SOME FINITE ANALOGUES OF THE POISSON SUMMATION FORMULA†

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1. Guinand (2) has obtained finite identities of the type

$$\frac{1}{n}\sum_{r=1}^{nN} f\left(\frac{mr}{n}\right)-\frac{1}{n}\int_0^{nN} f\left(\frac{mt}{n}\right)dt$$
$$=\frac{1}{m}\sum_{r=1}^{mN} g\left(\frac{nr}{m}\right)-\frac{1}{m}\int_0^{mM} g\left(\frac{nt}{m}\right)dt, \quad \text{(1.1)}$$

where m, n, N are positive integers and either

$$f(x)=g(x)=\psi(1+x)-\log x=\frac{\Gamma'(1+x)}{\Gamma(1+x)}-\log x \quad \text{(1.2)}$$

or

$$\begin{cases} f(x)=\dfrac{1}{x}\left\{\sum\limits_{1\leqq r\leqq x}' 1-x\right\}, \\ g(x)=\gamma+\log x-\sum\limits_{1\leqq r\leqq x}' \dfrac{1}{n}, \end{cases} \quad \text{(1.3)}$$

where γ is Euler's constant and the notation Σ' indicates that when x is integral the term $r = x$ is multiplied by $\frac{1}{2}$. Clearly there is no loss of generality in taking $N = 1$ in (1.1).

We should like to point out that identities of the form (1.1) can be obtained very easily in the following way. Following Mordell (3), let $f(x)$ be a function of x that satisfies the multiplication formula

$$\sum_{s=0}^{n-1} f\left(x+\frac{s}{n}\right)=C_n f(nx) \quad (n=1, 2, 3, \ldots), \quad \text{(1.4)}$$

where C_n is independent of x but may depend upon the function f. Also it follows from (1.4) that $C_{mn} = C_m C_n$ for all integral $m, n \geqq 1$. In a recent paper (1), the writer has shown that (1.4) implies

$$C_n\sum_{r=0}^{m-1} f\left(nx+\frac{nr}{m}\right)=C_m\sum_{s=0}^{n-1} f\left(mx+\frac{ms}{n}\right). \quad \text{(1.5)}$$

In the paper cited above, Mordell has noted that if $\{x\} = x-[x]$, the fractional part of the real variable x, then the function $f(\{x\})$ also satisfies (1.4). Thus if we define $\bar{f}(x)$ by means of $\bar{f}(x) = f(x)$ $(0 \leqq x < 1)$, $\bar{f}(x-1) = \bar{f}(x)$,

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then $\bar{f}(x)$ satisfies (1.4). We may accordingly assume that $f(x)$ in (1.4) has the period 1.

Now consider the sum

$$C_n \sum_{t=0}^{mN-1} f\left(nx+\frac{nt}{mN}\right),$$

where N is an arbitrary positive integer. Using (1.2) we obtain

$$\sum_{s=0}^{n-1} \sum_{t=0}^{mN-1} f\left(x+\frac{s}{n}+\frac{t}{mN}\right). \quad \text{.............................(1.6)}$$

We now assume that m and N are relatively prime and that $f(x)$ has the period 1. Then if r runs through a complete residue system (mod m) while u runs through a complete residue system (mod N) it follows that $t = rN+um$ runs through a complete residue system (mod mN). Consequently the expression (1.6) is equal to

$$\sum_{r=0}^{m-1} \sum_{s=0}^{n-1} \sum_{u=0}^{N-1} \left(x+\frac{r}{m}+\frac{s}{n}+\frac{u}{N}\right). \quad \text{.....................(1.7)}$$

If we prefer, each summation in (1.7) may be extended over a complete residue system modulo m, n or N, respectively. We have thus proved the formula

$$C_n \sum_{t=0}^{mN-1} f\left(nx+\frac{nt}{mN}\right) = \sum_{r=0}^{m-1} \sum_{s=0}^{n-1} \sum_{u=0}^{N-1} f\left(x+\frac{r}{m}+\frac{s}{n}+\frac{u}{N}\right). \quad \text{......(1.8)}$$

If we assume that n and N are relatively prime we get similarly

$$C_m \sum_{t=0}^{nN-1} f\left(mx+\frac{mt}{nN}\right) = \sum_{r=0}^{m-1} \sum_{s=0}^{n-1} \sum_{u=0}^{N-1} f\left(x+\frac{r}{m}+\frac{s}{n}+\frac{u}{N}\right). \quad \text{......(1.9)}$$

Comparing (1.9) with (1.8) we have

$$C_n \sum_{t=0}^{mN-1} f\left(nx+\frac{nt}{mN}\right) = C_m \sum_{t=0}^{nN-1} f\left(mx+\frac{mt}{nN}\right), \quad \text{..............(1.10)}$$

provided $(N, mn) = 1$ and $f(x)$ is of period 1.

Combining (1.5) and (1.10) we state

Theorem 1. *Let $f(x)$ satisfy* (1.4) *and have period* 1. *Then if m, n, N are positive integers such that $(N, mn) = 1$, it follows that*

$$C_n\left\{\sum_{r=0}^{m-1} f\left(nx+\frac{nr}{m}\right) - \frac{1}{N}\sum_{t=0}^{mN-1} f\left(nx-\frac{nt}{mN}\right)\right\}$$
$$= C_m\left\{\sum_{s=0}^{n-1} f\left(mx+\frac{ms}{n}\right) - \frac{1}{N}\sum_{t=0}^{nN-1} f\left(mx+\frac{mt}{nN}\right)\right\}. \quad \text{......(1.11)}$$

If in (1.11) we let $N\to\infty$ then, provided the integrals exist, we get

$$\lim_{N=\infty} \frac{1}{N}\sum_{t=0}^{mN-1} f\left(nx+\frac{nt}{mN}\right) = \int_0^m f\left(nx+\frac{nt}{m}\right)dt,$$
$$\lim_{N=\infty} \frac{1}{N}\sum_{t=0}^{nN-1} f\left(mx+\frac{mt}{nN}\right) = \int_0^n f\left(mx+\frac{mt}{n}\right)dt.$$

Thus Theorem 1 yields

Theorem 2. *Let* $f(x)$ *satisfy* (1.4) *and have period* 1. *Then if* m, n *are arbitrary positive integers, we have*

$$C_n\left\{\sum_{r=0}^{m-1} f\left(nx+\frac{nr}{m}\right)-\int_0^m f\left(nx+\frac{nt}{m}\right)dt\right\}$$
$$=C_m\left\{\sum_{s=0}^{n-1} f\left(mx+\frac{ms}{n}\right)-\int_0^n f\left(mx+\frac{mt}{n}\right)dt\right\}. \quad \text{...(1.12)}$$

2. As a simple application of Theorems 1 and 2 we may take $f(x) = \bar{B}_k(x)$, where $B_k(x)$ is the Bernoulli polynomial of degree k defined by

$$\frac{te^{xt}}{e^t-1}=\sum_{k=0}^{\infty} B_k(x)\frac{t^k}{k!}$$

and

$$\bar{B}_k(x) = B_k(x)(0 \leqq x < 1), \quad \bar{B}_k(x+1) = \bar{B}_k(x).$$

Since

$$\sum_{s=0}^{n-1} B_k\left(x+\frac{s}{n}\right)=n^{1-k}B_k(nx),$$

we have in this instance $C_n = n^{1-k}$.

In the second place, if we put

$$\zeta(\sigma, x)=\sum_{n=0}^{\infty}\frac{1}{(x+n)^{\sigma}} \quad (x>0,\ R(\sigma)>1),$$

then $\zeta(\sigma, x)$ satisfies (1.4) with $C_n = n^{\sigma}$. Thus in (1.11) and (1.12) we may take $f(x) = \zeta(\sigma, \{x\})$.

If $F'x) = f(x)$, then (1.2) implies

$$\sum_{s=0}^{n-1} F\left(x+\frac{s}{n}\right)=\frac{1}{n}C_nF(nx)+C'_n, \quad \text{........................(2.1)}$$

where C'_n is also independent of x. For example we have the well-known formula

$$\sum_{s=0}^{n-1} \Psi\left(x+\frac{s}{n}\right)=n\Psi(nx)-n\log n. \quad \text{.....................(2.2)}$$

It is easily verified that the function $\bar{F}(x) = F(\{x\})$ also satisfies (2.1).

It follows from (2.1) that

$$\frac{1}{m}C_m\sum_{s=0}^{n-1} F\left(mx+\frac{ms}{n}\right)+nC'_m=\frac{1}{n}C_n\sum_{r=0}^{m-1} F\left(nx+\frac{nr}{m}\right)+mC'_n. \quad \text{...(2.3)}$$

Also, exactly as in proving (1.10), we find that

$$\frac{1}{m}C_m\sum_{t=0}^{mN-1} \bar{F}\left(mx+\frac{mt}{n}\right)=\sum_{r=0}^{m-1}\sum_{s=0}^{n-1}\sum_{u=0}^{N-1} \bar{F}\left(x+\frac{r}{m}+\frac{s}{n}+\frac{u}{N}\right)-nNC'_m,$$

so that

$$\frac{1}{mN}C_m\sum_{t=0}^{mN-1} F\left(mx+\frac{mt}{n}\right)+nC'_m=\frac{1}{nN}C_n\sum_{t=0}^{mN} F\left(nx+\frac{nt}{m}\right)+mC'_n \quad \text{...........(2.4)}$$

provided $(N, mn) = 1$.

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We may therefore state the following theorems.

Theorem 3. *Let $F'(x) = f(x)$, where $f(x)$ satisfies* (1.4), *and put*

$$\bar{F}(x) = F(\{x\}) = F(x-[x]). \qquad \text{(2.5)}$$

Then if $(N, mn) = 1$ *it follows that*

$$\frac{1}{n}C_n\left\{\sum_{r=0}^{m-1}\bar{F}\left(nx+\frac{nr}{m}\right)-\frac{1}{N}\sum_{t=0}^{mN-1}\bar{F}\left(nx+\frac{nt}{m}\right)\right\}$$
$$=\frac{1}{m}C_m\left\{\sum_{s=0}^{n-1}\bar{F}\left(mx+\frac{ms}{n}\right)-\frac{1}{N}\sum_{t=0}^{nN-1}\bar{F}\left(mx+\frac{mt}{n}\right)\right\}. \qquad \text{(2.6)}$$

Moreover in the sums over r and s in (2.6), $\bar{F}$ *may be replaced by F.*

Theorem 4. *Let $F'(x) = f(x)$, where $f(x)$ satisfies* (1.4) *and define $\bar{F}(x)$ by means of* (2.5). *Then*

$$\frac{1}{n}C_n\left\{\sum_{r=0}^{m-1}F\left(nx+\frac{nr}{m}\right)-\int_0^m\bar{F}\left(nx+\frac{nt}{m}\right)dt\right\}$$
$$=\frac{1}{m}C_m\left\{\sum_{s=0}^{n-1}\bar{F}\left(mx+\frac{ms}{n}\right)-\int_0^n\bar{F}\left(mx+\frac{mt}{n}\right)dt\right\}. \qquad \text{(2.7)}$$

Moreover in each sum in (2.7) $\bar{F}$ *may be replaced by F.*

3. Since $\psi(x)$ satisfies (2.2) it is evident that (2.6) and (2.7) yield

$$\frac{1}{m}\sum_{r=0}^{m-1}\bar{\psi}\left(nx+\frac{nr}{m}\right)-\frac{1}{mN}\sum_{t=0}^{mN-1}\bar{\psi}\left(nx+\frac{nt}{m}\right)$$
$$=\frac{1}{n}\sum_{s=0}^{n-1}\bar{\psi}\left(mx+\frac{ms}{n}\right)-\frac{1}{nN}\sum_{t=0}^{nN-1}\bar{\psi}\left(mx+\frac{mt}{n}\right), \qquad \text{(3.1)}$$

$$\frac{1}{m}\sum_{r=0}^{m-1}\bar{\psi}\left(nx+\frac{nr}{m}\right)-\frac{1}{m}\int_0^m\bar{\psi}\left(nx+\frac{nt}{m}\right)dt$$
$$=\frac{1}{n}\sum_{s=0}^{n-1}\bar{\psi}\left(mx+\frac{ms}{n}\right)-\frac{1}{n}\int_0^n\bar{\psi}\left(mx+\frac{mt}{n}\right)dt, \qquad \text{(3.2)}$$

where $\bar{\psi}(x) = \psi(x-[x])$.

To get a result like (3.2) involving $\psi(x)$ we note first that by (2.2) and (2.3)

$$\frac{1}{m}\sum_{r=0}^{m-1}\psi\left(nx+\frac{nr}{m}\right)+\log n=\frac{1}{n}\sum_{s=0}^{n-1}\psi\left(mx+\frac{ms}{n}\right)+\log m. \qquad \text{(3.3)}$$

Secondly we have

$$\frac{1}{m}\int_0^m\psi\left(nx+\frac{nt}{m}\right)dt=\int_0^1\psi(nx+nt)dt$$
$$=\frac{1}{n}\log\frac{\Gamma(nx+n)}{\Gamma(nx)}=\log n+\sum_{s=0}^{n-1}\log\left(x+\frac{s}{n}\right). \qquad \text{(3.4)}$$

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If we put

$$G(x) = \psi(x) - \log x \qquad (3.5)$$

and make use of (3.3) and (3.4), we find after a little computation that

$$\frac{1}{m}\left\{\sum_{r=0}^{m-1} G\left(nx+\frac{nr}{m}\right) - \int_0^m G\left(nx+\frac{nt}{m}\right)dt\right\}$$
$$= \frac{1}{n}\left\{\sum_{s=0}^{n-1} G\left(mx+\frac{ms}{n}\right) - \int_0^n G\left(mx+\frac{mt}{n}\right)dt\right\}. \qquad (3.6)$$

It can be verified that (3.6) is equivalent to Theorem 2 of Guinand's paper.
Put

$$H(x) = G(x) - \bar{\psi}(x) = \psi(x) - \bar{\psi}(x) - \log x, \qquad (3.7)$$

which evidently implies that, for $x > 0$,

$$H(x) = \sum_{1 \leq k < x} \frac{1}{x-k} - \log x. \qquad (3.8)$$

Comparing (3.6) with (3.2), we have at once that

$$\frac{1}{m}\left\{\sum_{r=0}^{m-1} H\left(nx+\frac{nr}{m}\right) - \int_0^m H\left(nx+\frac{nt}{m}\right)dt\right\}$$
$$= \frac{1}{n}\left\{\sum_{s=0}^{n-1} H\left(mx+\frac{ms}{n}\right) - \int_0^n H\left(mx+\frac{mt}{n}\right)dt\right\}. \qquad (3.9)$$

The function $H(x)$ may be compared with $g(x)$ as defined in (1.3).
We have noted above that in Theorem 2 we may take $f(x) = \zeta(\sigma, \{x\})$.
Now for the function $\zeta(\sigma, x)$ we have first by (1.5)

$$n^\sigma \sum_{r=0}^{m-1} \zeta\left(\sigma, nx+\frac{nr}{m}\right) = m^\sigma \sum_{s=0}^{n-1} \zeta\left(\sigma, mx+\frac{ms}{n}\right). \qquad (3.10)$$

Secondly we have, for $\sigma \neq 1$, that

$$n^\sigma \int_0^m \zeta\left(\sigma, nx+\frac{nt}{m}\right)dt = \frac{1}{\sigma-1}\sum_{s=0}^{n-1}\left(mx+\frac{ms}{n}\right)^{1-\sigma}. \qquad (3.11)$$

Thus if we put

$$G_\sigma(x) = \zeta(\sigma, x) - \frac{x^{1-\sigma}}{1-\sigma}, \qquad (3.12)$$

it follows from (3.10) and 3.11) that

$$n^\sigma\left\{\sum_{r=0}^{m-1} G_\sigma\left(nx+\frac{nr}{m}\right) - \int_0^m G_\sigma\left(nx+\frac{nt}{m}\right)dt\right\}$$
$$= m^\sigma\left\{\sum_{s=0}^{n-1} G_\sigma\left(mx+\frac{mr}{n}\right) - \int_0^n G_\sigma\left(mx+\frac{mt}{n}\right)dt\right\}. \qquad (3.13)$$

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We may state

Theorem 5. *If*

$$\zeta(\sigma, x) = \sum_{k=0}^{\infty} \frac{1}{(x+k)^{\sigma}} \quad (R(\sigma)>1)$$

and $G_\sigma(x)$ *is defined by* (3.12), *then* (3.13) *holds for arbitrary positive integers m, n.*

By analytic continuation (3.13) holds for all $\sigma \neq 1$.

Since, by (1.12),

$$n^{\sigma}\left\{\sum_{r=0}^{m-1} \zeta\left(\sigma, \left\{nx+\frac{nr}{m}\right\}\right) - \int_0^m \zeta\left(\sigma, \left\{nx+\frac{nt}{m}\right\}\right)dt\right\}$$
$$= m^{\sigma}\left\{\sum_{s=0}^{n-1} \zeta\left(\sigma, \left\{mx+\frac{ms}{n}\right\}\right) - \int_0^n \zeta\left(\sigma, \left\{mx+\frac{mt}{n}\right\}\right)dt\right\},$$

comparison with (3.13) yields

$$n^{\sigma}\left\{\sum_{r=0}^{m-1} H_\sigma\left(nx+\frac{nr}{m}\right) - \int_0^m H_\sigma\left(nx+\frac{nt}{m}\right)dt\right\}$$
$$= m^{\sigma}\left\{\sum_{s=0}^{n-1} H_\sigma\left(mx+\frac{ms}{n}\right) - \int_0^n H_\sigma\left(mx+\frac{mt}{n}\right)dt\right\}, \quad \text{.........(3.14)}$$

where

$$H_\sigma(x) = \zeta(\sigma, \{x\}) - G_\sigma(x)$$
$$= \zeta(\sigma, \{x\}) - \zeta(\sigma, x) + \frac{x^{1-\sigma}}{1-\sigma}$$
$$= \sum_{1 \leq k < x} \frac{1}{(x-k)^{\sigma}} + \frac{x^{1-\sigma}}{1-\sigma}. \quad \text{..........................(3.15)}$$

The formula (3.14) may be compared with Theorem 5 of Guinand's paper. We remark that since

$$\zeta(1-k, x) = -\frac{1}{k}B_k(x) \quad (k = 1, 2, 3, \ldots),$$

(3.13) can be expressed in terms of Bernoulli polynomials.

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