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CONSTRUCTION OF LARGE SETS OF ALMOST DISJOINT STEINER TRIPLE SYSTEMS

C. C. LINDNER AND A. ROSA

1. Introduction. A *Steiner triple system* (briefly STS) is a pair (S, t) where S is a set and t is a collection of 3-subsets of S (called triples) such that every 2-subset of S is contained in exactly one triple of t. The number $|S|$ is called the order of the STS (S, t). It is well-known that there is an STS of order v if and only if $v \equiv 1$ or $3 \pmod 6$. Therefore in saying that a certain property concerning STS is true for all v it is understood that $v \equiv 1$ or $3 \pmod 6$. Two Steiner triple systems (S, t_1) and (S, t_2) are said to be *disjoint* if t_1 and t_2 have no triples in common, and are said to be *almost disjoint* if t_1 and t_2 have *exactly one* triple in common. In [**2**], Doyen has shown the existence of a pair of disjoint STS of order v for every $v \geqq 7$, and in [**6**] Lindner has shown the existence of a pair of almost disjoint STS of every order $v \geqq 3$. By a *large set* of STS of order v is meant a collection of distinct triple systems (S, t_1), (S, t_2), $\ldots$, (S, t_k) such that $t_1 \cup t_2 \cup \ldots \cup t_k = S_3$, the set of all 3-subsets of a v-set S. Since any STS of order v has $v(v-1)/6$ triples and the number of 3-subsets of a v-set is $v(v-1)(v-2)/6$ it follows that every large set of mutually disjoint STS of order v contains exactly $v - 2$ triple systems. The problem of constructing large sets of mutually disjoint STS is far from settled. Large sets of mutually disjoint STS have been constructed for the following orders: 9 (Kirkman [**4**]); 13, 15, 19, 21, 25, 31, 33, 43, 49, 61, 69 (Denniston [**1**]); and $3v$ whenever a large set of order v exists (Teirlinck [**10**]).

The purpose of this paper is to give a construction for a large set of mutually almost disjoint (MAD) STS of every order $v \geqq 3$. However, the situation with respect to the number of MAD STS in a large set is not as clear as in the case of disjoint STS. The construction used in this paper to construct large sets of MAD STS is obtained via the use of Steiner quadruple systems and is a slight modification of the construction given in [**7**] to construct almost disjoint STS. This construction always gives a large set of v MAD STS of order v. The authors have not been able to rule out the existence of large sets containing $v - 1$ or $v + 1$ MAD STS of order $v \geqq 13$. More will be said about this problem in a later section.

2. Construction of large sets of MAD Steiner triple systems. A *Steiner quadruple system* (briefly SQS) is a pair (Q, q) where Q is a set and q is a collection of 4-subsets of Q (called quadruples) such that every 3-subset of Q

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is contained in exactly one quadruple of q. The number $|Q|$ is called the order of the SQS (Q, q). It was shown by Hanani [**3**] that an SQS of order v exists if and only if $v \equiv 2$ or $4 \pmod 6$. If (Q, q) is an SQS and x is any element of Q we will denote $Q \backslash \{x\}$ by Q_x and the set of all triples $\{a, b, c\}$ such that $\{x, a, b, c\} \in q$ by $q(x)$. It is a routine matter to see that $(Q_x, q(x))$ is an STS. If x and y are any two distinct elements of Q we will denote by $(Q_{xy}, q(xy))$ the STS obtained from $(Q_x, q(x))$ by replacing y with x.

THEOREM 1. *Let (Q, q) be an SQS based on $Q = \{0, 1, \ldots, v\}$. Then the Steiner triple systems $(Q_{10}, q(10))$, $(Q_{20}, q(20)), \ldots, (Q_{v0}, q(v0))$ are mutually almost disjoint.*

Proof. We begin by noting the trivial fact that $Q_{10} = Q_{20} = \ldots = Q_{v0} = \{1, 2, \ldots, v\}$. We show that $(Q_{10}, q(10))$ and $(Q_{20}, q(20))$ are almost disjoint. To begin with, for some $a \in Q$, we must have $\{0, 1, 2, a\} \in q$. Hence $\{1, 2, a\}$ belongs to both $q(10)$ and $q(20)$. Let us show that this is the only triple common to $q(10)$ and $q(20)$. Let $\{x, y, z\} \in q(10)$ be any triple distinct from $\{1, 2, a\}$. If both 1 and 2 belong to $\{x, y, z\}$ then $\{x, y, z\} = \{1, 2, a\}$, a contradiction. If $\{1, 2\} \cap \{x, y, z\} = \emptyset$ then $\{x, y, z\} \in q(10)$ is possible only if $\{1, x, y, z\} \in q$ with all $x, y, z \neq 0$ which in turn implies $\{x, y, z\} \notin q(20)$. Hence we need only consider the cases where exactly one of 1, 2 belongs to $\{x, y, z\}$. If, say, $\{1, y, z\} \in q(10)$ then $\{0, 1, y, z\} \in q$ and so $\{1, y, z\} \notin q(20)$; if $\{2, y, z\} \in q(10)$ then $\{1, 2, y, z\} \in q$ and so $\{2, y, z\} \notin q(20)$. Hence $(Q_{10}, q(10))$ and $(Q_{20}, q(20))$ have the unique triple $\{1, 2, a\}$ in common, and the statement of the theorem follows.

THEOREM 2. *The triple systems $(Q_{10}, q(10))$, $(Q_{20}, q(20)), \ldots, (Q_{v0}, q(v0))$ form a large set of MAD STS.*

Proof. We begin by noting that if $\{x, y, z\} \subset \{1, 2, \ldots, v\}$ and $\{0, x, y, z\} \in q$ then the triple $\{x, y, z\}$ belongs to exactly three of the sets $q(10), q(20), \ldots, q(v0)$, namely to $q(x0)$, $q(y0)$ and $q(z0)$. On the other hand, if $\{w, x, y, z\} \in q$ with $w \neq 0$ then the triple $\{x, y, z\}$ belongs to exactly one set of triples, namely $q(w0)$. It follows that in counting the number of distinct triples belonging to $q(10) \cup q(20) \cup \ldots \cup q(v0)$, each triple $\{x, y, z\}$ where $\{0, x, y, z\} \in q$ is counted three times whereas all other triples are counted once. Hence

$$|q(10) \cup q(20) \cup \ldots \cup q(v0)| = v[v(v-1)/6] - 2[v(v-1)/6] = v(v-1)(v-2)/6$$

which is the number of all 3-subsets of a v-set. Hence the set of MAD STS $(Q_{10}, q(10))$, $(Q_{20}, q(20)), \ldots, (Q_{v0}, q(v0))$ is a large set completing the proof.

Let us call a set of MAD STS *maximal* (*=nonextendable*) if it is not a proper subset of some set of MAD STS.

THEOREM 3. *For $v \geqq 9$, the large set of MAD STS $(Q_{10}, q(10))$, $(Q_{20}, q(20)), \ldots, (Q_{v0}, q(v0))$ is a maximal set of MAD STS.*

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Proof. Let (Q, t) be an STS, $Q = \{1, 2, \ldots, v\}$ such that $(Q_{10}, q(10))$, $(Q_{20}, q(20)), \ldots, (Q_{v0}, q(v0))$, (Q, t) is a set of MAD STS. Then, since $(Q_{10}, q(10)), (Q_{20}, q(20)), \ldots, (Q_{v0}, q(v0))$ is a large set of MAD STS, we must have $t \subseteq q(10) \cup q(20) \cup \ldots \cup q(v0)$. Since t has exactly one triple in common with each of the $q(10), \ldots, q(v0)$, t contains at most v triples. But this is impossible, since, on the other hand, t contains exactly $v(v-1)/6$ triples and $v(v-1)/6 > v$ for $v \geqq 9$.

Remark. For $v = 7$ the proof of Theorem 3 fails because an STS of order 7 has 7 triples. In fact, the reader can easily construct for himself large sets of MAD STS of order 7 containing 7, 8, 9, 10, 11, 12, 13, 14 or 15 Steiner triple systems. Any set of less than 15 MAD STS can be extended to a unique maximal set of 15 MAD STS of order 7; as a matter of fact, the set of all 30 distinct STS of order 7 (see, e.g., [**8**] or [**9**]) can be partitioned into two maximal sets with 15 MAD STS each.

Theorems 1, 2 and 3 are trivially true for $v = 3$. Observe also that the proof of Theorem 3 is independent of the construction of the large set of v MAD STS.

3. The size of large sets of MAD STS. For $v = 3$ there is, of course, only one large set of STS. As mentioned in the previous Section, for $v = 7$ there are large sets of MAD STS containing 7, 8, 9, 10, 11, 12, 13, 14 or 15 STS, but a maximal set of MAD STS can contain only 15 STS. For $v = 9$, a complete analysis is still possible. It is well-known that any two STS of order 9 are isomorphic, and there are all together 840 distinct STS of order 9 based on the same set; given any STS of order 9 there are 216 STS almost disjoint with it (cf. [**5**]), and any two pairs of MAD STS of order 9 are isomorphic. It turns out that given any two almost disjoint STS of order 9, there are 44 further STS almost disjoint with both of them. An inspection of these STS reveals that there exist maximal sets of MAD STS of order 9 containing 4, 5 or 9 STS, respectively, there is up to an isomorphism only one large set of MAD STS (containing, of course, 9 STS), and any pair of almost disjoint STS can be extended to a unique large set of MAD STS.

For $v = 13$ it is easy to see that a large set of MAD STS must contain 12, 13, 14 or 15 STS. However, there are too many distinct triple systems of order 13 for a brute force attempt at constructing large sets of MAD STS containing 12, 14 or 15 STS.

THEOREM 4. *If $v \geqq 15$, then a large set of MAD STS of order v must contain $v - 1$ or v or $v + 1$ triple systems.*

Proof. To see that a large set (actually, any set) of MAD STS of order v can contain at most $v + 1$ triple systems we show that a set of $v + 1$ MAD STS cannot be extended. Suppose $(S, t_1), (S, t_2), \ldots, (S, t_{v+1})$ are $v + 1$ MAD STS of order v. In order for this set to be extended it must be the case that

$$v(v-1)(v-2)/6 - |t_1 \cup t_2 \cup \ldots \cup t_{v+1}| \geqq v(v-1)/6 - (v+1).$$

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However, $t_1 \cup t_2 \cup \ldots \cup t_{v+1}$ must contain at least $v(v-1)/6 + (v(v-1)/6 - 1) + (v(v-1)/6 - 2) + \ldots + (v(v-1)/6 - v) = v(v+1)(v-4)/6$ triples. Hence

$$v(v-1)(v-2)/6 - |t_1 \cup t_2 \cup \ldots \cup t_{v+1}| \leqq v(v-1)(v-2)/6 - v(v+1)(v-4)/6 = v.$$

However, for $v \geqq 15$,

$$v < v(v-1)/6 - (v+1).$$

In order to show that a large set of MAD STS of order v must contain at least $v - 1$ triple systems we show that no set of $v - 2$ MAD STS can form a large set. Again let $(S, t_1), (S, t_2), \ldots, (S, t_{v-2})$ be a set of MAD STS. Since at most $v - 4$ of these triple systems can intersect in the same triple it follows that the maximum number of triples in $t_1 \cup t_2 \cup \ldots \cup t_{v-2}$ is at most

$$[v(v-1)/6 + (v-5)(v(v-1)/6 - 1)] + [v(v-1)/6 + (v(v-1)/6 - 1)] = v(v-1)(v-2)/6 - v + 4.$$

Since a large set of MAD STS must contain $v(v-1)(v-2)/6$ triples, the proof is complete.

Corollary 5. *For $v \geqq 15$, a large set of MAD STS of order v is maximal.*

Proof. Since $v \geqq 15$, a large set of MAD STS contains $v - 1$, v or $v + 1$ triple systems. In each of the three cases, the proof of the impossibility to extend a large set is analogous to the proof of Theorem 3.

Incidentally, Corollary 5 holds also for $v = 9$.

4. Problems.

We close this paper with some problems.

(1) For $v = 13$, can large sets of MAD STS containing 12, 14 or 15 triple systems be constructed?

(2) For $v \geqq 15$, can large sets of MAD STS containing $v - 1$ or $v + 1$ triple systems be constructed?

(3) For $v \geqq 15$, every large set of MAD STS is maximal. Is it true that for every $v \geqq 9$ there is a maximal set of MAD STS which is not large? What can be said about the minimum cardinality of a maximal set of MAD STS of order v?

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Auburn University,
Auburn, Alabama;
McMaster University,
Hamilton, Ontario

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