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ON MAJORIZING AND CONE-ABSOLUTELY SUMMING MAPPINGS

YAU-CHUEN WONG

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Abstract

The notions of majorizing mappings and cone-absolutely summing mappings are studied in the locally convex Riesz space setting. It is shown that a locally convex Riesz space Y is an M-space in the sense of Jameson (1970) if and only if, for any locally convex space E, every continuous linear map from E into Y is majorizing. Another purpose of this note is to study the lattice properties of the vector space $\mathscr{L}^l(X, Y)$ of cone-absolutely summing mappings from one locally convex Riesz space into another Y. It is shown that if Y is both locally and boundedly order complete, then $\mathscr{L}^l(X, Y)$ is an l-ideal in $L^b(X, Y)$. This improves a result of Krengel.

1. Introduction

Schaefer (1972, 1974) and his school were the first who considered two important classes of continuous linear maps defined on Banach lattices: *majorizing mappings* and *cone-absolutely summing maps*. In terms of these two notions they gave new characterizations of AM-spaces and AL-spaces (see Schaefer, 1974, p. 248).

In this note we first generalize these two notions to the locally convex spaces setting, and then show that a locally convex Riesz space Y is an M-space in the sense of Jameson (1970) if and only if, for any locally convex space E, any continuous linear map from E into Y is majorizing (Theorem 2.3). This is a generalization of Schlotterbeck's result (see Schaefer, 1974, p. 243).

The final section is devoted to a study of the lattice properties of the vector space $\mathscr{L}^l(X, Y)$ of all cone-absolutely summing maps from one locally convex Riesz space X into another Y. It is shown in particular that if Y is both locally and boundedly order complete, then $\mathscr{L}^l(X, Y)$ is an l-ideal in $L^b(X, Y)$. This improves a result of Krengel (see May and Chivukula, 1972).

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2. A characterization of M-spaces

Let (Y, K) be a Riesz space (that is, vector lattice) with the positive cone K. A subset V of Y is *solid* if it follows from $|x| \leqslant |y|$ with $y \in V$ that $x \in V$, where $|x|$

is the absolute value of x. A seminorm q on Y is a *Riesz* (or *lattice*) *seminorm* if

$$|y| \leqslant |v| \text{ implies that } q(y) \leqslant q(v).$$

By a *locally convex Riesz space* (or *locally convex vector lattice*) we mean a Riesz space (Y, K) equipped with a Hausdorff locally convex topology $\mathscr{P}$ which admits a neighbourhood basis of 0 consisting of convex and solid sets. Clearly $(Y, K, \mathscr{P})$ is a locally convex Riesz space if and only if $\mathscr{P}$ is determined by a family of continuous Riesz seminorms. Throughout this paper Y' will denote the topological dual of $(Y, \mathscr{P})$, K^* will denote the set of all positive linear functionals on Y and $K' = K^* \cap Y'$.

Let E be a locally convex space and $(Y, K, \mathscr{P})$ a locally convex Riesz space. A linear map $T: E \to Y$ is said to be *majorizing* if for any continuous Riesz seminorm q on Y there is a continuous seminorm p on E such that the inequality

$$q\left(\sup_{1 \leqslant i \leqslant n} |Tx_i|\right) \leqslant \sup_{1 \leqslant i \leqslant n} p(x_i)$$

holds for any finite subset $\{x_1, \ldots, x_n\}$ of E. The set consisting of all majorizing linear maps from E into Y, denoted by $\mathscr{L}^m(E, Y)$, is obviously a vector subspace of the vector space $\mathscr{L}(E, Y)$ of all continuous linear maps from E into Y. Let $E^{(\mathbf{N})}$ be the countably algebraic direct sum and let $T^{(\mathbf{N})}$ be defined by

$$T^{(\mathbf{N})}([x_n, (\mathbf{N})]) = [Tx_n, (\mathbf{N})] \quad \text{for all } [x_n, (\mathbf{N})] \in E^{(\mathbf{N})},$$

where $\mathbf{N}$ is the set of all natural numbers. If (Y, K) is order complete, then $T \in \mathscr{L}^m(E, Y)$ if and only if $T^{(\mathbf{N})} \in \mathscr{L}(E_n^{(\mathbf{N})}, Y_\infty^{(\mathbf{N})})$ (see Wong, 1976, p. 76 and Lemma (2.2.8) (4)).

LEMMA 2.1. *Let E be a locally convex space, let $(Y, K, \mathscr{P})$ be a locally convex Riesz space and $T \in \mathscr{L}^m(E, Y)$. If Y has the property that each increasing $\mathscr{P}$-Cauchy sequence in Y has an upper bound (in particular, if $(Y, K, \mathscr{P})$ is boundedly σ-order-complete), then T maps each Cauchy sequence in E into a sequence which is majorized.*

PROOF. Let $\{x_n\}$ be a Cauchy sequence in E and let $y_n = \sup\{|Tx_i|: 1 \leqslant i \leqslant n\}$ for each natural number $n \geqslant 1$. We claim that $\{y_n\}$ is an increasing $\mathscr{P}$-Cauchy sequence in Y. If q is any $\mathscr{P}$-continuous Riesz seminorm on Y there is a continuous seminorm p on E such that

$$q(y_n) \leqslant \sup\{p(x_i): 1 \leqslant i \leqslant n\} \quad \text{for all } n \geqslant 1;$$

as $\{x_n\}$ is a Cauchy sequence in E there is for any $\varepsilon > 0$ an integer $m > 0$ such that

$$p(x_{m+i} - x_m) \leqslant \varepsilon \quad \text{for all } i \geqslant 1.$$

For each $i \geqslant 1$ we have

$$\begin{aligned}0 \leqslant y_{m+i}-y_m &= y_m \vee \sup\{|Tx_{m+j}|: 1 \leqslant j \leqslant i\}-y_m \\ &= 0 \vee \sup\{(|Tx_{m+j}|-y_m): 1 \leqslant j \leqslant i\} \\ &= 0 \vee \sup_{1\leqslant j\leqslant i} \inf_{1\leqslant k\leqslant m} (|Tx_{m+j}|-|Tx_k|) \\ &\leqslant \sup_{1\leqslant j\leqslant i} \inf_{1\leqslant k\leqslant m} |Tx_{m+j}-Tx_k| \leqslant \sup_{1\leqslant j\leqslant i} |Tx_{m+j}-Tx_m|,\end{aligned}$$

and thus

$$q(y_{m+i}-y_m) \leqslant q\Big(\sup_{1\leqslant j\leqslant i}|Tx_{m+j}-Tx_m|\Big) \leqslant \sup_{1\leqslant j\leqslant i} p(x_{m+j}-x_m) \leqslant \varepsilon \quad \text{for all } i \geqslant 1$$

as asserted.

By the hypothesis there is a $u \in K$ such that $y_n \leqslant u$ for all $n \geqslant 1$, and hence

$$|Tx_n| \leqslant y_n \leqslant u \quad \text{for all } n.$$

Therefore $\{Tx_n\}$ is majorized.

In particular, if $(Y, K, \mathscr{P})$ is complete then Y has the property mentioned in the preceding lemma. When E and Y are Fréchet spaces, Walsh (1973) has shown that T is majorizing if and only if T sends null sequences in E into majorized sequences in Y.

LEMMA 2.2. *Let E, F be locally convex spaces and suppose that Y, Z are locally convex spaces. Then the following assertions hold:*

(a) *If $S \in \mathscr{L}(E,F)$ and if $T \in \mathscr{L}^m(F, Y)$, then $T \circ S \in \mathscr{L}^m(E, Y)$.*

(b) *If $T \in \mathscr{L}^m(E, Y)$ and if $S \in \mathscr{L}(Y,Z)$ is positive, then $S \circ T \in \mathscr{L}^m(E,Z)$.*

PROOF. (a) Let q be any continuous Riesz seminorm on Y and r a continuous seminorm on F such that the inequality

$$q\Big(\sup_{1\leqslant i\leqslant n}|Ty_i|\Big) \leqslant \sup_{1\leqslant i\leqslant n} r(y_i) \tag{2.1}$$

holds for any finite subset $\{y_1, \ldots, y_n\}$ of F. The continuity of S ensures that there is a continuous seminorm p on E such that

$$r(Sx) \leqslant p(x) \quad \text{for all } x \in E. \tag{2.2}$$

Combining (2.1) and (2.2), we conclude that $T \circ S$ is majorizing.

(b) Let q be a continuous Riesz seminorm on Z and r a continuous Riesz seminorm on Y such that $q(Sy) \leqslant r(y)$ for all $y \in Y$. Since $T \in \mathscr{L}^m(E, Y)$ there is a continuous seminorm p on E such that the inequality

$$r\Big(\sup_{1\leqslant i\leqslant n}|Tx_i|\Big) \leqslant \sup_{1\leqslant i\leqslant n} p(x_i)$$

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holds for any finite subset $\{x_1, \ldots, x_n\}$ of E. As S is positive, we have

$$\pm S(Tx_j) \leqslant S\left(\sup_{1\leqslant i\leqslant n} |Tx_i|\right) \quad \text{for all } j \text{ with } 1 \leqslant j \leqslant n,$$

and thus

$$q\left(\sup_{1\leqslant i\leqslant n} |(S\circ T)\, x_i|\right) \leqslant q\left(S\left(\sup_{1\leqslant i\leqslant n} |Tx_i|\right)\right) \leqslant \sup_{1\leqslant i\leqslant n} p(x_i)$$

which shows that $S\circ T$ is majorizing.

A subset B of (Y,K) is called a *sublattice* if for any finite subset $\{y_1, \ldots, y_n\}$ of B it is true that $\sup_{1\leqslant i\leqslant n} y_i$ and $\inf_{1\leqslant i\leqslant n} y_i$ exist in B. Clearly the intersection of a family of sublattices is a sublattice, hence the smallest sublattice containing a given set B is denoted by $\mathrm{sl}(B)$. If B is convex then so is $\mathrm{sl}(B)$; if B is a symmetric sublattice, then the solid hull $\mathrm{S}(B)$ of B is the order-convex hull $[B]$ of B (where $[B] = (B+K)\cap(B-K)$) and $\mathrm{S}(B)$ is a convex sublattice. Following Jameson (1970), a locally convex Riesz space $(Y, K, \mathscr{P})$ is called an *M-space* if $\mathscr{P}$ admits a neighbourhood basis at 0 consisting of convex, solid sublattices. It is also clear that if U is a convex, solid $\mathscr{P}$-neighbourhood of 0 in Y, then U is a sublattice if and only if the gauge q_U of U is an M-seminorm in the following sense

$$q_U(x\vee y) = \sup\{q_U(x), q_U(y)\} \quad \text{for all } x, y \in K.$$

Therefore a locally convex Riesz space $(Y, K, \mathscr{P})$ is an M-space if and only if $\mathscr{P}$ is determined by a family of continuous M-seminorms. It is trivial that every AM-space is an M-space. In view of majorizing maps we are able to present a characterization of M-spaces as follows.

THEOREM 2.3. *For a locally convex Riesz space* $(Y, K, \mathscr{P})$ *the following statements are equivalent.*

(a) $(Y, K, \mathscr{P})$ *is an M-space.*

(b) *The identity map from* Y *onto* Y *is majorizing.*

(c) $\mathscr{L}(E, Y) = \mathscr{L}^m(E, Y)$ *for any locally convex space* E.

PROOF. The implications (a)$\Rightarrow$(b) and (c)$\Rightarrow$(b) are obvious, and the implication (b)$\Rightarrow$(c) follows from Lemma 2.2. Therefore we complete the proof by showing that (b) implies (a). To do this, let U be any convex solid $\mathscr{P}$-neighbourhood of 0 in Y and let V be a convex, solid $\mathscr{P}$-neighbourhood of 0 in Y such that

$$q_U\left(\sup_{1\leqslant i\leqslant n} |y_i|\right) \leqslant \sup_{1\leqslant i\leqslant n} p_V(y_i) \tag{2.3}$$

for any finite subset $\{y_1, \ldots, y_n\}$ of Y, where q_U (respectively p_V) is the gauge of U (respectively the gauge of V). From (2.3) it is easily seen that $\mathrm{sl}(V) \subset U$ since

$$\mathrm{sl}(V) = \left\{\sup_{1\leqslant i\leqslant n} \inf_{1\leqslant j\leqslant m} y_{ij} : y_{ij} \in V\right\}.$$

Since $\mathrm{sl}(V)$ is symmetric, the solid hull $\mathrm{S}(\mathrm{sl}(V))$ of $\mathrm{sl}(V)$ is a convex, solid and sublattice satisfying

$$V \subseteq \mathrm{S}(\mathrm{sl}(V)) \subseteq U.$$

Therefore $\mathscr{P}$ admits a neighbourhood basis at 0 consisting of convex, solid and sublattices, thus $(Y, K, \mathscr{P})$ is an M-space.

In particular, if $(Y, K, \|.\|)$ is a Banach lattice, then in view of the remark after Lemma 2.1, Y is isomorphic to an AM-space if and only if every null sequence in Y is order-bounded, therefore the preceding result is a generalization of Schlotterbeck's result (see Schaefer, 1974 (IV.2.8), p. 243).

3. Lattice properties of cone-absolutely summing mappings

Let $(E, C, \mathscr{T})$ be an ordered convex space and F a locally convex space. A linear map $T: E \to F$ is said to be *cone-absolutely summing* if for any continuous seminorm q on F there is a continuous seminorm p on E such that the inequality

$$\sum_{i=1}^{n} q(Tu_i) \leqslant p\left(\sum_{i=1}^{n} u_i\right)$$

holds for any finite subset $\{u_1, \ldots, u_n\}$ of C. The set consisting of all cone-absolutely summing maps from E into F, denoted by $\mathscr{L}^l(E, F)$, is a vector subspace of the space $L(E, F)$ of all linear maps from E into F. Clearly each element in $\mathscr{L}^l(E, F)$ is a continuous map from C into F, thus if E is locally decomposable (for definition, see Wong and Ng, 1973), then $\mathscr{L}^l(E, F) \subset \mathscr{L}(E, F)$. Other properties of cone-absolutely summing mappings can be found in Chapter 3 of Wong (1976).

The notions of cone-absolutely summing maps and majorizing maps were first considered by Schaefer and his school in the Banach lattices setting (see Schaefer, 1972, 1974). Walsh (1973) and others extended these notions to the case of locally o-convex spaces with closed and generating cones; Walsh (1973) was very successful in extending Schlotterbeck's results to a fairly general setting, and offering counter-examples to show that some of his results really involve the lattice structure intrinsically.

Let X and Y be locally convex Riesz spaces and let Y be order complete. Then $L^b(X, Y)$ is an order complete Riesz space under the canonical ordering, but $\mathscr{L}(X, Y)$ need not be a Riesz space (see Peressini, 1967 (IV.3.3)). If M is a vector subspace of $\mathscr{L}(X, Y)$, it is a difficult problem to determine under what conditions on M (or Y), M is a Riesz subspace of $L^b(X, Y)$. If Y is both locally and boundedly order complete, then $\mathscr{L}^l(X, Y)$ is an l-ideal (that is, solid subspace) in $L^b(X, Y)$ as shown by the following result.

THEOREM 3.1. *Let $(X, C, \mathscr{T})$ and $(Y, K, \mathscr{P})$ be locally convex Riesz spaces. If $(Y, K, \mathscr{P})$ is both locally and boundedly order-complete then $\mathscr{L}^l(X, Y)$ is an l-ideal in $L^b(X, Y)$ and hence is an order-complete Riesz space in its own right.*

PROOF. Let $T \in \mathscr{L}^l(X, Y)$. For any $\mathscr{P}$-continuous Riesz seminorm q on Y, there exists a $\mathscr{T}$-continuous Riesz seminorm p on X such that

$$\sum_{i=1}^{n} q(Tx_i) \leqslant p\left(\sum_{i=1}^{n} x_i\right) \tag{3.1}$$

holds for all finite subsets $\{x_1, \ldots, x_n\}$ of C. Let u be in C and let $u = \sum_{i=1}^{n} u_i$ with $u_i \in C$. As q is a Riesz seminorm, there is

$$q\left(\sum_{i=1}^{n} |Tu_i|\right) \leqslant \sum_{i=1}^{n} q(Tu_i) \leqslant p(u).$$

Hence the set $\{\sum_{i=1}^{n} |Tu_i|: u = \sum_{i=1}^{n} u_i, u_i \in C\}$ is a $\mathscr{P}$-bounded subset of Y. On the other hand, the Riesz decomposition property ensures that the set $\{\sum_{i=1}^{n} |Tu_i|: u = \sum_{i=1}^{n} u_i, u_i \in C\}$ is directed upwards. Therefore the bounded order-completeness of Y implies that the supremum

$$|T|u = \sup\left\{\sum_{i=1}^{n} |Tu_i|: u = \sum_{i=1}^{n} u_i, u_i \in C\right\}$$

exists for each $u \in C$. It then follows that $|T|$ exists in $L^b(X, Y)$. Consequently $\mathscr{L}^l(X, Y) \subset L^b(X, Y)$.

Further, we show that $|T| \in \mathscr{L}^l(X, Y)$. Let r be any $\mathscr{P}$-continuous Riesz seminorm on Y. By Lemma 1 of May and Chivukula (1972), there exists a $\mathscr{P}$-continuous Riesz seminorm q on Y such that

$$r(\sup B) \leqslant \sup\{q(b): b \in B\},$$

where B is any $\mathscr{P}$-bounded subset of Y which is directed upwards. In particular,

$$r(|T|u) \leqslant \sup\left\{q\left(\sum_{i=1}^{n} |Tu_i|\right): u = \sum_{i=1}^{n} u_i, u_i \in C\right\}$$

holds for all $u \in C$. As $T \in \mathscr{L}^l(X, Y)$, there exists a $\mathscr{T}$-continuous Riesz seminorm p on X such that the inequality (3.1) holds for all finite subsets $\{x_1, \ldots, x_n\}$ of C. If $\{w_1, \ldots, w_n\}$ is any finite subset of C, there are

$$\begin{aligned}
\sum_{j=1}^{n} r(|T|w_j) &\leqslant \sum_{j=1}^{n} \sup\left\{q\left(\sum_{i=1}^{m_j} |Tu_{ij}|\right): w_j = \sum_{i=1}^{m_j} u_{ij}, u_{ij} \in C\right\} \\
&= \sup\left\{\sum_{j=1}^{n} q\left(\sum_{i=1}^{m_j} |Tu_{ij}|\right): w_j = \sum_{i=1}^{m} u_{ij}, u_{ij} \in C\right\} \\
&\leqslant \sup\left\{\sum_{j=1}^{n} \sum_{i=1}^{m_j} q(Tu_{ij}): w_j = \sum_{i=1}^{m_j} u_{ij}, u_{ij} \in C\right\} \\
&\leqslant \sup\left\{q\left(\sum_{j=1}^{n} \sum_{i=1}^{m_j} u_{ij}\right): w_j = \sum_{i=1}^{m_j} u_{ij}, u_{ij} \in C\right\} \\
&= p\left(\sum_{j=1}^{n} w_j\right).
\end{aligned}$$

Therefore $|T|$ is cone-absolutely summing. Combining this with the above conclusion, we see that $\mathscr{L}^l(X, Y)$ is a Riesz space.

Finally, if $0 \leqslant S \leqslant T$, where $S \in L^b(X, Y)$ and $T \in \mathscr{L}^l(X, Y)$, then it is easily seen that $S \in \mathscr{L}^l(X, Y)$. This shows that $\mathscr{L}^l(X, Y)$ is an l-ideal in $L^b(X, Y)$ and therefore the proof is complete.

COROLLARY 3.2. *Let $(X, C, \mathscr{T})$ and $(Y, K, \mathscr{P})$ be locally convex Riesz spaces for which $(Y, K, \mathscr{P})$ is both locally and boundedly order complete. If $\mathscr{T}$ is the Dieudonné topology $\sigma_{\mathscr{S}}(X, X')$ (that is, the topology of uniform convergence on order-intervals in X'), then $\mathscr{L}(X, Y)$ is an l-ideal in $L^b(X, Y)$, and hence $\mathscr{L}(X, Y)$ is an order complete Riesz space.*

PROOF. In view of Theorem (3.2.12)(h) of Wong (1976), $\mathscr{L}(X, Y) = \mathscr{L}^l(X, Y)$, the result now follows from Theorem 3.1.

In particular, if $(X, C, \|.\|)$ is a normed vector lattice such that the norm $\|.\|$ is additive on C, then the norm-topology coincides with $\sigma_{\mathscr{S}}(X, X')$. Therefore the preceding corollary is an improvement of Theorem 1 of May and Chivukula (1972) as well as of Krengel's result (see Peressini, 1967 (IV.3.8), p. 174).

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Department of Mathematics
United College
The Chinese University of Hong Kong
Shatin, N.T.
Hong Kong

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