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NILPOTENTS IN FINITE SYMMETRIC INVERSE SEMIGROUPS

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1. Introduction

In semigroup theory as in other algebraic theories a significant part of the total effort is appropriately applied to the study of certain standard examples occurring, as it were, "in nature". The most obvious such semigroup is the full transformation semigroup $\mathscr{T}(X)$ (see [3]) and about this semigroup a great deal is known in both the finite and infinite cases.

Inverse semigroups (see [3, Chapter V]) are of interest not only as a naturally occurring special case of semigroups but also for their role in describing partial symmetries. Mathematically this property is expressed by the Vagner–Preston Theorem [3, Theorem V.1.10], by which every (finite) inverse semigroup is embedded in an appropriate (finite) *symmetric inverse semigroup* $\mathscr{I}(X)$, consisting of all partial one-to-one maps, or *subpermutations* (to use an attractive term from Cameron and Deza [1]) of X.

Since the theory of inverse semigroups is now extensive enough to have been the subject of a substantial book by Petrich [7], it is perhaps rather surprising that very little has been written on the symmetric inverse semigroup. Certainly the Green equivalences are well understood. (See [3, Example V.2].) A notable contribution was made by Liber [5], whose description of the congruences on $\mathscr{I}(X)$ is analogous to Mal'cev's work [6, see also 2, Section 10.8] on full transformation semigroups. Much more recently Howie and Marques-Smith [4] investigated certain properties of nilpotent elements in $\mathscr{I}(X)$ in the case where X is infinite. This paper is to some extent motivated by that work.

Let $Z=(1,\dots,n)$, let $I_n(=\mathscr{I}(Z))$ be the symmetric inverse semigroup on Z, let S_n be the symmetric group on Z and let $SP_n=I_n\backslash S_n$, the inverse semigroup of all *proper subpermutations* of Z.

The inverse semigroup SP_n has n $\mathscr{J}$-classes (or $\mathscr{D}$-classes, since $\mathscr{J}=\mathscr{D}$) $J_0,\dots,J_{n-1}$, where J_r $(r=0,\dots,n-1)$ consists of all subpermutations of *height* r, i.e. all α in SP_n for which

$$|\operatorname{dom}\alpha|=|\operatorname{im}\alpha|=r.$$

Notice that J_0 consists solely of 0, the unique subpermutation with empty domain, and that

$$|J_r|=\binom{n}{r}^2 r!$$

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In particular $|J_{n-1}|=n^2(n-1)!$ The $\mathscr{D}$-class J_{n-1} contains n $\mathscr{R}$-classes R_i (corresponding to the n different possible domains $Z\backslash\{i\}$ of cardinality $n-1$) and n $\mathscr{L}$-classes L_j (corresponding to the n different possible images $Z\backslash\{j\}$). Each of the n^2 $\mathscr{H}$-classes $H_{i,j}=R_i\cap L_j$ contains $(n-1)!$ elements.

Let N be the set of nilpotent elements in SP_n and let $N_1=N\cap J_{n-1}$. For a given subset A of SP_n we write $\langle A\rangle$ for the inverse subsemigroup of SP_n generated by A. The main result (Theorem 3.18) of Section 3 is that if n is even then

$$\langle N\rangle=\langle N_1\rangle=SP_n.$$

By contrast, if n is odd we still have $\langle N\rangle=\langle N_1\rangle$, but $\langle N\rangle$ is now a proper inverse subsemigroup $SP_n\backslash K$ of SP_n, where K consists of exactly half of the elements of J_{n-1}. Theorem 3.18 also specifies precisely which half of J_{n-1} lies in $\langle N\rangle$.

Since SP_n is finite, the ascent

$$N\subseteq N\cup N^2\subseteq N\cup N^2\cup N^3\subseteq\cdots$$

must stabilize at some k, the least integer for which

$$N\subseteq N^2\cup\cdots\cup N^k=\langle N\rangle.$$

It is shown in Section 4 that $k=2$ or 3 according as n is odd or even. By contrast, if l is the least integer for which

$$N_1\cup N_1^2\cup\cdots\cup N_1^l=\langle N_1\rangle(=\langle N\rangle)$$

we find that $l=n$.

2. Nilpotents in a finite symmetric inverse semigroup

For notation and for basic properties of symmetric inverse semigroups, see [3, Section V.1].

An element α of I_n is called *nilpotent* if $\alpha^k=0$ for some $k\geqq 1$. The *index* (*of nilpotency*) $i(\alpha)$ of $\alpha(\neq 0)$ is the unique k for which $\alpha^k=0$, $\alpha^{k-1}\neq 0$.

It is evident that all nilpotents in I_n lie in SP_n. The first step in our investigation is to give a set-theoretic characterization of nilpotent elements.

Lemma 2.1. *Let $\alpha\in J_r$, with $r<n$. Then α is nilpotent if and only if there exists no non-empty subset A of* $\operatorname{dom}\alpha$ *such that $A\alpha=A$.*

Proof. If $\alpha=0$ (the empty map) the result is trivial. We may therefore confine ourselves to elements α for which $\operatorname{dom}\alpha\neq\varnothing$. Certainly if there exists $A\neq\varnothing$ inside $\operatorname{dom}\alpha$ for which $A\alpha=A$ we have

$$A=A\alpha=A\alpha^2=\cdots$$

and so α is not nilpotent.

Conversely, suppose that no such A exists. Then

$$\operatorname{im}\alpha=(\operatorname{dom}\alpha)\alpha\neq\operatorname{dom}\alpha$$

and so $\operatorname{dom}\alpha^2\subset\operatorname{dom}\alpha$ (properly). We now show that for $k=2,3,\ldots$

$$\operatorname{dom}\alpha^k\neq\varnothing\Rightarrow\operatorname{dom}\alpha^{k+1}\subset\operatorname{dom}\alpha^k.$$

For suppose by way of contradiction that

$$\operatorname{dom}\alpha^{k+1}=\operatorname{dom}\alpha^k\neq\varnothing$$

for some $k\geqq 2$. Then

$$\operatorname{dom}\alpha^k=\operatorname{dom}(\alpha\cdot\alpha^k)=(\operatorname{im}\alpha\cap\operatorname{dom}\alpha^k)\alpha^{-1}. \tag{2.2}$$

Thus $|\operatorname{im}\alpha\cap\operatorname{dom}\alpha^k|=|\operatorname{dom}\alpha^k|$ and so, since the sets are finite,

$$\operatorname{im}\alpha\cap\operatorname{dom}\alpha^k=\operatorname{dom}\alpha^k. \tag{2.3}$$

Fom (2.2) and (2.3) it now follows that

$$(\operatorname{dom}\alpha^k)\alpha=\operatorname{dom}\alpha^k,$$

contrary to hypothesis. We thus have a strict descent

$$\operatorname{dom}\alpha\supset\operatorname{dom}\alpha^2\supset\cdots$$

and hence there exists $m\geqq 1$ such that $\operatorname{dom}\alpha^m=\varnothing$, i.e. such that $\alpha^m=0$.

Recall that if $\alpha\in J_r$ we say that α is of *height* r; let us alternatively write $h(\alpha)=r$. It is clear that for all α,β in I_n

$$h(\alpha\beta)\leqq\max\{h(\alpha),h(\beta)\}. \tag{2.4}$$

On the other hand $h(\alpha\beta)$ cannot be too small:

Lemma 2.5. *If $\alpha,\beta\in I_n$ then*

$$h(\alpha\beta)\geqq h(\alpha)+h(\beta)-n.$$

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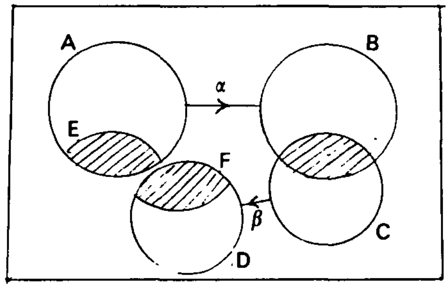
Proof. Suppose that $\operatorname{dom}\alpha = A$, $\operatorname{im}\alpha = B$, $\operatorname{dom}\beta = C$, $\operatorname{im}\beta = D$, $\operatorname{dom}\alpha\beta = E = (B\cap C)\alpha^{-1}$, $\operatorname{im}\alpha\beta = F = (B\cap C)\beta$. Then



$$Z\backslash\operatorname{dom}\alpha\beta = Z\backslash E = (Z\backslash A)\cup(A\backslash E).$$

Thus

$$\begin{aligned}|Z\backslash\operatorname{dom}\alpha\beta| &= |Z\backslash A| + |A\backslash E|\\ &= |Z\backslash A| + |B\backslash(B\cap C)| \text{ (since } \alpha \text{ is one-one)}\\ &\leqq |Z\backslash A| + |Z\backslash C|.\end{aligned}$$

Hence $n-h(\alpha\beta)\leqq n-h(\alpha)+n-h(\beta)$ and the result then follows.

As a consequence,

$$\alpha\in N_1^k \Rightarrow h(\alpha)\geqq k(n-1)-n(k-1)=n-k.$$

Lemma 2.5 also has consequences for indices of nilpotency:

Corollary 2.6. *If α is a nilpotent of height r in SP_n then $i(\alpha)\geqq n/(n-r)$.*

Proof. From the lemma it is easy to show inductively that for $k=1,2,\dots$

$$h(\alpha^k)\geqq kr-(k-1)n.$$

In particular, if $i(\alpha)=p$ then

$$0=h(\alpha^p)\geqq pr-(p-1)n$$

and the result then follows by rearrangement.

Specializing still further we have

Corollary 2.7. *If α is a nilpotent of height $n-1$ in SP_n then $i(\alpha)\geqq n$.*

Let $r\in\{1,\dots,n-1\}$ and let $a_1,\dots,a_{r+1}$ be distinct elements of $Z=\{1,\dots,n\}$. The element

$$\begin{pmatrix} a_1 & a_2 & \dots & a_r \\ a_2 & a_3 & & a_{r+1} \end{pmatrix}$$

(mapping a_i to a_{i+1} $(i=1,\dots,r)$) is clearly nilpotent of index $r+1$. Let us write it as $\|a_1a_2\dots a_{r+1}\|$ and call it a *primitive* nilpotent in SP_n.

Theorem 2.8. *Every non-zero nilpotent α in SP_n is a disjoint union $\alpha_1\cup\dots\cup\alpha_k$ of primitive nilpotents. Moreover, $k\leqq n-h(\alpha)$ and*

$$i(\alpha)=\max\{i(\alpha_1),\dots,i(\alpha_k)\}.$$

Proof. Certainly $\operatorname{dom}\alpha\neq(\operatorname{dom}\alpha)\alpha=\operatorname{im}\alpha$ by Lemma 2.1. Let $a_1\in\operatorname{dom}\alpha\setminus\operatorname{im}\alpha$ and consider the sequence

$$a_1, a_2=a_1\alpha, \quad a_3=a_2\alpha,\dots.$$

The sequence terminates when we reach an $a_{r+1}=a_r\alpha$ such that $a_{r+1}\notin\operatorname{dom}\alpha$. There can be no repetitions in the sequence: if $a_i=a_{i+j}=a_i\alpha^j$ $(j>0)$ then the non-empty set $\{a_i,\dots,a_{i+j-1}\}$ is invariant under α, which is impossible by Lemma 2.1. Hence the sequence *must* terminate in the way described.

If $r=h(\alpha)$ then α is the primitive nilpotent $\|a_1a_2\dots a_{r+1}\|$. Otherwise α is a disjoint union

$$\alpha=\|a_1a_2\dots a_{r+1}\|\cup\beta.$$

Since

$$\alpha^m=\|a_1a_2\dots a_{r+1}\|^m\cup\beta^m$$

for $m=1,2,\dots$ it follows that β is nilpotent. But $h(\beta)<h(\alpha)$ and so we may suppose inductively that β is a disjoint union of primitive nilpotents. Hence we may express α as $\alpha_1\cup\dots\cup\alpha_k$ as required.

Since each α_i is of the form $\|a_1a_2\dots a_{r+1}\|$ where $a_{r+1}\notin\operatorname{dom}\alpha$ it follows that there are at least k elements not in $\operatorname{dom}\alpha$; thus

$$k\leqq n-h(\alpha).$$

The final assertion of the theorem follows from the fact that

$$\alpha^m=\alpha_1^m\cup\dots\cup\alpha_k^m$$

for $m=1,2,\dots$.

Corollary 2.9. *Every nilpotent of height $n-1$ in SP_n is primitive and hence of index n.*

Proof. If α is nilpotent of height $n-1$ then $k \leqq n-(n-1)$ by Theorem 2.8. Hence α is primitive.

In the introduction the structure of the top $\mathscr{J}$-class J_{n-1} in SP_n was described. It can now be seen that each $\mathscr{H}$-class $H_{i,j}$ $(i \neq j)$ contains $(n-1)!$ elements of which exactly $(n-2)!$ are (primitive) nilpotents of the form

$$\|ja_2 \dots a_{n-1}i\|$$

with $a_2, \dots, a_{n-1} \in Z \setminus \{i, j\}$. Since there are $n(n-1)$ $\mathscr{H}$-classes containing nilpotents, the total number of nilpotents in J_{n-1} is $n!$.

We end this section with a result on conjugates of nilpotents

Theorem 2.10. *If β is a nilpotent in SP_n and $\alpha \in SP_n$ then $\alpha^{-1}\beta\alpha$ is a nilpotent.*

Proof. Suppose that $\beta^k = 0$. Then by elementary properties of the order $\leqq$ in an inverse semigroup (see [3, Section V.2])

$$(\alpha^{-1}\beta\alpha)^k = \alpha^{-1}\beta(\alpha\alpha^{-1})\beta \dots (\alpha\alpha^{-1})\beta\alpha \leqq \alpha^{-1}\beta^k\alpha = 0;$$

hence $(\alpha^{-1}\beta\alpha)^k = 0$.

3. The inverse semigroup generated by N_1

For each α in $H_{i,j}$ $(\subset J_{n-1})$ there exists a unique *completion* $\bar{\alpha}$ in S_n (the symmetric group on $Z = \{1, \dots, n\}$) defined by

$$i\bar{\alpha} = j, \quad x\bar{\alpha} = x\alpha \qquad (x \neq i).$$

The completion of the nilpotent $\|a_1a_2 \dots a_n\|$ in J_{n-1} is the cycle $(a_1a_2 \dots a_n)$ in S_n.

From Lemma 2.5 we can see that

$$H_{i,j}H_{k,l} \subseteq J_{n-2} \tag{3.1}$$

if $j \neq k$. Also

$$H_{i,j}H_{j,k} \subseteq H_{i,k}.$$

If $\alpha \in H_{i,j}$ and $\beta \in H_{j,k}$, then their product lies in J_{n-1}, and

$$\overline{\alpha\beta} = \bar{\alpha}\bar{\beta}.$$

For these reasons the question of whether an element of J_{n-1} is expressible as a product of elements of N_1 is closely bound up with the expression of elements of S_n as products of cycles of length n (what we shall call *n-cycles* for brevity). There follow some results concerning S_n; these may well be known but we have been unable to find them in print.

An element α of S_n partitions $Z=\{1,\ldots,n\}$ into two disjoint sets

$$\text{shift } \alpha=\{x: x\alpha\neq x\}, \quad \text{fix } \alpha=\{x: x\alpha=x\}.$$

When we refer to two permutations as disjoint we shall mean as usual that their shifts are disjoint.

We shall eventually prove that every even permutation α is expressible as a product of two cycles of length n. (These are not *disjoint* cycles, naturally.) It is convenient to prove something a little more precise: if fix $\alpha=\{x_1,\ldots,x_p\}$ we shall prove that $\alpha=\xi_1\xi_2$, where the n-cycle $\xi_1=(\ldots x_1x_2\ldots x_p)$ ends with the elements of fix α in some order and the n-cycle $\xi_2=(\ldots x_px_{p-1}\ldots x_1)$ ends with the elements of fix α in the reverse order. We refer to such a product as a *tidy* product of two n-cycles.

Lemma 3.2. *Every cycle of odd length $l\leqq n$ in S_n is a tidy product of two n-cycles.*

Proof. Let $\alpha=(a_1a_2\ldots a_l)$, where l is odd, and write fix $\alpha=\{x_1,\ldots,x_{n-l}\}$. Then

$$\alpha=(a_1a_3\ldots a_la_2a_4\ldots a_{l-1}x_1\ldots x_{n-l})(a_la_{l-1}\ldots a_1x_{n-l}\ldots x_1).$$

It is not possible here to drop the restriction that l be odd, for a product of two cycles of length n is necessarily an even permutation, while a cycle of even length is an odd permutation. However, we do have

Lemma 3.3. *If ξ_1,ξ_2 are disjoint cycles of even length then $\xi_1\xi_2$ is expressible as a tidy product $\eta_1\eta_2$ of two n-cycles.*

Proof. Write $\xi_1=(a_1a_2\ldots a_{2k})$, $\xi_2=(b_1b_2\ldots b_{2l})$. We may assume without loss of generality that $k\leqq l$. Let

$$\text{fix } \xi_1\xi_2=X\backslash(\{a_1,\ldots,a_{2k}\}\cup\{b_1,\ldots,b_{2l}\})=\{x_1,\ldots,x_p\},$$

where $p=n-2k-2l\geqq 0$. Let

$$\eta_1=(b_{2l}b_{2l-1}\ldots b_{2k+1}a_1b_1a_2b_2\ldots a_{2k}b_{2k}x_1x_2\ldots x_p),$$

$$\eta_2=(b_{2k+1}b_{2k+3}\ldots b_{2l-1}b_1a_2b_2\ldots a_{2k}b_{2k}a_1b_{2k+2}b_{2k+4}\ldots b_{2l}x_px_{p-1}\ldots x_1);$$

then it is not hard to verify that the tidy product $\eta_1\eta_2$ coincides with $\xi_1\xi_2$.

Lemma 3.4. *Let σ,τ be disjoint permutations in S_n and suppose that each of σ,τ is a tidy product of two n-cycles. Then $\sigma\tau$ is a tidy product $\zeta_1\zeta_2$ of two n-cycles.*

Proof. Suppose that shift $\sigma=\{a_1,\dots,a_k\}$, shift $\tau=\{c_1,\dots,c_l\}$, fix $\sigma\tau=\{x_1,\dots,x_p\}$. Then

$$\text{fix }\sigma=\{c_1,\dots,c_l,x_1,\dots,x_p\},\quad \text{fix }\tau=\{a_1,\dots,a_k,x_1,\dots,x_p\}.$$

We may assume that we have tidy products as follows:

$$\sigma=(a_1\dots a_kc_1\dots c_lx_1\dots x_p)(b_1\dots b_kx_p\dots x_1c_l\dots c_1) \tag{3.5}$$

$$\tau=(c_1\dots c_la_1\dots a_kx_1\dots x_p)(d_1\dots d_lx_p\dots x_1a_k\dots a_1); \tag{3.6}$$

here $(b_1,\dots,b_k)$, $(d_1,\dots,d_l)$ are permutations respectively of $(a_1,\dots,a_k)$, $(c_1,\dots,c_l)$. Then

$$\sigma\tau=(c_1\dots c_la_1\dots a_kx_1\dots x_p)(b_1\dots b_kd_1\dots d_lx_p\dots x_1), \tag{3.7}$$

a tidy product of two n-cycles. The verification of this is for the most part routine and we shall confine ourselves to noting some crucial points. If for brevity we write (3.5) as $\sigma=\xi_1\xi_2$ and (3.6) as $\tau=\eta_1\eta_2$, notice that

$$\begin{aligned} b_k&=x_p\xi_2^{-1}=x_p\sigma\xi_2^{-1}\quad(\text{since } x_p\in\text{fix }\sigma)\\ &=x_p\xi_1=a_1;\end{aligned}$$

similarly $d_l=c_1$. Also

$$a_k\sigma=c_1\xi_2=b_1$$

and similarly $c_l\tau=d_1$. Then for example we may consider a_i $(1\leqq i\leqq k-1)$. Using (3.5) and (3.6) we see that

$$\begin{aligned} a_i&\to a_{i+1}=b_r\text{ (say), where } r\neq k \text{ since } b_k=a_1\\ &\to b_{r+1}=a_s\text{ (say)}\\ &\to a_{s+1}\text{ (provided } s\neq k;\text{ if } s=k \text{ then } a_s\to x_1)\\ &\to a_s.\end{aligned}$$

On the other hand, using (3.7) we see that

$$a_i\to a_{i+1}=b_r\to b_{r+1}=a_s.$$

The other verifications are no harder than this.

As a consequence of the three lemmas proved above, we have

Theorem 3.8. *Every even permutation of S_n is a product of at most two n-cycles.*

Proof. Let α be an even permutation. We consider the standard decomposition of α into a product of disjoint cycles. If α is an n-cycle (which can happen if n is odd) then no further argument is necessary; otherwise the decomposition of α must involve an even number of cycles of even length. The result now follows from Lemmas 3.2, 3.3 and 3.4, and we may even conclude that the product $\alpha=\xi_1\xi_2$ is tidy.

Theorem 3.9. *Let n be even. Then every permutation in S_n can be expressed as a product of at most three n-cycles.*

Proof. In view of the last theorem we need only consider an odd permutation π. Then

$$\pi=[(12)(134\ldots n)][(134\ldots n)^{-1}(12)\pi]$$

$$=(12\ldots n)\pi',$$

where $\pi'=(134\ldots n)^{-1}(12)\pi$, being even × odd × odd, is even. The result now follows by Theorem 3.8.

It is obvious that "three" is best possible in this result, provided $n\geqq 4$. For a product of two n-cycles must be an even permutation, and clearly not every odd permutation is an n-cycle.

We now apply these group-theoretical results to the problem of finding $\langle N_1\rangle$. First, we have

Lemma 3.10. *Let n be odd and let $\alpha\in J_{n-1}$. Then $\alpha\in\langle N_1\rangle$ if and only if its completion $\bar{\alpha}$ is an even permutation of $\{1,\ldots,n\}$.*

Proof. Suppose first that $\bar{\alpha}$ is even, and let $\alpha\in H_{i,j}$. Then $\bar{\alpha}$ is either an n-cycle or a product of two n-cycles. In the former case $\alpha\in N_1\subseteq\langle N_1\rangle$. Otherwise $\bar{\alpha}=\xi\eta$. If we write $i\xi=p$ then we must have $p\eta=j$, since $i\bar{\alpha}=j$. Let $\gamma\in H_{i,p}$, $\delta\in H_{p,j}$ be such that $\bar{\gamma}=\xi$, $\bar{\delta}=\eta$. Then $\gamma,\delta\in N_1$ and $\alpha=\gamma\delta$ as required.

Conversely, suppose that $\alpha=\gamma_1\ldots\gamma_k$ for some $\gamma_1,\ldots,\gamma_k$ in N_1. Since $\alpha\in J_{n-1}$ we must have

$$\gamma_1\in H_{i,p_1},\ \gamma_2\in H_{p_1,p_2},\ldots,\gamma_k\in H_{p_{k-1},j}$$

for some $i,p_1,\ldots,p_{k-1},j$ in Z. Hence $\bar{\alpha}=\bar{\gamma}_1\ldots\bar{\gamma}_k$, a product of n-cycles. Since n is odd these n-cycles are even and so $\bar{\alpha}$ is even.

From the first part of this proof we have

Corollary 3.11. *If n is odd then*

$$\langle N_1\rangle\cap J_{n-1}\subseteq N_1\cup N_1^2.$$

For a given $H_{i,j}$ the set $\{\bar{\alpha}:\alpha\in H_{i,j}\}$ consists of all the permutations in S_n sending i to j. Exactly half of the elements in each $H_{i,j}$ are in $\langle N_1\rangle$, and so

$$|\langle N_1\rangle\cap J_{n-1}|=\tfrac{1}{2}|J_{n-1}|=\tfrac{1}{2}n^2(n-1)!$$

If n is even we have a different answer.

Lemma 3.12. *If n is even then $J_{n-1}\subset\langle N_1\rangle$.*

Proof. Let $\alpha\in H_{i,j}\subset J_{n-1}$. Then $\bar{\alpha}$, by Theorem 3.9, is a product of one, two or three n-cycles. Arguing as in the first part of the proof of Lemma 3.10, we deduce that α is a product of one, two or three elements of N_1.

By analogy with Corollary 3.11 we have

Corollary 3.13. *If n is even then*

$$J_{n-1}\subset N_1\cup N_1^2\cup N_1^3.$$

From (3.1) it is clear that $\langle N_1\rangle$ contains elements of height less than $n-1$. The following lemma is helpful.

Lemma 3.14. *Let $\alpha\in SP_n$, with $h(\alpha)\leqq n-2$. Then there exists ξ in N_1 and β in SP_n such that $h(\beta)=h(\alpha)+1$ and $\alpha=\beta\xi$.*

Proof. Write $h(\alpha)=h$ and

$$\alpha=\begin{pmatrix}a_1 & a_2\dots a_h\\ b_1 & b_2\dots b_h\end{pmatrix}.$$

Let $x\in X\setminus\{a_1,\dots,a_h\}$ and let

$$X\setminus\{b_1,\dots,b_h\}=\{y_1,\dots,y_{n-h}\},$$

where $n-h\geqq 2$ by assumption. Define

$$\beta=\begin{pmatrix}a_1 & a_2\dots a_{h-1} & a_h & x\ {}_{-h}\\ b_2 & b_3\dots b_h & y_1 & y_{n-h}\end{pmatrix},$$

$$\xi=\|y_1b_hb_{h-1}\dots b_1y_2\dots y_{n-h}\|;$$

then $h(\beta)=h+1$, $\xi\in N_1$, and $\alpha=\beta\xi$.

Next, we have

Lemma 3.15. $J_{n-2}\subseteq N_1^2$.

Proof. Let $\alpha \in J_{n-2}$. Suppose in fact that

$$\operatorname{dom}\alpha = X\backslash\{p, q\}, \quad \operatorname{im}\alpha = X\backslash\{r, s\},$$

with $p \neq q$, $r \neq s$. We can still "complete" α to make a permutation, but there are now two possible completions $\bar{\alpha}_1$, $\bar{\alpha}_2$, where

$$p\bar{\alpha}_1 = r, \qquad q\bar{\alpha}_1 = s,$$

$$p\bar{\alpha}_2 = s, \qquad q\bar{\alpha}_2 = r.$$

Since $\bar{\alpha}_2 = \bar{\alpha}_1(rs)$, exactly one of $\bar{\alpha}_1, \bar{\alpha}_2$ is an even permutation: let $\bar{\alpha}$ be the unique *even* completion of α. By Theorem 3.9 we have n-cycles ζ, τ such that $\bar{\alpha} = \zeta\tau$. This applies even if $\bar{\alpha}$ is itself an n-cycle, for that can happen if n is odd, and Lemma 3.2 makes it clear that a cycle of odd length n can be expressed as a product of two n-cycles. Now let

$$\gamma \in H_{p, p\zeta}, \quad \delta \in H_{q\zeta, q\bar{\alpha}}$$

be such that $\bar{\gamma} = \zeta$, $\bar{\delta} = \tau$. Then $\alpha = \gamma\delta \in N_1^2$ as required.

Remark 3.16. In the next section we shall require a slightly modified version of this result. In the above proof we took

$$\gamma = \zeta|(Z\backslash\{p\}), \quad \delta = \tau|(Z\backslash\{q\zeta\}).$$

If instead we take

$$\gamma' = \zeta|(Z\backslash\{p, q\}), \quad \delta' = \tau|(Z\backslash\{p\zeta, q\zeta\})$$

we again obtain $\alpha = \gamma'\delta'$, but γ', δ' are now in $N \cap J_{n-2}$. Thus

$$J_{n-2} \subseteq (N \cap J_{n-2})^2.$$

From Lemmas 3.14 and 3.15 we have

Corollary 3.17. *If $h(\alpha) = n - r$ then $\alpha \in N_1^r$.*

Our main result is now clear:

Theorem 3.18. *For $n \geqq 3$ let SP_n be the inverse semigroup of all proper subpermutations of $\{1, \ldots, n\}$, and let N_1 be the set of all nilpotents of height $n-1$ in SP_n.*

(i) *If n is even then*

$$\langle N_1 \rangle = N_1 \cup \cdots \cup N_1^n = \{0\} \cup N_1 \cup \cdots \cup N_1^{n-1} = SP_n.$$

(ii) *If n is odd then*

$$\langle N_1 \rangle = N_1 \cup \cdots \cup N_1^n = \{0\} \cup N_1 \cup \cdots \cup N_1^{n-1} = SP_n \backslash W_{n-1},$$

where W_{n-1} consists of those elements of height $n-1$ whose completions in S_n are odd permutations.

Remark 3.19. Because of Lemma 2.5 the result of Corollary 3.17 is best possible in the sense that $h(\alpha)=n-r$ implies that $\alpha\notin N_1^{r-1}$. Hence the result of Theorem 3.18 is also best possible. That is

$$\langle N_1\rangle \neq N_1\cup\cdots\cup N_1^{n-1},\quad \langle N_1\rangle\neq\{0\}\cup N_1\cup\cdots\cup N_1^{n-2}.$$

4. The inverse subsemigroup generated by all nilpotent elements

From (2.4) it is clear that no product of elements (nilpotent or otherwise) in $J_0\cup\cdots\cup J_{n-2}$ can lie in J_{n-1}. Since $\langle N_1\rangle$ contains $J_0\cup\cdots\cup J_{n-2}$ we thus have $\langle N\rangle=\langle N_1\rangle$ and so in one sense we gain nothing by using the whole of N as a set of generators rather than $N_1=N\cap J_{n-1}$. If we define $\Delta(\langle N\rangle)$ to be the unique k such that

$$\langle N\rangle=N\cup N^2\cup\cdots\cup N^k,\quad \langle N\rangle\neq N\cup N^2\cup\cdots\cup N^{k-1},$$

then from Theorem 3.18 we deduce that

$$\Delta(\langle N\rangle)\leqq n.$$

We shall see that this is in fact a very poor bound.

First, we have

Lemma 4.1. *Let $r\in\{2,\dots,n-1\}$. If $J_r\subseteq(N\cap J_r)^k$ then $J_{r-1}\subseteq(N\cap J_{r-1})^k$.*

Proof. Suppose that $J_r\subseteq(N\cap J_r)^k$, and let $\alpha\in J_{r-1}$. Let $p\in Z\backslash\operatorname{dom}\alpha$, $q\in Z\backslash\operatorname{im}\alpha$ and define $\alpha^*\in J_r$ by

$$x\alpha^*=x\alpha\quad(x\in\operatorname{dom}\alpha),\quad p\alpha^*=q.$$

By hypothesis, $\alpha^*=\gamma_1^*\dots\gamma_k^*$, a product of k nilpotents of height r. Write

$$p\gamma_1^*=t_2,\quad t_2\gamma_2^*=t_3,\dots,t_{k-1}\gamma_{k-1}^*=t_k;$$

then $t_k\gamma_k^*=p\gamma_1^*\dots\gamma_k^*=p\alpha^*=q$. Define

$$\gamma_1=\gamma_1^*|(\operatorname{dom}\gamma_1^*\backslash\{p\}),\quad \gamma_i=\gamma_i^*|(\operatorname{dom}\gamma_i^*\backslash\{t_i\})$$

$(i=2,\dots,k)$. Then $\gamma_1,\dots,\gamma_k$ are nilpotents in J_{r-1} and $\alpha=\gamma_1\dots\gamma_k$ as required.

If n is even then by Corollary 3.13

$$J_{n-1}\subseteq(N\cap J_{n-1})\cup(N\cup J_{n-1})^2\cup(N\cap J_{n-1})^3,$$

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and from the remark following the proof of Theorem 3.9 it is clear that 3 is best possible. If n is odd then Corollary 3.11 gives

$$\langle N\rangle \cap J_{n-1} \subseteq (N \cap J_{n-1}) \cup (N \cap J_{n-1})^2.$$

From the Remark 3.16 we know that for all n

$$J_{n-2} \subseteq (N \cap J_{n-2})^2.$$

Hence from Lemma 4.1 we may now deduce

Theorem 4.2. *For $n \geqq 3$ let SP_n be the inverse semigroup of all proper subpermutations of $\{1,\ldots,n\}$, and let N be the set of all nilpotents in SP_n. Let $\Delta(\langle N\rangle)$ be the unique k such that*

$$\langle N\rangle = N \cup N^2 \cup \cdots \cup N^k \neq N \cup N^2 \cup \cdots \cup N^{k-1}.$$

Then $\Delta(\langle N\rangle)=2$ or 3 according as n is odd or even.

REFERENCES

1. P. J. Cameron and M. Deza, on permutation geometries, *J. London Math. Soc.* (2) **20** (1979), 373–386.

2. A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups, Vol. II* (American Math. Soc., Providence, 1967).

3. John M. Howie, *An Introduction to Semigroup Theory* (Academic Press, London, 1976).

4. John M. Howie and M. Paula O. Marques-Smith, Inverse semigroups generated by nilpotent transformations, *Proc. Royal Soc. Edinburgh A*, **99** (1984), 153–162.

5. A. E. Liber, On symmetric generalized groups, *Mat. Sbornik* (*N.S.*) **33** (1953), 531–544 (Russian).

6. A. I. Mal'cev, Symmetric groupoids, *Mat. Sbornik* (*N.S.*) **31** (1952), 136–151 (Russian).

7. Mario Petrich, *Inverse Semigroups* (Wiley, New York, 1984).

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