ON AN INEQUALITY OF BOMBIERI

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Bieberbach's conjecture, proposed in 1916 and still unsolved, states that if $f(z) = z+a_2z^2+\cdots$ is holomorphic and univalent in the disc $|z| < 1$ then $|a_n| \leqq n$ for each $n \geqq 2$, with equality for some n only if $f(z)$ is the Koebe function

$$k(z) = z/(1-z)^2 = z+2z^2+3z^3+\cdots$$

or is obtained from this function by a rotation. Very recently Bombieri has succeeded in showing that if $f(z)$ is sufficiently close to the Koebe function, then $\mathscr{R}a_n \leqq n$ with equality only if $f(z) = k(z)$. This had previously been proved by Garabedian, Ross and Schiffer [3] for even values of n.

In the announcement [1] Bombieri proves this result for functions $f_\varepsilon(z)$ depending analytically on a parameter ε with $f_0(z) = k(z)$. The basis of the proof is the fact that the quadratic form

$$(1) \qquad R_N = \sum_{n=1}^{N-1} n(N-n)x_n^2+2\sum_{m+n<N}(N-m-n)x_m x_n$$

is positive definite, and Bombieri deduces this from the new integral inequality

$$(2) \qquad \int_{-1}^{1}(1-x^2)f^2(x)dx+2\iint_{x+y\geqq 0}(x+y)f(x)f(y)dxdy \geqq 0,$$

valid for all $f(x) \in L^2(-1, 1)$. In the present paper a different proof of the inequality (2) is given, using Legendre polynomials. Moreover it is shown that equality holds only if $f(x) = 0$ almost everywhere. The positive definiteness of the quadratic form R_N is also proved directly, without recourse to integrals, by using polynomials which are orthogonal over a finite set.

THEOREM. *If $f(x) \in L^2(-1, 1)$ and $0 \leqq \lambda \leqq 2$, then*

$$J(\lambda) = \int_{-1}^{1}(1-x^2)f^2(x)dx+\lambda\iint_{x+y\geqq 0}(x+y)f(x)f(y)dxdy \geqq 0,$$

with equality only if $f(x) = 0$ almost everywhere. If $\lambda < 0$ or $\lambda > 2$ there exist functions $f(x) \in L^2(-1, 1)$ for which $J(\lambda) < 0$.

Put

$$F(x) = -\int_x^1 f(\xi)d\xi.$$

Since $F(x)$ is absolutely continuous, it has a convergent expansion

$$F(x) = \sum_{n=0}^{\infty} a_n \bar{P}_n(x),$$

where

$$\bar{P}_n(x) = \left(\frac{2n+1}{2}\right)^{\frac{1}{2}} P_n(x)$$

is the normalised Legendre polynomial and

$$a_n = \int_{-1}^{1} F(x)\bar{P}_n(x)dx.$$

By integrating by parts and using the differential equation for the Legendre polynomials we obtain

$$\int_{-1}^{1} (1-x^2)f(x)\bar{P}'_n(x)dx = n(n+1)a_n.$$

In particular,

$$\int_{-1}^{1} (1-x^2)\bar{P}'_m(x)\bar{P}'_n(x)dx = n(n+1)\delta_{mn}.$$

Thus the polynomials $\bar{P}'_n(x)$ $(n \geqq 1)$ form an orthogonal system with respect to the weight function $w(x) = 1-x^2$. Since the interval is finite, this system is complete. Therefore we have the Parseval relation

$$J_1 \equiv \int_{-1}^{1} (1-x^2)f^2(x)dx = \sum_{n=1}^{\infty} n(n+1)a_n^2.$$

On the other hand, by integrating by parts we get

$$\begin{aligned}
J_2 &= \iint_{x+y\geqq 0} (x+y)f(x)f(y)dxdy \\
&= \int_{-1}^{1} xf(x)\int_{-x}^{1} f(y)dydx + \int_{-1}^{1} f(x)\int_{-x}^{1} yf(y)dydx \\
&= -\int_{-1}^{1} xf(x)F(-x)dx + F(x)\int_{-x}^{1} yf(y)dy\Big|_{-1}^{1} + \int_{-1}^{1} F(x)xf(-x)dx \\
&= \int_{-1}^{1} x[F(x)f(-x)-f(x)F(-x)]dx \\
&= -\int_{-1}^{1} x[F(x)F(-x)]'dx \\
&= \int_{-1}^{1} F(x)F(-x)dx,
\end{aligned}$$

since $F(1) = 0$. Since $\bar{P}_n(-x) = (-1)^n\bar{P}_n(x)$ it follows from the Parseval relation for the system of Legendre polynomials that

$$J_2 = \sum_{n=0}^{\infty} (-1)^n a_n^2.$$

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Thus

$$J(\lambda) = J_1+\lambda J_2$$
$$= \lambda a_0^2+\sum_{n=1}^{\infty}[n(n+1)+\lambda(-1)^n]a_n^2.$$

This shows that $J(\lambda) \geqq 0$ if $0 \leqq \lambda \leqq 2$. Moreover if $J(\lambda) = 0$ and $0 \leqq \lambda < 2$ then $a_n = 0$ $(n \geqq 1)$ and hence $f(x) = 0$ a.e. If $J(\lambda) = 0$ and $\lambda = 2$ then $a_0 = 0$ and $a_n = 0$ $(n \geqq 2)$. The vanishing of a_n for $n \geqq 2$ implies that $f(x)$ is constant a.e. Since

$$2^{\frac{1}{2}}a_0 = \int_{-1}^{1} F(x)dx$$
$$= -\int_{-1}^{1} f(x)dx-\int_{-1}^{1} xf(x)dx$$
$$= -\int_{-1}^{1}(1+x)f(x)dx,$$

this constant must be 0.

Now write

$$h_N = \sum_{n=1}^{N}\frac{2n+1}{n(n+1)} \approx 2\log N \qquad \text{for } N \to \infty$$

and let $f(x)$ be the polynomial defined by taking

$$a_n = \begin{cases} (2n+1)^{\frac{1}{2}}/n(n+1) & \text{for } 1 \leqq n \leqq N, \\ 0 & \text{for } n > N. \end{cases}$$

Since $F(1) = 0$ we have

$$a_0 = -\sum_{n=1}^{N}(2n+1)^{\frac{1}{2}}a_n = -h_N.$$

Hence

$$J(\lambda) = h_N+\lambda[h_N^2+O(1)].$$

If $\lambda < 0$ it follows that $J(\lambda) < 0$ for sufficiently large N. On the other hand, if we define a_n in the same way as before for $n \geqq 2$ but take

$$a_1 = -3^{-\frac{1}{2}}(h_N-\tfrac{3}{2}),$$

then $a_0 = 0$ and

$$J(\lambda) = \tfrac{2}{3}(h_N-\tfrac{3}{2})^2+h_N-\tfrac{3}{2}+\lambda\sum_{n=1}^{N}(-1)^n a_n^2$$
$$= \tfrac{1}{3}(2-\lambda)(h_N-\tfrac{3}{2})^2+h_N-\tfrac{3}{2}+\lambda O(1).$$

If $\lambda > 2$ it follows that $J(\lambda) < 0$ for sufficiently large N. This completes the proof of the theorem.

To prove the positive definiteness of the quadratic form R_N directly, without using the inequality (2), we need the analogues of the Legendre polynomials for a discrete variable. Čebyšev has defined, for each positive

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integer N, a finite sequence of polynomials $t_n(x)$ $(n = 0, 1, \cdots, N-1)$ with the following properties (see [2]):

(i) $t_n(x)$ has degree n in x,

(ii) Orthogonality

$$\sum_{x=0}^{N-1} t_m(x)t_n(x) = \beta_n \delta_{mn}, \tag{3}$$

where

$$\beta_n = N(N^2-1^2)(N^2-2^2)\cdots(N^2-n^2)/(2n+1),$$

(iii) Symmetry

$$t_n(N-1-x) = (-1)^n t_n(x), \tag{4}$$

(iv) Difference equation

$$\Delta[x(x-N)\Delta t_n(x-1)]-n(n+1)t_n(x) = 0. \tag{5}$$

We will also require an orthogonality property of the first differences $\Delta t_n(x)$. By partial summation we get for $m, n = 1, \cdots, N-1$

$$\begin{aligned}\sum_{x=0}^{N-1} x(x-N)\Delta t_m(x-1)\Delta t_n(x-1) &= -\sum_{x=0}^{N-1} t_m(x)\Delta[x(x-N)\Delta t_n(x-1)]\\ &= -n(n+1)\sum_{x=0}^{N-1} t_m(x)t_n(x)\\ &= -n(n+1)\beta_n\delta_{mn}.\end{aligned} \tag{6}$$

Let $f(x)$ be the uniquely determined polynomial of degree $< N-1$ such that $f(n-1) = x_n$ $(n = 1, \cdots, N-1)$. There exists a unique polynomial $F(x)$ of degree $< N$ such that $F(0) = 0$ and

$$f(x) = \Delta F(x) = F(x+1)-F(x).$$

We can write

$$F(x) = \sum_{n=0}^{N-1} a_n \beta_n^{-\frac{1}{2}} t_n(x) \tag{7}$$

with suitable coefficients a_n. Then

$$f(x) = \sum_{n=1}^{N-1} a_n \beta_n^{-\frac{1}{2}} \Delta t_n(x). \tag{8}$$

We wish to evaluate the sum

$$\begin{aligned}S(\lambda) &= S_1+\lambda S_2\\ &= \sum_{x=1}^{N-1} x(N-x)f^2(x-1)+\lambda \sum_{\substack{x+y<N\\ x,y>0}} (N-x-y)f(x-1)f(y-1).\end{aligned}$$

By (8) and (6)

$$\begin{aligned} S_1 &= \sum_{m,n=1}^{N-1} a_m a_n \beta_m^{-\frac{1}{2}} \beta_n^{-\frac{1}{2}} \sum_{x=1}^{N-1} x(N-x)\Delta t_m(x-1)\Delta t_n(x-1) \\ &= \sum_{n=1}^{N-1} n(n+1)a_n^2. \end{aligned}$$

By the definition of $F(x)$ and by partial summation

$$\begin{aligned} S_2 &= \sum_{x=1}^{N-2} (N-x)f(x-1) \sum_{y=1}^{N-1-x} f(y-1) - \sum_{x=1}^{N-2} f(x-1) \sum_{y=1}^{N-1-x} yf(y-1) \\ &= \sum_{x=1}^{N-2} (N-x)f(x-1)F(N-1-x) - \sum_{x=1}^{N-2} (N-1-x)f(N-2-x)F(x) \\ &= \sum_{x=0}^{N-2} (N-1-x)[f(x)F(N-2-x) - f(N-2-x)F(x)] \\ &= \sum_{x=1}^{N-1} x[f(N-1-x)F(x-1) - f(x-1)F(N-1-x)] \\ &= -\sum_{x=1}^{N-1} x\Delta[F(x-1)F(N-x)] \\ &= \sum_{x=0}^{N-1} F(x)F(N-1-x). \end{aligned}$$

Therefore, by the symmetry property (4) and the orthogonality property (3),

$$S_2 = \sum_{n=0}^{N-1} (-1)^n a_n^2.$$

Thus

$$S(\lambda) = \lambda a_0^2 + \sum_{n=1}^{N-1} [n(n+1) + \lambda(-1)^n] a_n^2.$$

Hence $S(\lambda) \geqq 0$ if $0 < \lambda \leqq 2$, with equality only when $a_0 = 0$ and $a_n = 0$ $(2 \leqq n \leqq N-1)$. The vanishing of a_n for $n \geqq 2$ implies that $f(x)$ is a constant. Since

$$\begin{aligned} a_0 &= N^{-\frac{1}{2}} \sum_{x=0}^{N-1} F(x) \\ &= N^{-\frac{1}{2}} \sum_{x=0}^{N-1} (N-1-x)f(x) \end{aligned}$$

it follows that $f(x) \equiv 0$. In particular, for $\lambda = 2$, this shows that the quadratic form R_N is positive definite.

[*Added in proof*: The complete proof of Bombieri's contribution to the Bieberbach conjecture has now appeared in *Inventiones Math.* 4 (1967), 26–27.]

References

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