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SMALL SUBHARMONIC FUNCTIONS

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Some time ago Barry [*Proc. London Math. Soc.* 12 (1962), 445-495] established the right connection between the smallest and largest values of small subharmonic functions on certain circles about the origin. The behaviour of functions extremal for this connection is investigated.

1. Introduction and results

1.1. Suppose that $u(z)$ is subharmonic in the plane and that $B(r, u)$ (or simply $B(r)$) denotes the maximum of $u(z)$ on $|z| = r$, while $A(r, u)$ (or $A(r)$) denotes the infimum of $u(z)$ on $|z| = r$. Of the results that relate $A(r)$ and $B(r)$, one of the most appealing is Kjellberg's [6] elegant formulation of the $\cos \pi\lambda$ theorem:

given any λ *satisfying* $0 < \lambda < 1$ *either*

$$A(r) > \cos \pi\lambda B(r)$$

for certain arbitrarily large values of r *or, if this is false, then*

$$\lim_{r\to\infty} \frac{B(r)}{r^{\lambda}}$$

exists and is positive or $+\infty$.

When $\lambda = 0$ the result is true (setting aside the case of constant u)

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but vacuously so, which is somewhat unsatisfying. Of course Kjellberg's result is a sharpening of the classical $\cos \pi\rho$ theorem, which asserts that if $u(z)$ has order ρ, $0 \le \rho \le 1$, then

$$A(r) > \{\cos \pi\rho + o(1)\}B(r) \tag{1.1}$$

on a sequence of $r \to \infty$; and when $\rho = 0$ (1.1) does not admit a sharpening simply by dispensing with the $o(1)$ term. One is led to ask whether a refinement of (1.1) is possible when $\rho = 0$ and, if so, whether this refinement can be sharpened to produce a result like Kjellberg's. The first question has been answered in detail by Barry and the second forms the subject of this note. Among other results Barry proves the following theorem, though his statement of it is a little different (see also, Fenton [4]).

THEOREM A (Barry [1, pp. 470, 492]). *If* $u(z)$ *is a non-constant subharmonic function and satisfies*

$$\overline{\lim_{r\to\infty}} \frac{B(r)}{(\log r)^p} \le \sigma < \infty \tag{1.2}$$

for some $p > 1$ *then, for certain arbitrarily large values of* r,

$$A(r) > B(r) - \{\sigma + o(1)\}\Psi_p(r) , \tag{1.3}$$

where

$$\Psi_p(r) = \mathrm{Re}\{(\log r)^p - (\log r + i\pi)^p\} . \tag{1.4}$$

We shall prove

THEOREM 1. *Suppose that* $u(z)$ *is subharmonic and let* σ *and* p *be positive numbers, with* $1 < p \le 3$. *Then either*

$$A(r) > B(r) - \sigma\Psi_p(r) \tag{1.5}$$

for certain arbitrarily large values of r *or, if this is false, then*

$$\alpha = \lim_{r\to\infty} \frac{B(r) - \sigma(\log r)^p}{\log r} \tag{1.6}$$

exists and is $+\infty$ *or finite.*

It will be seen later that any value of $\alpha > -\infty$ is possible in (1.6). Should α be finite then u is certainly of order 0 and we may introduce the usual auxiliary function

$$u_1(z) = \int_0^\infty \log\left|1 + \frac{z}{t}\right| d\mu^*(t) \ . \tag{1.7}$$

Here $\mu^*(t) = \mu(|z| < t)$, μ being the Riesz mass of u . (Implicit in this is the assumption, which may be made without loss of generality, that u is harmonic at 0 .) Concerning u_1 we have

THEOREM 2. *If* α *is finite in* (1.6) *then*

$$\lim_{r\to\infty} \frac{u_1(r)-\sigma(\log r)^p}{\log r} = \alpha \ .$$

The case $p = 2$ of Theorems 1 and 2 was proved by the author elsewhere [3] by different methods.

When $p > 3$ the result is less precise.

THEOREM 3. *Suppose that* u *is subharmonic and let* σ *and* p *be positive numbers, with* $p > 3$. *Then either* (1.5) *holds for certain arbitrarily large values of* r *or, if this is false, then*

$$\varliminf_{r\to\infty} \frac{B(r)-\sigma(\log r)^p}{(\log r)^{p-2}} \ge -\tfrac{1}{2}\pi^2\sigma p(p-1) \ . \tag{1.8}$$

The constant on the right hand side of (1.8) can be improved slightly but cannot be replaced by any number greater than $-\frac{1}{48}\pi^2\sigma p(p-1)$ – actually any value of the lower limit greater than $-\frac{1}{48}\pi^2\sigma p(p-1)$ is possible. More interesting is that when $p > 3$ the extremal functions are not merely not regular, in fact there is no restriction on their upper growth: later we shall describe a construction which produces, when $p > 3$, a subharmonic function for which (1.5) fails for all large r and for which the lower limit in (1.8) is negative, while $B(r)$ grows at any prescribed rate on a sequence of $r \to \infty$. This makes a surprising counterpoint to the heuristic "regularity principle".

1.2. To prove Theorem 1 we suppose that (1.5) fails for all large r , also that

$$\alpha = \varliminf \left(B(r)-\sigma(\log r)^p\right)/\log r < \infty \ , \tag{1.9}$$

and aim to establish the existence of the limit (1.6). The proof is incidentally complicated by the fact that the function occurring indirectly in (1.3), namely $\operatorname{Re}(\operatorname{Log} z)^p$, is not in general subharmonic in the plane. In order not to obscure the simplicity of the proof, which rests on the methods developed by Hellsten, Kjellberg and Norstad [5], we shall consider a related case from which Theorem 1 follows with slight changes. First let us recall a certain representation formula. If $u(z)$ is subharmonic in $|z| < R$, harmonic off the negative axis and such that $u(z) = u(\bar{z})$ then

$$(1.10) \qquad u(r) = \int_0^R Q(r, t)u(-t)dt + \int_{-\pi}^{\pi} T(r, \phi)u\left(Re^{i\phi}\right)d\phi ,$$

for $0 < r < R$. $Q(r, t)$ and $T(r, \phi)$ are non-negative functions which need not be specified here, though it will be convenient to know that

$$(1.11) \qquad \int_0^R Q(r, t)dt < 1 \quad \text{for} \quad 0 < r < R .$$

Details may be found in the paper by Hellsten *et al* [5]. Another formula is also useful. Suppose that $v(z)$ is subharmonic in $|z| < R$, continuous on $|z| = R$ from within $|z| \le R$, and such that $v\left(Re^{i\theta}\right)$ is symmetric in θ and decreasing on $[0, \pi]$. By rotating the Riesz mass of v onto the negative real axis we shall produce a function $u(z)$ satisfying the conditions sufficient for (1.10) to hold. Moreover u has the same boundary values as v and $u(-r) \le A(r, v) \le B(r, v) \le u(r)$, for $0 \le r \le R$. By (1.10) therefore,

$$(1.12) \qquad B(r, v) \le \int_0^R Q(r, t)A(t, v)dt + \int_{-\pi}^{\pi} T(r, \phi)v\left(Re^{i\phi}\right)d\phi ,$$

for $0 < r < R$.

2. A preliminary result

2.1. Let $w(z)$ be a non-constant subharmonic function of order 0 which is harmonic off the negative real axis and harmonic at 0 , locally bounded below, and satisfies $w(0) = 0$ and

$$(2.1) \qquad w(r) - w(-r) = \{C+o(1)\} \log r \quad \text{as} \quad r \to \infty ,$$

which C is a non-negative constant. We shall prove

LEMMA 1. *Suppose that* $u(z)$ *is subharmonic and satisfies*

$$A(r, u) \le B(r, u) - \sigma\big(w(r)-w(-r)\big) \tag{2.2}$$

for all large r . *Then* $\alpha = \lim\big(B(r, u)-\sigma w(r)\big)/\log r$ *exists and is* $+\infty$ *or finite. If* $\alpha < +\infty$ *then* $\alpha = \lim\big(u_1(r)-\sigma w(r)\big)/\log r$, *where* u_1 *is given by* (1.7).

The proof is quite long. Suppose that

$$\alpha = \underline{\lim}\ \big(B(r, u)-\sigma w(r)\big)/\log r < \infty , \tag{2.3}$$

for otherwise there is nothing to prove. We assume throughout that $u(0) = 0$, which entails no loss of generality.

The case when $w(r) = O(\log r)$ can be set aside since then (2.3) implies $B(r, u) = O(\log r)$ also and Lemma 1 follows.

The remainder of the proof concerns the case $w(r) \ne O(\log r)$ and we begin with the observation that $B(r, u) \ne O(\log r)$. For, from Fenton [3],

$$\int_0^t \big(u_1(x)-u_1(-x)\big)\ \frac{dx}{x} = \int_0^\infty \mu^*(s)\log\left|\frac{t+s}{t-s}\right|\ \frac{ds}{s} , \tag{2.4}$$

where μ^* is the radial mass distribution for u , and so (2.2) has the consequence

$$\sigma \int_0^\infty \xi^*(s)\log\left|\frac{t+s}{t-s}\right|\ \frac{ds}{s} \le \int_0^\infty \mu^*(s)\log\left|\frac{t+s}{t-s}\right|\ \frac{ds}{s} + O(1) ,$$

where ξ^* is the radial mass distribution for w . But this is impossible if μ^* is bounded and ξ^* is not. (To see this the formula

$$\int_0^\infty \log\left|\frac{t+s}{t-s}\right|\ \frac{ds}{s} = \int_0^\infty \log\left|\frac{s+1}{s-1}\right|\ \frac{ds}{s} = \tfrac{1}{2}\pi^2 , \tag{2.5}$$

which is needed later, may be helpful.)

Given $\alpha' > \alpha$ define

$$U(z) = \begin{cases} 0 , & |z| \le 1 , \\ \max\{0,\ u(z)-\alpha' \log|z|-B(1, u)-1\} , & |z| > 1 , \end{cases}$$

which is subharmonic. Since $u(z)$ has lower order 0 and $u(z) \ne O(\log|z|)$ it follows from the $\cos \pi\lambda$ theorem that there are

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circles $|z| = \rho$ of arbitrarily large radius on which $U(z) > 0$. Select one such ρ sufficiently large that (2.2) holds for $r > \rho$ and define

$$V(z) = \begin{cases} U(z)\ , & |z| \le \rho\ , \\ u(z) - \alpha' \log|z| - B(1, u) - 1\ , & |z| > \rho\ , \end{cases} \tag{2.6}$$

which is also subharmonic. Choose α'' satisfying $\alpha < \alpha'' < \alpha'$, let R be any large number such that

$$B(R, u) < \sigma w(R) + \alpha'' \log R \tag{2.7}$$

and define

$$V_R^*(z) = \max\{V^*(z, R) - K, \sigma w(z)\} \quad (|z| \le R)\ .$$

Here $V^*(z, R)$ is the auxiliary function associated with V in $|z| < R$ introduced by Hellsten *et al* [5, where it appears as $V^*(z)$], and

$$K = [\max_{0 \le t \le \rho} A(t, V) - \min_{0 \le t \le \rho} \sigma w(-t)]^+\ .$$

The salient properties of $V^*(z, R)$ are: $V^*(z, R)$ is subharmonic in $|z| < R$, harmonic off the negative axis and has constant boundary values $B(R, V)$ on $\{|z| = R\}\backslash\{-R\}$; also

$$V^*(-r, R) \le A(r, V) \le B(r, V) \le V^*(r, R)$$

for $0 \le r < R$. As the reader will easily verify

$$V_R^*(-r) \le V_R^*(r) - \sigma(w(r) - w(-r)) \quad \text{for} \quad 0 \le r < R\ ;$$

and it follows from (2.1) and (2.7) that when R is large $V_R^*(z) = \sigma w(z)$ on $|z| = R$, provided only that $\alpha' - \alpha'' > C$. On subtracting (1.10) applied to $w(z)$ from (1.12) applied to $V_R^*(z)$ we obtain

$$[V_R^*(r) - w(r)]^+ \le \int_0^R Q(r, t)\,[V_R^*(t) - \sigma w(t)]^+ dt$$

for $0 < r < R$. Taking account of (1.11) we deduce that $[V_R^*(r) - \sigma w(r)]^+$ attains its largest value either at 0 or at R, at both of which it is 0. Thus $V_R^*(r) \le \sigma w(r)$ for $0 \le r \le R$ and therefore

$$B(r, u) - \alpha' \log r - B(1, u) - 1 - K \le V^*(r, R) - K \le \sigma w(r)\ , \tag{2.8}$$

for $\rho < r \le R$. The outer inequality is independent of R.

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Allowing α' to approach $\alpha + C$ then

(2.9) $$\overline{\lim}\ \big(B(r, u)-\sigma w(r)\big)/\log r \le \alpha + C\ .$$

A little more is needed to get the limit when $C \ne 0$ (and this is deferred for a moment) but we can form $u_1(z)$ given by (1.7); and it is easy to show that

$$\lim_{R\to\infty} V^*(r, R) = V_1(r) = u_1(r) - \alpha' \log r + O(1)\ ,$$

where V_1 is obtained from V as u_1 is obtained from u, by rotating its mass onto the negative real axis. Combined with (2.8) this leads to

(2.10) $$\overline{\lim}\ \big(u_1(r)-\sigma w(r)\big)/\log r \le \alpha + C\ .$$

The proof of Lemma 1 when $C = 0$ is thus complete once it is shown that $\alpha = -\infty$ is impossible. This is contained in

LEMMA 2. *Let u_1 and w be functions subharmonic in the plane, harmonic away from the negative axis and harmonic at* 0 , *and of order* 0 . *Suppose that $u_1(-r) \le u_1(r) - \sigma\big(w(r)-w(-r)\big)$ for $r \ge r_0$. Then*

(2.11) $$L(r) \equiv \int_0^\infty \log\left|\frac{r+s}{r-s}\right| \frac{ds}{s} \int_0^\infty \big(u_1(x)-\sigma w(x)\big)\log\left|\frac{s+x}{s-x}\right| \frac{dx}{x}$$
$$\ge \int_0^r \frac{\nu(t)}{t}\,dt + O(1)\ ,$$

where $\nu(t)$ is increasing for $r \ge r_0$. If in addition (2.10) *holds then*

(2.12) $$-\infty < A \equiv \lim_{r\to\infty} \nu(r) \le \tfrac{1}{4}\pi^4(\alpha+C)\ .$$

We shall prove Lemma 2 now and later return to the case $C \ne 0$ of Lemma 1. We make the harmless assumption that $u_1(0) = 0 = w(0)$.

On integrating (2.4) and reversing the order of integration we obtain

(2.13) $$\int_0^r \frac{dt}{t} \int_0^t \{u_1(x)-u_1(-x)\}\ \frac{dx}{x} = \int_0^\infty \left\{\int_0^s \frac{\mu^*(x)}{x}\,dx\right\}\log\left|\frac{r+s}{r-s}\right| \frac{ds}{s}\ .$$

Moreover from the formula on page 2 of Boas' book [2], for $s > 0$,

$$(2.14) \qquad \int_0^s \frac{\mu^*(x)}{x}\,dx = \frac{1}{\pi}\int_0^\pi u_1\left(se^{i\theta}\right)d\theta$$

$$= \frac{1}{\pi^2}\int_0^\infty \{u_1(x)+u_1(-x)\}\log\left|\frac{s+x}{s-x}\right|dx \ ,$$

and on substituting this into (2.13) we obtain

$$\int_0^r \frac{dt}{t}\int_0^t \{u_1(x)-u_1(-x)\}\,\frac{dx}{x}$$

$$= \frac{1}{\pi^2}\int_0^\infty \log\left|\frac{r+s}{r-s}\right|\,\frac{ds}{s}\int_0^\infty \{u_1(x)+u_1(-x)\}\log\left|\frac{s+x}{s-x}\right|\,\frac{dx}{x} \ .$$

There is a similar equation for w and on subtracting the two and taking account of the hypotheses of Lemma 2 we deduce that the left-hand side of (2.11) is at least

$$\int_0^r \frac{\nu(t)}{t}\,dt + O\left\{\int_0^\infty \log\left|\frac{r+s}{r-s}\right|\,\frac{ds}{s}\int_0^{r_0}\log\left|\frac{x+s}{x-s}\right|\,\frac{dx}{x}\right\} = \int_0^r \frac{\nu(t)}{t}\,dt + O(1) \ ,$$

after noting (2.5), where

$$(2.15) \qquad \nu(t) = \frac{\pi^2}{2}\int_0^t \{u_1(x)-u_1(-x)-\sigma\left(w(x)-w(-x)\right)\}\,\frac{dx}{x} \ .$$

$\nu(t)$ is evidently increasing for $t \geq r_0$.

For the second part of Lemma 2 choose $\alpha''' > \alpha$ arbitrarily and let $K' > 0$ be such that

$$(2.16) \qquad \begin{cases} u_1(x) - \sigma w(x) \leq K' \ , & \text{for } 0 \leq x \leq 1 \ , \\ u_1(x) - \sigma w(x) \leq (\alpha'''+C)\log x + K' \ , & \text{for } x > 1 \ . \end{cases}$$

It follows from (2.5) that

$$L(r) \leq \tfrac{1}{4}K'\pi^4 + (\alpha'''+C)\int_0^\infty \log\left|\frac{r+s}{r-s}\right|\,\frac{ds}{s}\int_{1/s}^\infty \{\log s + \log x\}\log\left|\frac{x+1}{x-1}\right|\,\frac{dx}{x}$$

$$= (\alpha'''+C)\int_0^\infty \{\log s + \log r\}\log\left|\frac{s+1}{s-1}\right|\,\frac{ds}{s}\int_{1/rs}^\infty \log\left|\frac{x+1}{x-1}\right|\,\frac{dx}{x} + O(1)$$

$$= \tfrac{1}{4}\pi^4\left(\alpha'''+C+o(1)\right)\log r + O(1) \ ,$$

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which gives (2.12) and completes the proof of Lemma 2.

2.2. It remains to improve (2.10) when $C \neq 0$. We have

$$u_1(r) = \int_0^\infty \frac{r\mu^*(t)}{t(t+r)}\,dt = \int_0^\infty \left\{\int_0^t \frac{\mu^*(s)}{s}\,ds\right\} \frac{r}{(t+r)^2}\,dt \,. \tag{2.17}$$

Also

$$\int_0^t \frac{\mu^*(s)}{s}\,ds = \int_0^t \frac{\sigma\xi^*(s)}{s}\,ds + \frac{4}{\pi^4}\int_0^\infty \frac{t\nu(s)}{s(s+t)}\,ds - \frac{2}{\pi^4}\int_0^\infty \nu'(s)\log\left|\frac{t+s}{t-s}\right| ds \,, \tag{2.18}$$

where ξ^* is the radial mass distribution associated with w and ν is given by (2.15). For, from (2.14),

$$\int_0^t \frac{\mu^*(s)}{s}\,ds = \frac{2}{\pi^2}\int_0^\infty u_1(s)\log\left|\frac{t+s}{t-s}\right| \frac{ds}{s} - \frac{1}{\pi^2}\int_0^\infty \{u_1(s)-u_1(-s)\}\log\left|\frac{t+s}{t-s}\right| \frac{ds}{s} \tag{2.19}$$

and the first of the integrals on the right hand side of (2.19) can be written as

$$\int_0^\infty \frac{t}{s(s+t)}\,ds \int_0^\infty \mu^*(x)\log\left|\frac{s+x}{s-x}\right| \frac{dx}{x} = \int_0^\infty \frac{t}{s(s+t)}\,ds \int_0^s \{u_1(x)-u_1(-x)\}\,\frac{dx}{x} \,,$$

in view of (1.7) and (2.4). The same holds when w replaces u_1 and on combining the two equations we obtain (2.18).

Substituting (2.18) into (2.17):

$$u_1(r) = \sigma w(r) + \left(\frac{4A}{\pi^4} + o(1)\right)\log r - \frac{2}{\pi^4}\int_0^\infty \frac{r}{(t+r)^2}\,dt \int_0^\infty \nu'(s)\log\left|\frac{t+s}{t-s}\right| ds \,.$$

This last integral is bounded by

$$\int_0^\infty |\nu'(s)|\,ds \int_0^\infty \frac{rt}{(t+r)^2}\log\left|\frac{t+s}{t-s}\right| \frac{dt}{t} \le \tfrac{1}{4}\int_0^\infty |\nu'(s)|\,ds \int_0^\infty \log\left|\frac{t+s}{t-s}\right| \frac{dt}{t} = \tfrac{1}{8}\pi^2 \int_0^\infty |\nu'(s)|\,ds \,,$$

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which is finite since ν' is ultimately positive, and thus

$$(2.20)\qquad u_1(r) = \sigma\omega(r) + \left(\frac{4A}{\pi^4} + o(1)\right) \log r .$$

The regular growth of $B(r, u)$ now follows by a standard argument. Since $\nu(r)$ is bounded, it follows from (2.1) that, given $\varepsilon > 0$,

$$u_1(-r) > u_1(r) - (C+\varepsilon) \log r$$

for all r outside a set $E \subseteq (1, \infty)$ such that $\int_E (\log t/t)dt < \infty$.

Hence, for all large r outside E,

$$B(r, u) \geq A(r, u) + (C-\varepsilon) \log r \geq u_1(-r) + (C-\varepsilon) \log r$$
$$> u_1(r) - 2\varepsilon \log r > \sigma\omega(r) + \left(\frac{4A}{\pi^4} - 3\varepsilon\right) \log r .$$

Given $\delta > 0$, the interval $I_r = [r \log r/(\delta+\log r), r]$ must contain (when r is large) at least one point, r' say, outside E, since $\int_{I_R} (\log t/t)dt \to \delta$ as $r \to \infty$. Hence, for all large r,

$$(2.21)\qquad B(r, u) \geq B(r', u) \geq \sigma\omega(r') - \left(\frac{4A}{\pi^4} - 3\varepsilon\right) \log r$$
$$= \sigma\omega(r) - \left(\frac{4A}{\pi^4} - 3\varepsilon + O(\delta)\right) \log r$$

since (as is shown below), for $2r_1 > r_2 > r_1$,

$$(2.22)\qquad w(r_2) - w(r_1) = O\left(\left(\frac{r_2}{r_1} - 1\right)(\log r_1)^2\right) .$$

Combining (2.20) and (2.21) we obtain finally

$$B(r, u) = \sigma\omega(r) - \left(\frac{4A}{\pi^4} + o(1)\right) \log r ,$$

which, taken with (2.20), proves Lemma 1.

To justify (2.22), observe that (2.1) combined with (2.13) applied to w gives $\int_0^s (\xi^*(x)/x)dx = O(\log x)^3$, so $\xi^*(x) = O(\log x)^2$. But from the usual representation formula

$$w(r_2) - w(r_1) = (r_2 - r_1) \int_0^\infty \frac{\xi^*(s)}{(s+r_1)(s+r_2)}\, ds$$
$$\leq O\left\{\left(\frac{r_2}{r_1} - 1\right) \int_0^\infty \frac{(\log r_1 + \log r)^2}{(1+s)^2}\, ds\right\},$$

and (2.22) follows.

3. Proofs of Theorems 1 and 2

Theorem 1 is easily proved once $\operatorname{Re}(\operatorname{Log} z)^p$ is replaced by a suitable subharmonic function.

From the expansion

$$(3.1)\quad W_p(re^{i\theta}) \equiv \operatorname{Re}(\operatorname{Log} re^{i\theta})^p = (\log r)^p - \tfrac{1}{2}p(p-1)\theta^2(\log r)^{p-2} + \dots ,$$

which is valid for $|\theta| \leq \pi$ and $r > e^\pi$, it follows that $W_p(z)$ is continuous in $|z| > e^\pi$. Also, since W_p is the real part of an analytic function in $|z| > 1$ cut along the negative axis, it is harmonic there and so will be subharmonic for all large z if it satisfies the submean value property at all points of the negative axis sufficiently far from 0. Now the right hand side of (3.1) is harmonic for $0 < \theta < 2\pi$ and $r > e^{2\pi}$ and is, when r is large, a decreasing function of θ on $(0, 2\pi)$. It follows that in a neighbourhood of any point of the negative axis far from 0, W_p dominates a harmonic function with which it agrees at the point, and this implies the submean value property. Finally, as an inspection of (3.1) shows, $W_p(z) \to \infty$ as $|z| \to \infty$ and thus the function obtained by replacing W_p in a disc $\Delta = \{|z| < \rho_0\}$ by the Poisson integral of its boundary values will be subharmonic when ρ_0 is large enough. Call this function $W_{p\Delta}(z)$, let $\xi_{p\Delta}$ be its Riesz mass and let $\xi^*_{p\Delta}(t) = \xi_{p\Delta}\{|z| < t\}$. Since $W_{p\Delta}(z) = W_p(z)$ for $|z| \geq \rho_0$ it follows from Jensen's theorem that

$$\xi^*_{p\Delta}(t) = \xi^*_p(t) \equiv p(\log t)^{p-1} - \tfrac{1}{6}\pi^2 p(p-1)(p-2)(\log t)^{p-3} + \dots$$

for $t > \rho_0$. Now

$$(3.2) \qquad W_{p\Delta}(z) = \int_{|\zeta|<\infty} \log\left|1 + \frac{z}{\zeta}\right| d\xi_{p\Delta}$$

$$= \int_{\rho_0}^{\infty} \log\left|1 + \frac{z}{t}\right| d\xi^*_p(t) + \int_{|\zeta|=\rho_0} \log\left|1 + \frac{z}{\zeta}\right| d\xi_{p\Delta} .$$

It follows that if, given $R_0 > \rho_0$, we define

$$w_p(z) = \int_{R_0}^{\infty} \log\left|1 + \frac{z}{t}\right| d\xi^*_p(t) ,$$

we shall have

$$(3.3) \qquad w_p(z) = \mathrm{Re}(\mathrm{Log}\, z)^p - \left(\xi^*_p(R_0) + o(1)\right)\log|z| \quad (|z| \to \infty)$$

and

$$(3.4) \qquad w_p(r) - w_p(-r) = \Psi_p(r) - \frac{2}{r}\left\{\int_{\rho_0}^{R_0} t\,d\xi^*_p(t) + G(\rho_0, r)\right\} ,$$

where $G(\rho_0, r)$ is independent of R_0 and satisfies $|G(\rho_0, r)| < \rho_0\xi^*_p(\rho_0+)$. Fix $R_0 > \rho_0$ large enough that the second term on the right hand side of (3.4) is negative. We have

$$(3.5) \qquad w_p(r) - w_p(-r) = \left(\tfrac{1}{2}p(p-1)\pi^2 + o(1)\right)(\log r)^{p-2}$$

cf. (2.1), and Theorems 1 and 2 follow from Lemma 1, taking $w = w_p$.

4. Proof of Theorem 3

Suppose that, for some $p > 3$,

$$A(r, u) \le B(r, u) - \sigma\, \mathrm{Re}\left\{(\log r)^p - (\log r + i\pi)^p\right\}$$

for all large r, so that, with w_p as defined in the preceding section,

$$(4.1) \qquad A(r, u) \le B(r, u) - \sigma\left(w_p(r) - w_p(-r)\right)$$

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for all large r also, say for $r \geq r_0$. Suppose further that, on a sequence $R_n \to \infty$,

(4.2) $$B(R_n, u) \leq \sigma(\log R_n)^p + \alpha(\log R_n)^{p-2},$$

for some real number α. For each positive integer n define

$$Y_n(z) = u^*(z, R_n) + \varepsilon w_{p-2}(z) + \sigma\Delta(z) - u(0),$$

where ε is a positive number and

$$\Delta(z) = \int_{R_0}^{\max(2r_0, R_0)} \log\left|1 + \frac{z}{t}\right| d\xi_p^*(t) + 6r_0\left\{\int_{2r_0}^{\infty} \frac{1}{t}\, d\xi_p^*(t)\right\}\log\left|\frac{z+2r_0}{2r_0}\right| .$$

($\Delta(z)$ is included because

$$\Delta(-r) - \Delta(r) \leq w_p(-r) - w_p(r) \quad \text{for} \quad 0 \leq r \leq r_0,$$

which has the effect of extending (4.1) to $r > 0$.)

From (4.2) and (3.5) we obtain

$$\begin{aligned} B(R_n, Y_n) &= Y_n(R_n) \\ &\leq \sigma(\log R_n)^p + (\alpha+\varepsilon+o(1))(\log R_n)^{p-2} \\ &< \sigma w_p(-R_n) + \left(\alpha+\varepsilon+\tfrac{1}{2}\pi^2\sigma p(p-1)+o(1)\right) \log R_n . \end{aligned}$$

Thus if $\alpha < -\frac{1}{2}\pi^2\sigma p(p-1)$ then $\varepsilon > 0$ can be chosen so that, on $|z| = R_n$, $Y_n(z) - \sigma w_p(z) < 0$. From (1.10), then,

$$Y_n(r) - \sigma w_p(r) \leq \int_0^{R_n} Q(r, t)\{Y_n(t) - \sigma w_p(t)\}dt ,$$

for $0 < r < R_n$, and arguing as before we deduce that $Y_n(r) \leq \sigma w_p(r)$ for $0 \leq r \leq R_n$. This leads to

$$B(r, u) \leq \sigma w_p(r) - (\varepsilon+o(1))w_{p-2}(r)$$

and in turn to

$$u_1(r) \leq \sigma w_p(r) - (\varepsilon+o(1))w_{p-2}(r) .$$

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But then $\lim(u_1(r)-\sigma w_p(r))/\log r = -\infty$, which contradicts the second part of Lemma 2. This proves Theorem 3.

5. Examples

(i) To see that any value of $\alpha > -\infty$ is possible in Theorem 1, take $u(z) = 2W_{p\Delta}(z)$ in case $\alpha = +\infty$, where $W_{p\Delta}$ is given by (3.2), while if α is real, take

$$u(z) = \begin{cases} 0 , & |z| < 1 , \\ \max\{0, W_{p\Delta}(z)+\alpha \log|z| - B(1, W_{p\Delta})-1\} , & |z| > 1 . \end{cases}$$

(ii) We now construct the example referred to following the statement of Theorem 3.

Let $W_{p\Delta}(z)$ be the subharmonic function given by (3.2) and, given any $R > 0$, define $H_R(z)$ to be the harmonic function in $|z| < R$ with boundary values $H_R(Re^{i\theta}) = W_{p\Delta}(R) - W_{p\Delta}(Re^{i\theta})$ on $|\theta| \le \frac{1}{2}\pi$ and $H_R(Re^{i\theta}) = 0$ on $\frac{1}{2}\pi < |\theta| \le \pi$. Since $W_{p\Delta}$ agrees with $\operatorname{Re}(\operatorname{Log} z)^p$ for $|z| \ge \rho_0$ we have $H_R(z) \ge 0$ when R is large and, as $R \to \infty$,

$$H_R(0) = \left(\tfrac{1}{48}\pi^2 p(p-1)+o(1)\right)(\log R)^{p-2} ; \tag{5.1}$$

also

$$\max_{|z|\le\sqrt{R}} |H_R(z)-H_R(0)| = o(1) . \tag{5.2}$$

Let ε be any positive number, let $R_0 = R_1 = \rho_0 + 2$ and, supposing $R_0, \ldots, R_n$ defined such that $R_1 < R_2 < \ldots < R_n$, define, for $n \ge 1$,

$$u_n(z) = W_{p\Delta}(z) + H_{R_n}(z) - \sum_{j=1}^{n} H_{R_j}(0) - n$$
$$+ \varepsilon H_{R_n}(0)(\log R_n)^{-1}\max\left\{\log\left|\frac{z}{R_{n-1}-1}\right|, 0\right\} ,$$

for $|z| < R_n$. Since $H_{R_n}(r) \ge H_{R_n}(-r)$,

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(5.3) $$u_n(-r) - u_n(r) \le -\Psi_p(r) \quad \text{for} \quad \rho_0 \le r < R_n \ .$$

Moreover

(a) On $|z| = R_n - 1$:

$$u_n(z) > W_{p\Delta}(z) - \sum_{j=1}^{n} H_{R_j}(0) - n \ ,$$

while if R_{n+1} is large enough we have, from (5.2),

$$u_{n+1}(z) \le W_{p\Delta}(z) - \sum_{j=1}^{n} H_{R_j}(0) - n - \tfrac{1}{2} \ .$$

Thus if R_{n+1} is large enough, $u_{n+1}(z) < u_n(z)$ on $|z| = R_n - 1$.

(b) On $|z| = R_n$:

$$u_n(z) \le W_{p\Delta}(R_n) - \sum_{j=1}^{n} H_{R_j}(0) + \varepsilon H_{R_n}(0) - n \ ,$$

while, again if R_{n+1} is large enough,

(5.4) $$u_{n+1}(z) \ge W_{p\Delta}(-R_n) - \sum_{j=1}^{n} H_{R_j}(0) - n - 2 + \varepsilon H_{R_{n+1}}(0)\left(\log R_{n+1}\right)^{-1}\log\left(\frac{R_n}{R_n - 1}\right) .$$

In view of (5.1) and the fact that $p > 3$, the right hand side of (5.4) tends to $+\infty$ as R_{n+1} tends to infinity. It is thus possible to choose R_{n+1} sufficiently large that $u_{n+1}(z) > u_n(z)$ on $|z| = R_n$ - in fact $u_{n+1}(z)$ can be made as large as we please on $|z| = R_n$ - and with such an R_{n+1} chosen,

$$v_n(z) = \begin{cases} u_n(z) \ , & |z| \le R_n - 1 \ , \\ \max\{u_n(z), \ u_{n+1}(z)\} \ , & R_n - 1 < |z| \le R_n \ , \\ u_{n+1}(z) \ , & R_n \le |z| < R_{n+1} \ , \end{cases}$$

is subharmonic. Evidently the construction can be continued indefinitely giving a subharmonic function $u(z)$ such that $u(z) = u_1(z)$ in $|z| < R_1 - 1$, and, for any $n \geq 1$, $u(z) = v_n(z)$ for $R_n - 1 \leq |z| < R_{n+1} - 1$. From (5.3) it follows that (1.5) fails for $\sigma u(z)$ for $|z| > \rho_0$. Further

$$B(R_n-1, \sigma u) = \sigma u_n(R_n-1)$$
$$< \sigma W_{p\Delta}(R_n) - \sigma(1-\varepsilon)H_{R_n}(0) ,$$

so that $\underline{\lim}\left(\sigma u(r)-\sigma(\log r)^p\right)/(\log r)^{p-2} < -\frac{1}{48}(1-\varepsilon)\sigma p(p-1)$. Finally, as was mentioned above, the sequence R_n can be chosen so that $u(R_n) = u_{n+1}(R_n)$ is as large as we please. This completes the analysis of the example.

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