appropriate duals of these results are given.

We will work in the category of spaces with base point and having the homotopy type of countable CW-complexes. All maps and homotopies are to respect base points. For simplicity, we shall frequently use the same symbol for a map and its homotopy class. Given spaces X, Y we denote the set of homotopy classes of maps from X to Y by $[X, Y]$. We also have an isomorphism $\tau: [\Sigma X, Y] \to [X, \Omega Y]$.

Given a map $f: X \times Y \to Z$, the axes of f are the maps $fi_1: X \to Z$, $fi_2: Y \to Z$ where i_1, i_2 are the imbeddings of X, Y in $X \times Y$. Let $g: X \to Z$, $h: Y \to Z$ be maps. We observe that there exists a map $f: X \times Y \to Z$ with g and h as its axes if and only if $\nabla(g \vee h): X \vee Y \to Z$ extends to $X \times Y$ where $\nabla: Z \vee Z \to Z$ is the folding map.

1. Let $f: A \times B \to X$ be a map and let Y be an H-space with multiplication $\phi: Y \times Y \to Y$. A map $g: X \to Y$ is said to be primitive with respect to f if the following diagram homotopy-commutes

$$\begin{tikzcd} A \times B \arrow[r, "f"] \arrow[d, "gfi_1 \times gfi_2"'] & X \arrow[d, "g"] \\ Y \times Y \arrow[r, "\phi"'] & Y \end{tikzcd}$$

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where $i_1 : A \to A \times B$, $i_2 : B \to A \times B$ are the imbeddings. It is easily checked that this means that $gf = gfi_1\pi_1 + gfi_2\pi_2$ in $[A \times B, Y]$ where $\pi_1 : A \times B \to A$, $\pi_2 : A \times B \to B$ are the projections. If X is also an H-space and f is the multiplication on X, then primitivity of g with respect to f merely means that g is an H-map.

We recall that in [1], Arkowitz defined a generalized Whitehead product $[\ ,\ ] : [\Sigma A, X] \times [\Sigma B, X] \to [\Sigma(A \wedge B), X]$. Suppose X is an H-space. Then in [2], he defined a generalized Samelson product $<,> : [A, X] \times [B, X] \to [A \wedge B, X]$. These operations are related in the following way. Suppose α is an element of $[\Sigma A, X]$ and β is an element of $[\Sigma B, X]$ where A, B, X are any three spaces. Then $\tau[\alpha, \beta] = <\tau(\alpha), \tau(\beta)>$.

Before stating our results, we need one more construction, namely the Hopf construction. We can either consider this as well-known or refer the reader to [8] where this was defined and used. However, for the sake of completeness, we shall define our version of this construction briefly here. Let A and B be spaces. We consider $A \vee B \xrightarrow{j} A \times B \xrightarrow{q} A \wedge B$ as a cofibration where $A \wedge B$ is the smashed product, and j is the usual inclusion of the wedge product in the cartesian product. Then one can show that there exists a map $p : \Sigma(A \times B) \to \Sigma(A \vee B)$ such that $p(\Sigma j) \simeq 1_{\Sigma(A \vee B)}$. In fact, let $i_1 : A \to A \vee B$, $i_2 : B \to A \vee B$ be the inclusions, and let $\pi_1 : A \times B \to A$, $\pi_2 : A \times B \to B$ be the projections. Let $p_1 = i_1\pi_1$, $p_2 = i_2\pi_2$. Then we can and shall take $p = \nabla(\Sigma p_1 \vee \Sigma p_2)\phi'$ where $\phi' : \Sigma(A \times B) \to \Sigma(A \times B) \vee \Sigma(A \times B)$ is the suspension structure, and ∇ is the folding map. The exact sequence of the cofibration now shows that $(\Sigma q)^{\#}$ is a monomorphism. Hence there exists a unique element $[d]$ of $[\Sigma(A \wedge B), \Sigma(A \times B)]$ satisfying the relation $1_{\Sigma(A \times B)} = d(\Sigma q) + (\Sigma j)p$. Given a space X and a map $f : A \times B \to X$, we define a map $J(f) : \Sigma(A \wedge B) \to \Sigma X$ by $J(f) = (\Sigma f)d$. We call $J(f)$ the map obtained from f by the Hopf construction. It is the unique element satisfying the relation $\Sigma f = J(f)\,\Sigma q + \Sigma(fj)p$. If $f_1 : X \to Y$ is another map, then $J(f_1 f) = (\Sigma f_1)\, J(f)$.

Given $f : A \times B \to X$, let $P(f) = \Sigma X \underset{J(f)}{\cup} C\Sigma(A \wedge B)$, that is, $P(f)$ is the cofibre of the cofibration $J(f) : \Sigma(A \wedge B) \to \Sigma X$. If $\phi : X \times X \to X$ is an H-space multiplication, then $P(\phi)$ is the projective plane of the H-space. In general, if $f : A \times B \to X$ is a map, let $\ell : \Sigma X \to P(f)$ denote the inclusion. Now let Y be a space and $g : X \to \Omega Y$ a map. Then in [5], Hilton proved the following theorem.

THEOREM 1. $g : X \to \Omega Y$ is primitive with respect to f if and only if $\tau^{-1}(g) : \Sigma X \to Y$ extends to $P(f)$.

In particular, it can be checked that $\tau(\ell) : X \to \Omega P(f)$ is always primitive with respect to f. Thus if $\phi : X \times X \to X$ is an H-space multiplication, then $\tau(\ell) : X \to \Omega P(\phi)$ is always an H-map. Now let A_1, B_1 be spaces and $h_1 : A_1 \to A$, $h_2 : B_1 \to A$ be maps. Let $T : A \times A \to A \times A$ be the switching map. Let $f : A \times A \to X$ be a map. Then in [6], Hilton proved the following theorem.

THEOREM 2. If $\tau(\ell) f (h_1 \times h_2) \simeq \tau(\ell) f T(h_1 \times h_2)$, then there exists a map $\Sigma A_1 \times \Sigma B_1 \to P(f)$ with axes $\ell\, \Sigma(fi_1h_1)$, $\ell\, \Sigma(fi_2h_2)$ where $i_1 : A \to A \times A$, $i_2 : A \to A \times A$ are the inclusions in the first and second coordinates respectively.

This is the theorem we wish to generalize. We shall show that it is not necessary to assume that $A = B$, that is, we may assume that f is a map $f : A \times B \to X$ for any spaces A, B. In order to do this, we cannot, of course, use the switching map T. We shall give another formulation which will imply Theorem 2 in case $A = B$. In this reformulation, the argument will be reversible, so that we actually obtain an "if and only if" result.

We first state and prove our result and then show how it implies Theorem 2. Let $f : A \times B \to X$ be a map, and let $h_1 : A_1 \to A$, $h_2 : B_1 \to B$ be maps, where A_1, B_1 are any spaces. Let $i_1 : A \to A \times B$, $i_2 : B \to A \times B$ be the inclusions and $\ell : \Sigma X \to P(f)$ the usual inclusion. Then $\tau(\ell) : X \to \Omega P(f)$ is primitive with respect to f. Let $c : \Omega P(f) \times \Omega P(f) \to \Omega P(f)$ be the basic commutator.

THEOREM 3. There exists a map $\Sigma A_1 \times \Sigma B_1 \to P(f)$ with axes $\ell\, \Sigma(fi_1h_1)$, $\ell\, \Sigma(fi_2h_2)$ if and only if $c(\tau(\ell)\, fi_1h_1 \times \tau(\ell)\, fi_2h_2) \simeq *$.

Proof. Let us consider the map $\nabla\{\ell\, \Sigma(fi_1h_1) \vee \ell\, \Sigma(fi_2h_2)\}$. $\Sigma A_1 \vee \Sigma B_1 \to P(f)$. In order to obtain the required map with the prescribed axes, we need to show that this map extends to $\Sigma A_1 \times \Sigma B_1$. Let $k_1 : \Sigma A_1 \to \Sigma A_1 \vee \Sigma B_1$, $k_2 : \Sigma B_1 \to \Sigma A_1 \vee \Sigma B_1$ be the inclusions. Then we have the generalized Whitehead product $k = [k_1, k_2] : \Sigma(A_1 \wedge B_1) \to \Sigma A_1 \vee \Sigma B_1$. We may consider $\Sigma(A_1 \wedge B_1) \xrightarrow{k} \Sigma A_1 \vee \Sigma B_1 \to (\Sigma A_1 \vee \Sigma B_1) \cup_k C\, \Sigma(A_1 \wedge B_1)$ as a cofibration, if necessary by replacing the situation by a homotopically equivalent situation, using standard constructions of homotopy theory. Then clearly $\nabla\{\ell\,\Sigma(fi_1h_1)\vee\ell\,\Sigma(fi_2h_2)\}$ extends to $(\Sigma A_1\vee\Sigma B_1)\bigcup_k C\,\Sigma(A_1\wedge B_1)$ if and only if $\nabla\{\ell\,\Sigma(fi_1h_1)\vee\ell\,\Sigma(fi_2h_2)\}\,[k_1,k_2]\simeq *$, that is, if and only if $[\ell\,\Sigma(fi_1h_1),\ \ell\,\Sigma(fi_2h_2)]=0$. Now $\tau[\ell\,\Sigma(fi_1h_1),\ \ell\,\Sigma(fi_2h_2)]=$ $<\tau(\ell)\,fi_1h_1,\ \tau(\ell)\,fi_2h_2>$ and $q^{\#}<\tau(\ell)\,fi_1h_1,\ \tau(\ell)\,fi_2h_2>=c(\tau(\ell)fi_1h_1\times\tau(\ell)\,fi_2h_2)$ where $c:\Omega P(f)\times\Omega P(f)\to\Omega P(f)$ is the basic commutator and $q:A_1\times B_1\to A_1\wedge B_1$ is the projection onto the smashed product. Since τ is an isomorphism, and $q^{\#}$ is a monomorphism by Lemma 4.1 of [4], it follows that $\nabla\{\ell\,\Sigma(fi_1h_1)\vee\ell\,\Sigma(fi_2h_2)\}$ extends to $(\Sigma A_1\vee\Sigma B_1)\bigcup_k C\,\Sigma(A_1\wedge B_1)$ if and only if $c(\tau(\ell)\,fi_1h_1\times\tau(\ell)fi_2h_2)\simeq *$. Now, by Corollary 4.3 of [1], $(\Sigma A_1\vee EB_1)\bigcup_k C\,\Sigma(A_1\wedge B_1)$ is homotopically equivalent to $\Sigma A_1\times\Sigma B_1$. This proves the theorem.

Remark 1. Suppose $A=B$ and $T:A\times A\to A\times A$ is the switching map. Suppose $\tau(\ell)\,f(h_1\times h_2)\simeq\tau(\ell)\,f\,T(h_1\times h_2)$. Since $\tau(\ell)$ is primitive with respect to f, we have $\tau(\ell)\,f=\tau(\ell)\,fi_1\pi_1+\tau(\ell)\,fi_2\pi_2$. Hence $\tau(\ell)\,f(h_1\times h_2)=\tau(\ell)fi_1h_1\pi_1+\tau(\ell)\,fi_2h_2\pi_2$, and $\tau(\ell)\,f\,T(h_1\times h_2)=$ $\tau(\ell)\,fi_1\pi_1\,T(h_1\times h_2)+\tau(\ell)\,fi_2\pi_2\,T(h_1\times h_2)=\tau(\ell)fi_1h_2\pi_2+\tau(\ell)\,fi_2h_1\pi_1$. Thus $\tau(\ell)\,f\,T(h_1\times h_2)\,i_1=\tau(\ell)\,fi_2h_1$ and $\tau(\ell)\,f(h_1\times h_2)\,i_1=\tau(\ell)\,fi_1h_1$. Since $\tau(\ell)\,f(h_1\times h_2)\simeq\tau(\ell)\,f\,T(h_1\times h_2)$, we have $\tau(\ell)\,fi_1h_1\simeq\tau(\ell)\,fi_2h_1$. Similarly $\tau(\ell)\,fi_2h_2\simeq\tau(\ell)\,fi_1h_2$. Since $\tau(\ell)\,f(h_1\times h_2)\simeq\tau(\ell)\,f\,T(h_1\times h_2)$, we have $\tau(\ell)\,fi_1h_1\pi_1+\tau(\ell)\,fi_2h_2\pi_2=\tau(\ell)\,fi_1h_2\pi_2+\tau(\ell)\,fi_2h_1\pi_1$, and hence $\tau(\ell)\,fi_1h_1\pi_1+\tau(\ell)\,fi_2h_2\pi_2-\tau(\ell)\,fi_2h_1\pi_1-\tau(\ell)\,fi_1h_2\pi_2=0$. Since $\tau(\ell)\,fi_1h_1=\tau(\ell)\,fi_2h_1$ and $\tau(\ell)\,fi_2h_2=\tau(\ell)\,fi_1h_2$, this means that $\tau(\ell)\,fi_1h_1\pi_1+\tau(\ell)\,fi_2h_2\pi_2-\tau(\ell)\,fi_1h_1\pi_1-\tau(\ell)\,fi_2h_2\pi_2=0$. Hence $c(\tau(\ell)\,fi_1h_1\times\tau(\ell)\,fi_2h_2)=c(\tau(\ell)\,fi_1h_1\pi_1\times\tau(\ell)\,fi_2h_2\pi_2)\Delta=$ $\tau(\ell)\,fi_1h_1\pi_1+\tau(\ell)\,fi_2h_2\pi_2-\tau(\ell)\,fi_1h_1\pi_1-\tau(\ell)\,fi_2h_2\pi_2=0$. It follows then from Theorem 3 that there exists the required map with the prescribed axes. Thus Theorem 3 implies Theorem 2.

Remark 2. Suppose $\phi:X\times X\to X$ is an H-space multiplication. Then $\phi\,i_1\simeq 1_X$, $\phi\,i_2\simeq 1_X$ and $\tau(\ell):X\to\Omega P(\phi)$ is an H-map. Hence $c(\tau(\ell)\times\tau(\ell))\simeq\tau(\ell)\,c$ where we denote the commutator of X by c also. Thus there exists a map $\Sigma A_1\times\Sigma B_1\to P(\phi)$ with axes $\ell(\Sigma h_1)$, $\ell(\Sigma h_2)$ if and only if $\tau(\ell)\,c(h_1\times h_2)\simeq *$. If X is homotopy associative and right translation is a homotopy equivalence, then Stasheff showed in [9]

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that there exists a map $\gamma : \Omega P(\phi) \to X$ such that $\gamma\,\tau(\ell) \simeq 1_X$. Thus the required map exists if and only if $c(h_1 \times h_2) \simeq *$. In [6], Hilton remarked that if X is a countable connected CW-complex and is an associative H-space, then inverses exist, and right translation is always a homotopy equivalence. Thus we obtain a generalization of Theorem 1.9 of [9].

2. We now discuss the dual of the above. In [6], Hilton showed that Theorem 2 does not dualise. However, Theorem 3 does have a formal dual of a sort. In fact, if in Theorem 3 we ask for extensions of maps, not to $\Sigma A_1 \times \Sigma B_1$ but to the homotopically equivalent space $(\Sigma A_1 \vee \Sigma B_1) \cup_k C\, \Sigma(A_1 \wedge B_1)$, then clearly everything dualises. In fact, in order to dualise, we should talk about extensions of certain maps and not about maps with prescribed axes, and dualise to compressions of certain maps and not refer to axes at all. Let us discuss this briefly.

We first dualise the Hopf construction. Let A and B be spaces. We can consider $A \flat B \xrightarrow{i} A \vee B \xrightarrow{j} A \times B$ as a fibration where $A \flat B$ is the flat product and j is the usual inclusion. Then we can find a map $\chi : \Omega(A \times B) \to \Omega(A \vee B)$ such that $(\Omega j)\,\chi \simeq 1_{\Omega(A \times B)}$. In fact we can and shall take $\chi = \Omega(i_1\pi_1) + \Omega(i_2\pi_2)$ where $i_1 : A \to A \vee B$, $i_2 : B \to A \vee B$ are the inclusions and $\pi_1 : A \times B \to A$, $\pi_2 : A \times B \to B$ are the projections. The exact sequence of the fibration now shows that $(\Omega i)_\#$ is a monomorphism, and that there exists a unique element $[g]$ of $[\Omega(A \vee B), \Omega(A \flat B)]$ such that $1_{\Omega(A \vee B)} = (\Omega i)g + \chi(\Omega j)$. Now for any space X and a map $f : X \to A \vee B$, we can form the map $H(f) = g(\Omega f) : \Omega X \to \Omega(A \flat B)$. We shall call $H(f)$ the element obtained from f by the co-Hopf construction. The element $H(f)$ satisfies $\Omega f = (\Omega i)\, H(f) + \chi\,\Omega(jf)$.

We can now form the dual of $P(f)$. In fact, recall that if $g : X \to Y$ is a map, the dual of attaching a cone to Y by means of g is the space $X \cap_g P\,Y = \{ (x, \ell) \in X \times Y^I$ such that $g(x) = \ell(o)$, $\ell(1) = * \}$. Thus suppose $f : X \to A \vee B$ is a map. The co-Hopf construction gives a map $H(f) : \Omega X \to \Omega(A \flat B)$. Let $P'(f) = \Omega X \cap_{H(f)} P\Omega(A \flat B)$. The projection $\Omega X \times \Omega(A \flat B)^I \to \Omega X$ induces a projection $\ell' : P'(f) \to \Omega X$. Observe that if $\phi' : X \to X \vee X$ is a comultiplication of an H'-space, then $P'(\phi')$ is called the co-projective plane of X in [6]. Let Y be an H'-space and $g : Y \to X$ a map. We say that g is primitive with respect to f if the following diagram homotopy commutes.

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$$\begin{tikzcd} Y \arrow[r, "\phi'"] \arrow[d, "g"] & Y \vee Y \arrow[d] \\ X \arrow[r, "f"'] & A \vee B \end{tikzcd} \qquad p_1fg \vee p_2fg$$

where $p_1 : A \vee B \to A$, $p_2 : A \vee B \to B$ are induced by the projections, and $\phi' : Y \to Y \vee Y$ is the H'-space comultiplication on Y. Then in [6], Hilton proved the following theorem.

THEOREM 4. <u>A map</u> $g : \Sigma Z \to X$ <u>is primitive with respect to</u> f <u>if and only if</u> $\tau(g) : Z \to \Omega X$ <u>lifts to</u> $P'(f)$.

In particular, it is easily checked that $\tau^{-1}(\ell') : \Sigma P'(f) \to X$ is always primitive with respect to f. We now recall that in [1], Arkowitz defined a dual product $[\,,\,]' : [X, \Omega A] \times [X, \Omega B] \to [X, \Omega(A \flat B)]$, which is the dual of the generalized Whitehead product. Also, if X is an H'-space, in [3] he defined a "flat product" $<\,,\,>' : [X, A] \times [X, B] \to [X, A \flat B]$. These are related as follows. If α is an element of $[X, \Omega A]$ and β is an element of $[X, \Omega B]$, then
$\tau < \tau^{-1}(\alpha), \tau^{-1}(\beta) >' = [\alpha, \beta]'$.

Now suppose $f : X \to A \vee B$ is a map and $\ell' : P'(f) \to \Omega X$ is the projection. Let $h_1 : A \to A_1$, $h_2 : B \to B_1$ be maps. Let
$\pi_1 : \Omega A_1 \times \Omega B_1 \to \Omega A_1$, $\pi_2 : \Omega A_1 \times \Omega B_1 \to \Omega B_1$ be the projections.
Let $[\pi_1, \pi_2]' : \Omega A_1 \times \Omega B_1 \to \Omega(A_1 \flat B_1)$ be the dual product. We can consider $[\pi_1, \pi_2]'$ as a fibration, if necessary by replacing it by a homotopically equivalent situation. Then the fibre of $[\pi_1, \pi_2]'$ is
$(\Omega A_1 \times \Omega B_1) \underset{[\pi_1, \pi_2]'}{\cap} P\Omega(A_1 \flat B_1)$. Consider the map
$\{\Omega(h_1p_1f)\ell' \times \Omega(h_2p_2f)\ell'\} \quad \Delta : P'(f) \to \Omega A_1 \times \Omega B_1$ where
$p_1 : A \vee B \to A$, $p_2 : A \vee B \to B$ are induced by the projections.
Let $c' : \Sigma P'(f) \to \Sigma P'(f) \vee \Sigma P'(f)$ be the basic co-commutator.
Then we have the following result.

THEOREM 5. $\{\Omega(h_1p_1f)\ell' \times \Omega(h_2p_2f)\ell'\} \quad \Delta : P'(f) \to \Omega A_1 \times \Omega B_1$ <u>can be compressed into</u> $(\Omega A_1 \times \Omega B_1) \underset{[\pi_1, \pi_2]'}{\cap} P\Omega(A_1 \flat B_1)$ <u>if and only if</u> $\{h_1p_1f\tau^{-1}(\ell') \vee h_2p_2f\tau^{-1}(\ell')\}\, c' \simeq *$.

<u>Proof</u>. The proof is an exact dual of that of Theorem 3, and we leave it to the reader to provide it.

By specializing f we obtain appropriate duals of the results referred to in Remarks 1 and 2.

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