Mathematical Proceedings of the Cambridge Philosophical Society

VOL. 177 JULY 2024 PART 1

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doi: 10.1017/S0305004124000100

First published online 12 July 2024

Differential forms on universal $K3$ surfaces

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(*Received* 14 *August* 2023*; accepted* 20 *March* 2024)

Abstract

We give a vanishing and classification result for holomorphic differential forms on smooth projective models of the moduli spaces of pointed $K3$ surfaces. We prove that there is no nonzero holomorphic k-form for $0 < k < 10$ and for even $k > 19$. In the remaining cases, we give an isomorphism between the space of holomorphic k-forms with that of vector-valued modular forms ($10 \leq k \leq 18$) or scalar-valued cusp forms (odd $k \geq 19$) for the modular group. These results are in fact proved in the generality of lattice-polarisation.

2020 Mathematics Subject Classification: 14J28 (Primary); 14J15, 11F55 (Secondary)

1. *Introduction*

Let $\mathcal{F}_{g,n}$ be the moduli space of n-pointed $K3$ surfaces of genus $g > 2$, i.e., primitively polarised of degree $2g - 2$. It is a quasi-projective variety of dimension $19 + 2n$ with a natural morphism $\mathcal{F}_{g,n} \to \mathcal{F}_g$ to the moduli space $\mathcal{F}_g$ of $K3$ surfaces of genus g, which is generically a $K3^n$-fibration. In this paper we study holomorphic differential k-forms on a smooth projective model of $\mathcal{F}_{g,n}$. They do not depend on the choice of a smooth projective model, and thus are fundamental birational invariants of $\mathcal{F}_{g,n}$. We prove a vanishing result for about half of the values of the degree k, and for the remaining degrees give a correspondence with modular forms on the period domain.

†Supported by KAKENHI 21H00971 and 20H00112.

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Our main result is stated as follows.

THEOREM 1·1. *Let $\bar{\mathcal{F}}_{g,n}$ be a smooth projective model of $\mathcal{F}_{g,n}$ with $g > 2$. Then we have a natural isomorphism:*

$$H^0(\bar{\mathcal{F}}_{g,n}, \Omega^k) \simeq \begin{cases} 0 & 0 < k \leq 9 \\ M_{\wedge^k,k}(\Gamma_g) & 10 \leq k \leq 18 \\ 0 & k > 19,\ k \in 2\mathbb{Z} \\ S_{19+m}(\Gamma_g, \det) \otimes \mathbb{C}\mathcal{S}_{n,m} & k = 19+2m,\ 0 \leq m \leq n \end{cases}. \tag{1·1}$$

Here Γ_g is the modular group for $K3$ surfaces of genus g, which is defined as the kernel of $\mathrm{O}^+(L_g) \to \mathrm{O}(L_g^\vee/L_g)$ where $L_g = 2U \oplus 2E_8 \oplus \langle 2-2g \rangle$ is the period lattice of $K3$ surfaces of genus g. In the second case, $M_{\wedge^k,k}(\Gamma_g)$ stands for the space of vector-valued modular forms of weight $(\wedge^k, k)$ for Γ_g (see [4]). In the last case, $S_{19+m}(\Gamma_g, \det)$ stands for the space of scalar-valued cusp forms of weight $19+m$ and determinant character for Γ_g, and $\mathcal{S}_{n,m}$ stands for the right quotient $\mathfrak{S}_n/(\mathfrak{S}_m \times \mathfrak{S}_{n-m})$, which is a left $\mathfrak{S}_n$-set. Theorem 1·1 is actually formulated and proved in the generality of lattice-polarisation (Theorem 2·6).

In the case of the top degree $k = 19+2n$, namely for canonical forms, the isomorphism (1·1) is proved in [2]. Theorem 1·1 is the extension of this result to all degrees $k < 19+2n$. The spaces in the right-hand side of (1·1) can also be geometrically explained as follows. In the case $k \leq 18$, $M_{\wedge^k,k}(\Gamma_g)$ is identified with the space of holomorphic k-forms on a smooth projective model of $\mathcal{F}_g$, pulled back by $\mathcal{F}_{g,n} \to \mathcal{F}_g$. In the case $k = 19+2m$, $S_{19+m}(\Gamma_g, \det)$ is identified with the space of canonical forms on $\bar{\mathcal{F}}_{g,m}$, and the tensor product $S_{19+m}(\Gamma_g, \det) \otimes \mathbb{C}\mathcal{S}_{n,m}$ is the direct sum of pullback of such canonical forms by various projections $\mathcal{F}_{g,n} \to \mathcal{F}_{g,m}$. Therefore Theorem 1·1 can be understood as a kind of classification result which says that except for canonical forms, there are essentially no new differential forms on the tower $(\mathcal{F}_{g,n})_n$ of moduli spaces. In fact, this is how the proof proceeds.

The space $S_l(\Gamma_g, \det)$ is nonzero for every sufficiently large l, so the space $H^0(\bar{\mathcal{F}}_{g,n}, \Omega^k)$ for odd $k \geq 19$ is typically nonzero (at least when k is large). On the other hand, it is not clear at present whether $M_{\wedge^k,k}(\Gamma_g) \neq 0$ or not in the range $10 \leq k \leq 18$. This is a subject of study in the theory of vector-valued orthogonal modular forms.

The isomorphism (1·1) in the case $k = 19+2m$ is an $\mathfrak{S}_n$-equivariant isomorphism, where $\mathfrak{S}_n$ acts on $H^0(\bar{\mathcal{F}}_{g,n}, \Omega^k)$ by its permutation action on $\mathcal{F}_{g,n}$, while it acts on $S_{19+m}(\Gamma_g, \det) \otimes \mathbb{C}\mathcal{S}_{n,m}$ by its natural left action on $\mathcal{S}_{n,m}$. Therefore, taking the $\mathfrak{S}_n$-invariant part, we obtain the following simpler result for the unordered pointed moduli space $\mathcal{F}_{g,n}/\mathfrak{S}_n$, which is birationally a $K3^{[n]}$-fibration over $\mathcal{F}_g$.

COROLLARY 1·2. *Let $\overline{\mathcal{F}_{g,n}/\mathfrak{S}_n}$ be a smooth projective model of $\mathcal{F}_{g,n}/\mathfrak{S}_n$. Then we have a natural isomorphism:*

$$H^0(\overline{\mathcal{F}_{g,n}/\mathfrak{S}_n}, \Omega^k) \simeq \begin{cases} 0 & 0 < k \leq 9 \\ M_{\wedge^k,k}(\Gamma_g) & 10 \leq k \leq 18 \\ 0 & k > 19,\ k \in 2\mathbb{Z} \\ S_{19+m}(\Gamma_g, \det) & k = 19+2m,\ 0 \leq m \leq n \end{cases}.$$

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The universal $K3$ surface $\mathcal{F}_{g,1}$ is an analogue of elliptic modular surfaces ([6]), and the moduli spaces $\mathcal{F}_{g,n}$ for general n are analogues of the so-called Kuga varieties over modular curves ([7]). Starting with the case of elliptic modular surfaces [6], holomorphic differential forms on the Kuga varieties have been described in terms of elliptic modular forms: [7] for canonical forms, and [1] for the case of lower degrees (somewhat implicitly). Theorem 1·1 can be regarded as a $K3$ version of these results.

As a final remark, in view of the analogy between universal $K3$ surfaces and elliptic modular surfaces, invoking the classical fact that elliptic modular surfaces have maximal Picard number ([6]) now raises the question if $H^{k,0}(\bar{\mathcal{F}}_{g,n})\oplus H^{0,k}(\bar{\mathcal{F}}_{g,n})$ is a sub $\mathbb{Q}$-Hodge structure of $H^k(\bar{\mathcal{F}}_{g,n},\mathbb{C})$. This is independent of the choice of a smooth projective model $\bar{\mathcal{F}}_{g,n}$.

The rest of this paper is devoted to the proof of Theorem 1·1. In Section 2·1 we compute a part of the holomorphic Leray spectral sequence associated to a certain type of $K3^n$-fibration. This is the main step of the proof. In Section 2·2 we study differential forms on a compactification of such a fibration. In Section 2·3 we deduce (a generalised version of) Theorem 1·1 by combining the result of Section 2·2 with some results from [2–5]. Sometimes we drop the subscript X from the notation Ω^k_X when the variety X is clear from the context.

2. *Proof*

2·1. *Holomorphic Leray spectral sequence*

Let $\pi: X\to B$ be a smooth family of $K3$ surfaces over a smooth connected base B. In this subsection X and B may be analytic. We put the following assumption:

Condition 2·1. In a neighbourhood of every point of B, the period map is an embedding.

This is equivalent to the condition that the differential of the period map

$$T_bB\to \mathrm{Hom}(H^{2,0}(X_b),H^{1,1}(X_b))$$

is injective for every $b\in B$, where X_b is the fiber of π over b.

For a natural number $n>0$ we denote by $X_n=X\times_B\cdots\times_B X$ the n-fold fiber product of X over B, and let $\pi_n\colon X_n\to B$ be the projection. We denote by Ω_{π_n} the relative cotangent bundle of π_n, and $\Omega^p_{\pi_n}=\wedge^p\Omega_{\pi_n}$ for $p\geq 0$ as usual.

PROPOSITION 2·2. *Let $\pi: X\to B$ be a K3 fibration satisfying Condition 2·1. Then we have a natural isomorphism:*

$$(\pi_n)_*\Omega^k_{X_n}\simeq\begin{cases}\Omega^k_B & k\leq \dim B\\ 0 & k>\dim B,\ \ k\not\equiv \dim B \mod 2\\ K_B\otimes(\pi_n)_*\Omega^{2m}_{\pi_n} & k=\dim B+2m,\ 0\leq m\leq n\end{cases}.$$

This assertion amounts to a partial degeneration of the holomorphic Leray spectral sequence. Recall ([8, section 5·2]) that $\Omega^k_{X_n}$ has the holomorphic Leray filtration $L^\bullet\Omega^k_{X_n}$ defined by

$$L^l\Omega^k_{X_n}=\pi_n^*\Omega^l_B\wedge\Omega^{k-l}_{X_n},$$

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whose graded quotients are naturally isomorphic to

$$\mathrm{Gr}_L^l \Omega_{X_n}^k = \pi_n^* \Omega_B^l \otimes \Omega_{\pi_n}^{k-l}.$$

This filtration induces the holomorphic Leray spectral sequence

$$(E_r^{l,q}, d_r) \;\Rightarrow\; E_\infty^{l+q} = R^{l+q}(\pi_n)_* \Omega_{X_n}^k$$

which converges to the filtration

$$L^l R^{l+q}(\pi_n)_* \Omega_{X_n}^k \;=\; \mathrm{Im}(R^{l+q}(\pi_n)_* L^l \Omega_{X_n}^k \to R^{l+q}(\pi_n)_* \Omega_{X_n}^k).$$

By [**8**, proposition 5·9], the E_1 page coincides with the collection of the Koszul complexes associated to the variation of Hodge structures for π_n:

$$(E_1^{l,q}, d_1) = (\mathcal{H}^{k-l,l+q} \otimes \Omega_B^l, \bar{\nabla}). \tag{2·1}$$

Here $\mathcal{H}^{*,*}$ are the Hodge bundles associated to the fibration $\pi_n\colon X_n \to B$, and

$$\bar{\nabla}\colon \mathcal{H}^{*,*} \otimes \Omega_B^* \to \mathcal{H}^{*-1,*+1} \otimes \Omega_B^{*+1}$$

are the differentials in the Koszul complexes (see [**8**, section 5·1·3]). For degree reasons, the range of (l, q) in the E_1 page satisfies the inequalities

$$0 \le l \le \dim B, \quad 0 \le k-l \le 2n, \quad 0 \le l+q \le 2n.$$

The first two can be unified:

$$\max(0, k-2n) \le l \le \min(\dim B, k), \quad 0 \le l+q \le 2n. \tag{2·2}$$

We calculate the E_1 to E_2 pages on the edge line $l+q=0$.

LEMMA 2·3. *The following holds:*

(1) $E_1^{l,-l} = 0$ *when* $l \le \min(\dim B, k)$ *with* $l \not\equiv k$ *mod 2;*

(2) $E_2^{l,-l} = 0$ *when* $l < \min(\dim B, k)$*;*

(3) *For* $l_0 = \min(\dim B, k)$ *we have* $E_1^{l_0,-l_0} = E_2^{l_0,-l_0} = \cdots = E_\infty^{l_0,-l_0}$.

Proof. By (2·1), we have $E_1^{l,-l} = \mathcal{H}^{k-l,0} \otimes \Omega_B^l$. By the Künneth formula, the fiber of $\mathcal{H}^{k-l,0}$ over a point $b \in B$ is identified with

$$H^{k-l,0}(X_b^n) = \bigoplus_{(p_1,\cdots,p_n)} H^{p_1,0}(X_b) \otimes \cdots \otimes H^{p_n,0}(X_b), \tag{2·3}$$

where $(p_1, \cdots, p_n)$ ranges over all indices with $\sum_i p_i = k-l$ and $0 \le p_i \le 2$.

(1) When $k-l$ is odd, every index $(p_1, \cdots, p_n)$ in (2·3) must contain a component $p_i = 1$. Since $H^{1,0}(X_b) = 0$, we see that $H^{k-l,0}(X_b^n) = 0$. Therefore $\mathcal{H}^{k-l,0} = 0$ when $k-l$ is odd.

(3) Let $l_0 = \min(\dim B, k)$. By the range (2·2) of (l, q), we see that for every $r \ge 1$ the source of d_r that hits $E_r^{l_0,-l_0}$ is zero, and the target of d_r that starts from $E_r^{l_0,-l_0}$ is also zero. This proves our assertion.

(2) Let $l < \min(\dim B, k)$. In view of (1), we may assume that $l = k-2m$ for some $m > 0$. By (2·2), the source of d_1 that hits $E_1^{l,-l}$ is zero. We shall show that $d_1\colon E_1^{l,-l} \to E_1^{l+1,-l}$ is

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injective. By (2·1), this morphism is identified with

$$\bar{\nabla}\colon \mathcal{H}^{2m,0} \otimes \Omega_B^l \to \mathcal{H}^{2m-1,1} \otimes \Omega_B^{l+1}. \tag{2·4}$$

By the Künneth formula as in (2·3), the fibers of the Hodge bundles $\mathcal{H}^{2m,0}$, $\mathcal{H}^{2m-1,1}$ over $b \in B$ are respectively identified with

$$H^{2m,0}(X_b^n) = \bigoplus_{|\sigma|=m} H^{2,0}(X_b)^{\otimes\sigma}, \tag{2·5}$$

$$\begin{aligned} H^{2m-1,1}(X_b^n) &= \bigoplus_{|\sigma'|=m-1} \bigoplus_{i\notin\sigma'} H^{2,0}(X_b)^{\otimes\sigma'} \otimes H^{1,1}(X_b) \\ &= \bigoplus_{|\sigma|=m} \bigoplus_{i\in\sigma} H^{2,0}(X_b)^{\otimes\sigma-\{i\}} \otimes H^{1,1}(X_b). \end{aligned} \tag{2·6}$$

In (2·5), σ ranges over all subsets of $\{1, \cdots, n\}$ consisting of m elements, and $H^{2,0}(X_b)^{\otimes\sigma}$ stands for the tensor product of $H^{2,0}(X_b)$ for the jth factors X_b of X_b^n over all $j \in \sigma$. The notations σ', σ in (2·6) are similar, and $H^{1,1}(X_b)$ in (2·6) is the $H^{1,1}$ of the ith factor X_b of X_b^n.

Let us write $V = H^{2,0}(X_b)$ and $W = (T_bB)^\vee$ for simplicity. The homomorphism (2·4) over $b \in B$ is written as

$$\bigoplus_{|\sigma|=m} \left(V^{\otimes\sigma} \otimes \wedge^l W \to \bigoplus_{i\in\sigma} V^{\otimes\sigma-\{i\}} \otimes H^{1,1}(X_b) \otimes \wedge^{l+1} W \right). \tag{2·7}$$

By [8, lemma 5·8], the (σ, i)-component

$$V^{\otimes\sigma} \otimes \wedge^l W \to V^{\otimes\sigma-\{i\}} \otimes H^{1,1}(X_b) \otimes \wedge^{l+1} W \tag{2·8}$$

factorises as

$$\begin{aligned} V^{\otimes\sigma} \otimes \wedge^l W &\to V^{\otimes\sigma-\{i\}} \otimes H^{1,1}(X_b) \otimes W \otimes \wedge^l W \\ &\to V^{\otimes\sigma-\{i\}} \otimes H^{1,1}(X_b) \otimes \wedge^{l+1} W, \end{aligned}$$

where the first map is induced by the adjunction $V \to H^{1,1}(X_b) \otimes W$ of the differential of the period map for the ith factor X_b, and the second map is induced by the wedge product $W \otimes \wedge^l W \to \wedge^{l+1} W$. By linear algebra, this composition can also be decomposed as

$$\begin{aligned} V^{\otimes\sigma} \otimes \wedge^l W &\to V^{\otimes\sigma-\{i\}} \otimes V \otimes W^\vee \otimes \wedge^{l+1} W \\ &\to V^{\otimes\sigma-\{i\}} \otimes H^{1,1}(X_b) \otimes \wedge^{l+1} W, \end{aligned} \tag{2·9}$$

where the first map is induced by the adjunction $\wedge^l W \to W^\vee \otimes \wedge^{l+1} W$ of the wedge product, and the second map is induced by the adjunction $V \otimes W^\vee \to H^{1,1}(X_b)$ of the differential of the period map. By our initial Condition 2·1, the second map of (2·9) is injective. Moreover, since $l + 1 \leq \dim W$ by our assumption, the wedge product $\wedge^l W \times W \to \wedge^{l+1} W$ is nondegenerate, so its adjunction $\wedge^l W \to W^\vee \otimes \wedge^{l+1} W$ is injective. Thus the first map of (2·9) is also injective. It follows that (2·8) is injective. Since the map (2·7) is the direct sum of its (σ, i)-components, it is injective. This finishes the proof of Lemma 2·3.

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We can now complete the proof of Proposition 2·2.

Proof of Proposition 2·2. By Lemma 2·3 (2), we have $E_\infty^{l,-l}=0$ when $l<l_0=\min(\dim B,k)$. Together with Lemma 2·3 (3), we obtain

$$(\pi_n)_*\Omega^k_{X_n}=E^0_\infty=E^{l_0,-l_0}_\infty=E^{l_0,-l_0}_1.$$

When $k\leq \dim B$, we have $l_0=k$, and $E_1^{l_0,-l_0}=\Omega^k_B$ by (2·1). When $k>\dim B$, we have $l_0=\dim B$, and $E_1^{l_0,-l_0}=\mathcal{H}^{k-\dim B,0}\otimes K_B$ by (2·1). When $k-\dim B$ is odd, this vanishes by Lemma 2·3 (1).

In the case $k=\dim B+2m$, the vector bundle $\mathcal{H}^{2m,0}\otimes K_B=(\pi_n)_*\Omega^{2m}_{\pi_n}\otimes K_B$ can be written more specifically as follows. For a subset σ of $\{1,\cdots,n\}$ with cardinality $|\sigma|=m$, we denote by $X_\sigma\simeq X_m$ the fiber product of the ith factors $X\to B$ of $X_n\to B$ over all $i\in\sigma$. We denote by

$$X_n\stackrel{\pi_\sigma}{\to}X_\sigma\stackrel{\pi^\sigma}{\to}B$$

the natural projections. The Künneth formula (2·5) says that

$$(\pi_n)_*\Omega^{2m}_{\pi_n}\simeq\bigoplus_{|\sigma|=m}\pi^\sigma_*K_{\pi^\sigma}.$$

Combining this with the isomorphism

$$\pi^\sigma_*K_{X_\sigma}\simeq K_B\otimes\pi^\sigma_*K_{\pi^\sigma} \tag{2·10}$$

for each X_σ, we can rewrite the isomorphism in the last case of Proposition 2·2 as

$$(\pi_n)_*\Omega^{\dim B+2m}_{X_n}\simeq\bigoplus_{|\sigma|=m}\pi^\sigma_*K_{X_\sigma}. \tag{2·11}$$

2·2. *Extension over compactification*

Let $\pi:X\to B$ be a $K3$ fibration as in Section 2·1. We now assume that X, B are quasi-projective and π is a morphism of algebraic varieties. We take smooth projective compactifications of X_n, X_σ, B and denote them by $\bar{X}_n, \bar{X}_\sigma, \bar{B}$ respectively.

PROPOSITION 2·4. *We have*

$$H^0(\bar{X}_n,\Omega^k)\simeq\begin{cases}H^0(\bar{B},\Omega^k) & k\leq\dim B\\ 0 & k>\dim B,\ \ k\not\equiv\dim B \mod 2\\ \oplus_\sigma H^0(\bar{X}_\sigma,K_{\bar{X}_\sigma}) & k=\dim B+2m,\ 0\leq m\leq n\end{cases}$$

In the last case, σ ranges over all subsets of $\{1,\cdots,n\}$ with $|\sigma|=m$. The isomorphism in the first case is given by the pullback by $\pi_n: X_n\to B$, and the isomorphism in the last case is given by the direct sum of the pullbacks by $\pi_\sigma: X_n\to X_\sigma$ for all σ.

Proof. The assertion in the case $k>\dim B$ with $k\not\equiv\dim B$ mod 2 follows directly from the second case of Proposition 2·2. Next we consider the case $k\leq\dim B$. We may assume that $\pi_n: X_n\to B$ extends to a surjective morphism $\bar{X}_n\to\bar{B}$. Let ω be a holomorphic k-form on $\bar{X}_n$. By the first case of Proposition 2·2, we have $\omega|_{X_n}=\pi_n^*\omega_B$ for a holomorphic

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k-form ω_B on B. Since ω is holomorphic over $\bar{X}_n$, ω_B is holomorphic over $\bar{B}$ as well by a standard property of holomorphic differential forms. (Otherwise ω must have pole at the divisors of $\bar{X}_n$ dominating the divisors of $\bar{B}$ where ω_B has pole.) Therefore the pullback $H^0(\bar{B}, \Omega^k) \to H^0(\bar{X}_n, \Omega^k)$ is surjective.

Finally, we consider the case $k = \dim B + 2m$, $0 \le m \le n$. Let ω be a holomorphic k-form on $\bar{X}_n$. By (2·11), we can uniquely write $\omega|_{X_n} = \sum_\sigma \pi_\sigma^* \omega_\sigma$ for some canonical forms ω_σ on X_σ.

Claim 2.5. For each σ, ω_σ is holomorphic over $\bar{X}_\sigma$.

Proof. We identify X_n with the fiber product $X_\sigma \times_B X_\tau$ where $\tau = \{1, \cdots, n\} - \sigma$ is the complement of σ. We may assume that this fiber product diagram extends to a commutative diagram of surjective morphisms

$$\begin{array}{ccc} \bar{X}_n & \xrightarrow{\pi_\tau} & \bar{X}_\tau \\ {\scriptstyle \pi_\sigma}\downarrow & & \downarrow{\scriptstyle \pi^\tau} \\ \bar{X}_\sigma & \xrightarrow{\pi^\sigma} & \bar{B} \end{array}$$

between smooth projective models. We take an irreducible subvariety $\tilde{B} \subset \bar{X}_\tau$ such that $\tilde{B} \to \bar{B}$ is surjective and generically finite. Then $\pi_\tau^{-1}(\tilde{B}) \subset \bar{X}_n$ has a unique irreducible component dominating $\tilde{B}$. We take its desingularisation and denote it by Y. By construction $\pi_\sigma|_Y \colon Y \to \bar{X}_\sigma$ is dominant (and so surjective) and generically finite. On the other hand, for any $\sigma' \neq \sigma$ with $|\sigma'| = m$, the projection $\pi_{\sigma'}|_Y \colon Y \dashrightarrow X_{\sigma'}$ is not dominant. Indeed, such σ' contains at least one component $i \in \tau$, so if $Y \dashrightarrow X_{\sigma'}$ was dominant, then the ith projection $Y \dashrightarrow X$ would be also dominant, which is absurd because it factorises as $Y \to \tilde{B} \subset \bar{X}_\tau \dashrightarrow X$.

We pullback the differential form $\omega = \pi_\sigma^* \omega_\sigma + \sum_{\sigma' \neq \sigma} \pi_{\sigma'}^* \omega_{\sigma'}$ to Y and denote it by $\omega|_Y$. Since ω is holomorphic over $\bar{X}_n$, $\omega|_Y$ is holomorphic over Y. Since $\pi_{\sigma'}^* \omega_{\sigma'}|_Y$ is the pullback of the canonical form $\omega_{\sigma'}$ on $X_{\sigma'}$ by the non-dominant map $Y \dashrightarrow X_{\sigma'}$, it vanishes identically. Hence $\pi_\sigma^* \omega_\sigma|_Y = \omega|_Y$ is holomorphic over Y. Since $\pi_\sigma|_Y \colon Y \to \bar{X}_\sigma$ is surjective, this implies that ω_σ is holomorphic over $\bar{X}_\sigma$ as before.

The above argument will be clear if we consider over the generic point η of B: we restrict ω to the fiber of $(X_\eta)^n \to (X_\eta)^\tau$ over the geometric point $\tilde{B}$ of $(X_\eta)^\tau$ over η.

By Claim 2.5, the pullback

$$(\pi_\sigma^*)_\sigma \colon \bigoplus_{|\sigma| = m} H^0(\bar{X}_\sigma, K_{\bar{X}_\sigma}) \to H^0(\bar{X}_n, \Omega^{\dim B + 2m})$$

is surjective. It is also injective as implied by (2·11). This proves Proposition 2·4.

2·3. *Universal K3 surface.*

Now we prove Theorem 1·1, in the generality of lattice-polarisation. Let L be an even lattice of signature $(2, d)$ which can be embedded as a primitive sublattice of the $K3$ lattice $3U \oplus 2E_8$. We denote by

$$\mathcal{D} = \{ \mathbb{C}\omega \in \mathbb{P}L_{\mathbb{C}} \mid (\omega, \omega) = 0, (\omega, \bar{\omega}) > 0 \}^+$$

the Hermitian symmetric domain associated to L, where $+$ means a connected component.

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Let $\pi: X \to B$ be a smooth projective family of $K3$ surfaces over a smooth quasi-projective connected base B. We say ([3]) that the family $\pi: X \to B$ is *lattice-polarised with period lattice* L if there exists a sub local system Λ of $R^2\pi_*\mathbb{Z}$ such that each fiber Λ_b is a hyperbolic sublattice of the Néron-Severi lattice $NS(X_b)$ and the fibers of the orthogonal complement $\Lambda^\perp$ are isometric to L. Then we have a period map

$$\mathcal{P}: B \to \Gamma\backslash\mathcal{D}$$

for some finite-index subgroup Γ of $\mathrm{O}^+(L)$. By Borel's extension theorem, $\mathcal{P}$ is a morphism of algebraic varieties.

Let us put the assumption

$$\mathcal{P} \text{ is birational and } -\mathrm{id} \notin \Gamma. \tag{2·12}$$

For such a family $\pi: X \to B$, if we shrink B as necessary, then $\mathcal{P}$ is an open immersion and Condition 2·1 is satisfied. For example, the universal $K3$ surface $\mathcal{F}_{g,1} \to \mathcal{F}_g$ for $g > 2$ restricted over a Zariski open set of $\mathcal{F}_g$ satisfies this assumption with $L = L_g$ and $\Gamma = \Gamma_g$ (see Section 1 for these notations).

As in Section 1, we denote by $M_{\wedge^k,k}(\Gamma)$ the space of vector-valued modular forms of weight $(\wedge^k, k)$ for Γ, $S_l(\Gamma, \det)$ the space of scalar-valued cusp forms of weight l and character det for Γ, and $\mathcal{S}_{n,m} = \mathfrak{S}_n/(\mathfrak{S}_m \times \mathfrak{S}_{n-m})$.

THEOREM 2·6. *Let $\pi: X \to B$ be a lattice-polarised K3 family with period lattice L of signature (2, d) with $d \geq 3$ and monodromy group Γ satisfying (2·12). Then we have an $\mathfrak{S}_n$-equivariant isomorphism*

$$H^0(\bar{X}_n, \Omega^k) \simeq \begin{cases} 0 & 0 < k < d/2 \\ M_{\wedge^k,k}(\Gamma) & d/2 \leq k < d \\ 0 & k > d,\ k-d \notin 2\mathbb{Z} \\ S_{d+m}(\Gamma, \det) \otimes \mathbb{C}\mathcal{S}_{n,m} & k = d+2m,\ 0 \leq m \leq n \end{cases}.$$

Proof. When $k \leq d$, we have $H^0(\bar{X}_n, \Omega^k) \simeq H^0(\bar{B}, \Omega^k)$ by Proposition 2·4. Then $\bar{B}$ is a smooth projective model of the modular variety $\Gamma\backslash\mathcal{D}$. By a theorem of Pommerening [5], the space $H^0(\bar{B}, \Omega^k)$ for $k < d$ is isomorphic to the space of Γ-invariant holomorphic k-forms on $\mathcal{D}$, which in turn is identified with the space $M_{\wedge^k,k}(\Gamma)$ of vector-valued modular forms of weight $(\wedge^k, k)$ for Γ (see [4]). The vanishing of this space in $0 < k < d/2$ is proved in [4, theorem 1·2] in the case when L has Witt index 2, and in [4, theorem 1·5 (1)] in the case when L has Witt index ≤ 1.

The vanishing in the case $k > d$ with $k \not\equiv d$ mod 2 follows from Proposition 2·4. Finally, we consider the case $k = d+2m$, $0 \leq m \leq n$. By Proposition 2·4, we have a natural $\mathfrak{S}_n$-equivariant isomorphism

$$H^0(\bar{X}_n, \Omega^{d+2m}) \simeq \bigoplus_{|\sigma|=m} H^0(\bar{X}_\sigma, K_{\bar{X}_\sigma}),$$

where $\mathfrak{S}_n$ permutes the subsets σ of $\{1, \cdots, n\}$. Here note that the stabiliser of each σ acts on $H^0(\bar{X}_\sigma, K_{\bar{X}_\sigma})$ trivially by (2·10). Therefore, as an $\mathfrak{S}_n$-representation, the right-hand side can be written as

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$$H^0(\bar{X}_m, K_{\bar{X}_m}) \otimes \left(\bigoplus_{|\sigma|=m} \mathbb{C}\sigma \right) \simeq H^0(\bar{X}_m, K_{\bar{X}_m}) \otimes \mathbb{C}\mathcal{S}_{n,m}.$$

Finally, we have $H^0(\bar{X}_m, K_{\bar{X}_m}) \simeq S_{d+m}(\Gamma, \det)$ by [**3**, theorem 3·1].

Remark 2·7. The case $k \geq d$ of Theorem 2·6 holds also when $d = 1, 2$. We put the assumption $d \geq 3$ for the requirement of the Koecher principle from [**5**]. Therefore, in fact, only the case $(d, k) = (2, 1)$ with Witt index 2 is not covered.

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