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ON THE DIMENSION OF A COMPLETE METRIZABLE TOPOLOGICAL VECTOR SPACE

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The purpose of this note is to prove a result which is known to hold for Fréchet spaces [1, Chapitre II, §5, Exercise 24]. M. M. Day [2, p. 37] attributes the Banach space case to H. Löwig, although the earliest version that we have been able to find is that given by G. W. Mackey in [7, Theorem I-1]. Recently H. E. Lacey has given an elegant proof for Banach spaces [5]. It is perhaps interesting to note that the non-locally convex case can be deduced from these known results which are established by duality arguments.

PROPOSITION. *The (vector space) dimension of an infinite dimensional complete metrizable topological vector space is at least c, the cardinality of the real numbers. If in addition the space is separable, its dimension is precisely c.*

Proof. Let E be an infinite dimensional complete metrizable topological vector space with topology ξ and let p be an F-norm on E which defines ξ [4, §15.11]. Choose a sequence (x_n) of linearly independent elements of E and for each n choose $\alpha_n > 0$ such that $p(\alpha_n x_n) \le 1/2^n$. Put $y_n = \alpha_n x_n$ $(n \in \mathbb{N})$ and let B be the absolutely convex envelope of $\{y_n : n \in \mathbb{N}\}$.

We show first that B is ξ-bounded. Suppose that (λ_n) is a scalar sequence with only finitely many non-zero terms and such that $\sum_{n=1}^{\infty} |\lambda_n| \le 1$. For each $m \in \mathbb{N}$,

$$p\left(\sum_{n=m}^{\infty} \lambda_n y_n\right) \le \sum_{n=m}^{\infty} p(\lambda_n y_n) \le \sum_{n=m}^{\infty} p(y_n) \le \frac{1}{2^{m-1}}.$$

Given $\varepsilon > 0$, we may therefore choose $M > 1$ such that $p(\sum_{n=M}^{\infty} \lambda_n y_n) \le \varepsilon/2$ for all such (λ_n). Certainly $A = \{\sum_{n=1}^{M-1} \mu_n y_n : \sum_{n=1}^{M-1} |\mu_n| \le 1\}$ is ξ-bounded [3, 7.3], and so there exists $\beta \ge 1$ such that $A \subseteq \beta\{x \in E : p(x) \le \varepsilon/2\}$. Thus for any sequence (λ_n) of the above type,

$$p\left(\frac{1}{\beta} \sum_{n=1}^{\infty} \lambda_n y_n\right) \le p\left(\frac{1}{\beta} \sum_{n=1}^{M-1} \lambda_n y_n\right) + p\left(\frac{1}{\beta} \sum_{n=M}^{\infty} \lambda_n y_n\right)$$

$$\le \frac{\varepsilon}{2} + p\left(\sum_{n=M}^{\infty} \lambda_n y_n\right) \le \varepsilon$$

and consequently $B \subseteq \beta\{x \in E : p(x) \le \varepsilon\}$.

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The ξ-closure D of B must also be ξ-bounded and absolutely convex [3, 5.2, 6.2]. Let H be the linear span of D. The gauge of D is a norm on H and, since D is absorbed by each ξ-neighbourhood of the origin, the resulting norm topology η is finer than the topology induced on H by ξ. We now show that H is complete under η. Let (z_n) be an η-Cauchy sequence. It is also an ξ-Cauchy sequence and so converges under ξ to $z_0 \in E$ say. Since $\{z_n : n \in \mathbb{N}\}$ is absorbed by D and since D is ξ-closed, it follows that $z_0 \in H$. Because η has a base of neighbourhoods of the origin which are ξ-closed sets, we may now deduce from [4, §18.4(4)] that $z_n \to z_0$ under η.

Since each x_n is an element of H, the Banach space $H(\eta)$ is infinite dimensional. We therefore have $\dim E \geq \dim H \geq c$. If E is also separable its cardinality is c. This implies that the dimension of E cannot exceed c [6, Satz 2] and so must be precisely c.

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