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Compositio Math. **146** (2010), 259–270.

doi:10.1112/S0010437X09004369

FOUNDATION COMPOSITIO MATHEMATICA

The London Mathematical Society

https://doi.org/10.1112/S0010437X09004369 Published online by Cambridge University Press

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Abstract

In this article we study the Kähler–Ricci flow, the corresponding parabolic Monge–Ampère equation and complete non-compact Kähler–Ricci flat manifolds. Our main result states that if (M, g) is sufficiently close to being Kähler–Ricci flat in a suitable sense, then the Kähler–Ricci flow has a long time smooth solution $g(t)$ converging smoothly uniformly on compact sets to a complete Kähler–Ricci flat metric on M. The main step is to obtain a uniform C^0-estimate for the corresponding parabolic Monge–Ampère equation. Our results on this can be viewed as parabolic versions of the main results of Tian and Yau [*Complete Kähler manifolds with zero Ricci curvature. II*, Invent. Math. **106** (1990), 27–60] on the elliptic Monge–Ampère equation.

1. Introduction

Let (M^n, g_0) be a complete non-compact Kähler manifold with complex dimension n. Consider the following Kähler–Ricci flow on M:

$$\begin{cases} \dfrac{\partial g_{i\bar{j}}}{\partial t} = -R_{i\bar{j}} \\ g_{i\bar{j}}(x, 0) = (g_0)_{i\bar{j}}. \end{cases} \tag{1}$$

We are interested in studying when (1) admits a long time solution $g(t)$ converging smoothly on M to a complete Kähler metric $g(\infty)$. Such a limit $g(\infty)$ must be Kähler–Einstein with zero scalar curvature by (1). We are thus interested in studying when (M^n, g_0) converges to a Kähler–Ricci flat metric under (1). When M is compact, Cao [Cao85] established that a necessary and sufficient condition for such convergence is that

$$(R_0)_{i\bar{j}} = (f_0)_{i\bar{j}} \tag{2}$$

for a smooth potential function f_0 on M where $(R_0)_{i\bar{j}}$ is the Ricci tensor of g_0. This re-establishes the famous Calabi conjecture first proved by Yau [Yau78]. In Theorem 1 we establish a non-compact version of Cao's result. We prove that when (M^n, g_0) is complete, non-compact with bounded curvature, with volume growth $V_{x_0}(r) \leqslant Cr^{2n}$ for some x_0 and C for all r, and satisfies a certain Sobolev inequality, then:

Received 14 July 2008, accepted in final form 20 March 2009, published online 11 December 2009.
2000 Mathematics Subject Classification 53C55 (primary), 58J35 (secondary).
Keywords: non-compact Kähler–Einstein metrics, Kähler–Ricci flow, parabolic Monge–Ampère equation.
The first author's research was partially supported by NSERC grant no. #327637-06. The second author's research was partially supported by Hong Kong RGC General Research Fund #GRF 2160357.
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https://doi.org/10.1112/S0010437X09004369 Published online by Cambridge University Press

under the above conditions, the Kähler–Ricci flow (1) has a long time solution $g(t)$ converging smoothly on M provided (2) is satisfied and $|f_0|(x) \leqslant C/(1+\rho_0^{2+\epsilon}(x))$ for some C, $\epsilon > 0$ and all x.

See Theorem 1 for details. The result is motivated by the work of Tian–Yau [TY86, TY90] on the existence of complete non-compact Kähler–Ricci flat manifolds. In particular in [TY90] they roved the existence of a Kähler–Ricci flat metric in the complement of a smooth divisor D in a compact Kähler manifold M under the following conditions: D is neat, almost ample and admissible so that D admits a Kähler–Einstein metric with positive scalar curvature and $-K_M - \beta L_D$ for some $\beta > 1$, where K_M is the canonical line bundle of M and L_D is the line bundle associated with D (see [TY90] for details). Their method was to first construct a Kähler metric satisfying the conditions in Theorem 1, then solve the elliptic complex Monge–Ampère equation to obtain a Kähler–Ricci flat metric. Our results can thus be viewed as a parabolic version of the result on the elliptic Monge–Ampère equation in [TY90].

Related results on the convergence of the Kähler–Ricci flow to Kähler–Einstein metrics with negative scalar curvature were obtained in [Cao85, Cha04]. In [Cao85], it was proved that (1) converges after re-scaling to a Kähler–Einstein metric with negative scalar curvature provided that $(R_0)_{i\bar{j}} + (g_0)_{i\bar{j}} = (f_0)_{i\bar{j}}$ for smooth f_0. A non-compact version of this result was proved in [Cha04].

2. The main result

Let (M^n, g_0) be a complete non-compact Kähler manifold with complex dimension n such that (2) holds for some smooth potential f_0 on M. When $f_0 = 0$, then (M^n, g_0) is Kähler–Einstein with zero scalar curvature. We are thus interested in the behavior of the Kähler–Ricci flow on complete Kähler manifolds which are close to being Kähler–Einstein. We prove the following theorem.

Theorem 1. *Let (M^n, g_0) be a complete non-compact Kähler manifold with bounded curvature and $n \geqslant 3$. Assume the following:*

(a) *the Ricci tensor of g_0 has a smooth potential f_0, (i.e. (2) holds for some smooth f_0), such that f_0 satisfies*

$$|f_0|(x) \leqslant \frac{C_1}{1+\rho_0^{2+\epsilon}(x)} \tag{3}$$

for some C_1, $\epsilon > 0$, and all $x \in M$ where $\rho_0(x)$ is the distance function from a fixed $o \in M$;

(b) *the following Sobolev inequality is true*

$$\left(\int_M |\phi|^{2n/(n-1)}\, dV_0\right)^{(n-1)/n} \leqslant C_2 \int_M |\nabla_0 \phi|^2\, dV_0 \tag{4}$$

for some $C_2 > 0$ and all $\phi \in C_0^\infty(M)$;

(c) *there exists a constant $C_3 > 0$ such that*

$$V_0(r) \leqslant C_3 r^{2n} \tag{5}$$

for some $C_3 > 0$ and all r where $V_0(r)$ is the volume of the geodesic ball with radius r centered at some $o \in M$.

Then (1) has a long time solution $g(t)$. Moreover, as $t \to \infty$, $g(t)$ converges uniformly on compact sets in the C^∞ topology on M to a complete Kähler–Ricci flat metric g_∞ on M which is uniformly equivalent to g_0.

Here and below ∇_t and Δ_t denote the covariant derivative and Laplacian with respect to $g(t)$.

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In order to prove the theorem, as in [Cao85] we use the following parabolic Monge–Ampère equation corresponding to (1):

$$\begin{cases} \dfrac{\partial u}{\partial t} = \log \dfrac{\det((g_0)_{k\bar{l}} + u_{k\bar{l}})}{\det((g_0)_{k\bar{l}})} - f_0 \\ u(x,0) = 0. \end{cases} \tag{6}$$

The relationship between the two equations can be described as follows, see [Cha04, Proof of Lemma 4.1]. If (6) has a smooth solution u on $M \times [0,T)$, then $g_{i\bar{j}} = (g_0)_{i\bar{j}} + u_{i\bar{j}}$ is a smooth solution to (1). Conversely, if (1) has a solution g, then

$$u(x,t) = \int_0^t \log \frac{\det(g_{i\bar{j}})}{\det((g_0)_{i\bar{j}})}(x,s)\, ds - t f_0(x) \tag{7}$$

is a solution to (6).

Since the curvature of g_0 is bounded by some constant k_0, by [Shi97, Theorem 1.1] (1) has a solution $g(t)$ on $M \times [0,T]$ where T depends only on k_0 and n. Moreover, the curvature tensor $Rm(t)$ of $g(t)$ satisfies:

$$|\nabla_t^m Rm(t)| \leqslant \frac{C(m,k_0,n)}{t^m} \tag{8}$$

for all $m \geqslant 0$ and $T > t > 0$, where $C(m,k_0,n)$ is a constant depending only on m, k_0 and n.

Remark 1. Let $[0,T_{\max})$, with $T_{\max} \leqslant \infty$, be the maximal time interval such that (1) has a smooth solution in $M \times [0,T_{\max})$. By the estimates in [Shi89, Shi97], for any $0 < T_0 < T_1 < T_{\max}$ the curvature tensor $g(t)$ is uniformly bounded in $M \times [0,T_1]$ and the covariant derivatives of the curvature tensor are uniformly bounded in $M \times [T_0,T_1]$.

Let $u(x,t)$ be as in (7). The major step in proving the main Theorem 1 is to obtain a uniform C^0 bound on u. Once this is obtained, the higher order estimates for u can be obtained by somewhat more standard estimates for (6) (see Lemma 5). To get the C^0 estimate for u, we introduce the function $f = -u_t$ and derive initial estimates for f. We will do this using maximum principle arguments. While there are various versions of the maximum principle which can be used here (see [EH91, NT04, Shi97]), the version in Ecker–Huisken [EH91] seems to be most suitable in our setting. The following is a consequence of their more general result.

LEMMA 1 (Ecker–Huisken [EH91, Theorem 4.3]). *Let $g(t)$ be a solution of (1) on $M \times [0,T]$ with uniformly bounded curvature tensor. Let h be a smooth function on $M \times [0,T]$ such that*

$$\frac{\partial h}{\partial t} \leqslant \Delta_t h + \langle \mathbf{a}, \nabla_t h \rangle_t$$

for some vector field $\mathbf{a}$ which is uniformly bounded on $M \times [0,T]$. Suppose that h satisfies

$$\int_0^T \left(\int_M \exp(-\alpha\rho_t^2) |\nabla_t h|^2 \, dV_t \right) dt < \infty$$

for some $\alpha > 0$, where ρ_t is the distance function of $g(t)$ from a fixed point $o \in M$. If $h \leqslant 0$ at $t = 0$, then $h \leqslant 0$ in $M \times [0,T]$.

COROLLARY 1. *Let $g(t)$ be as in Lemma 1. Let h be a smooth function on $M \times [0,T]$ such that $\partial h/\partial t \leqslant \Delta_t h$ and $h(\partial h/\partial t) \leqslant h \Delta_t h$. Suppose that*

$$\int_0^T \left(\int_M \exp(-\alpha\rho_t^2) h^2 \, dV_t \right) dt + \int_M \exp(-\alpha\rho_0^2) h^2 \, dV_0 = A < \infty$$

for some $\alpha > 0$. Then $\sup_{M \times [0,T]} h \leqslant \sup_{x \in M} h(x,0)$.

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Proof. We may assume that $\sup_{x\in M} h(x,0)<\infty$. Since $g(t)$ has uniformly bounded curvature, by [Shi97, Lemma 4.5] there exists a smooth exhaustion function η independent of t such that for some positive constants K_1, K_2, K_3, depending only on n, T and the uniform bound on $Rm(x,t)$ on $M\times[0,T]$, η satisfies

(1) $K_1(\rho_t+1)\leqslant\eta\leqslant K_2(\rho_t+1)$;

(2) $|\nabla_t\eta|, |\nabla_t^2\eta|\leqslant K_3$;

for all $t\in[0,T]$. Now take a smooth function $\psi:\mathbb{R}\to\mathbb{R}$ which is equal to one on $[-1,1]$ and has support in $[-2,2]$. Define the function $\varphi_R:=\psi(\eta/R)$. Then φ_R is equal to one on $B_0(R)$ and zero outside $B_0(2R)$, and there exists $C>0$ such that for all R, $\sup_{M\times[0,T]}|\nabla_t\varphi_R|_t\leqslant C/R$. We have

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\int_M \varphi_R^2 h^2\, dV_t\right)\\ &\quad\leqslant \int_M 2\varphi_R^2 h\Delta_t h\, dV_t-\int_M \varphi_R^2 h^2 R(t)\, dV_0\\ &\quad\leqslant 4\int_M |\nabla_t\varphi_R|_t^2 h^2\, dV_t-\int_M \varphi_R^2|\nabla_t h|_t^2\, dV_t+C_1\int_M \varphi_R^2 h^2\, dV_t \end{aligned}\tag{9}$$

for some constant C_1 depending only on the bound of the curvature of $g(t)$. Here $R(t)$ is the scalar curvature of $g(t)$. Integrating from zero to T, and using the assumption on h and the fact that $Rm(t)$ is uniformly bounded we have

$$\int_0^T\int_{B_0(R)}|\nabla_t h|_t^2\, dV_t\leqslant C_2\exp(\beta R^2)$$

for some constants C_2 and $\beta>0$ depending only on the bound of $Rm(t)$, α and A. Hence, there exists $\gamma>0$ depending only on T and the bound of $Rm(t)$, such that

$$\int_0^T\int_{B_0(R)\setminus B_0(R/2)}\exp(-\gamma\rho_t^2)|\nabla_t h|_t^2\, dV_t\leqslant C_2\exp(-R^2)$$

for all $R>0$. Therefore,

$$\int_0^T\int_M \exp(-\gamma\rho_t^2)|\nabla_t h|_t^2\, dV_t<\infty.$$

Applying Lemma 1 to the function $h-\sup_{x\in M} h(x,0)$, the result follows. □

LEMMA 2. *Let (M^n, g_0) be a complete non-compact Kähler manifold with bounded curvature such that (2) holds for a smooth bounded potential f_0. Suppose that $g(t)$ is a smooth solution to (1) on $M\times[0,T]$ such that $g(t)$ has uniformly bounded curvature in $M\times[0,T]$. If $u(x,t)$ is the corresponding solution to (6), then $f(x,t)=-u_t(x,t)$ satisfies the following.*

(i) *We have $\sup_{M\times[0,T]}(f^2+t|\nabla_t f|^2)\leqslant\sup_M f_0^2$.*

(ii) *For each $0<t\leqslant T$, the covariant derivatives of $f(x,t)$ relative to $g(t)$ are bounded on M by constants depending only on bounds for the curvature tensor of $g(t)$ and its covariant derivatives.*

(iii) *If, in addition, $|f_0(x)|\leqslant C/(1+\rho_0(x))^N$ for some $N>1$, where ρ_0 is the distance function relative some $o\in M$ with respect to $g(0)$, then there is a constant C' depending only on T, a bound on the curvature tensor of g in $M\times[0,T]$, and a bound on $|\rho_0^N f_0(x)|$ such that $|f(x,t)|\leqslant C'/(1+\rho_t(x))^N$ where $\rho_t(x)$ is the distance relative to o with respect to $g(t)$.*

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Proof. We first prove part (i). First note that $f(x,0)=f_0(x)$. By (1), (6) and the assumption that $g(t)$ has bounded curvature in $M\times[0,T]$, we see that $f=-u_t$ is uniformly bounded in $M\times[0,T]$. Now differentiating (6) gives

$$\left(\frac{\partial}{\partial t}-\Delta_t\right)f=0. \tag{10}$$

By Corollary 1, we have

$$\sup_{x\in M}|f(x,t)|\leqslant\sup_M|f_0|. \tag{11}$$

Direct computation shows

$$\frac{\partial}{\partial t}|\nabla_t f|^2=\Delta_t|\nabla_t f|^2-|f_{ij}|^2-|f_{i\bar{j}}|^2.$$

Hence, we have

$$\frac{\partial}{\partial t}(|\nabla_t f|^2+1)^{1/2}\leqslant\Delta_t(|\nabla_t f|^2+1)^{1/2}$$

and for any $T>t_0>0$

$$\frac{\partial}{\partial t}((t-t_0)|\nabla_t f|^2+f^2)\leqslant\Delta_t((t-t_0)|\nabla_t f|^2+f^2)$$

for $T\geqslant t\geqslant t_0$, see [Cho01]. By the proof of Corollary 1, we see that

$$\int_0^T\int_M\exp(-\alpha\rho_t^2)|\nabla_t f|^2\,dV_t<\infty.$$

for some $\alpha>0$.

By part (ii), which is proved below, we see that $\sup_{x\in M}|\nabla_{t_0}f|^2(x,t_0)$ is bounded. Hence, one can apply Lemma 1 to $(|\nabla_t f|^2+1)^{1/2}$ to conclude that

$$\sup_{M\times[t_0,T]}|\nabla_t f|^2<\infty.$$

By applying Lemma 1 to $(t-t_0)|\nabla_t f|^2+f^2$, we conclude that

$$\sup_{M\times[t_0,T]}(t-t_0)|\nabla_t f|^2+f^2\leqslant\sup_{x\in M}f^2(x,t_0)$$

from which we conclude, by applying Lemma 1 to f and letting $t_0\to0$, that

$$\sup_{M\times[t_0,T]}(t|\nabla_t f|^2+f^2)\leqslant\sup_{x\in M}f_0^2.$$

Now part (ii) can be shown as follows. By (8), for fixed $T>t>0$ all of the covariant derivatives of $Rm(t)$ are bounded. On the other hand, $\Delta_t f(t)=R(t)$. By lifting this equation to the tangent space by the exponential map, and using Schauder estimates we conclude that part (ii) is true by (11).

We now prove part (iii). Let η be the smooth exhaustion function as in the proof of Corollary 1. Direct computation gives

$$\begin{aligned}\frac{d}{dt}(\eta^{2N}f^2)&\leqslant\eta^{2N}\Delta_t f^2\\&=\Delta_t(\eta^{2N}f^2)-(\Delta_t\eta^{2N})f^2-2\langle\nabla_t\eta^{2N},\nabla_t f^2\rangle_t\\&=\Delta_t(\eta^{2N}f^2)-2N\eta^{2N-1}(\Delta_t\eta)f^2\\&\quad-2N(2N-1)\eta^{2N-2}|\nabla_t\eta|_t^2f^2-4N\eta^{2N-1}\langle\nabla_t\eta,\nabla_t f^2\rangle_t\end{aligned}$$

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$$
\begin{aligned}
&\leqslant \Delta_t(\eta^{2N}f^2) + C_1\eta^{2N}f^2 \\
&\quad - 4N\eta^{2N-1}\left\langle \nabla_t\eta, \left(\frac{\nabla_t(\eta^{2N}f^2) - \nabla_t(\eta^{2N})f^2}{\eta^{2N}}\right)\right\rangle_t \\
&\leqslant \Delta_t(\eta^{2N}f^2) + C_2\eta^{2N}f^2 - 4N\eta^{-1}\langle\nabla_t\eta, \nabla_t(\eta^{2N}f^2)\rangle_t
\end{aligned} \tag{12}
$$

for constants $C_1, C_2 > 0$ depending only on n, N, T and the bound on $Rm(x,t)$ on $M \times [0,T]$.

Let $h = \exp(-C_2 t)\eta^{2N}f^2$, then by (12)

$$
\frac{\partial}{\partial t}h \leqslant \Delta_t h - 4N\eta^{-1}\langle\nabla_t\eta, \nabla_t h\rangle_t. \tag{13}
$$

By our hypothesis on f_0 and the first property for η we have that h is bounded at $t=0$. Let $h_1 = h - \sup_M h(0)$. Then h_1 satisfies (13) while $h_1(0) \leqslant 0$. By Lemma 1 we conclude that $h(t) \leqslant 0$ for all $t \in [0,T]$. By the first property for η we conclude that part (iii) is true. □

Remark 2. We make the following remarks.

(i) Since $f(t)$ is a potential for the Ricci tensor of $g(t)$, the lemma implies that for $t > 0$ small enough, $g(t)$ also satisfies the conditions in Theorem 1.

(ii) By Remark 1, if we take $t_0 > 0$ small enough to be the initial time, we may assume that the curvature tensor of $g(t)$ and all of its covariant derivatives are uniformly bounded on $M \times [0,T]$ for any $T < T_{\max}$.

(iii) Hence, each covariant derivative of $f(x,t)$ is uniformly bounded on $M \times [0,T]$ for any $T < T_{\max}$, and since $f = -u_t$, this is also true for u.

In the following, we assume these are all true for the solution $g(t)$.

To obtain a C^0 estimate of u, we begin by showing that the evolving L_p norms of f remain bounded independent of t for certain p.

Lemma 3. *Let (M, g_0) be a complete non-compact Kähler manifold with bounded curvature satisfying conditions* (a) *and* (c) *in Theorem 1 with $n \geqslant 3$ and let $g(t)$ be the solution of the Kähler–Ricci flow (1) defined on $M \times [0, T_{\max})$. For any $p_0 > n$ with $(p_0+1)(2+\epsilon)/(n+p_0) > 2$ and $p^* = n(p_0+1)/(n+p_0) > 2n/(2+\epsilon)$, we have*

$$
\sup_{t\in[0,T_{\max})} \int_M |f(t)|^{p^*}\, dV_t < \infty.
$$

Proof. Let $p_0 > n$ be such that

$$
\frac{p_0+1}{n+p_0}(2+\epsilon) > 2
$$

and $p^* = n(p_0+1)/(n+p_0) > 2n/(2+\epsilon)$. Such a p_0 exists because $n \geqslant 3$.

By the assumption on f_0, the volume growth of (M, g_0) and Lemma 2, we see that for any $T_{\max} > t > 0$ and $p \geqslant p^*$,

$$
\int_M |f(t)|^p\, dV_t < \infty. \tag{14}
$$

Here we have used the fact that for any finite $T < T_{\max}$, $g(t)$ is uniformly equivalent to g for $0 \leqslant t \leqslant T$.

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For any even integer $p > p^* + 2$, we have

$$
\begin{aligned}
\frac{d}{dt}\int_M |f|^p dV_t &= \frac{d}{dt}\int_M f^p\,dV_t \\
&= p\int_M f^{p-1}\Delta_t f\,dV_t - \int_M f^p\Delta_t f\,dV_t \\
&= -p(p-1)\int_M f^{p-2}|\nabla_t f|_t^2\,dV_t + p\int_M f^{p-1}|\nabla_t f|_t^2\,dV_t \\
&\leqslant -(p(p-1) - pC_1)\int_M f^{p-2}|\nabla_t f|_t^2\,dV_t, \qquad (15)
\end{aligned}
$$

where $C_1 = \sup_M |f_0|$ and we have used the fact that for any t, $|\nabla f|_t$ is bounded on M and the fact that $p - 2 > p^*$ to justify the integration by parts. Hence, if $p \geqslant 1 + C_1$, then

$$
\int_M |f(t)|^p\,dV_t \leqslant \int_M |f_0|^p\,dV_0 < \infty. \qquad (16)
$$

We have to improve (16).

Let $v = \max\{f, 0\}$ and for a fixed $R > 0$ let $\varphi = \varphi_R$ be the cutoff function on M defined in the proof of Corollary 1 (thus, φ is independent of t). For any $p \geqslant p^* - 1 > 1$,

$$
\begin{aligned}
\int_0^T\int_M \varphi^2 v^p \frac{\partial}{\partial t} f\,dV_t\,dt &= \int_0^T\int_M \varphi^2 v^p\Delta_t f\,dV_t\,dt \\
&= -\int_0^T\int_M p v^{p-1}\varphi^2|\nabla_t v|_t^2\,dV_t\,dt - 2\int_0^T\int_M v^p\varphi\langle\nabla_t\varphi, \nabla_t v\rangle_t\,dV_t\,dt \\
&\leqslant \frac{1}{p}\int_0^T\int_M v^{p+1}|\nabla_t\varphi|_t^2\,dV_t\,dt. \qquad (17)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\int_0^T\int_M \varphi^2 v^p\frac{\partial f}{\partial t}\,dV_t\,dt &= -\int_M \varphi^2\int_0^T v^p\left(\frac{\partial v}{\partial t}\right)e^{f_0 - f}\,dt\,dV_0 \\
&= \frac{1}{p+1}\int_M \varphi^2\int_0^T \frac{\partial}{\partial t}[v^{p+1}e^{f_0-f}]\,dt\,dV_0 \\
&\quad + \frac{1}{p+1}\int_M \varphi^2\int_0^T v^{p+1}e^{f_0-f}\frac{\partial f}{\partial t}\,dt\,dV_0. \qquad (18)
\end{aligned}
$$

Combining this with (17), we have

$$
\begin{aligned}
\int_M \varphi^2 v^{p+1}e^{f_0-f}\,dV_t|_{t=T} &\leqslant \int_M \varphi^2 v^{p+1}\,dV_t|_{t=0} + \frac{p+1}{p}\int_0^T\int_M v^{p+1}|\nabla_t\varphi|_t^2\,dV_t\,dt \\
&\quad - \int_M \varphi^2 e^{f_0}\int_0^T v^{p+1}\sum_{k=0}^{\infty}\frac{(-f)^k}{k!}\frac{\partial v}{\partial t}\,dt\,dV_0 \\
&= \int_M \varphi^2 v^{p+1} dV_t|_{t=0} + \frac{p+1}{p}\int_0^T\int_M v^{p+1}|\nabla_t\varphi|_t^2\,dV_t\,dt \\
&\quad - \int_M \varphi^2 e^{f_0}\sum_{k=0}^{\infty}\frac{(-1)^k v^{p+k+2}}{k!(p+k+2)}\,dV_0|_{t=T} \\
&\quad + \int_M \varphi^2 e^{f_0}\sum_{k=0}^{\infty}\frac{(-1)^k v^{p+k+2}}{k!(p+k+2)}\,dV_0|_{t=0}. \qquad (19)
\end{aligned}
$$

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Now

$$-\int_M \varphi^2 e^{f_0} \sum_{k=0}^{\infty} \frac{(-1)^k v^{p+k+2}}{k!(p+k+2)}\, dV_0|_{t=T} + \int_M \varphi^2 e^{f_0} \sum_{k=0}^{\infty} \frac{(-1)^k v^{p+k+2}}{k!(p+k+2)}\, dV_0|_{t=0}$$
$$= -\int_M \varphi^2 v^{p+2} e^{f_0} \sum_{k=0}^{\infty} \frac{(-1)^k v^k}{k!(p+k+2)}\, dV_0|_{t=T}$$
$$+ \int_M \varphi^2 v^{p+2} e^{f_0} \sum_{k=0}^{\infty} \frac{(-1)^k v^k}{k!(p+k+2)}\, dV_0|_{t=0}$$
$$\leqslant C_2 \int_M v^{p+2}\, dV_t|_{t=T} + C_3 \int_M v^{p+2}\, dV_t|_{t=0}$$
$$\leqslant C_2 \int_M v^{p+2} dV_t|_{t=T} + C_4(p) \tag{20}$$

where C_2, C_3, C_4 are independent of T. Here we have used the fact that $f(t)$ is uniformly bounded in spacetime. Combine (19) and (20) to give

$$\int_M \varphi^2 v^{p+1}\, dV_t|_{t=T} \leqslant C_5 + \frac{p+1}{p}\int_0^T \int_M v^{p+1} |\nabla_t \varphi|_t^2\, dV_t\, dt + C_2 \int_M v^{p+2}\, dV_t|_{t=T} \tag{21}$$

for some constants C_2, C_5 independent of T. Letting $R \to \infty$, gives

$$\int_M v^{p+1}\, dV_t|_{t=T} \leqslant C_5 + C_2 \int_M v^{p+2}\, dV_t|_{t=T}. \tag{22}$$

Similarly one can prove that if $w = \max\{-f, 0\}$, then

$$\int_M w^{p+1}\, dV_t|_{t=T} \leqslant C_5 + C_2 \int_M w^{p+2}\, dV_t|_{t=T} \tag{23}$$

by modifying C_5 and C_2 if necessary, while still independent of T. Hence, we have

$$\int_M |f|^{p+1}\, dV_t|_{t=T} \leqslant 2C_5 + C_2 \int_M |f|^{p+2}|_{t=T} \tag{24}$$

for all $p \geqslant p^* - 1$. By iteration and (16), we conclude that

$$\int_M |f|^{p^*}\, dV_t|_{t=T} \leqslant C_5 \tag{25}$$

for some constant C_5 independent of T. □

In the next lemma we show that $|u|$ remains uniformly bounded along (6) independent of t. Our approach can be described roughly as follows. In [TY90] the elliptic Monge–Ampère equation

$$0 = \log \frac{\det(g_{k\bar{l}} + u_{k\bar{l}})}{\det(g_{k\bar{l}})} - f \tag{26}$$

was studied on a complete non-compact Kähler manifold $(M, g_{i\bar{j}})$ where f is a given function on M. In particular, an *a priori* C^0 estimate was established for u provided that g and f satisfy basically the same hypothesis as in Theorem 1. This was done using a non-compact version of the Nash–Moser iteration for (26) established in [Yau78] in the compact case. Now for each t we may treat (6) as an elliptic Monge–Ampère equation as in (26) simply by subtracting u_t

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from both sides of (6). Doing this for each t, our hypothesis on (M, g_0) together with (25) will essentially allow us to proceed as in [TY90] to estimate the C^0 norm of u independent of t. This method of using estimates for the elliptic Monge–Ampère equation to derive estimates for the corresponding parabolic equation was first done in [Cao85] in the compact case.

LEMMA 4. *Let (M, g_0) be as in Theorem 1 and let $g(t)$ be the solution of the Kähler–Ricci flow (1) in $M \times [0, T_{\max})$. Then*

$$\sup_{M\times[0,T_{\max})} |u| < \infty.$$

Proof. Fix some $T_{\max} > t \geqslant 0$ and let ω_0 be the Kähler form corresponding to $g(0)$ and let $\omega = \omega_0 + (\sqrt{-1}/2)\partial\bar{\partial}u$ be the Kähler form corresponding to $g(t)$. Then by (6)

$$\omega^n = e^{f_0 - f}\omega_0^n$$

and

$$\begin{aligned}(e^{f_0-f} - 1)\omega_0^n &= (\omega^n - \omega_0^n) \\ &= \frac{\sqrt{-1}}{2}\partial\bar{\partial}u \wedge \bigg(\sum_{j=0}^{n-1} \omega_0^j \wedge \omega^{n-j-1}\bigg). \end{aligned} \tag{27}$$

(See [Cao85, (1.13)].) For any $p > 1$, multiply both sides by $-\phi^2|u|^p \operatorname{sign}(u)$ and integrate, where ϕ is a smooth function with compact support. After integrating by parts and some computation (see [Cao85] for a similar computation in the compact case), we have

$$\begin{aligned} 2\int_M &\phi^2|u|^p|e^{f_0-f} - 1|\,dV_0 \\ &\geqslant \frac{p}{(p+1)^2}\int_M |\nabla|u|^{(p+1)/2}\phi|^2\,dV_0 - \frac{3}{p}\int_M |u|^{p+1}|\nabla\phi|^2\,dV_0 \end{aligned} \tag{28}$$

where we have used the fact that $(\sqrt{-1}/2)\partial u \wedge \bar{\partial}u \wedge \omega_0^j \wedge \omega^{n-j-1} \geqslant 0$ for all j.

Note that for all t, $|u|$ also decays as $\rho_t^{-2-\epsilon}$ by Lemma 2. Let ϕ be such that $\phi = 1$ in $B_0(r)$ and $\phi = 0$ outside $B_0(2r)$ such that $|\nabla_0\phi| \leqslant C/r$ for some constant C independent of r. Using the fact that the curvature of $g(t)$ is bounded in $M \times [0, T]$ for all $T < T_{\max}$, the last term in the right-hand side of (28) will tend to zero as $r \to \infty$ provided that $p > p^*$. Hence, using the Sobolev inequality (4) we obtain the following for $p \geqslant p_0 > p^*$ (here p_0 and p^* are as in Lemma 3):

$$\begin{aligned}\bigg(\int_M |u|^{(p+1)\kappa}\,dV_0\bigg)^{1/\kappa} &\leqslant C_1 p\int_M |u|^p|e^{f_0-f} - 1|\,dV_0 \\ &\leqslant C_2 p\int_M |u|^p|f_0 - f|\,dV_0 \\ &\leqslant C_3 p\bigg(\int_M |u|^{p+1}\,dV_0\bigg)^{p/(p+1)}\bigg(\int_M |f_0 - f|^{p+1}\,dV_0\bigg)^{1/(p+1)} \\ &\leqslant C_4 p\bigg(\int_M |u|^{p+1}\,dV_0\bigg)^{p/(p+1)} \\ &\leqslant C_4 p\bigg(\int_M |u|^{p+1}\,dV_0 + 1\bigg). \end{aligned} \tag{29}$$

Here $C_1, \ldots, C_4$ are constants independent of t, p and $\kappa = n/(n-1) > 1$ and we have used Lemma 3 and the fact that $f(t)$ is uniformly bounded on space and time. Take $p = p_0$,

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we also have

$$\begin{aligned}\left(\int_M |u|^{(p_0+1)\kappa}\,dV_0\right)^{1/\kappa} &\leqslant C_2 p_0 \int_M |u|^{p_0}|f_0-f|\,dV_0\\ &\leqslant C_2 p_0\left(\int_M |u|^{(p_0+1)\kappa}\,dV_0\right)^{p_0/(p_0+1)\kappa}\\ &\quad\times\left(\int_M |f_0-f|^{(p_0+1)\kappa/(p_0+1)\kappa-p_0}\,dV_0\right)^{1-p_0/(p_0+1)\kappa}\\ &= C_2 p_0\left(\int_M |u|^{(p_0+1)\kappa}\,dV_0\right)^{p_0/(p_0+1)\kappa}\left(\int_M |f_0-f|^{p^*}\,dV_0\right)^{1/p^*}. \qquad (30)\end{aligned}$$

Hence, by Lemma 3, we have

$$\left(\int_M |u|^{(p_0+1)\kappa}\,dV_0\right)^{1/(p_0+1)\kappa} \leqslant C_5 \qquad (31)$$

for some constant C_5 independent of t. By (29) and Young's inequality we have that for $p \geqslant p_0$:

$$\begin{aligned}\int_M |u|^{(p+1)\kappa}\,dV_0 + 1 &\leqslant (C_4 p)^\kappa\left[\int_M |u|^{p+1}\,dV_0 + 1\right]^\kappa + 1\\ &\leqslant (C_4 p)^\kappa\left[\int_M |u|^{p+1}\,dV_0 + 2\right]^\kappa\\ &\leqslant (2C_4 p)^\kappa\left[\int_M |u|^{p+1}\,dV_0 + 1\right]^\kappa. \qquad (32)\end{aligned}$$

Hence, we have

$$\begin{aligned}\left[\int_M |u|^{(p+1)\kappa}\,dV_0 + 1\right]^{1/\kappa(p+1)} &\leqslant (2C_4 p)^{1/(p+1)}\left[\int_M |u|^{p+1}\,dV_0 + 1\right]^{1/(p+1)}\\ &\leqslant (2C_4(p+1))^{1/(p+1)}\left[\int_M |u|^{p+1}\,dV_0 + 1\right]^{1/(p+1)}. \qquad (33)\end{aligned}$$

That is to say, for all $q \geqslant p_0 + 1$,

$$\left[\int_M |u|^{\kappa q}\,dV_0 + 1\right]^{1/\kappa q} \leqslant (2C_4 q)^{1/q}\left[\int_M |u|^{q}\,dV_0 + 1\right]^{1/q}. \qquad (34)$$

By iteration (see [TY90]), it is straightforward to show that

$$\sup_M |u| \leqslant C_6\left[\int_M |u|^{(p_0+1)\kappa}\,dV_0 + 1\right]^{1/(p_0+1)\kappa} \leqslant C_7$$

for some constants $C_6, C_7 > 0$ independent of t. Here we have used (31). This completes the proof of the lemma. □

Once we obtain a C^0 bound for u, then we may obtain bounds on the higher-order derivatives of u as in [Cha04, § 5].[1] See also [Cao85, Yau78].

[1] The equation treated in [Cha04] was actually (6) with an additional term $-u$ on the right: the equation for negative Kähler–Einstein metrics. Despite the difference in these equations, the *a priori* estimates for higher-order derivatives of u in terms of the C^0 norm of u follow from essentially the exact same calculations. We refer also to [Cao85] for similar *a priori* estimates for (6) in the compact case.

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LEMMA 5. *Let $g(t)$ be the solution to (1) on $M \times [0,T]$ such that the curvature of $g(t)$ and all of its covariant derivatives are uniformly bounded. Assume that the initial Ricci tensor has a bounded potential f_0 and let u and f be as before. Assume also that all of the covariant derivatives of $f(t)$ with respect to $g(t)$ are uniformly bounded in $M \times [0,T]$. Suppose that $\sup_{M\times[0,T]} |u| = A$. Then for any $k \geqslant 1$, there is a constant C depending only on A, k, f_0 and $g(0)$ such that*

$$|\nabla_0^k u| \leqslant C.$$

COROLLARY 2. *Let (M^n, g_0) be a complete non-compact Kähler manifold with bounded curvature such that (2) is satisfied for a smooth bounded potential f_0. Then (1) has a long time smooth solution.*

Proof. Let u and f be as before. By Lemma 2(i) we have $|f(x,t)| \leqslant \sup_M |f_0|$. Hence, $|u(x,t)| \leqslant t \sup_M |f_0|$ for all $(x,t) \in M \times [0, T_{\max})$ and $\sup_M u(x,t)$ cannot blowup in finite time. By Lemma 5, the curvature tensor of $g(t)$ cannot blowup in finite time. By [Shi89], we conclude that $T = \infty$ and thus (1) has a long time smooth solution. □

We now prove Theorem 1.

Proof of Theorem 1. By Lemmas 3, 4, and 5, we conclude that for all $t \geqslant 0$, $g(t)$ is uniformly equivalent to g_0 independent of t, and that for any sequence $t_k \to \infty$ some subsequence of $u(x,t_k)$ (which we still denote by $u(x,t_k)$) converges in the C^∞ sense on compact subsets of M to a smooth limit v on M. Thus, by Lemma 2(i), we conclude that $\frac{\partial u}{\partial t}(x,t_k) = -f(x,t_k)$ converges uniformly on M to a constant c. Lemma 3 and the fact that M has infinite volume imply that c must be zero. Hence, $g_{i\bar{\jmath}} + v_{i\bar{\jmath}}$ is a smooth complete Kähler–Ricci flat metric on M. On the other hand, Lemma 6 below implies that the limit metric $g_{i\bar{\jmath}} + v_{i\bar{\jmath}}$ is independent of the t_k. We conclude that $u(x,t)$ converges to v in the C^∞ sense on compact subsets of M. This completes the proof of the theorem. □

The following was basically proved in [Cha05], and says that bounded limits of (6) are unique.

LEMMA 6. *Let g and h be two equivalent complete Kähler metrics on a non-compact complex manifold M such that:*

(i) $h_{i\bar{\jmath}} = g_{i\bar{\jmath}} + v_{i\bar{\jmath}}$ *for some smooth bounded function v;*

(ii) *g has nonnegative Ricci curvature; and*

(iii)

$$\log \frac{\det(g_{i\bar{\jmath}} + v_{i\bar{\jmath}})}{\det(g_{i\bar{\jmath}})} = 0.$$

Then $g = h$.

Proof. We sketch the proof. Since

$$\begin{aligned} 0 &= \int_0^1 \frac{\partial}{\partial s} \log\det(g_{i\bar{\jmath}} + s v_{i\bar{\jmath}}) \\ &= \left(\int_0^1 g^{i\bar{\jmath}}(s)\, ds\right) v_{i\bar{\jmath}} \end{aligned}$$

where $g_{i\bar{\jmath}}(s) = g_{i\bar{\jmath}} + s v_{i\bar{\jmath}}$. Hence, v satisfies $a^{i\bar{\jmath}} v_{i\bar{\jmath}} = 0$ for some Kähler metric $a_{i\bar{\jmath}}$ which is uniformly equivalent to g. As g has nonnegative Ricci curvature, v must be constant by [Gri92, Sal92]. □

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References

Cao85 H.-D. Cao, *Deformation of Kähler metrics to Kähler Einstein metrics on compact Kahler manifolds*, Invent. Math. **81** (1985), 359–372.

Cha04 A. Chau, *Convergence of the Kähler Ricci flow on non-compact Kähler manifolds*, J. Differential Geom. **66** (2004), 211–232.

Cha05 A. Chau, *Stability of the Kähler–Ricci flow at complete noncompact Kähler Einstein metrics*, Contemp. Math. **367** (2005), 43–62.

Cho01 B. Chow, *A gradient estimate for the Ricci Kähler flow*, Ann. Global Anal. Geom. **19** (2001), 321–325.

EH91 K. Ecker and G. Huisken, *Interior estimates for hypersurfaces moving by mean curvature*, Invent. Math. **105** (1991), 547–569.

Gri92 A. A. Grigor'yan, *The heat equation on non-compact Riemannian manifolds*, Math. USSR Sbornik **72** (1992).

NT04 L. Ni and L.-F. Tam, *Kähler Ricci flow and the Poincaré–Lelong equation*, Comm. Anal. Geom. **12** (2004), 111–141.

Sal92 L. Saloff-Coste, *Uniformly elliptic operators on Riemannian manifolds*, J. Differential Geom. **36** (1992), 417–450.

Shi89 W.-X. Shi, *Ricci deformation of the metric on complete noncompact Riemannian manifolds*, J. Differential Geom. **30** (1989), 223–301.

Shi97 W.-X. Shi, *Ricci Flow and the uniformization on complete non compact Kahler manifolds*, J. Differential Geom. **45** (1997), 94–220.

TY86 G. Tian and S. T. Yau, *Existence of Kähler–Einstein metrics on complete Kähler manifolds and their applications to algebraic geometry*, in *Mathematical Aspects of String Theory,* ed. S. T. Yau (World Scientific, Singapore, 1986).

TY90 G. Tian and S. T. Yau, *Complete Kähler manifolds with zero Ricci curvature. II*, Invent. Math. **106** (1990), 27–60.

Yau78 S.-T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation. I*, Comm. Pure Appl. Math. **31** (1978), 339–411.

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