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Zero-one dual characters of flagged Weyl modules*

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Abstract. **We prove a criterion of when the dual character $\chi_D(x)$ of the flagged Weyl module associated to a diagram D in the grid $[n]\times[n]$ is zero-one, that is, the coefficients of monomials in $\chi_D(x)$ are either 0 or 1. This settles a conjecture proposed by Mészáros–St. Dizier–Tanjaya. Since Schubert polynomials and key polynomials occur as special cases of dual flagged Weyl characters, our approach provides a new and unified proof of known criteria for zero-one Schubert/key polynomials due to Fink–Mészáros–St. Dizier and Hodges–Yong, respectively.**

1 Introduction

The goal of this paper is to confirm a conjectured criterion for zero-one dual characters of flagged Weyl modules due to Mészáros, St. Dizier and Tanjaya [25, Conjecture 3.9]. A polynomial is called zero-one (or, multiplicity-free) if its coefficients are equal to either 0 or 1. It is well known that dual flagged Weyl characters encompass Schubert and key polynomials as special cases. So our result leads to a unified proof for previously known criteria respectively for zero-one Schubert polynomials by Fink, Mészáros and St. Dizier [10] and zero-one key polynomials by Hodges and Yong [14].

Motivations of investigating zero-one dual characters of flagged Weyl modules are multifold. It was proved by Fink, Mészáros and St. Dizier [9] that the supports of the dual character of a flagged Weyl module are exactly the same as the integer points in its Newton polytope. Hence a zero-one dual flagged Weyl character is completely determined by the integer points in its Newton polytope.

As aforementioned, Schubert and key polynomials appear as special cases of dual characters of flagged Weyl models, see for example [9] and references therein. A criterion for zero-one Schubert polynomials was found by Fink, Mészáros and St. Dizier [10], whose proof makes use of Magyar's orthodontia formula for Schubert polynomials [22, 23]. By the work of Knutson and Miller [21], Schubert polynomials are the multidgree polynomials of matrix Schubert varieties. Particularly, zero-one Schubert polynomials correspond to multiplicity-free matrix Schubert varieties, and in this case, one could capture the entire information of K-polynomials (namely, Grothendieck polynomials) from that of zero-one Schubert polynomials [4–6, 20, 24, 27]. Moreover, for a permutation whose associated Schubert polynomial is zero-one, the CDG generators of the defining ideal of its matrix Schubert variety form a diagonal Gröbner basis [12, 15].

A criterion (first announced in [13, Theorem 4.10]) for zero-one key polynomials was proved by Hodges and Yong [14] by employing two combinatorial models for key polynomials: the quasi-key model by Assaf and Searles [3] and the Kohnert diagram model by Kohnert [17]. The algebraic motivation of studying zero-one key polynomials stems from the classification of Levi-spherical Schubert varieties [11, 13].

Another notable subfamily of dual flagged Weyl characters are for northwest diagrams. This subfamily also includes Schubert and key polynomials as special cases. It was shown by Armon, Assaf, Bowling and Ehrhard [1] that the dual flagged Weyl character for a northwest diagram coincides with the Kohnert polynomial defined in [2]. As remarked in [14], the techniques they developed for zero-one key polynomials do not seem to extend to zero-one dual flagged Weyl characters for northwest diagrams.

We proceed by describing the criterion for zero-one dual characters of flagged Weyl modules, as stated in [25, Conjecture 3.9]. Let $[n]=\{1,2,\ldots,n\}$. A subset D of boxes in the grid $[n]\times[n]$ is called a diagram. The flagged

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Weyl module $\mathcal{M}_D$ associated to a diagram D is a module of the group B of invertible upper-triangular matrices over $\mathbb{C}$ [18, 19, 22], see Section 2 for the precise definition. The dual character of $\mathcal{M}_D$, denoted $\chi_D(x)$, is a polynomial in $x_1, \ldots, x_n$, which coincides with the Schubert polynomial $\mathfrak{S}_w(x)$ when D is the Rothe diagram of a permutation w, and the key polynomial $\kappa_\alpha(x)$ when D is the skyline diagram of a composition α. The flagged Weyl modules have inspired considerable recent research interests, see for example [1, 7–10, 23, 25, 26, 28].

Following the terminology in [25], a *multiplicitous* configuration refers to one of the six instances of configurations listed in Figure 1. Here, a crossing "×" indicates the absence of a box, a blank square indicates the presence of a box, and a star "∗" indicates no restriction on the presence or absence of a box.

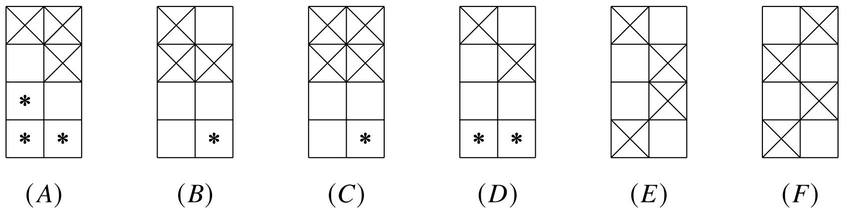

Figure 1. Multiplicitous configurations.

A diagram D is called *multiplicitous* if it contains one of multiplicitous configurations as a subdiagram, up to possibly swapping the order of the columns. This means that there exist row indices $i_1 < i_2 < i_3 < i_4$ and column indices $j_1 < j_2$ such that the subdiagram of D, which is restricted to rows $\{i_1, i_2, i_3, i_4\}$ and columns $\{j_1, j_2\}$, is either a multiplicitous configuration or a configuration obtained from a multiplicitous configuration by swapping its two columns. We call a diagram D *multiplicity-free* if it is not multiplicitous. So a multiplicity-free diagram avoids a total of 12 configurations.

In light of an observation in [10, Theorem 5.8], it is readily verified that if the dual character $\chi_D(x)$ is zero-one, then D must be multiplicity-free [25, Proposition 3.11]. This paper aims to prove the reverse direction which appears as [25, Conjecture 3.9].

***Theorem* 1.1** *([25, Conjecture 3.9]) If D is a multiplicity-free diagram, then the dual character $\chi_D(x)$ is zero-one.*

Combining Theorem 1.1 with the above mentioned [25, Proposition 3.11] gives a full criterion for zero-one dual Weyl characters.

***Corollary* 1.2** *The dual character $\chi_D(x)$ is zero-one if and only if D is a multiplicity-free diagram.*

The above corollary on the one hand specializes to the criterion for zero-one Schubert polynomials [10, Theorem 1] when D is the Rothe diagram of a permutation, and on the other hand to the criterion for zero-one key polynomials [14, Theorem 1.1] when D is the skyline diagram of a composition.

This paper is structured as follows. In Section 2, we review the construction of the flagged Weyl module $\mathcal{M}_D$, as well as some facts about its dual character $\chi_D(x)$. Particularly, it will be seen that the coefficient of a monomial x^a in $\chi_D(x)$ is equal to the dimension of the eigenspace in $\mathcal{M}_D$ corresponding to x^a. Section 3 is devoted to several lemmas concerning the properties of multiplicity-free diagrams. In Section 4, we finish the proof of Theorem 1.1 which will be achieved by showing that when a diagram D is multiplicity-free, each eigenspace in $\mathcal{M}_D$ has dimension exactly equal to one. The arguments are combinatorial.

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2 Dual characters of flagged Weyl modules

In this section, we give an overview of some basic information about flagged Weyl modules. We mainly follow the notation in [9, 10].

For the square grid $[n] \times [n]$, let us use (i, j) to denote the box in row i and column j in the matrix coordinate. Recall that a diagram D is a subset of boxes in $[n] \times [n]$. Write $D = (D_1, D_2, \ldots, D_n)$, where, for $1 \le j \le n$, D_j stands for the j-th column of D. We shall often represent D_j by a subset of $[n]$, that is, $i \in D_j$ if and only if the box (i, j) belongs to D. For example, the diagram in Figure 2 can be written as $D = (\{2, 3, 4\}, \emptyset, \{1, 2\}, \{3\})$. As before, we use crossings to represent the absence of boxes.

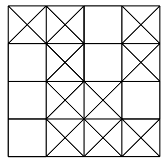

Figure 2. A diagram in $[4] \times [4]$.

For two k-element subsets $R = \{r_1 < \cdots < r_k\}$ and $S = \{s_1 < \cdots < s_k\}$ of $[n]$, write $R \le S$ if $r_i \le s_i$ for $1 \le i \le k$. This defines a partial order, called the *Gale order*, on all k-element subsets of $[n]$. For a given subset S of $[n]$, the collection of subsets $R \le S$ constitutes the basis of a matroid, called Schubert matroid, on the ground set $[n]$ [9]. It is worth mentioning that the Gale order plays an important role in the study of the positroid decomposition of (positive) Grassmannians [16, 29]. For two diagrams $C = (C_1, \ldots, C_n)$ and $D = (D_1, \ldots, D_n)$, write $C \le D$ if $C_j \le D_j$ for each $1 \le j \le n$.

Let $\mathrm{GL}(n, \mathbb{C})$ be the general linear group of $n \times n$ invertible matrices over $\mathbb{C}$, and B the Borel subgroup of $\mathrm{GL}(n, \mathbb{C})$ consisting of all upper-triangular matrices. Let Y be the upper-triangular matrix of variables y_{ij} with $1 \le i \le j \le n$:

$$Y = \begin{bmatrix} y_{11} & y_{12} & \ldots & y_{1n} \\ 0 & y_{22} & \ldots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & y_{nn} \end{bmatrix}.$$

Denote by $\mathbb{C}[Y]$ the linear space of polynomials over $\mathbb{C}$ in the variables $\{y_{ij}\}_{i \le j}$. Define the (right) action of B on $\mathbb{C}[Y]$ by $f(Y) \cdot b = f(b^{-1} \cdot Y)$, where $b \in B$ and $f \in \mathbb{C}[Y]$. For two subsets R and S of $[n]$ with the same cardinality, we use Y_S^R to represent the submatrix of Y obtained by restricting to rows indexed by R and columns indexed by S. It is not hard to check that $\det\left(Y_S^R\right) \neq 0$ if and only if $R \le S$. Given two diagrams $C = (C_1, \ldots, C_n)$ and $D = (D_1, \ldots, D_n)$ with $C \le D$, denote

$$\det\left(Y_D^C\right) = \prod_{j=1}^{n} \det\left(Y_{D_j}^{C_j}\right).$$

The flagged Weyl module $\mathcal{M}_D$ for D is a subspace of $\mathbb{C}[Y]$ defined by

$$\mathcal{M}_D = \mathrm{Span}_{\mathbb{C}} \left\{ \det\left(Y_D^C\right) : C \le D \right\},$$

which is a B-module with the action inherited from the action of B on $\mathbb{C}[Y]$.

Let X be the diagonal matrix with diagonal entries $x_1, \ldots, x_n$. The character of $\mathcal{M}_D$ is defined by

$$\mathrm{char}(\mathcal{M}_D)(x_1, \ldots, x_n) = \mathrm{tr}(X \colon \mathcal{M}_D \to \mathcal{M}_D).$$

It can be directly checked that $\det\left(Y_D^C\right)$ is an eigenvector of X with eigenvalue

$$\prod_{j=1}^{n} \prod_{i \in C_j} x_i^{-1}.$$

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The dual character is defined as

$$\chi_D(x) := \mathrm{char}(\mathcal{M}_D)(x_1^{-1}, \ldots, x_n^{-1}).$$

Hence the coefficient of a monomial x^a in $\chi_D(x)$ is equal to the dimension of the corresponding eigenspace:

$$[x^a]\, \chi_D(x) = \dim \mathrm{Span}_{\mathbb{C}} \left\{ \det\left(Y_D^C\right) : C \leq D,\, x^C = x^a \right\}, \tag{1}$$

where

$$x^C = \prod_{j=1}^{n} \prod_{i \in C_j} x_i.$$

In particular, the set of monomials appearing in $\chi_D(x)$ is $\left\{x^C : C \leq D\right\}$.

***Proposition* 2.1** *The dual character $\chi_D(x)$ is zero-one if and only if each eigensapce on the right-hand side of (1) has dimension one.*

We end this section with a combinatorial expression for the generator $\det\left(Y_D^C\right)$. A *flagged filling* of a diagram $D = (D_1, \ldots, D_n)$ is a filling of the boxes of D with positive integers such that

(1) each box receives exactly one integer, and the entries in each column are distinct;
(2) the entry in the box $(i, j) \in D$ cannot exceed i.

Let $F = (F_1, \ldots, F_n)$ be a flagged filling of D, where, for $1 \leq j \leq n$, F_j denotes the j-th column of F. Define the inversion number $\mathrm{inv}(F)$ of F as follows. Let $w = w_1 \cdots w_m$ be the word obtained by reading the entries of F_j from top to bottom. Denote by $\mathrm{inv}(F_j)$ the inversion number of w, namely,

$$\mathrm{inv}(F_j) = \#\{(r, s) \colon 1 \leq r < s \leq m,\ w_r > w_s\}.$$

In the case when D_j is empty, we adopt the convention that $\mathrm{inv}(F_j) = 0$. Define

$$\mathrm{inv}(F) = \mathrm{inv}(F_1) + \cdots + \mathrm{inv}(F_n).$$

For the flagged filling depicted in Figure 3, The column reading words are 231, 213, 1 and 32 from left to right, and

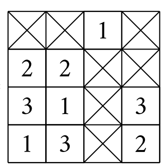
$F =$ 

Figure 3. A flagged filling.

so we have $\mathrm{inv}(F) = 2 + 1 + 0 + 1 = 4$. Assign a weight to F in the following way:

$$y^F = \prod_{(i,j) \in D} y_{c_{ij} i},$$

where c_{ij} is the entry of F filled in the box (i, j). For example, the flagged filling in Figure 3 gives

$$y^F = y_{11}\, y_{22}^2\, y_{13}\, y_{33}^2\, y_{14}\, y_{24}\, y_{34}.$$

For $C \leq D$, let $\mathcal{F}_D(C)$ denote the set of flagged fillings $F = (F_1, \ldots, F_n)$ of D such that for $1 \leq j \leq n$, the set of entries in F_j is exactly C_j. The following proposition appears as [28, Lemma 2.2].

***Proposition* 2.2** *For $C \leq D$, we have*

$$\det\left(Y_D^C\right) = \sum_{F \in \mathcal{F}_D(C)} \mathrm{sgn}(F) \cdot y^F, \tag{2}$$

where $\mathrm{sgn}(F) = (-1)^{\mathrm{inv}(F)}$.

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As explained in [28, Remark 2.3], the formula in (2) may not be cancellation-free, that is, there may exist distinct fillings in $\mathcal{F}_D(C)$ that contribute the same weight, but have opposite signs.

3 Multiplicity-free diagrams

In this section, we investigate the distribution of boxes in a multiplicity-free diagram. This will be a step stone in the proof that each eigenspace for a multiplicity-free diagram has dimension exactly equal to one.

Clearly, the dual character $\chi_D(x)$ is independent of the order of columns of D. We say that two diagrams are *equivalent* if one can be obtained from the other by reordering the columns. We shall designate a particular representative in the class of diagrams equivalent to D. Such a representative is called the *normalization* of D.

3.1 Normalization of diagrams

Let $D = (D_1, \ldots, D_n)$ be a diagram. The normalization, denoted $\mathrm{Norm}(D)$, of D is the unique diagram equivalent to D by rearranging the columns in the following way. For $1 \le j \le n$, let $\overline{D}_j = ([n] \setminus D_j) \cup \{\infty\}$. For $D_{j_1} \ne D_{j_2}$, suppose that

$$\overline{D}_{j_1} = \{a_1 < \cdots < a_s < a_{s+1} = \infty\}, \quad \overline{D}_{j_2} = \{b_1 < \cdots < b_t < b_{t+1} = \infty\}.$$

Define $D_{j_1} \prec D_{j_2}$ if $\overline{D}_{j_1} <_{\text{lex}} \overline{D}_{j_2}$ in the lexicographic order, that is, there exists some index k such that $a_i = b_i$ for $1 \le i < k$ and $a_k < b_k$. The usage of ∞ in $\overline{D}_j$ is to ensure that any two distinct columns of D are comparable. Now $\mathrm{Norm}(D)$ is defined by rearranging the columns of D increasingly from left to right with respect to $\prec$. See Figure 4 for an illustration of the normalization of a diagram.

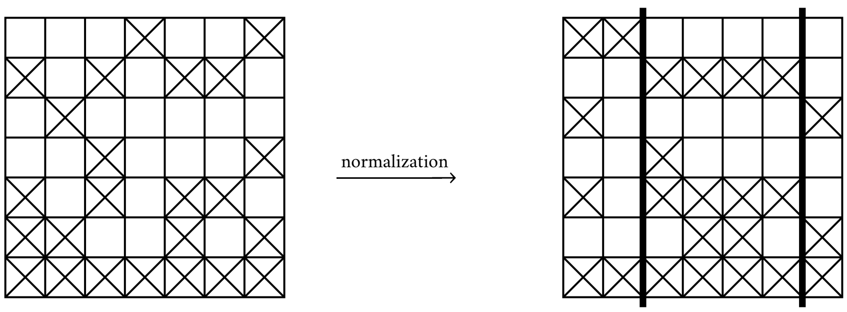

Figure 4. An illustration of normalization.

A diagram D is called *normalized* if $D = \mathrm{Norm}(D)$. Given a normalized diagram D, we partition the columns of D into distinct regions, according to the positions of the first crossing in each column. Here, the crossings in each column are counted from top to bottom. Precisely, columns of D belong to the same region if the first crossings in these columns lie in the same row. For example, the right normalized diagram in Figure 4 is partitioned into three regions, as divided by lines in boldface.

3.2 Columns in the same region

In this subsection, we analyze the distribution of boxes of a diagram $D = (D_1, \ldots, D_n)$ in the same region. We could always assume that each column in $[n] \times [n]$ has at least two "×" since otherwise one may embed D into a larger grid $[m] \times [m]$ with $m > n$. With this in mind, we distinguish the columns of D into three types, according to the number of boxes below the second crossing. For $1 \le j \le n$, we say that

- D_j is of Type I if there is no box of D_j below the second crossing;
- D_j is of Type II if there is exactly one box of D_j below the second crossing;
- D_j is of Type III if there are at least two boxes of D_j below the second crossing.

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Define the *signature* of D_j as the number of boxes of D_j lying between the first and the second crossings. For example, the third column in the right diagram in Figure 4 is a Type II column with signature equal to 1. When D is a normalized diagram, the signatures of columns in the same region are weakly increasing from left to right.

***Lemma* 3.1** *Let $D = (D_1, \dots, D_n)$ be a normalized multiplicity-free diagram. Consider the columns of D in the same region. Suppose that a column D_j does not reach the maximum signature among columns in the region. Then D_j must be of* Type I.

Proof Since D_j does not reach the maximum signature, there exists a column $D_{j'}$ with $j < j'$ in the same region such that $D_{j'}$ has a larger signature than D_j. Assume that the first two crossings in column j are at the positions (i_1, j) and (i_2, j). Since D_j and D'_j lie in the same region, the first crossing in column j' lies in row i_1. Suppose that the second crossing in column j' lies in row i_3. Then we have $i_3 > i_2$, and so the box (i_2, j') belongs to D'_j. See Figure 5 for an illustration.

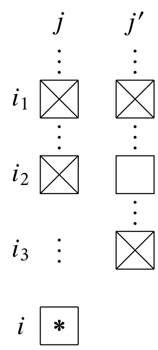

Figure 5. An illustration for the proof of Lemma 3.1.

Let (i, j) be any box below (i_2, j). We conclude that (i, j) does not belong to D_j since otherwise the subdiagram of D, which is restricted to rows $\{i_1, i_2, i\}$ and columns $\{j, j'\}$, would form the configuration obtained by swapping the columns of (A) in Figure 1. This implies that D_j is a column of Type I. ∎

From Lemma 3.1, we see that if there are Type II or Type III columns occurring in the same region, then they must be columns with the maximum signature. We discuss this in detail in the next lemma.

***Lemma* 3.2** *Let $D = (D_1, \dots, D_n)$ be a normalized multiplicity-free diagram. Consider the columns in the same region. Then*

(1) Type II *columns and* Type III *columns cannot occur simultaneously;*
(2) *If there is a* Type II *column occurring, then all* Type II *columns have the same configuration (In this case, the region starts with some* Type I *columns, followed by some copies of a* Type II *column, see the left picture in Figure 6 for an illustration);*
(3) *If there is a* Type III *column occurring, then this is the only* Type III *column in the region (In this case, the region starts with some* Type I *columns, followed by a single* Type III *column, see the right picture in Figure 6 for an illustration).*

Proof (1) Suppose to the contrary that there are both a Type II column (say, D_{j_1}) and a Type III column (say, D_{j_2}) occurring in the same region. We only consider the case for $j_1 < j_2$, and the arguments for $j_1 > j_2$ are the same. Since column j_1 and column j_2 are in the same region, their first crossings lie in the same row, say row i_1. By Lemma 3.1, D_{j_1} and D_{j_2} have the same signature, and so the second crossings in column j_1 and column j_2 are also in the same row, say row i_2. Let (i, j_1) be the unique box of D_{j_1} below row i_2. Since D_{j_2} is of Type III, it has at least two boxes below row i_2. We have the following observation.

(O) For $i_2 < h < i$, there is no box (h, j_2) belonging to D_{j_2}, since otherwise the subdiagram, restricted to rows $\{i_2, h, i\}$ and columns $\{j_1, j_2\}$, would become the configuration obtained by swapping the columns of (A) in Figure 1.

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The arguments are divided into two cases.

Case 1: The box (i, j_2) belongs to D_{j_2}. By the observation in (O), there exists another box (ℓ, j_2) of D_{j_2} with $\ell > i$. So the subdiagram, restricted to rows $\{i_1, i_2, i, \ell\}$ and columns $\{j_1, j_2\}$, is the configuration by swapping the columns of (C) in Figure 1, leading to a contradiction.

Case 2: The box (i, j_2) does not belong to D_{j_2}. Still, by the observation in (O), one can choose a box (ℓ, j_2) of D_{j_2} with $\ell > i$. In this case, the subdiagram, restricted to rows $\{i_2, i, \ell\}$ and columns $\{j_1, j_2\}$, forms the configuration (A) in Figure 1, yielding a contradiction. This concludes the assertion in (1).

(2) Let D_{j_1} and D_{j_2} with $j_1 < j_2$ be two columns of Type II. As in (1), let $(i_1, j_1), (i_2, j_1)$ (respectively, $(i_1, j_2), (i_2, j_2)$) be boxes containing the first two crossings in column j_1 (respectively, column j_2). The unique box of D_{j_1} below row i_2 must be in the same row as the unique box of D_{j_2} below row i_2, since otherwise there would result in a subdiagram of D which is the same as the configuration (A) (possibly by swapping the columns) in Figure 1.

(3) Suppose to the contrary that there are two Type III columns (say, D_{j_1} and D_{j_2} with $j_1 < j_2$) occurring. Again, let us use $(i_1, j_1), (i_2, j_1)$ (respectively, $(i_1, j_2), (i_2, j_2)$) to denote positions of the first two crossings in column j_1 (respectively, column j_2). In D_{j_1} (respectively, D_{j_2}), locate the top most two boxes below row i_2, denoted (h_1, j_1) and (h_2, j_1) with $i_2 < h_1 < h_2$ (respectively, (ℓ_1, j_2) and (ℓ_2, j_2) with $i_2 < \ell_1 < \ell_2$). If $h_1 = \ell_1$, then the subdiagram, restricted to rows $\{i_1, i_2, h_1, h_2\}$ and columns $\{j_1, j_2\}$, is the configuration (C) in Figure 1. If $h_1 \neq \ell_1$, then the subdiagram, restricted to rows $\{i_2, h_1, \ell_1\}$ and columns $\{j_1, j_2\}$, is the configuration (A) (possibly after a swapping of columns) in Figure 1. In both situations, we are led to a contradiction. ∎

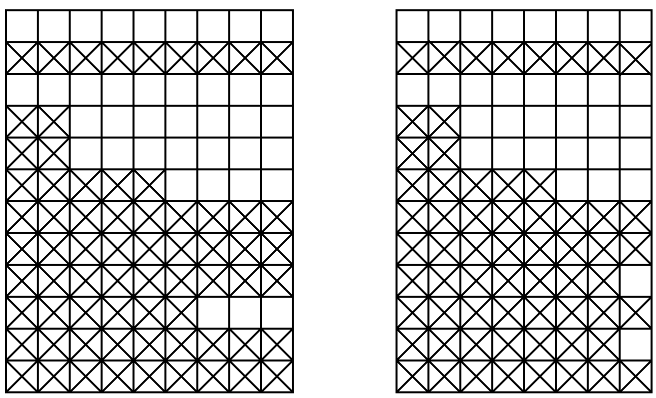

Figure 6. Illustrations of the regions in (2) and (3) of Lemma 3.2, respectively.

3.3 Columns in distinct regions

We now investigate the columns in distinct regions. Both lemmas in this subsection will be used in Subsections 4.2 and 4.3.

***Lemma* 3.3** *Let $D = (D_1, \ldots, D_n)$ be a normalized multiplicity-free diagram. Suppose that D_{j_1} and D_{j_2} with $j_1 < j_2$ lie in distinct regions, and the first crossing of column j_1 (respectively, column j_2) is in row i_1 (respectively, row i_2). Suppose further that $\emptyset \neq D_{j_1} = \{d_1 < d_2 < \cdots < d_k\}$, and that $D_{j_2} \neq [i_2 - 1]$ (that is, there is at least one box of D_{j_2} lying below row i_2). Then we have $i_2 > d_{k-1}$ (here we set $d_0 = 0$). In other words, the first crossing in column j_2 is below row d_{k-1}.*

Before giving a proof of this lemma, let us explain that the assumption that $D_{j_2} \neq [i_2 - 1]$ is natural. Suppose that D has a column $D_j = [m]$. Such a column is called a standard interval in [23]. Then there is only one choice of C_j such that $C_j \leq D_j$, namely, $C_j = D_j = [m]$. So the diagram D' obtained by removing D_j from D inherits all information of D in the sense that $\chi_D(x) = x_1 \cdots x_m \chi_{D'}(x)$. So, to prove Theorem 1.1, we may without loss of generality require that D has no standard intervals.

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Proof The assertion is trivial when $k = 1$. Assume now that $k \geq 2$. Suppose to the contrary that $i_2 \leq d_{k-1}$. We shall deduce the contradiction that $D_{j_2} = [i_2 - 1]$. Since D_{j_1} and D_{j_2} lie in distinct regions, we have $i_1 < i_2$. Our aim is to verify that any box in column j_2, lying below row i_2, contains a crossing. There are two cases.

Case 1: The box (i_2, j_1) belongs to D_{j_1}. In this case, we have $d_k > i_2$. To avoid the configuration (D) in Figure 1, the box (d_k, j_2) must contain a crossing, as illustrated in Figure 7. Consider the box (i, j_2) with $i > d_k$. Since each

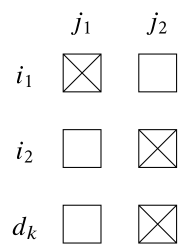

Figure 7. An illustration for the proof of Case 1.

box in column j_1, lying below row d_k, contains a crossing, to avoid the configuration (E) in Figure 1, the box (i, j_2) must contain a crossing.

It remains to consider the box (r, j_2) with $i_2 < r < d_k$. We assert that $(r, j_2) \notin D_{j_2}$. Suppose otherwise that $(r, j_2) \in D_{j_2}$. If (r, j_1) belongs to D_{j_1}, then the boxes of D in rows $\{i_1, i_2, r\}$ and columns $\{j_1, j_2\}$ form the configuration (D) in Figure 1, and if (r, j_1) does not belong to D_{j_1}, then the boxes of D in rows $\{i_1, i_2, r, d_p\}$ and columns $\{j_1, j_2\}$ form the configuration by swapping the columns of (F) in Figure 1. In either case, we are led to a contradiction. This concludes the assertion.

Case 2: The box (i_2, j_1) contains a crossing. In this case, we have $d_{k-1} > i_2$. To avoid the configuration (B) in Figure 1, the box (d_{k-1}, j_2) must contain a crossing. Moreover, to avoid the configuration (A) in Figure 1, any box in column j_2, which lies below row d_{k-1}, contains a crossing. The boxes in rows $\{i_1, i_2, d_{k-1}, d_k\}$ and columns $\{j_1, j_2\}$ look as depicted in Figure 8.

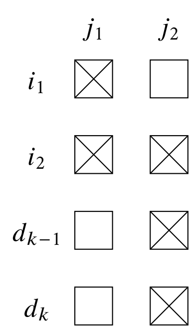

Figure 8. An illustration for the proof of Case 2.

We still need to check that any box in column j_2, which lies between row i_2 and row d_{k-1}, contains a crossing. Suppose otherwise that there exists a box (i, j_2) with $i_2 < i < d_{k-1}$ belonging to D_{j_2}. Consider the box (i, j_1). If it belongs to D_{j_1}, then the boxes in rows $\{i_1, i_2, i, d_{k-1}\}$ and columns $\{j_1, j_2\}$ would form the configuration (B) in Figure 1, and otherwise, the boxes in rows $\{i_2, i, d_{k-1}\}$ and columns $\{j_1, j_2\}$ would form the configuration by swapping the columns of (A) in Figure 1. This verifies that any box (i, j_2) with $i_1 < i < d_{p-1}$ contains a crossing. So the proof is complete. ∎

***Lemma* 3.4** *Take the same assumptions as in Lemma 3.3 (so $i_2 > d_{k-1}$).*

(1) If $i_2 < d_k - 1$, then $D_{j_2} = [i_2 - 1] \cup \{d_k\}$.

(2) If $i_2 = d_k - 1$, then $[i_2 - 1] \cup \{d_k\} \subseteq D_{j_2}$.

Proof Keep in mind that $i_2 > d_{k-1}$.

(1) First, to avoid the configuration obtained by swapping the columns of (A) in Figure 1, the box $(d_k - 1, j_2)$ must contain a crossing. So the boxes in rows $\{d_{k-1}, i_2, d_k - 1, d_k\}$ and columns $\{j_1, j_2\}$ are as illustrated in Figure 9.

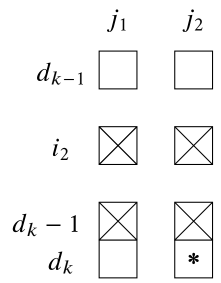

Figure 9. An illustration for the proof of (1).

For the same reason as above, any box in column j_2, lying between row i_2 and row $d_k - 1$, contains a crossing. Let us look at the box (d_k, j_2). We assert that this box belongs to D_{j_2}. Suppose otherwise that (d_k, j_2) contains a crossing. To avoid the configuration by swapping the columns of (A) in Figure 1, any box in column j_2 lying below row d_k contains a crossing. This, along with the above analysis, implies that $D_{j_2} = [i_2 - 1]$, contradicting the assumption that $D_{j_2} \neq [i_2 - 1]$. So the assertion is verified.

Now, because $(d_k, j_2) \in D_{j_2}$, any box (i, j_2) with $i > d_k$ contains a crossing since otherwise the boxes in rows $\{i_2, d_k - 1, d_k, i\}$ and columns $\{j_1, j_2\}$ would yield the configuration by swapping the columns of (C) in Figure 1. This shows that $D_{j_2} = [i_2 - 1] \cup \{d_k\}$.

(2) We need to check that $(d_k, j_2) \in D_{j_2}$. Suppose otherwise that $(d_k, j_2) \notin D_{j_2}$. Since $D_{j_2} \neq [i_2 - 1]$, there exists a box (i, j_2) with $i > d_k$ belonging to D_{j_2}. Then the subdiagram, restricted to rows $\{d_k - 1, d_k, i\}$ and columns $\{j_1, j_2\}$, becomes the configuration (A) in Figure 1, leading to a contradiction. ∎

4 Proof of Theorem 1.1

To complete the proof of Theorem 1.1, by Proposition 2.1, it is equivalent to showing that when $D = (D_1, \ldots, D_n)$ is multiplicity-free, the eigenspace

$$\mathrm{Span}_{\mathbb{C}} \left\{ \det\left(Y_D^C\right) : C \leq D,\ x^C = x^a \right\}$$

has dimension one. We do this by showing that $\det\left(Y_D^C\right) = \det\left(Y_D^{C'}\right)$ for $C, C' \leq D$ with $x^C = x^{C'}$. In light of Proposition 2.2, this will be achieved by establishing a bijection between the sets $\mathcal{F}_D(C)$ and $\mathcal{F}_D(C')$ which preserves both the sign and the weight.

***Theorem* 4.1** *Let $D = (D_1, \ldots, D_n)$ be a normalized multiplicity-free diagram, and let $C = (C_1, \ldots, C_n)$ and $C' = (C'_1, \ldots, C'_n)$ be two distinct diagrams, that are less than or equal to D, such that $x^C = x^{C'}$. Then there is a sign- and weight-preserving bijection between the sets $\mathcal{F}_D(C)$ and $\mathcal{F}_D(C')$.*

Without loss of generality, we may make the following extra assumptions on D:

(C 1): each column in $[n] \times [n]$ has at least two crossings.
(C 2): D has no column of the form $[m]$ where $m \geq 1$;

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The reasons for (C 1) and (C 2) have been explained respectively in the beginning of Subsection 3.2 and below Lemma 3.3.

The construction of the bijection is an iterative procedure, based on an operation, denoted Φ, acting on flagged fillings. The operation depends only on D, C and C'. Roughly speaking, for $F \in \mathcal{F}_D(C)$, the operation Φ interchanges the columns of F by sliding certain entries along the same rows. After applying Φ, we get a new flagged filling $\Phi(F)$, which is "closer" to a flagged filling in $\mathcal{F}_D(C')$ in the sense that there are more columns whose entries match the subsets in C'. Then we replace F by a flagged filling, denoted $\hat{\Phi}(F)$, which is obtained from $\Phi(F)$ by ignoring certain columns that have been adjusted by Φ, and we continue to apply Φ to $\hat{\Phi}(F)$. Eventually, we arrive at a flagged filling belonging to $\mathcal{F}_D(C')$, which is defined as the image of F.

The description of Φ will be distinguished into three cases, depending on the configuration of the first region of D. Specifically, according to Lemma 3.2, each region of D is one of the following three types: it contains

(R 1): only Type I columns, or
(R 2): Type I columns followed by some copies of a Type II column, or
(R 3): type I columns followed by exactly one Type III column.

In the remainder of this section, let $F = (F_1, \ldots, F_n)$ be any fixed filling in $\mathcal{F}_D(C)$. Our task is to construct $\Phi(F)$, as well as $\hat{\Phi}(F)$ obtained from $\Phi(F)$ by ignoring some specific columns.

4.1 The first region of D is of Type (R 1)

In this case, the first region of D contains only Type I columns. Suppose that there are m columns in the first region. We may further suppose that for $1 \le j \le m$, D_j is not empty (since empty columns can be obviously ignored). Recall that D satisfies the requirement (C 2) given below Theorem 4.1. Therefore, for $1 \le j \le m$, the signature of D_j is positive. As an illustration, Figure 10 lists the columns in the first region. Here we have erased some rows, lying on the bottom, which contain only crossings.

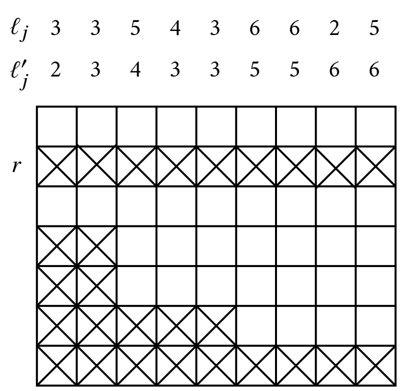

Figure 10. Columns in a Type (R 1) region and their labels.

To describe the operation Φ, we label the m columns in two ways. Suppose that the first crossing in each column is in row r. Let $1 \le j \le m$. Assume that (n_j, j) is the box right above the second crossing in column j (which is the lowest box of D_j). Note that $n_1 \le n_2 \le \cdots \le n_m$. For the region in Figure 10, we have $m = 9$ and $(n_1, \ldots, n_9) = (3, 3, 5, 5, 5, 6, 6, 6, 6)$. Observe that $C_j = [n_j] \setminus \{\ell_j\}$ for some $r \le \ell_j \le n_j$, and $C'_j = [n_j] \setminus \{\ell'_j\}$ for some $r \le \ell'_j \le n_j$. By assigning D_j with a label ℓ_j or ℓ'_j, we obtain two kinds of labelings for the m columns. See Figure 10 for an illustration of the labelings, from which we could recover C_j and C'_j.

***Lemma* 4.2** *As multisets, $\{\ell_1, \ldots, \ell_m\}$ is the same as $\{\ell'_1, \ldots, \ell'_m\}$.*

Proof For notational simplicity, we denote $p = n_m$. Write

$$x^C = x^{C'} = x_1^{a_1} \cdots x_p^{a_p} \, x_{p+1}^{a_{p+1}} \cdots x_n^{a_n}.$$

Consider the factor $x_1^{a_1} \cdots x_p^{a_p}$. It can be seen that

$$\begin{aligned} x_1^{a_1} \cdots x_p^{a_p} &= \frac{\prod_{j=1}^m x_1 \cdots x_{n_j}}{x_{\ell_1} \cdots x_{\ell_m}} \prod_{j=m+1}^{n} \prod_{\substack{t \in C_j \\ 1 \le t \le p}} x_t \\ &= \frac{\prod_{j=1}^m x_1 \cdots x_{n_j}}{x_{\ell'_1} \cdots x_{\ell'_m}} \prod_{j=m+1}^{n} \prod_{\substack{t \in C'_j \\ 1 \le t \le p}} x_t . \end{aligned}$$

By Lemma 3.3, for $m + 1 \le j \le n$, the first crossing in column j is strictly lower than row $p - 1$, and thus we have $[p-1] \subseteq D_j$, and hence $[p-1] \subseteq C_j$ and $[p-1] \subseteq C'_j$. Therefore,

$$\begin{aligned} x_1^{a_1} \cdots x_p^{a_p} &= \frac{\prod_{j=1}^m x_1 \cdots x_{n_j}}{x_{\ell_1} \cdots x_{\ell_m}} (x_1 \cdots x_{p-1})^{n-m} \prod_{j=m+1}^{n} \prod_{p \in C_j} x_p \\ &= \frac{\prod_{j=1}^m x_1 \cdots x_{n_j}}{x_{\ell'_1} \cdots x_{\ell'_m}} (x_1 \cdots x_{p-1})^{n-m} \prod_{j=m+1}^{n} \prod_{p \in C'_j} x_p . \end{aligned}$$

From the above, we know that $\{m + 1 \le j \le n \colon p \in C_j\}$ and $\{m + 1 \le j \le n \colon p \in C'_j\}$ have the same cardinality. So we have $x_{\ell_1} \cdots x_{\ell_m} = x_{\ell'_1} \cdots x_{\ell'_m}$, which justifies $\{\ell_1, \ldots, \ell_m\} = \{\ell'_1, \ldots, \ell'_m\}$ as multisets. ∎

The operation Φ will be performed on the columns of F in the first region, based on a column-interchanging procedure. Let $1 \le j_1 < j_2 \le m$. Note that if (i, j_1) is a box of F_{j_1}, then (i, j_2) is a box of F_{j_2}. We say that F_{j_1} and F_{j_2} are *interchangeable* if the resulting filling by swapping the entries in the same rows of F_{j_1} and F_{j_2} (leaving the entries of F_{j_2} below row n_{j_1} unchanged) is still a flagged filling. Equivalently, the entries of F_{j_1} have no intersection with the entries of F_{j_2} that lie below row n_{j_1}.

***Lemma* 4.3** *If $\ell_{j_2} \le n_{j_1}$, then F_{j_1} and F_{j_2} are interchangeable.*

Proof Recall that for $1 \le j \le m$, $C_j = [n_j] \setminus \{\ell_j\}$. When $\ell_{j_2} \le n_{j_1}$, the elements in the interval $[n_{j_1} + 1, n_{j_2}]$ all belong to C_{j_2}, and they occupy the boxes of F_{j_2} below row n_{j_1}. So, after the column-interchanging, the resulting filling is still a flagged filling. ∎

We can now describe the construction of $\Phi(F)$ in the Type (R 1) case.

Step 1. Denote $c = \max\{\ell_1, \ldots, \ell_m\}$. Let $A = \{1 \le j \le m \colon \ell_j = c\}$, $B = \{1 \le j \le m \colon \ell'_j = c\}$ and $S = A \cap B$. By Lemma 4.2, we have $|A| = |B|$. Assume that $A \setminus S = \{j_1 < \cdots < j_t\}$ and $B \setminus S = \{j'_1 < \cdots < j'_t\}$. For every $1 \le s \le t$, swap the entries in the same rows of F_{j_s} and $F_{j'_s}$ (clearly, this is independent of the order of s).

We explain that after Step 1, the resulting filling is also a flagged filling. Consider two cases: $j_s > j'_s$ and $j_s < j'_s$. In the former case, we have $\ell_{j_s} = c = \ell'_{j'_s} \le n_{j'_s}$, and in the latter case, we have $\ell_{j'_s} < c = \ell_{j_s} \le n_{j_s}$. Both cases satisfy the requirement in Lemma 4.3, and thus F_{j_s} and $F_{j'_s}$ are interchangeable.

Step 2. After Step 1, the columns labeled with c are indexed by the set B. Ignore such columns, and then return to Step 1 where the operation is applied to the remaining columns in the first region.

The above algorithm eventually terminates. Define $\Phi(F)$ to be the resulting flagged filling. Figure 11 gives an illustration of the construction of Φ. Note that for $1 \le j \le m$, the label of the j-th column in $\Phi(F)$ is equal to ℓ'_j.

Assume that $\Phi(F) = (F'_1, \ldots, F'_n)$. For $1 \le j \le m$, the set of entries in F'_j is equal to C'_j, while for $m < j \le n$, we have $F'_j = F_j$. This means that $\Phi(F)$ is "closer" to a filling in $\mathcal{F}_D(C')$. We set $\hat{\Phi}(F)$ to be the flagged filling obtained from $\Phi(F)$ by ignoring the columns in the first region.

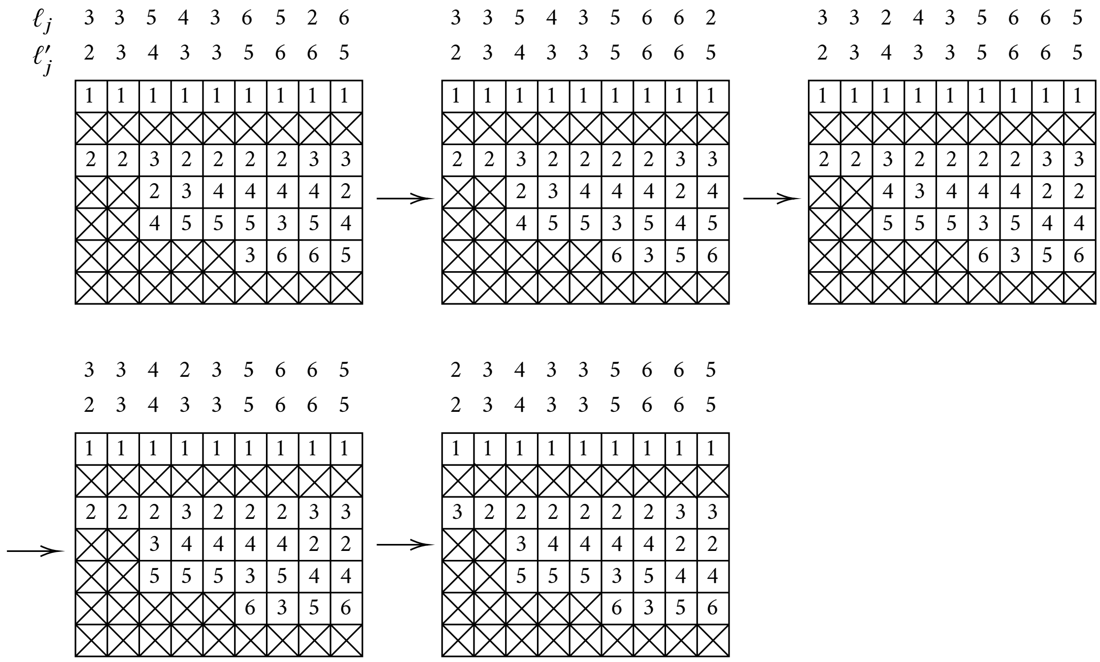

Figure 11. An illustration of the construction of Φ in the Type (R 1) case.

4.2 The first region of D is of Type (R 2)

Still, let $(D_1, \ldots, D_m)$ be the columns of D in the first region. As in Subsection 4.1, we assume that each D_j for $1 \leq j \leq m$ is not empty. We also adopt the notation n_j which means that (n_j, j) is the box right above the second crossing in column j. Note that when D_j is a column of Type II, (n_j, j) is not the lowest box of D_j.

Imitating Subsection 4.1, we label each column D_j for $1 \leq j \leq m$ in two ways, also denoted by ℓ_j and ℓ'_j. Let us first define ℓ_j. Consider $[n_j] \setminus C_j$. If D_j is of Type I, then $[n_j] \setminus C_j$ contains exactly a single element, just as in Subsection 4.1. While, if D_j is of Type II, then $[n_j] \setminus C_j$ either contains a single element or is the empty set. Clearly, $[n_j] \setminus C_j = \emptyset$ if and only if $C_j = [n_j]$. When $[n_j] \setminus C_j$ contains a single element, let ℓ_j be this element, and when $[n_j] \setminus C_j = \emptyset$, let $\ell_j = \emptyset$. In completely the same manner, we may define ℓ'_j with respect to C'_j. Figure 12 illustrates the columns, as well as their labels, in the first (Type (R 2)) region of a filling F in $\mathcal{F}_D(C)$.

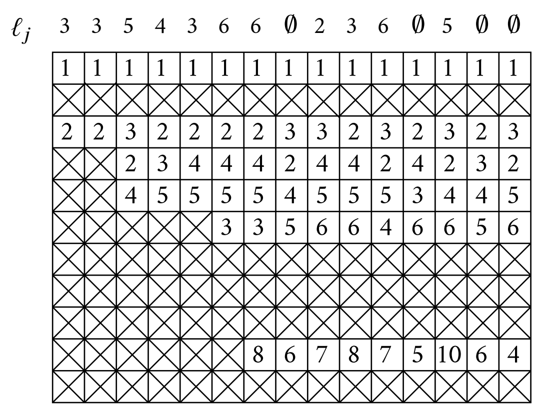

Figure 12. An illustration of columns and their labels in a Type (R 2) region.

***Lemma* 4.4** *As multisets,* $\{\ell_1, \ldots, \ell_m\}$ *is the same as* $\{\ell'_1, \ldots, \ell'_m\}$.

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Proof The proof is similar to (actually slightly easier than) that of Lemma 4.2. Let $p = n_m$, and write $x^C = x^{C'} = x_1^{a_1} \cdots x_p^{a_p} x_{p+1}^{a_{p+1}} \cdots x_n^{a_n}$. By Lemma 3.3, for $m+1 \le j \le n$, the first crossing in column j is lower than row p, which implies $[p] \subseteq D_j$ and in turn $[p] \subseteq C_j$ and $[p] \subseteq C'_j$. Now we see that

$$x_1^{a_1} \cdots x_p^{a_p} = \frac{\prod_{j=1}^m x_1 \cdots x_{n_j}}{x_{\ell_1} \cdots x_{\ell_m}} (x_1 \cdots x_p)^{n-m} = \frac{\prod_{j=1}^m x_1 \cdots x_{n_j}}{x_{\ell'_1} \cdots x_{\ell'_m}} (x_1 \cdots x_p)^{n-m},$$

where we set $x_\emptyset = 1$. It follows that $x_{\ell_1} \cdots x_{\ell_m} = x_{\ell'_1} \cdots x_{\ell'_m}$, and so $\{\ell_1, \dots, \ell_m\} = \{\ell'_1, \dots, \ell'_m\}$ as multisets. ∎

Unlike Subsection 4.1, the construction of $\Phi(F)$ in the Type (R 2) case requires two algorithms. The first algorithm acts on the columns in the first region in the same way as the algorithm in Subsection 4.1. Here the label $\emptyset$ is regarded as the infinity which is greater than any positive integer. This procedure will be best understood via an example in Figure 13.

After applying the above algorithm, the label of D_j for $1 \le j \le m$ becomes ℓ'_j. Denote by $F^{(1)}$ the resulting flagged filling. Suppose that $F^{(1)} = (F_1^{(1)}, \dots, F_n^{(1)})$ belongs to $\mathcal{F}_D(C^{(1)})$ where $C^{(1)} = (C_1^{(1)}, \dots, C_n^{(1)})$. Clearly, we have $F_j^{(1)} = F_j$ for $m < j \le n$. For $1 \le j \le m$, we have $C_j^{(1)} = C'_j$ when D_j is either a Type I column or a Type II column labeled with $\emptyset$.

We still need to consider the type II columns in the first region whose labels are not $\emptyset$. Let $p = n_m$. By (2) in Lemma 3.2, the lowest boxes in the Type II columns lie in the same row, say row q. Our next task is to adjust the entries in row q that are greater than p. Note that such entries lie in the Type II columns whose labels are not $\emptyset$. In Figure 13, the entries greater than p lying in row q are displayed in boldface, where $p = 6$ and $q = 10$.

To deal with the entries in row q that are greater than p, we need additionally to invoke the regions (after the first region) whose first crossings lying in or above row $q-1$. According to Lemma 3.3, the first crossing in each such region is below row p. Suppose that there are $k-1$ ($k \ge 1$) such regions. By Lemma 3.4, each column in these regions contains a box of D in row q. Together with the first region, we next consider the first k regions of D. Assume that the first k regions have a total of m' columns. For $1 \le r \le k$, assume that the first crossing in the r-th region is in row i_r. Note that $1 \le i_1 < \cdots < i_k \le q-1$

The description of the second algorithm will be divided into two cases, according to the relative values of i_k and $q-1$.

Case 1. $i_k < q-1$. By (1) in Lemma 3.4, for $m < j \le m'$, D_j has exactly one box below the first crossing which lies in row q. The second algorithm acts on the entries in the row q of $F^{(1)}$, restricted to the first k regions, that are greater than p. Suppose that there are d such entries, say $(a_1, \dots, a_d)$ listed from left to right, and that for $1 \le t \le d$, a_t lies in column j_t. An illustration is given in Figure 14, where we set $k = 3$ and $d = 10$, and the entries in $(a_1, \dots, a_{10})$ are drawn in boldface.

Notice that for $1 \le t \le d$, a_t can be obviously read off from $C_{j_t}^{(1)}$. To be precise, suppose that D_{j_t} is in the r-th region. For $r = 1$, a_t is the unique element in $C_{j_t}^{(1)}$ that is greater than p, while for $2 \le r \le k$, a_t is the unique element in $C_{j_t}^{(1)} = C_{j_t}$ that is greater than or equal to i_r. We define b_t by replacing $C_{j_t}^{(1)}$ with C'_{j_t}.

***Lemma* 4.5** *As multisets, we have* $\{a_1, \dots, a_d\} = \{b_1, \dots, b_d\}$.

Proof Note that the first crossing in any column of D after the k-th region lie in or below row q. The proof is then done by using completely similar arguments to those for Lemma 4.2. ∎

Define $\Phi(F)$ to be the flagged filling obtained from $F^{(1)}$ by replacing the entry a_t with b_t for $1 \le t \le d$. Write $\Phi(F) = (F'_1, \dots, F'_n)$. Then, for $1 \le j \le m'$, the set of entries in F'_j is equal to C'_j, while for $m' < j \le n$, we have $F'_j = F_j$. Let $\hat{\Phi}(F)$ be obtained from $\Phi(F)$ by ignoring the first m' columns.

Case 2. $i_k = q-1$. By (2) in Lemma 3.4, each column of D in the k-th region has a box in row q. Focus on the columns of $F^{(1)}$ in the first $k-1$ regions (columns in the k-th region will be kept unchanged). We consider the entries in the q-th row. In the same way as in Case 1, we may define the sequences $(a_1, \dots, a_d)$ and $(b_1, \dots, b_d)$. See Figure 15 for an illustration, where $k = 4$, and the entries in the sequence $(a_1, \dots, a_d)$ are shown in boldface. In this figure,

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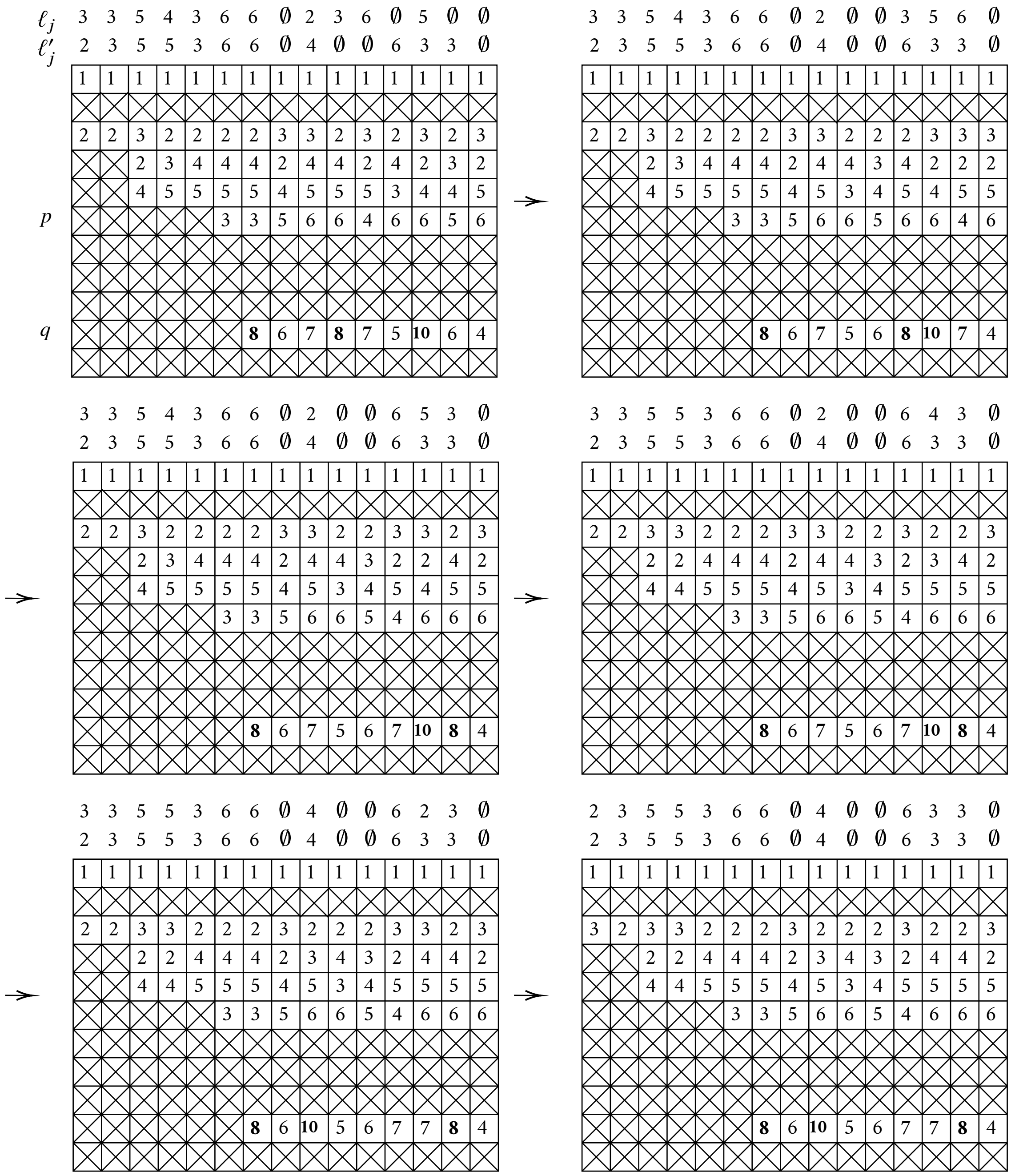

Figure 13. Illustration of the first algorithm in the Type (R 2) case.

we use italics to distinguish the entries equal to $q-1$ or q, and the reason for this will be clear later.

***Lemma* 4.6** *The multiset obtained from* $\{a_1,\ldots,a_d\}$ *by removing the elements equal to* $q-1$ *or* q *is the same as the multiset obtained from* $\{b_1,\ldots,b_d\}$ *by removing the elements equal to* $q-1$ *or* q*.*

Proof The proof is the analogous to that for Lemma 4.4, and so is omitted. ∎

It should be pointed out that the subset of $\{a_1,\ldots,a_d\}$ consisting of $q-1$ and q is not necessarily equal to the subest of $\{b_1,\ldots,b_d\}$ consisting of $q-1$ and q, because of the existence of the k-th region (namely, the region with topmost crossings in row $q-1$).

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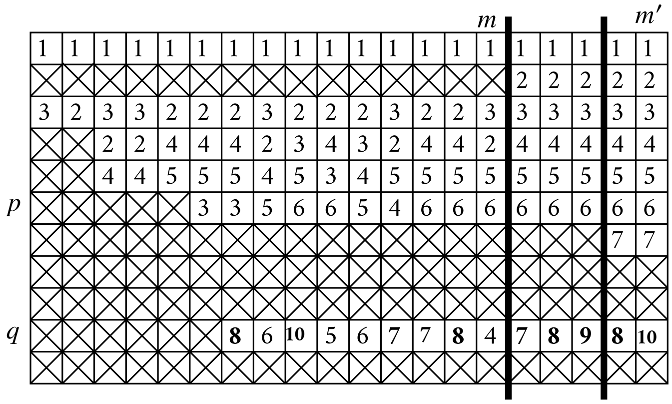

Figure 14. An illustration for Case 1 in the second algorithm.

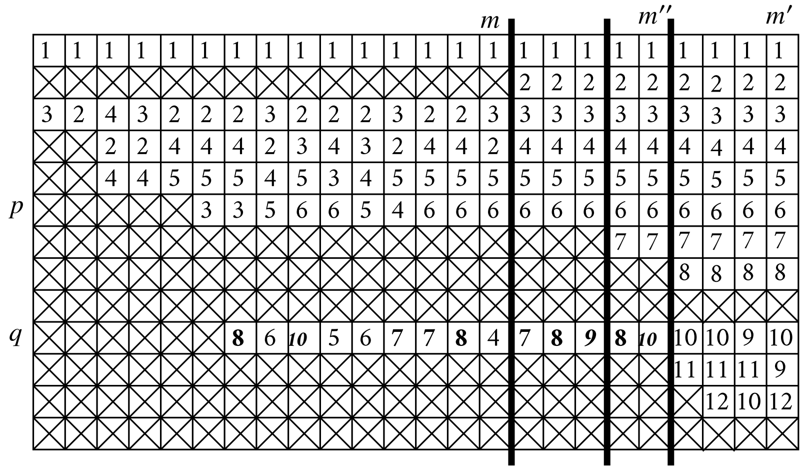

Figure 15. An illustration for Case 2 in the second algorithm.

We construct $(a'_1, \ldots, a'_d)$ by reordering the elements of $(a_1, \ldots, a_d)$ as follows. First, shuffle the subword including elements not equal to $q-1$ or q and the subword including $q-1$ and q, such that the elements not equal to $q-1$ or q occupy the same positions as in $(b_1, \ldots, b_d)$ (this is well defined according to Lemma 4.6). Then, rearrange the elements not equal to $q-1$ or q such that they have the same order as in $(b_1, \ldots, b_d)$. For example, assuming $(b_1, \ldots, b_d) = (7, 8, 9, 8, 7, 9, 8, 10, 7, 8)$, the sequence $(a_1, \ldots, a_d)$ in Figure 15 will be changed into $(a'_1, \ldots, a'_d)$ as illustrated in Figure 16.

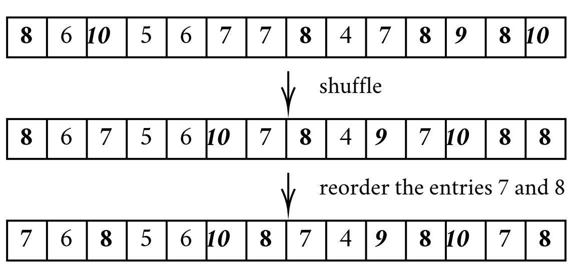

Figure 16. An illustration from $(a_1, \ldots, a_d)$ to $(a'_1, \ldots, a'_d)$.

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Define $\Phi(F)$ as the flagged filling obtained from $F^{(1)}$ by replacing the entry a_t with a_t' for $1 \leq t \leq d$. Write $\Phi(F) = (F_1', \ldots, F_n')$. Suppose that there are m'' columns in the first $k-1$ regions. For $m'' < j \leq n$, we clearly have $F_j' = F_j$. While, for $1 \leq j \leq m''$, when F_j' does not contain $q-1$ or q, the set of entries in F_j' is equal to C_j'.

The construct of $\hat{\Phi}(F)$ is as follows. First, ignore the "well-behaved" columns, namely, the columns in the first $k-1$ regions of $\Phi(F)$ not containing $q-1$ or q. Then, merge the remaining columns (namely, columns containing $q-1$ or q) into the k-th region, in such a way that we erase all crossings above row $q-1$. Specifically, for $1 \leq j \leq m''$, if F_j' contains $q-1$ or q, then we replace D_j by $D_j \cup [q-2] = [q-2] \cup \{q\}$, and correspondingly, we replace $C_j^{(1)}$ and C_j' respectively by $C_j^{(1)} \cup [p-2]$ and $C_j' \cup [p-2]$. For the portion in Figure 15, the illustration that the columns containing $q-1$ or q are merged into the k-th region is given in Figure 17.

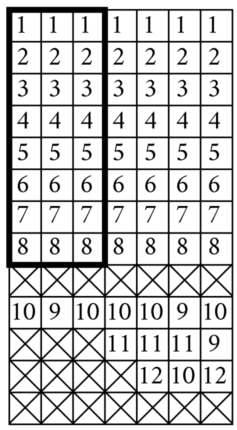

Figure 17. The merging procedure for Figure 15.

We remark that all the entries above row $q-1$ in the merged columns, which are framed by lines in boldface in Figure 17, will keep unchanged in the next round of the iteration. So the merging operation merely plays a role that the columns in the first $k-1$ regions, containing $q-1$ or q, are viewed as columns in the k-th region, so that our algorithms could be implemented in the next iteration.

4.3 The first region of D is of Type (R 3)

In this case, the construction of $\Phi(F)$ as well as $\hat{\Phi}(F)$ is nearly the same as that for the Type (R 2) case in Subsection 4.2. So the description will be sketched. All but one of the notation (namely, the notation p) will be fully consistent with what we used in Subsection 4.2.

Let $(D_1, \ldots, D_m)$ be the first region of D. Keep in mind that D_j for $1 \leq j < m$ are Type I columns, and D_m is a Type III column. For $1 \leq j \leq m$, let n_j be the row index such that (n_j, j) is the box right above the second crossing in column j. Note that $[n_j] \setminus C_j$ (respectively, $[n_j] \setminus C_j'$) contains a single element, or is equal to $\emptyset$ (this possibly occurs only when $j = m$), which is defined as the label ℓ_j (respectively, ℓ_j').

Perform the first algorithm as in Subsection 4.2 to interchange the columns of F in the first region. The resulting flagged filling is denoted $F^{(1)} = (F_1^{(1)}, \ldots, F_n^{(1)})$. Suppose that $F^{(1)}$ belongs to $\mathcal{F}_D(C^{(1)})$ where $C^{(1)} = (C_1^{(1)}, \ldots, C_n^{(1)})$. Then $C_j^{(1)} = C_j'$ for $1 \leq j < m$. Moreover, one has $F_j^{(1)} = F_j$ for $m < j \leq n$. We next deal with the column $F_m^{(1)}$, parallel to what we do in the second algorithm in Subsection 4.2.

Since D_m is a Type III column, there are at least two boxes of D_m lying below the second crossing. Let $p < q$ be the row indices such that (p, m) and (q, m) are the lowest two boxes of D_m. Unlike in Subsection 4.2, we will no longer have the relation $p = n_m$. Suppose that there are k regions whose first crossings lie in or above row $q-1$.

For $1 \leq r \leq k$, assume that the first crossing in the r-th region is in row i_r. We consider two cases.

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Case 1. $i_k < q-1$. The left picture in Figure 18 is an instance of this case. Define $\Phi(F)$ and $\hat{\Phi}(F)$ by applying the same procedure as in Case 1 of Subsection 4.2.

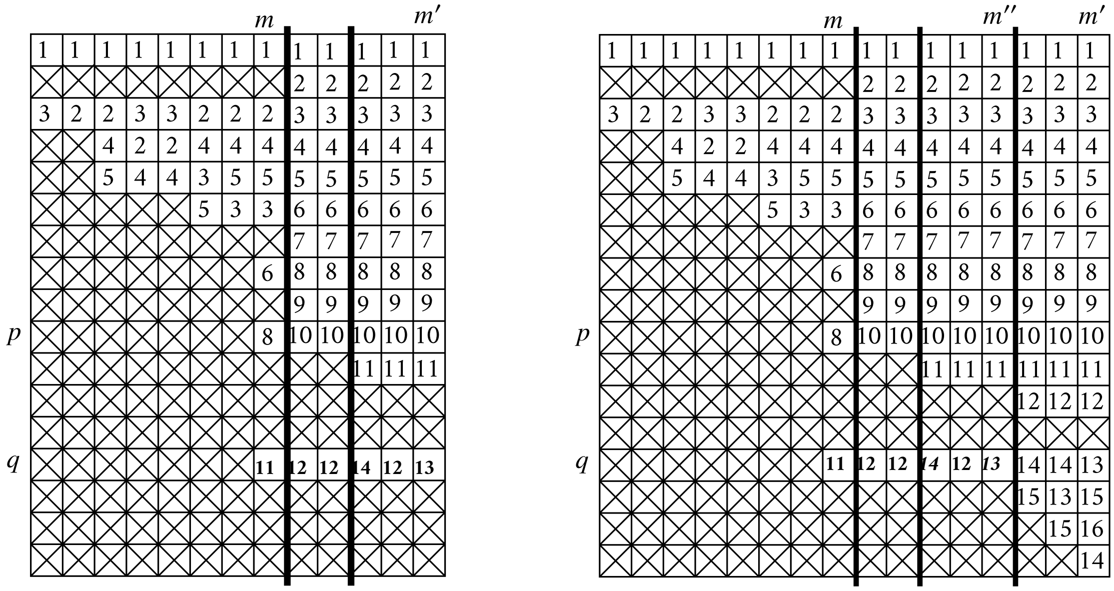

Figure 18. Illustrations for Type (R 3) case with $i_k < q-1$ or $i_k = q-1$.

Case 2. $i_k = q-1$. This case is demonstrated in the right picture in Figure 18. Define $\Phi(F)$ and $\hat{\Phi}(F)$ by applying the same procedure as in Case 2 of Subsection 4.2.

4.4 Proof of Theorem 4.1

Starting with $F \in \mathcal{F}_D(C)$, we iterate the operation Φ and eventually arrive at a flagged filling, denoted $\Omega(F)$, in $\mathcal{F}_D(C')$. We explain that Ω is a bijection from $\mathcal{F}_D(C)$ to $\mathcal{F}_D(C')$ that preserves both the sign and weight. To check that Ω is a bijection, the key is to make clear which and how columns are swapped in each step. By our construction, this only depends on the diagrams D, C and C', independent of the flagged filling F. Moreover, it is easy to see that the operation in each step in Subsections 4.1, 4.2, 4.3 may be reversed.

We next check that Ω preserves the sign and weight of F. It suffices to verify that F and $\Phi(F)$ have the same sign and weight. Since the entries (if moved) are slid in the same row, we have $y^F = y^{\Phi(F)}$. To see that F and $\Phi(F)$ have the same sign, recall that there are two kinds of operations in our construction.

(1) Two columns F_{j_1} and F_{j_2} $(j_1 < j_2)$ of F are exchanged. Suppose that F_{j_1} has column reading word $u = a_1a_2\cdots a_s$, and F_{j_1} has column reading word $v = b_1b_2\cdots b_t$. Note that $s \le t$. After column-exchanging, the reading words in columns j_1 and j_2 become $u' = b_1b_2\cdots b_s$ and $v' = a_1a_2\cdots a_s\, b_{s+1}\cdots b_t$, respectively. Notice that any element in $\{b_{s+1}, \ldots, b_t\}$ is bigger than any element in $\{a_1, \ldots, a_s\}$ or $\{b_1, \ldots, b_s\}$. This implies that $\mathrm{inv}(u) + \mathrm{inv}(v) = \mathrm{inv}(u') + \mathrm{inv}(v')$.

(2) Some entries in row q are reordered. In this case, the inversion number of each column reading word is unchanged since the moved entries in row q are bigger than any entry above row q in the corresponding columns.

By the above analysis, we obtain that Ω is a sign- and weight-preserving bijection. This allows us to conclude the proof of Theorem 4.1.

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References

[1] S. Armon, S. Assaf, G. Bowling and H. Ehrhard, Kohnert's rule for flagged Schur modules, J. Algebra 617 (2023), 352–381.
[2] S. Assaf and D. Searles, Kohnert polynomials, Exp. Math. 31 (1) (2022), 93–119.
[3] S. Assaf and D. Searles, Kohnert tableaux and a lifting of quasi-Schur functions, J. Combin. Theory Ser. A 156 (2018), 85–118.
[4] M. Brion, Multiplicity-free subvarieties of flag varieties, Commutative algebra (Grenoble/Lyon, 2001), Contemp. Math., vol. 331, Amer. Math. Soc., Providence, RI, 2003, pp. 13–23.
[5] F. Castillo, Y. Cid-Ruiz, F. Mohammadi and J. Montaño, K-polynomials of multiplicity-free varieties, arXiv:2212.13091, 2022.
[6] C. Eur and M. Larson, K-theoretic positivity for matroids, arXiv:2311.11996, 2023
[7] N.J.Y. Fan and P.L. Guo, Vertices of Schubitopes, J. Combin. Theory Ser. A 177 (2021), 105311, 20pp.
[8] N.J.Y. Fan and P.L. Guo, Upper bounds of Schubert polynomials, Sci. China Math. 65 (2022), 1319–1330.
[9] A. Fink, K. Mészáros and A. St. Dizier, Schubert polynomials as integer point transforms of generalized permutahedra, Adv. Math. 332 (2018), 465–475.
[10] A. Fink, K. Mészáros and A. St. Dizier, Zero-one Schubert polynomials, Math. Z. 297 (2021), 1023–1042.
[11] Y. Gao, R. Hodges and A. Yong, Classification of Levi-spherical Schubert varieties. Selecta Math. (N.S.) 29 (2023), Paper No. 55, 40 pp.
[12] Z. Hamaker, O. Pechenik and A. Weigandt, Gröbner geometry of Schubert polynomials through ice, Adv. Math. 398 (2022), Paper No. 108228, 29 pp.
[13] R. Hodges and A. Yong, Coxeter combinatorics and spherical Schubert geometry, J. Lie Theory 32 (2022), 447–474.
[14] R. Hodges and A. Yong, Multiplicity-free key polynomials, Ann. Comb. 27 (2023), 387–411.
[15] P. Klein, Diagonal degenerations of matrix Schubert varieties, Algebraic Combin. 6 (2023), 1073–1094.
[16] A. Knutson, T. Lam and D.E. Speyer, Positroid varieties: juggling and geometry, Compositio Math. 149 (2013), 1710–1752.
[17] A. Kohnert, Weintrauben, Polynome, Tableaux, Bayreuth Math. Schrift. 38(1990), 1–97.
[18] W. Kraskiewicz and P. Pragacz, Foncteurs de Schubert, C. R. Acad. Sci. Paris Sér. I Math. 304 (1987), 209–211.
[19] W. Kraśkiewicz and P. Pragacz, Schubert functors and Schubert polynomials, European J. Combin. 25 (2004), 1327–1344.
[20] A. Knutson, Frobenius splitting and Möbius inversion, arXiv:0902.1930v1, 2009.
[21] A. Knutson and E. Miller, Gröbner geometry of Schubert polynomials, Ann. Math. 161 (2005), 1245–1318.
[22] P. Magyar, Schubert polynomials and Bott–Samelson varieties, Comment. Math. Helv. 73 (1998), 603–636.
[23] K. Mészáros, L. Setiabrata and A. St. Dizier, An orthodontia formula for Grothendieck polynomials, Trans. Amer. Math. Soc. 375 (2022), 1281–1303.
[24] K. Mészáros, L. Setiabrata and A. St. Dizier, On the support of Grothendieck polynomials, Ann. Comb. 29 (2025), no. 2, 541–562.
[25] K. Mészáros, A. St. Dizier and A. Tanjaya, Principal specialization of dual characters of flagged Weyl modules, Electron. J. Combin. 28 (2021), Paper No. 4.17, 12 pp.
[26] K. Mészáros and A. Tanjaya, Inclusion-exclusion on Schubert polynomials, Algebr. Comb. 5 (2022), 209–226.
[27] O. Pechenik and M. Satriano, Proof of a conjectured Möbius inversion formula for Grothendieck polynomials, Selecta Math. (N.S.) 30 (2024), Paper No. 83, 8 pp.
[28] S.C.Y. Peng, Z. Lin and S.C.C. Sun, Upper bounds of dual flagged Weyl characters, Adv. Appl. Math. 160 (2024), 102752, 12pp.
[29] A. Postnikov, Total positivity, Grassmannians, and networks, http://arxiv.org/abs/math/0609764, 2006.

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