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CONVEX FUNCTIONS ON BANACH SPACES NOT CONTAINING ℓ_1

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ABSTRACT. There is a sizeable class of results precisely relating boundedness, convergence and differentiability properties of continuous convex functions on Banach spaces to whether or not the space contains an isomorphic copy of ℓ_1. In this note, we provide constructions showing that the main such results do not extend to natural broader classes of functions.

Introduction. Following [3], we will say a Banach space is *sequentially reflexive* if Mackey and norm convergence coincide sequentially in its dual space. In addition to showing that Asplund spaces are sequentially reflexive, [3] also shows that weak Hadamard and Fréchet differentiability coincide for continuous convex functions on sequentially reflexive spaces (and thus on all Asplund spaces which was quite unexpected; see also [4]). Using Rosenthal's ℓ_1 theorem, [10] shows that a Banach space X is sequentially reflexive if and only if $X \not\supset \ell_1$ (meaning no subspace of X is isomorphic to ℓ_1). Sequential reflexivity has turned out to be an extremely useful notion in convex analysis. Indeed, in addition to its implications in the study of differentiability properties of convex functions [3, 4], it has applications to boundedness and convergence properties of convex functions [2, 6, 7].

Because the notion of uniform convergence on bounded sets plays a fundamental role in convex analysis and optimization (see [1]), it is natural to ask when it is implied by weaker forms of convergence. We studied this question in [7], where among other results it was shown that on each sequentially reflexive space, every sequence of lsc convex functions converging uniformly on weakly compact sets to a continuous affine function converges uniformly on bounded sets. However, it was not known if this result still holds when the limit function is only a continuous convex or even Lipschitz convex function—Theorem 1 (a) below shows, in a decisive fashion, that it fails even for norms.

Throughout, we will work with real Banach spaces. We use B_X and S_X to denote the closed unit ball and unit sphere of X. Definitions of additional basic concepts used but not defined here can be found in [1], [8], or [11]. We also often use the Eberlein-Šmulian theorem (see [8]) without specific reference to it.

THEOREM 1. *Let X be a nonreflexive Banach space. Then:*

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(a) *There is a non-increasing sequence of equi-Lipschitz norms converging uniformly on weakly compact sets to a norm, but such that the convergence is not uniform on bounded sets.*

(b) *There are norms μ and ν on X such that $\mu-\nu$ is weak Hadamard but not Fréchet differentiable at some point.*

(c) *There are continuous convex functions f and h on X such that $h-f$ is bounded on weakly compact sets, but not on bounded sets.*

Before proceeding with the proof of Theorem 1, let us recall that a function f is *weak Hadamard differentiable* at x if there is a $\Lambda \in X^*$ such that

$$\lim_{t\downarrow 0}[f(x+th)-f(x)-\Lambda(th)]/t = 0$$

with the limit being uniform for h in weakly compact sets. Notice that Theorem 1(b) and (c) sharply contrast the following known result which is a restatement of parts of [4, Theorem 2] and [6, Theorem 2.4].

THEOREM 2. *For a Banach space X the following are equivalent.*

(a) *X is sequentially reflexive or equivalently $X \not\supset \ell_1$.*

(b) *Weak Hadamard and Fréchet differentiability coincide for continuous convex functions on X.*

(c) *Every continuous convex function bounded on weakly compact subsets of X is bounded on bounded subsets of X.*

The following lemma is a key component in the proof of Theorem 1. Notice that the system it provides is stronger than a sequence in the dual space that is weak* null but not norm null as given by the Josefson-Nissenzweig theorem in the dual of each infinite dimensional Banach space (see [8, p. 219]). However, its proof shares similarities with the manner in which the Josefson-Nissenzweig theorem is derived from [9, Corollary 1].

LEMMA 3. *If X is not reflexive and $X \not\supset \ell_1$, then there is a system $\{x_n, \Lambda_n\} \subset B_X \times X^*$ such that $\Lambda_n \xrightarrow{w^*} 0$ and*

(a) *$1-\epsilon \le \Lambda_k(x_n) \le 1$ for $k \le n$,*

(b) *$\Lambda_k(x_n) = 0$ whenever $k > n$.*

PROOF. First suppose that B_{X^*} is w^*-sequentially compact. For $\epsilon > 0$ given, we shall construct a system $\{u_n, \phi_n\} \subset S_X \times S_{X^*}$ such that

$$1-\epsilon \le \phi_k(u_n) \le 1 \text{ if } k \le n; \tag{1}$$

$$\phi_k(u_n) = 0 \text{ if } k > n. \tag{2}$$

Because X is not reflexive, we can choose $\Phi \in S_{X^{**}} \setminus X$ such that $d(\Phi, X) > 1-\epsilon$. By Goldstine's theorem one can fix a net $\{x_\alpha\} \subset S_X$ such that $x_\alpha \xrightarrow{w^*} \Phi$. Let $\phi_1 \in S_{X^*}$ satisfy $\phi_1(\Phi) > 1-\epsilon$; fix α_1 such that $\phi_1(x_\alpha) > 1-\epsilon$ for $\alpha \ge \alpha_1$, and let $u_1 = x_{\alpha_1}$. Suppose $u_1, \ldots, u_n$ and $\phi_1, \ldots, \phi_n$ have been chosen appropriately. To choose ϕ_{n+1},

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let $E_n := \operatorname{span}\{u_1, \ldots, u_n\}$. Then E_n is w^*-closed, and by the separation theorem one chooses $\phi_{n+1} \in S_{X^*}$ such that $\phi_{n+1}(E_n) = \{0\}$, and $\phi_{n+1}(\Phi) > 1 - \epsilon$. Fix $\alpha_{n+1} \geq \alpha_n$ such that $\phi_{n+1}(x_\alpha) > 1 - \epsilon$ for $\alpha \geq \alpha_{n+1}$ and let $u_{n+1} = x_{\alpha_{n+1}}$. It follows from the construction that $\{u_n, \phi_n\}$ satisfy (1) and (2). By w^*-sequential compactness, $\phi_{n_k} \xrightarrow{w^*} \phi$ for some ϕ. We let $x_k := u_{n_k}$ and $\Lambda_k := \phi_{n_k} - \phi$. Observe that $\phi(u_n) = 0$ for each n, so it follows that the system $\{x_k, \Lambda_k\}$ satisfies the conditions of the lemma.

It remains to prove the lemma in the case B_{X^*} is not w^*-sequentially compact. Because $X \not\supset \ell_1$ and B_{X^*} is not w^*-sequentially compact, [9, Corollary 1] shows that c_0 is isomorphic to a quotient of X. Let T be the quotient map of X onto Y where Y is isomorphic to c_0. From the first part of the proof, there is a system $\{y_n, y_n^*\} \subset Y \times Y^*$ satisfying the conclusion of the lemma. Choose u_n such that $\{u_n\}_{n=1}^\infty$ is bounded and $Tu_n = y_n$. Let $K > 0$ be such that $Ku_n \in B_X$ for each n. For $x_n := Ku_n$ and $\Lambda_n := \frac{1}{K} y_n^* \circ T$, it follows that $\{x_n, \Lambda_n\}$ has the desired properties. ∎

PROOF OF THEOREM 1. (a) If $X \supset \ell_1$, then X is not sequentially reflexive [10] and so there is a sequence $\{\Lambda_n\} \subset S_{X^*}$ that converges to 0 uniformly on weakly compact sets. Hence the sequence of norms $\|\cdot\|_n$ defined by $\|x\|_n := \|x\| + \sup_{k \geq n} |\Lambda_k(x)|$ decreases to $\|\cdot\|$ uniformly on weakly compact sets but not on bounded sets. So we may assume $X \not\supset \ell_1$ and that $\{x_n, \Lambda_n\}$ is a system in $B_X \times X^*$ as given by Lemma 3 with $\epsilon = 1/4$. Thus in particular,

$$3/4 \leq \Lambda_k(x_n) \leq 1 \text{ if } k \leq n; \tag{3}$$

$$\Lambda_k(x_n) = 0 \text{ if } k > n. \tag{4}$$

Now one can use the system $\{x_n, \Lambda_n\}$ to define norms

$$\nu(x) := \|x\| + \sup_n \sup_{k \leq n} \Big| \Big(\Lambda_n - \frac{1}{2} \Lambda_k \Big)(x) \Big|,$$
$$\nu_n(x) := \max\{\nu(x), \|x\| + \sup_{k \geq n} |\Lambda_k(x)|\}.$$

Notice that these norms are uniformly bounded on B_X because $\{\Lambda_n\}$ is norm bounded.

Let us first check that ν_n converges to ν uniformly on weakly compact sets. Indeed, because the sequence $\{\nu_n\}$ is nonincreasing, if it did not converge on some weakly compact set, one could find $w_n \xrightarrow{w} \bar{w}$ such that $\limsup_{n\to\infty} \nu_n(w_n) - \nu(w_n) > 0$. However, for any fixed $\epsilon > 0$, because $\Lambda_k \xrightarrow{w^*} 0$, we can find k_0 such that $|\Lambda_{k_0}(\bar{w})| < \epsilon$. Thus for some n_0, $|\Lambda_{k_0}(w_n)| < \epsilon$ for all $n \geq n_0$. Let $N = \max\{k_0, n_0\}$. Then for $n \geq N$, we have

$$\nu(w_n) \geq \|w_n\| + \sup_{m \geq N} \Big| \Big(\Lambda_m - \frac{1}{2} \Lambda_{k_0} \Big)(w_n) \Big|$$
$$\geq \|w_n\| + \sup_{m \geq N} |\Lambda_m(w_n)| - \epsilon.$$

Using this with the definition of ν_n, one sees that $\nu(w_n) \geq \nu_n(w_n) - \epsilon$ for $n \geq N$. Thus we conclude the convergence is uniform on weakly compact sets.

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To see that the convergence is not uniform on bounded sets, using (3) and (4) it follows for $k \le n$ that

$$0 \le \Big(\Lambda_n - \frac{1}{2}\Lambda_k\Big)(x_m) \le 1 - \frac{3}{8} \le \frac{5}{8} \text{ if } n \le m; \tag{5}$$

$$-\frac{1}{2} \le \Big(\Lambda_n - \frac{1}{2}\Lambda_k\Big)(x_m) = 0 - \frac{1}{2}\Lambda(x_k) \le 0 \text{ if } n > m. \tag{6}$$

Therefore, $\nu(x_n) \le 5/8 + \|x_n\|$ for all n, while $\nu_n(x_n) \ge \Lambda_n(x_n) + \|x_n\| \ge 3/4 + \|x_n\|$ for all n. Hence the convergence is not uniform on the bounded set $\{x_n\}_{n=1}^{\infty}$. This proves (a).

To prove (b), if $X \supset \ell_1$, then [4, Theorem 2] shows there is a *norm* on X for which weak Hadamard and Fréchet differentiability do not coincide. So we may suppose $X \not\supset \ell_1$, and we write $X = H \times \mathbb{R}$. Now let $\{x_n, \Lambda_n\} \subset B_H \times H^*$ be a system given by Lemma 3 that satisfies (3) and (4). We let $\phi_{n,k}(x) := \Lambda_n(x) - 2\Lambda_k(x)$ for $k \le n$. Using (3) and (4) as in (5) and (6), one obtains

$$-2 \le \phi_{n,k}(x_m) \le 0 \text{ for all } m, n \text{ and } k \le n. \tag{7}$$

Motivated by [4], we let $\gamma_1 = 1/2$ and $\gamma_n = 1 - 1/n$ for $n \ge 2$, and we define functions f and g on $H \times \mathbb{R}$ by

$$f(x,t) := \sup_n \sup_{k \le n} |\phi_{n,k}(x) + t\gamma_n|,$$
$$g(x,t) := \sup_n |\Lambda_n(x) + t\gamma_n|.$$

The desired equivalent norms are now defined for $(x,t) \in H \times \mathbb{R}$ by

$$\mu(x,t) := \max\{f(x,t), g(x,t), \frac{1}{2}(\|x\| + |t|)\},$$
$$\nu(x,t) := \max\{f(x,t), \frac{1}{2}(\|x\| + |t|)\}.$$

We shall show that $\mu - \nu$ is weak Hadamard differentiable at $(0,1)$ but not Fréchet differentiable there. First observe that $f(0,1) = g(0,1) = 1$ while $1/2(\|0\| + |1|) = 1/2$. Therefore, the norm term in the definition of μ and ν is locally inactive around $(0,1)$ and so it suffices to show that $h - f$ is weak Hadamard but not Fréchet differentiable at $(0,1)$ where $h(x,t) := \max\{f(x,t), g(x,t)\}$.

Let us now show that $h - f$ has weak Hadamard derivative 0 at $(0,1)$. Because $(h-f)(0,1) = 0$ and $h - f \ge 0$, if $h - f$ were not weak Hadamard differentiable at $(0,1)$, one could find $t_n \downarrow 0$, $\epsilon > 0$ and a weakly convergent sequence $\{(w_n, r_n)\}$, such that

$$\limsup_{n\to\infty} \frac{h(t_n w_n, 1 + t_n r_n) - f(t_n w_n, 1 + t_n r_n)}{t_n} > \epsilon,$$

and consequently

$$\limsup_{n\to\infty} \frac{g(t_n w_n, 1 + t_n r_n) - f(t_n w_n, 1 + t_n r_n)}{t_n} > \epsilon. \tag{8}$$

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As in (a), one can fix k_0 and n_0, with $n_0 \geq k_0$ such that $|\Lambda_{k_0}(w_n)| < \epsilon/2$ for all $n \geq n_0$. Now fix $N \geq n_0$ such that

$$|\Lambda_m(t_n w_n)| + |t_n r_n| < \frac{1}{2n_0} \text{ for } n \geq N. \tag{9}$$

Then for $n \geq N$ and $m \leq n_0$ one has

$$|\Lambda_m(t_n w_n) + \gamma_n + t_n r_n \gamma_m| \leq |\gamma_m| + \frac{1}{2n_0} \leq 1 - \frac{1}{2n_0}. \tag{10}$$

For each $n \geq N$, using (10) with the definition of g yields

$$g(t_n w_n, 1 + t_n r_n) = \max\Big\{1 - \frac{1}{2n_0}, \sup_{m > n_0} |\Lambda_m(t_n w_n) + \gamma_m(1 + t_n r_n)|\Big\}. \tag{11}$$

Because $\phi_{m,m} \xrightarrow{w^*} 0$, using (9) for each $n \geq N$ one obtains

$$\begin{aligned} f(t_n w_n, 1 + t_n r_n) &\geq \lim_{m\to\infty} |\phi_{m,m}(t_n w_n) + \gamma_m + t_n r_n \gamma_m| \\ &\geq 1 - |t_n r_n| \\ &> 1 - \frac{1}{2n_0}. \end{aligned} \tag{12}$$

The definition of f, (11) and (12), for $n \geq N$, imply that

$$\begin{aligned} f(t_n w_n,\ 1 + t_n r_n) &\geq \max\Big\{1 - \frac{1}{2n_0}, \sup_{m>n_0} |\Lambda_m(t_n w_n) - 2\Lambda_{k_0}(t_n w_n) + \gamma_m(1 + t_n r_n)|\Big\} \\ &\geq \max\Big\{1 - \frac{1}{2n_0}, \sup_{m>n_0} [|\Lambda_m(t_n w_n) + \gamma_m(1 + t_n r_n)| - t_n\epsilon]\Big\} \\ &\geq g(t_n w_n,\ 1 + t_n r_n) - t_n\epsilon. \end{aligned}$$

This contradicts (8) and so we conclude that $h - f$ has weak Hadamard derivative 0 at 0.

Now we show that $h-f$ is not Fréchet differentiable at $(0, 1)$. First, by (3), $h(\frac{2}{n}x_n, 1) \geq \frac{2}{n}\Lambda_n(x_n) + 1 - \frac{1}{n} \geq 1 + \frac{1}{2n}$. However, since $\phi_{n,n}$ is weak* null (7) implies that $f(\frac{2}{n}x_n, 1) = 1$ for $n \geq 3$. Therefore,

$$\limsup_{n\to\infty} \frac{h(\frac{2}{n}x_n, 1) - f(\frac{2}{n}x_n, 1)}{\frac{2}{n}} \geq \frac{1}{4}.$$

Hence $h - f$ does not have Fréchet derivative 0 at $(0, 1)$, which proves (b).

Finally, to prove (c), according to Theorem 2 (see [6, Theorem 2.4]) we may assume $X \not\supset \ell_1$. Let $\{x_n, \Lambda_n\} \subset B_X \times X^*$ be a system as in Lemma 3 which satisfies (3) and (4). Let $\phi_{n,k} = \Lambda_n - \frac{2}{n}\Lambda_k$ for $1 \leq k \leq n$, and let $a_n = \Lambda_n(x_n)$. Using this, one defines real functions f_n by $f_n(t) = 0$ if $t \leq a_n - \frac{1}{n}$ and $f_n(t) = n!\,(t + \frac{1}{n} - a_n)$ if $t \geq a_n - \frac{1}{n}$. We define g_n by $g_n(t) = 0$ if $t \leq a_n - \frac{1}{2n}$ and $g_n(t) = n!\,(t + \frac{1}{2n} - a_n)$ if $t \geq a_n - \frac{1}{2n}$. The desired functions f and h are defined as follows:

$$\begin{aligned} f(x) &= \sup_n \sup_{k\leq n} \Big\{f_n\big(\phi_{n,k}(x)\big)\Big\}, \\ g(x) &= \sup_n \Big\{g_n\big(\Lambda_n(x)\big)\Big\} \text{ and} \\ h(x) &= \max\{f(x), g(x)\}. \end{aligned}$$

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First, we show that f and g (and hence h) are continuous convex functions. Indeed, it is clear that f and g are convex because they are suprema of such functions. Also, because $\Lambda_n \xrightarrow{w^*} 0$, one verifies as in the proof of [6, Lemma 2.1] that f and g are locally a maximum of finitely many Lipschitz functions and therefore continuous.

If $h - f$ were not bounded on some weakly compact set, one could find a weakly convergent sequence $\{w_n\}$ such that $\limsup_{n\to\infty}(h - f)(w_n) = \infty$, which implies $\limsup_{n\to\infty}(g - f)(w_n) = \infty$, as $h - f \geq 0$. However, as in (a), one finds k_0 and n_0 so that $|\Lambda_{k_0}(w_n)| < \frac{1}{4}$ for $n \geq n_0$. Let $N = \max\{k_0, n_0\}$. Then for $m, n \geq N$, it follows that

$$\begin{aligned}|\phi_{m,k_0}(w_n) - \Lambda_m(w_n)| &= \left|\left(\Lambda_m - \frac{2}{m}\Lambda_{k_0}\right)(w_n) - \Lambda_m(w_n)\right| \\ &= \frac{2}{m}|\Lambda_{k_0}(w_n)| \leq \frac{1}{2m}.\end{aligned}$$

Thus, $\phi_{m,k_0}(w_n) \geq \Lambda_m(w_n) - \frac{1}{2m}$ for $m, n \geq N$. Observe that the definitions of f_m and g_m imply that $f_m(s) \geq g_m(t)$ if $s \geq t - \frac{1}{2m}$. Consequently $f_m\big(\phi_{m,k_0}(w_n)\big) \geq g_m\big(\Lambda_m(w_n)\big)$ for $m, n \geq N$. Hence, letting $M := \sup_n |\Lambda_n(w_n)|$, for $k \geq N$ we have

$$\begin{aligned}g(w_n) &\leq \max\Big\{g_1\big(\Lambda_1(w_n)\big), \ldots, g_{N-1}\big(\Lambda_{N-1}(w_n)\big), f(w_n)\Big\} \\ &\leq \max\{N!\, M, f(w_n)\}.\end{aligned}$$

Thus $\limsup_{n\to\infty}(g - f)(w_n) \leq N!\, M$, and we conclude $h - f$ is bounded on each weakly compact set.

Finally, let us show that $h - f$ is unbounded on $\{x_n\}_{n=1}^{\infty}$. Indeed, $\Lambda_n(x_n) = a_n$ and so $g(x_n) \geq g_n\big(\Lambda_n(x_n)\big) = (n-1)!\,/2$. On the other hand, using (3) and (4), we obtain:

$$\begin{aligned}\phi_{n,k}(x_m) \leq 0 \text{ if } m < n \text{ and so } f_n\big(\phi_{n,k}(x_m)\big) = 0; \\ \phi_{n,k}(x_n) < a_n - 1/n \text{ and so } f_n\big(\phi_{n,k}(x_n)\big) = 0;\end{aligned}$$

and finally $\phi_{n,k}(x_m) < \Lambda_m(x_n) \leq 1$ for $n < m$, and so

$$\begin{aligned}f_n\big(\phi_{n,k}(x_m)\big) &< n!\,(1 + 1/n - 3/4) \\ &= (n-1)! + n!\,/4 \\ &\leq (m-2)! + (m-1)!\,/4.\end{aligned}$$

Therefore $f(x_m) \leq (m-2)! + (m-1)!\,/4$ and $g(x_m) - f(x_m) \geq (m-1)!\,/4 - (m-2)!$ which tends to ∞ as $m \longrightarrow \infty$. ∎

REMARK 4. (a) One can also construct an *increasing* sequence of norms as in Theorem 1(a). However, our proof requires a more complicated system than given by Lemma 3, so we have chosen to omit the details.

(b) An underlying theme from [4] and [6] is that many constructions involving convex functions are not as easy as it might first appear. We should also emphasize this here. Indeed, it may seem that it should be easy to construct a difference of continuous convex functions as in Theorem 1(c). However, as soon as the difference of continuous convex functions is unbounded on a bounded set, at least one of the continuous convex functions must be unbounded on a bounded set. From this, [6, Lemma 2.3] immediately produces a sequence in the dual sphere that converges weak* to 0. Therefore, the highly nontrivial Josefson-Nissenzweig theorem is a direct corollary of Theorem 1(c). This provides justification to our use of deep structural properties in Banach spaces, namely Rosenthal's ℓ_1 theorem and Hagler and Johnson's [9, Corollary 1].

We should add that Theorem 1(a) largely answers the main open question in our article [7]. However, one issue remains: in a sequentially reflexive space, whenever the directional derivative for a convex continuous function is approached uniformly on weak compact sets, is it approached uniformly on bounded sets? In other words, if the function is *directionally weak Hadamard differentiable* is it perforce *directionally Fréchet differentiable* even at points of non differentiability? This question was first answered by John Giles and Scott Sciffer, who informed us—immediately upon receiving an earlier version of this note which did not contain any of the results listed hereunder—that they have constructed an example showing the answer is negative on c_0. This motivated us to re-examine the consequences of Theorem 1(a), and indeed one can use it to provide a negative answer to the preceding question on all nonreflexive spaces:

COROLLARY 5. *On each nonreflexive space there is a Lipschitz convex function that is directionally weak Hadamard differentiable at* 0, *but such that it is not directionally Fréchet differentiable at* 0.

PROOF. By Theorem 1(a), there is a sequence of uniformly bounded norms $\{\nu_n\}_{n=1}^\infty$ decreasing to a norm ν uniformly on weakly compact sets, but not uniformly on bounded sets. Therefore, we can find $\delta > 0$ and $\{x_n\}_{n=1}^\infty$ bounded such that $\nu_n(x_n) > \nu(x_n) + 2\delta$. Now let

$$f(x) := \max\Big\{\nu(x), \sup_n \Big[\nu_n(x) - \frac{\delta}{n}\Big]\Big\},$$

and define $f_n(x) := n[f(0 + x/n) - f(0)]$. Then $f_n(x) = nf(x/n)$, and $d^+f(0)(x) = \lim_{n\to\infty} f_n(x)$ where $d^+f(0)(x)$ denotes the directional derivative of f at 0 in the direction x (see [11, Section 1] for basic properties of directional derivatives). We shall show that f_n converges to ν uniformly on weakly compact sets, but not on bounded sets; consequently f is directionally weak Hadamard differentiable at 0, with $d^+f(0)(x) = \nu(x)$.

Let W be a weakly compact set and let $\epsilon > 0$. We fix $N \in \mathbb{N}$ such that $\nu_n(w) \le \nu(w)+\epsilon$ for all $n \ge N$ and $w \in W$. Then choose $M \in \mathbb{N}$ such that

$$\nu_n(w/M) - \frac{\delta}{N} \le 0 \text{ for all } w \in W,\ n \in \mathbb{N}.$$

Using this for $w \in W$ and $n \ge \max\{M, N\}$, we have

$$\begin{aligned}\nu(w) \le f_n(w) &= n \max\{\nu(w/n), \sup_k [\nu_k(w/n) - \delta/k]\}\\ &\le n \max\{\nu(w/n), \max_{k\le N}[\nu_k(w/n) - \delta/N], \sup_{k>N}[\nu_k(w/n) - \delta/k]\}\end{aligned}$$

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$$\leq n \max\Big\{\nu(w/n),\ 0,\ \sup_{k\geq N} \frac{1}{n}\nu_k(w)\Big\}$$
$$\leq n \max\Big\{\nu(w/n), \frac{1}{n}\big(\nu(w)+\epsilon\big)\Big\}$$
$$= \nu(w)+\epsilon.$$

Finally, we shall show the limit is not uniform on the bounded set $\{x_n\}_{n=1}^{\infty}$ (and so f is not directionally Fréchet differentiable at 0). Indeed:

$$f_n(x_n) \geq n[\nu_n(x_n/n) - \delta/n] = \nu_n(x_n) - \delta \geq \nu(x_n) + \delta.$$

So we've shown all we wish to show. ∎

As a consequence of Corollary 5, one also obtains the following result that is slightly weaker than Theorem 1(b), in that the functions obtained are not norms.

COROLLARY 6. *If X is a nonreflexive Banach space, then there is a difference of two Lipschitz convex functions that is weak Hadamard but not Fréchet differentiable at* 0.

PROOF. Let f be a function as guaranteed by Corollary 5. Then one can verify that $f(x) - d^+f(0)(x)$ is the desired difference of Lipschitz convex functions. ∎

We close, by considering briefly a different class of spaces. As in [5], we shall say that X has the *DP** *property* if weak* and Mackey convergence coincide sequentially in X^*. Also, recall that a Banach space has the *Schur property* if its weakly compact sets are norm compact. Notice that all spaces with Schur property trivially have the DP* property, while the converse fails. Indeed, any space with the Grothendieck and Dunford Pettis properties, such as ℓ_∞ has the DP* property; see [8, 5] for more. In fact, the relation between the Schur property and the DP* property is analogous to the relation between reflexivity and sequential reflexivity. Moreover, the results of [4] and [6] combine to immediately provide the following result which parallels Theorem 2.

THEOREM 7. *For a Banach space X, the following are equivalent.*
(a) X has the DP* property.
(b) Gateaux and weak Hadamard differentiability coincide for continuous convex functions on X.
(c) Each continuous convex function on X is bounded on weakly compact sets.

In contrast to this, we have the following analog of Corollaries 5 and 6.

PROPOSITION 8. *Suppose X does not have the Schur property. Then:*
(a) there is a continuous convex function for which weak Hadamard directional differentiability and (Gateaux) directional differentiability do not agree;
(b) there is a difference of Lipschitz convex functions that is Gateaux but not weak Hadamard differentiable at 0.

PROOF. Let $\{w_n\} \subset S_X$ converge weakly to 0. Because $\{w_n\}_{n=1}^{\infty}$ is not relatively norm compact, it is easy to construct a system $\{x_n, \phi_n\} \subset X \times X^*$ where $\{x_n\}$ is a subsequence of $\{w_n\}$, $\{\phi_n\}$ is norm bounded, and $\phi_n(x_n) = 2$ while $\phi_n(x_m) = 0$ for $n > m$ (see for instance the proof of [5, Theorem 3.4]). Now we define $f(x) := \sup_n[\phi_n(x) - 1/n]$.

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An argument similar to the proof of Corollary 5 shows $d^+f(0)(x) = \limsup_{n\to\infty} \phi_n(x)$. Consequently, $d^+f(0)(x_n) = 0$ for each n because $\lim_{m\to\infty} \phi_m(x_n) = 0$ for each n. However, $nf(x_n/n) \geq 1$ and so the difference quotients do not converge uniformly to the directional derivative on the weakly compact set $\{x_n\}_{n=1}^{\infty} \cup \{0\}$. This proves (a). To prove part (b), simply consider $f(x) - d^+f(0)(x)$. ∎

By comparing Proposition 8 with Corollaries 5 and 6, one might guess an analog for Theorem 1(c): on any space which does not have the Schur property, there is a difference of continuous convex functions that is unbounded on some weakly compact set. Curiously, this is not the case. Indeed, if X has the DP* property and f and g are continuous convex functions, then Theorem 7 ensures that f and g are bounded on weakly compacts sets and hence $f - g$ is also. This serves as a further reminder of the subtleties one can encounter when dealing with convex functions.

References

1. G. Beer, *Topologies on Closed and Closed Convex Sets*, Kluwer Academic Publishers, The Netherlands, 1993.

2. G. Beer and J. Borwein, *Mosco and slice convergence of level sets and graphs of linear functionals*, J. Math. Anal. Appl. **175**(1993), 53–67.

3. J. Borwein, *Asplund spaces are "sequentially reflexive"*, CORR 91–14, University of Waterloo, 1991.

4. J. Borwein and M. Fabian, *On convex functions having points of Gateaux differentiability which are not points of Fréchet differentiability*, Canad. J. Math. **45**(1993), 1121–1134.

5. J. Borwein, M. Fabian and J. Vanderwerff, *Locally Lipschitz functions and bornological derivatives*, CECM Research Report 93–012, Simon Fraser University, 1993.

6. J. Borwein, S. Fitzpatrick and J. Vanderwerff, *Examples of convex functions and classifications of normed spaces*, J. Convex Analysis **1**(1994), 61–73.

7. J. Borwein and J. Vanderwerff, *Epigraphical and uniform convergence of convex functions*, Trans. Amer. Math. Soc. **348**(1996), 1617–1631.

8. J. Diestel, *Sequences and Series in Banach Spaces*, Graduate Texts in Mathematics **92**, Springer-Verlag, Berlin–New York–Tokyo, 1984.

9. J. Hagler and W. B. Johnson, *On Banach spaces whose dual balls are not weak* sequentially compact*, Israel J. Math. **28**(1977), 325–330.

10. P. Ørno, *On J. Borwein's concept of sequentially reflexive Banach spaces*, Banach Bulletin Board, 1991.

11. R. R. Phelps, *Convex Functions, Monotone Operators and Differentiability*, Lecture Notes in Mathematics **1364**, Springer–Verlag, 1989.

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