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WHEN IS A MATRIX POSITIVE

BY
DANIEL BRAND

Throughout this note we shall use the following conventions and notations: All matrices have entries in the field of complex numbers. I denotes the identity matrix with compatible dimensions. A^* is the conjugate transpose of a matrix A. A being self adjoint means $A=A^*$. $A\geq 0$ (A is positive) means: $v^*Av\geq 0$ for all vectors v. $A>0$ means: $A\geq 0$ and A is invertible. For an $n\times m$ matrix A, whose (i,j)-the entry is a $a_{i,j}$, we write $A=(a_{i,j})_{i=1;n}^{j=1;m}$. For an $n\times n$ matrix $S=(s_{i,j})_{i=1;n}^{j=1;n}$ with $n\geq 2$ we let

$$S_1=(s_{i,j})_{i=1;n-1}^{j=1;n-1} \qquad S_2=(s_{i,n})_{i=1;n-1}$$
$$S_3=(s_{n,j})^{j=1;n-1} \qquad S_4=s_{n,n}$$

i.e.

$$S=\left(\begin{array}{c|c} S_1 & S_2 \\ \hline S_3 & S_4 \end{array}\right)$$

DEFINITION. For an $n\times n$ matrix S, with $n\geq 2$, we define

$$D(S)=S_1S_4-S_2S_3.$$

(S_1S_4 is a matrix times a scalar, S_2S_3 is an ordinary matrix product.)

Note that $D(S)$ is an $(n-1)\times(n-1)$ matrix, and it becomes $\det S$ in the case $n=2$.

THEOREM 1. *Let S be a self adjoint $n\times n$ matrix, with $n\geq 2$. Then $S\geq 0$ if and only if $S_1\geq 0$, $S_4\geq 0$, and $D(S)\geq 0$.*

Proof. The relation

$$\begin{pmatrix} I & -S_2/S_4 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix}\begin{pmatrix} I & 0 \\ -S_2^*/S_4 & 1 \end{pmatrix}=\begin{pmatrix} D(S)/S_4 & 0 \\ S_3-S_2^* & S_4 \end{pmatrix}$$

shows that if $S_4\neq 0$ then $\det D(S)=S_4^{n-2}\det S$. It is also true if $S_4=0$ with the convention that $0^0=1$. But this fact is not necessary in the proof. Replacing S by $S-aI$ we find

(1) $$\det D(S-aI)=(S_4-a)^{n-2}\det(S-aI).$$

Note that

(2) $$D(S-aI)=a^2I-a(S_1+S_4I)+D(S)$$

which is an analogue of the characteristic polynomial.

(I) Assume that $S_1\geq 0$, $S_4\geq 0$, $D(S)\geq 0$. Suppose that S would have a negative characteristic value a. Then (1) gives

(3) $$\det D(S-aI)=0.$$

By hypothesis, $a^2I>0$, $-a(S_1+S_4I)+D(S)\geq 0$. So from (2), $D(S-aI)$ is invertible, contradicting (3). Hence $S\geq 0$.

(II) Assume $S\geq 0$. Then immediately $S_1\geq 0$ and $S_4\geq 0$. Suppose that $D(S)$ is not positive. We will show that then S must have a negative characteristic value.

Since S is self adjoint, so is $D(S-xI)$ for any real x. Let f be the real valued function $f(x)$=least characteristic value of $D(S-xI)$ for real x. Characteristic values of $D(S-xI)$ are roots of the corresponding characteristic polynomial. Coefficients of this polynomial are continuous functions of x, hence so are the roots, so f is continuous.

From the assumption that $D(S)$ is not positive, $f(0)<0$. From (2), $f(x)\geq 0$ for some negative x, sufficiently large in magnitude. Hence for some $a<0$, $f(a)=0$. So $D(S-aI)$ is not invertible, so by (1) a is a characteristic value of S, which is a contradiction. Therefore $D(S)\geq 0$. Q.E.D.

We can now give an alternative proof of the following theorem (see [1]).

THEOREM 2. *Let* $A=(a_{i,j})_{i=1;n}^{j=1;n}$, $B=(b_{i,j})_{i=1;n}^{j=1;n}$ *be positive matrices. Then their Hadamard product* $H=(a_{i,j}b_{i,j})_{i=1;n}^{j=1;n}$ *is positive.*

Proof. By induction on n.

(I) $n=1$ Obvious.

(II) Suppose true for all matrices of dimension less than n, $n\geq 2$.

H is self adjoint, and by induction hypothesis $H_1\geq 0$, $H_4\geq 0$. So we need to show $D(H)\geq 0$.

From Theorem 1, $0\leq D(A)=(a_{i,j}a_{n,n}-a_{i,n}a_{n,j})_{i=1;n-1}^{j=1;n-1}$. Applying induction hypothesis to $D(A)$ and B_1, and then multiplying by $B_4\geq 0$, gives an inequality, which when added to the analogous inequality with A and B interchanged, yields

$$2(a_{i,j}b_{i,j}a_{n,n}b_{n,n})_{i=1;n-1}^{j=1;n-1}\geq(a_{i,n}a_{n,j}b_{i,j}b_{n,n}+a_{i,j}a_{n,n}b_{i,n}b_{n,j})_{i=1;n-1}^{j=1;n-1}. \tag{4}$$

We show that

$$(4)\geq 2(a_{i,n}b_{i,n}a_{n,j}b_{n,j})_{i=1;n-1}^{j=1;n-1}. \tag{5}$$

Apply induction hypothesis to $D(B)$ and $A_2A_3=A_2A_2^*\geq 0$ to get one inequality, and an analogous one with roles of A and B interchanged. Adding these two establishes (5), from which follows $D(H)\geq 0$. Whence $H\geq 0$. Q.E.D.

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REFERENCE

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DEPARTMENT OF COMPUTER SCIENCE
UNIVERSITY OF TORONTO
TORONTO, ONTARIO