1 Particle Dynamics

Particle dynamics is the starting point for the vector formulation of dynamics. It was first formulated by Isaac Newton through three fundamental laws (***Newton's laws of motion***), which are the core of the *Philosophiæ Naturalis Principia Mathematica* (Isaac Newton's major work). These laws have been reformulated many times since they were published, not only to adapt the language to a more modern and understandable form (and that includes using mathematical formulations not existing in Newton's times) but also to overcome several axiomatic issues as "are those laws independent? (is the first law a particular case of the second law?)," "are they laws or definitions? (is the second one a law or just a definition of force?)." Though these laws are widely known and used, their deep understanding is not straightforward. Concepts such as mass and force, which are at the core of that formulation, are not at all simple, and have been criticized by many scientists over the centuries.

Ernst Mach's axiomatization of Newtonian mechanics (presented in his book *The Science of Mechanics*) is among the most important reformulations, and it is the one we will present in this chapter. It does not mean that it is the "final" formulation: it has also been subjected to criticism by other scientists, and reformulated in turn!

Mach's new axiomatization is based on three **empirical propositions**. We will show that, together with the consideration of the properties of space and time and a precise definition of the concepts mass and force, those principles lead to ***Newton's laws of motion***. Once this is proved, we will stick to Newton's formulation.

The fundamental law governing particle dynamics (Newton's second law) is presented both in Galilean and non-Galilean reference frames. A discussion on the frames which appear to behave as Galilean ones (according to the scope of the problem under study) is also included.

Finally, we present the most usual interactions acting on particles and provide a formulation for the gravitational attraction, forces associated with springs and dampers, friction, and a description of constraint forces.

1.1 Fundamental Assumptions Underlying Newtonian Dynamics

Newtonian dynamics rely on a few definitions and laws (assumptions that cannot be proved) from which many useful theorems may be deduced. There is not a unique way of formulating those laws. Though Newton was the first to lay the framework for

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classical mechanics, other scientists have proposed different systems of axioms and definitions to overcome the drawbacks of Newton's formulation.

Whatever the formulation is, there are, however, two principles which are assumed in Newtonian mechanics, and which deserve some comments: the ***principle of causality*** and the ***principle of absolute simultaneity***.

Principle of Causality

Dynamics studies the motion of material objects as a function of the physical factors affecting them. In the Newtonian formulation, those physical factors are mainly their *mass* and the *forces* acting on the objects. Whatever the forces may be, all definitions assume implicitly that there is a correlation between them and the object's motion. This correlation is sometimes described in a simplistic way as a causality relationship: "Dynamics is the study of the motion of bodies *caused by* the action of forces."

The equations of Newtonian dynamics are differential equations relating the second order derivative of the objects generalized coordinates to the coordinates and their first derivatives, all of them at a same time instant. This simultaneity of motion variables makes it difficult (and formally impossible) to distinguish "causes" and "effects," as the former should actually precede the latter. That distinction becomes a convention.

Principle of Absolute Simultaneity

The ***principle of absolute simultaneity***[1] states, in short, that a sequence (order of succession) of events is the same for all observers (or independent from the reference frame). In Newton's conception of the universe, **absolute time** goes hand in hand with **absolute space**: they constitute an immutable stage where physical events occur, they are independent external realities.

Other Assumptions

Last but not least: as in any scientific theory, Newtonian dynamics have a limited field of application (directly related to the experimental limitations of the seventeenth century!). Outside that field, it yields inaccurate results. The main limitations are:

- Low-speed dynamics: objects moving with speeds comparable to the speed of light cannot be treated successfully with this theory.
- Medium length scale dynamics: molecular dynamics and long-reach astronomy are also out of scope.
- Electromagnetic phenomena are excluded.

[1] Batlle, J.A. and Barjau Condomines, A. (2019) *Rigid Body Kinematics*, Cambridge University Press, chapter 1.

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1.2 Galilean and Non-Galilean Reference Frames

When we enter the field of Newtonian dynamics, we are confronted with a surprising fact: all reference frames are not equivalent for the formulation of the dynamical equations (or of the fundamental laws of Newtonian dynamics).

The methods presented in kinematics (composition of movements, rigid body kinematics. . .) apply equally in all reference frames.

The situation in dynamics is quite different: as dynamics intends to relate movement and "causes," as the movement depends in principle on the reference frame, the "causes" may vary from one frame to another.

What can be the "cause" of motion (or motion change) of a particle?[2] Certainly, the existence of other material objects, which may interact with the particle (that is, have an effect on its motion). But the observation of reality suggests other factors that might have a consequence on the particle motion. Given a frame where the observations or experiments are performed, those factors are:

- the particular position of the interacting system in the reference frame;
- the particular orientation of the interacting system in the reference frame;
- the particular time instant of the observations.

The following example is an illustration of those three possibilities. We assume that the reader is acquainted with some very basic aspects of particle dynamics (to be presented later on in this chapter).

► **Example 1.1** Let's consider the following situations:

(a) Particle **P** is initially at rest on a small smooth horizontal surface (whose area is much smaller than that of the Earth) close to the Earth's surface (Fig. 1.1a). If we consider only the horizontal motion (thus reducing the problem from 3D to 2D), the absence of roughness is equivalent to zero horizontal interaction, and **P** behaves as a free particle.

(b) Particle **P** is initially launched with a given speed in any horizontal direction and from any location on a smooth horizontal surface close to the Earth's surface (Fig. 1.1b). Under these circumstances, **P** is again a free particle.

(c) Particle **P** is in a smooth slot in a support and attached to two identical springs with ends fixed to that support (Fig. 1.1c). If we consider only the rectilinear motion in the slot, the particle and the springs constitute the interacting system. The support will be glued anywhere and with any orientation on a horizontal surface close to the Earth's surface.

[2] Newton did not formulate the laws for "particles" but for "bodies." However, updated formulations of those laws talk of "particles" (short denomination for a mass-point object). Further on, we will present the concept "particle" as the simplest model for a material object without any considerations on its size, as far as its orientation is irrelevant for the situation under study.

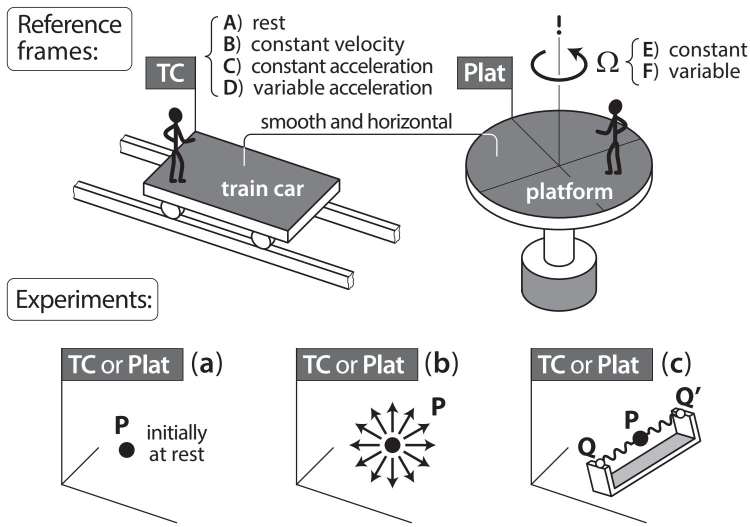

Fig. 1.1

In situations (1) and (2), we want to observe the evolution of the initial state (initial position and velocity of the particle). In situation (3) we will be interested in the equilibrium position of the particle.

The reference frame will be that of the smooth surface (which coincides with the springs support in the third situation). Let's consider different choices:

A. Reference frame at rest with respect to the Earth.
B. Reference frame with a uniform rectilinear motion with respect to the Earth.
C. Reference frame with a rectilinear motion with constant acceleration with respect to the Earth.
D. Reference frame with a rectilinear motion with variable acceleration with respect to the Earth.
E. Reference frame rotating with constant angular velocity about a fixed axis with respect to the Earth.
F. Reference frame rotating with variable angular velocity about a fixed axis with respect to the Earth.

The reference frames **A** to **D** can be pictured as a train car with different motions on a horizontal straight railway, while the reference frames **E** and **F** can be assimilated to a platform rotating about an Earth-fixed axis.

Train cars and rotating platforms are observation reference frames familiar enough to the reader, so that the result of those three thought experiments can be correctly guessed. For instance, leaving the particle **P** at rest on a train car in different positions will have no consequences, whereas the time instant when this is done does have an influence on

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Table 1.1

Reference frame	Position dependence	Orientation dependence	Time dependence
A	NO	NO	NO
B	NO	NO	NO
C	NO	**YES**	NO
D	NO	**YES**	**YES**
E	**YES**	**YES**	NO
F	**YES**	**YES**	**YES**

the observations when the train car has a variable velocity relative to the Earth. However, when leaving **P** at rest on a rotating platform, different initial positions yield different results: for instance, if located just on the platform center, the particle stays at rest.

Table 1.1 summarizes the influence of position, orientation and time instant for experiments (1), (2), and (3) in reference frames **A** to **F**. A "YES" means that at least one of the experiments does show a dependence on position/orientation/time instant. ◀

Example 1.1 suggests that there are reference frames (**A** and **B**) which do not have any influence on the motion of particles because, *as far as mechanical phenomena are concerned*:

- All positions are equivalent: space is **homogeneous**.
- All orientations are equivalent: space is **isotropic**.
- All time instants are equivalent: time is **uniform**.

Those reference are called **inertial** or **Galilean reference frames**.

Example 1.1 also suggests the existence of reference frames (**C** to **F**) where some of those properties are not fulfilled: those are non-inertial or non-Galilean reference frames.

The formulation of the dynamics of a mechanical system will be simpler in a Galilean reference frame because it will have to take into account just the interactions between material objects (which will not depend on location, orientation, and time instant). However, proving the existence of at least one Galilean reference frame is strictly impossible, hence it is taken as a principle: the ***principle of existence of a Galilean reference frame***.[3]

[3] In Newton's *Principia Mathematica*, Galilean reference frames do not appear explicitly. Instead, Newton shows the difference between *true motion* and *relative motion*, which correspond to motion with respect to a Galilean and to a non-Galilean reference frame, respectively.

1.3 Dynamics of a Free Particle: Newton's First Law (Principle of Inertia)

The simplest dynamical problem is that of a **free particle** (a particle free from interactions) observed from a Galilean reference frame. A free particle is an idealization: all real particles interact with other material objects located at a finite distance from them. Solving the dynamics of a free particle amounts to a thought experiment, but it is an interesting and useful one.

The properties of space and time in a Galilean reference frame partially restrict the possible motions of a free particle. If it is initially at rest, it will remain at rest (Fig. 1.2a): not doing so would imply choosing a direction for the initial motion, and that would contradict the isotropy of space. If its initial velocity is nonzero, its motion will have to be rectilinear (if it does not stop) though not necessarily uniform (Fig. 1.2b), as a curvature in its trajectory would again imply choosing a direction (the intrinsic normal direction). Summarizing: the properties of space and time in a Galilean reference frame (RGal) restrict the possible motion of the free particle according to $\mathrm{a}^{\mathrm{n}}_{\mathrm{RGal}}(\mathbf{P}) = 0$.

However, those properties do not imply that $\mathrm{a}^{\mathrm{s}}_{\mathrm{RGal}}(\mathbf{P}) = 0$. The particle could either increase or decrease its speed or even stop without contradicting any of them: this would be equivalent to imagine that space has a constant intrinsic friction equal in all locations and directions.

Stating that $\mathrm{a}^{\mathrm{s}}_{\mathrm{RGal}}(\mathbf{P}) = 0$, then, is formulating a principle (or a law): ***Newton's first law*** (***principle of inertia***). That principle is usually expressed mathematically as

$$\bar{\mathbf{a}}_{\mathrm{RGal}}(\mathbf{P}_{\mathrm{free}}) = \bar{0}. \tag{1.1}$$

This law is based on everyday experience, and had already been formulated (among others, by Galileo): if there are no interactions, rectilinear motion with constant speed lasts forever (Fig. 1.2b).

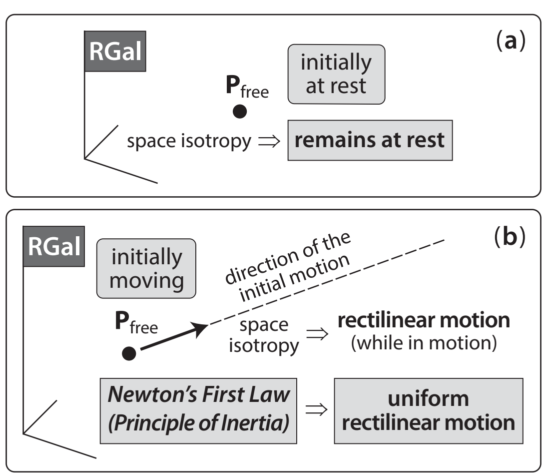

Fig. 1.2

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The analytical expression of Newton's first law (Eq. (1.1)) shows that there is an infinite number of Galilean reference frames (provided that we accept the principle of existence of one Galilean reference frame): all those having a uniform translation relative to the postulated one. The proof is straightforward through a composition of accelerations: between any of those reference frames and the initial one, the transportation and the Coriolis accelerations are zero.

1.4 Dynamics of Interacting Particles

Galileo's Principle of Relativity

We have just proved that all Galilean reference frames share the law governing the dynamics of a free particle.[4] When seeking for the law governing the dynamics of interacting particles, a logical question is: Will the Galilean reference frames also share that law? Answering that question is equivalent to establishing a principle of relativity (a principle defining the set of reference frames for which a scientific law is valid).

In Newtonian mechanics, that principle is the special principle of relativity, also known as ***Galileo's principle of relativity***, as it was first enunciated by Galileo in 1632. A modern formulation of that principle is: The laws governing all mechanical phenomena are the same in all reference frames where the ***Newton's first law*** is fulfilled. A consequence of that principle is that Galilean reference frames are not distinguishable through mechanical observations.[5]

Mach's Empirical Propositions

Though the vector formulation of dynamics proposed by Newton is the most widespread one and its application is certainly more straightforward (as interaction forces are a more practical representation of interactions than interaction accelerations because of their symmetry, as will be seen later on), it has many drawbacks from an axiomatic point of view. It relies on the concepts *mass* and *force*, but the definition of the former is unclear and that of the latter is actually implicit.

The two axioms introduced so far (Newton's first law and the principle of relativity) do not mention mass or force, while Newton's second and third laws do explicitly

[4] Actually, Newton's first law provides an alternative definition for Galilean reference frames (initially defined as those where space is homogeneous and isotropic, and time is uniform): a reference frame R is Galilean when Newton's first law is fulfilled in R.

[5] This is actually how Galileo presents the principle: He proposes to "shut yourself up with some friend in the main cabin below decks on some large ship," proceed to observe several different mechanical phenomena, and then repeat the same observations but "have the ship proceed with any speed you like, so long as the motion is uniform and not fluctuating this way and that." He then affirms that "you will discover not the least change in all the effects named, nor could you tell from any of them whether the ship was moving or standing still," which is equivalent to saying that the laws governing the mechanical phenomena are exactly the same in all reference frames whose relative motion is a uniform translation (Galilei, G. [1953] *Dialogue Concerning the Two Chief World Systems*, trans. Stillman Drake).

include these concepts. Mach's formulation relies exclusively on interaction accelerations, and for this reason it will be presented before completing Newton's axioms.

Ernst Mach's axiomatization is based on three **empirical propositions** and two **definitions**. It is an alternative formulation whose application is not straightforward, but it has two important advantages:

- it is based on accelerations, which are observable (and measurable) variables, and for that reason they correspond to "real" concepts;
- it allows us to define in a rigorous way *mass* and *force*.

The **first empirical proposition** considers two isolated interacting particles **P** and **Q**. If observed from a Galilean reference frame, their accelerations will be exclusively the consequence of their mutual interaction. If $\bar{\mathbf{a}}_{\mathrm{RGal}}^{\mathbf{Q}\to\mathbf{P}}$ and $\bar{\mathbf{a}}_{\mathrm{RGal}}^{\mathbf{P}\to\mathbf{Q}}$ are the acceleration of **P** under the action of **Q** and that of **Q** under the action of **P**, respectively, the proposition states that those accelerations will be either attractive or repulsive, in the direction of $\overline{\mathbf{QP}}$ (Fig. 1.3). Mathematically, this can be expressed as

$$\frac{\bar{\mathbf{a}}_{\mathrm{RGal}}^{\mathbf{Q}\to\mathbf{P}}}{\left|\bar{\mathbf{a}}_{\mathrm{RGal}}^{\mathbf{Q}\to\mathbf{P}}\right|} = -\frac{\bar{\mathbf{a}}_{\mathrm{RGal}}^{\mathbf{P}\to\mathbf{Q}}}{\left|\bar{\mathbf{a}}_{\mathrm{RGal}}^{\mathbf{P}\to\mathbf{Q}}\right|}, \quad \overline{\mathbf{QP}} \times \bar{\mathbf{a}}_{\mathrm{RGal}}^{\mathbf{Q}\to\mathbf{P}} = \bar{0}. \tag{1.2}$$

This proposition is followed by the definition of **inertial mass-ratio** of two particles $(\mathrm{m}_{\mathbf{Q}}/\mathrm{m}_{\mathbf{P}})$:

$$\frac{\mathrm{m}_{\mathbf{Q}}}{\mathrm{m}_{\mathbf{P}}} \equiv \frac{\left|\bar{\mathbf{a}}_{\mathrm{RGal}}^{\mathbf{Q}\to\mathbf{P}}\right|}{\left|\bar{\mathbf{a}}_{\mathrm{RGal}}^{\mathbf{P}\to\mathbf{Q}}\right|}. \tag{1.3}$$

If we choose a particular value $\mathrm{m}_{\mathbf{P}}$ for a particle, the mass of the other particles may be determined straightaway.

Equation (1.3) provides an interpretation for the inertial mass. As the product $\mathrm{m}_{\mathbf{P}}\left|\bar{\mathbf{a}}_{\mathrm{RGal}}^{\mathbf{Q}\to\mathbf{P}}\right|$ is equal to $\mathrm{m}_{\mathbf{P}}\left|\bar{\mathbf{a}}_{\mathrm{RGal}}^{\mathbf{Q}\to\mathbf{P}}\right|$, it follows that the particle which acquires a higher acceleration has a lower mass, and vice versa. Hence, the inertial mass is a measure of the resistance of a particle to change its state of motion.

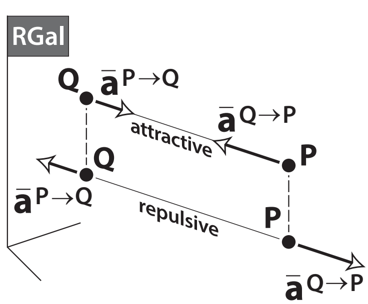

Fig. 1.3

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The **second empirical proposition** is a principle of separation, and it guarantees that the determination of the ratio between the interaction accelerations of any pair of isolated particles (hence that of the mass-ratios) is unique through two statements:

A. the acceleration-ratio associated with a pair of particles is constant and independent from the interaction phenomenon

$$\frac{\left|\bar{\mathbf{a}}_{\mathrm{RGal}}^{\mathbf{Q}\to\mathbf{P}}\right|}{\left|\bar{\mathbf{a}}_{\mathrm{RGal}}^{\mathbf{P}\to\mathbf{Q}}\right|} = \text{constant} \Leftrightarrow \frac{\mathrm{m}_{\mathbf{Q}}}{\mathrm{m}_{\mathbf{P}}} = \text{constant}, \tag{1.4}$$

B. the acceleration-ratio is separable, that is, it can be expressed as the product of two independent acceleration-ratios.

This last idea is illustrated in Fig. 1.4 and can be formulated mathematically as:

$$\frac{\left|\bar{\mathbf{a}}_{\mathrm{RGal}}^{\mathbf{Q}\to\mathbf{P}}\right|}{\left|\bar{\mathbf{a}}_{\mathrm{RGal}}^{\mathbf{P}\to\mathbf{Q}}\right|} = \frac{\left|\bar{\mathbf{a}}_{\mathrm{RGal}}^{\mathbf{Q}\to\mathbf{S}}\right|}{\left|\bar{\mathbf{a}}_{\mathrm{RGal}}^{\mathbf{S}\to\mathbf{Q}}\right|} \cdot \frac{\left|\bar{\mathbf{a}}_{\mathrm{RGal}}^{\mathbf{S}\to\mathbf{P}}\right|}{\left|\bar{\mathbf{a}}_{\mathrm{RGal}}^{\mathbf{P}\to\mathbf{S}}\right|} \Leftrightarrow \frac{\mathrm{m}_{\mathbf{Q}}}{\mathrm{m}_{\mathbf{P}}} = \frac{\mathrm{m}_{\mathbf{Q}}}{\mathrm{m}_{\mathbf{S}}} \cdot \frac{\mathrm{m}_{\mathbf{S}}}{\mathrm{m}_{\mathbf{P}}}. \tag{1.5}$$

The **third empirical proposition** states that the accelerations that any number of particles (**Q**, **S**, **T**...) induce in a particle **P** are independent from each other. It is actually a principle of superposition, it is illustrated in Fig. 1.5 and can be formulated mathematically as:

$$\bar{\mathbf{a}}_{\mathrm{RGal}}^{(\mathbf{Q},\mathbf{S})\to\mathbf{P}} = \bar{\mathbf{a}}_{\mathrm{RGal}}^{\mathbf{Q}\to\mathbf{P}} + \bar{\mathbf{a}}_{\mathrm{RGal}}^{\mathbf{S}\to\mathbf{P}}. \tag{1.6}$$

The combination between the principle of separation and the principle of superposition for the interaction accelerations yields a principle of superposition for the mass of particles. If **QP** is the particle obtained when **P** and **Q** share the same location (two

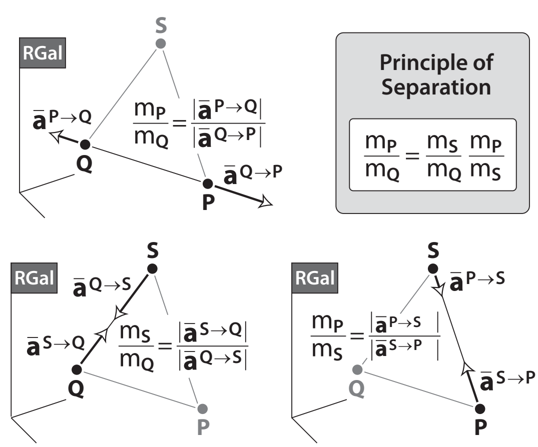

Fig. 1.4

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dimensionless particles become one single particle in that case), its mass is the addition of those of **P** and **Q**: $m_{\mathbf{PQ}} = m_{\mathbf{P}} + m_{\mathbf{Q}}$.

Finally, Mach introduces the concept of interaction force through a definition:

$$\overline{F}_{\mathbf{Q}\to\mathbf{P}} \equiv m_{\mathbf{P}}\overline{\mathbf{a}}_{\mathrm{RGal}}^{\mathbf{Q}\to\mathbf{P}}. \tag{1.7}$$

The definition of interaction force is extremely useful: It describes the interaction between two particles through just one single magnitude $\left(\text{as } \left|\overline{F}_{\mathbf{Q}\to\mathbf{P}}\right| = \left|\overline{F}_{\mathbf{P}\to\mathbf{Q}}\right|\right)$. This is not possible when we use accelerations as main magnitude to describe that interaction: as the interacting particles acquire different accelerations $\left(\left|\overline{\mathbf{a}}_{\mathrm{RGal}}^{\mathbf{Q}\to\mathbf{P}}\right| \neq \left|\overline{\mathbf{a}}_{\mathrm{RGal}}^{\mathbf{P}\to\mathbf{Q}}\right|\right)$, we require two related magnitudes for a complete description (Fig. 1.6).

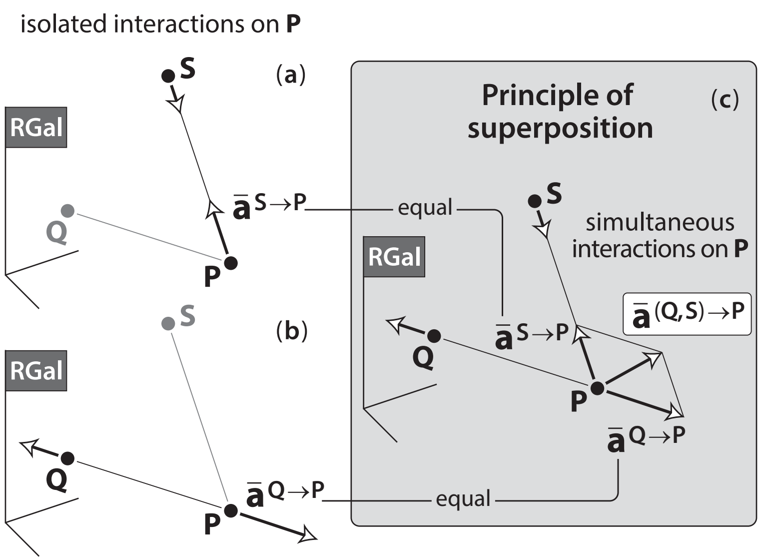

Fig. 1.5

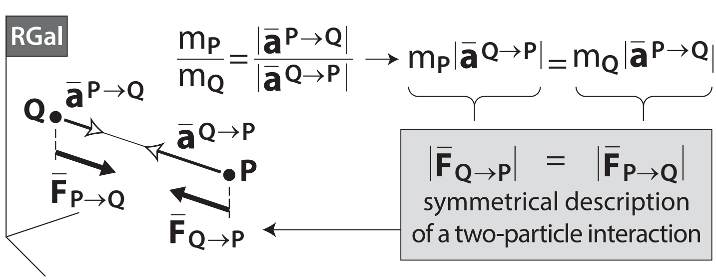

Fig. 1.6

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1.5 Closing the Formulation of Dynamics: From Mach's Axiomatics to Newton's Laws, and Principle of Determinacy

Newton's laws of motion can be derived from Mach's propositions and definitions.

Newton's Third Law

The first empirical proposition (Eq. (1.2)) and the definition of force (Eq. (1.7)) lead to:

$$\overline{\mathrm{F}}_{\mathbf{P}\to\mathbf{Q}} \equiv \mathrm{m}_{\mathbf{Q}}\overline{\mathbf{a}}_{\mathrm{RGal}}^{\mathbf{P}\to\mathbf{Q}} = -\overline{\mathrm{F}}_{\mathbf{Q}\to\mathbf{P}}, \quad \text{with } \overline{\mathrm{F}}_{\mathbf{Q}\to\mathbf{P}} \times \overline{\mathbf{QP}} = \overline{0}, \tag{1.8}$$

which is ***Newton's third law*** (***principle of action and reaction***). Note that the interaction forces between **P** and **Q** fulfill:

$$\overline{\mathrm{F}}_{\mathbf{P}\to\mathbf{Q}} + \overline{\mathrm{F}}_{\mathbf{Q}\to\mathbf{P}} = \overline{0}, \quad \overline{\mathbf{OQ}} \times \overline{\mathrm{F}}_{\mathbf{P}\to\mathbf{Q}} + \overline{\mathbf{OP}} \times \overline{\mathrm{F}}_{\mathbf{Q}\to\mathbf{P}} = \overline{0}, \tag{1.9}$$

where **O** is any point. The cross product $\overline{\mathbf{OP}} \times \overline{\mathrm{F}}_{\mathbf{Q}\to\mathbf{P}}$ defines the **moment** (or **torque**) $\overline{\mathrm{M}}(\mathbf{O})$ of force $\overline{\mathrm{F}}_{\mathbf{Q}\to\mathbf{P}}$ about point **O**. Equations (1.9) show that the resultant force and the resultant moment (about any point **O**) of an action–reaction pair of forces are always a zero (Fig. 1.7).

Newton's third law provides powerful information about the interaction force between two particles, and it is considered as the main law by many authors precisely because it introduces the symmetry in the description of the interaction mentioned earlier (description of the interaction through one single magnitude).

Newton's Second Law

The definition of force (Eq. (1.7)) together with Mach's third empirical proposition (superposition principle, Eq. (1.6)) lead to:

$$\sum_{\mathbf{Q}} \overline{\mathbf{F}}_{\mathbf{Q}\to\mathbf{P}} = \mathrm{m}_{\mathbf{P}}\overline{\mathrm{a}}_{\mathrm{RGal}}(\mathbf{P}), \tag{1.10}$$

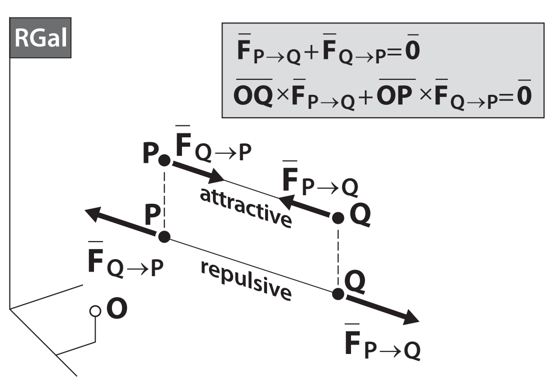

Fig. 1.7

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which is ***Newton's second law*** (***fundamental law of dynamics***).[6] This formulation contains a principle of superposition for the interaction forces.

Principle of Determinacy

Whether we adopt Newton's or Mach's axiomatics, the principle of determinacy has to be added to the other laws to complete the formulation of dynamics. It is actually known as the ***Newtonian principle of determinacy***, though he did not postulate it as a law but commented on it as an observation. A possible formulation is: "In an isolated system (undergoing no other interactions but those between the system's particles), the particles' accelerations at each time instant depend exclusively on the system's mechanical state (position and velocity of every particle) at that same time instant."[7] In other words, the initial mechanical state determines univocally its future evolution. Hence, this principle states that isolated mechanical systems have no memory.

This principle may seem incompatible with everyday experiences, such as driving a vehicle, because it is without discussion that its future evolution depends on the driver and not on the present mechanical state. However, driving a vehicle (or controlling a system, in general) consists basically of varying the interactions – internal and external – acting on the system.[8] It may also seem incompatible with the existence of "memory" materials capable of recovering past shapes, but we must keep in mind that those are very complex systems – with interactions between deformable bodies – whose mechanical description is usually over-simplified.

1.6 Usual Galilean Reference Frames

We have defined the Galilean reference frames in three different ways:

- those where space is homogeneous and isotropic, and time is uniform;
- those where the law of inertia is fulfilled;
- those where Newton's second law is fulfilled.

Checking the first two conditions is impossible, but the third condition provides a practical check: when studying a particular problem, a reference frame can be considered to be Galilean if Newton's second law is fulfilled within the degree of accuracy sought for in that problem (that is, if the measured acceleration can be justified exclusively from interaction forces within that degree of accuracy). Hence, accepting whether a reference frame is Galilean is directly associated with the degree of accuracy

[6] This formulation corresponds actually to what appear as Law II (dealing with the interaction of two isolated particles) and Corollary I (a principle of superposition) in Newton's *Principia Mathematica*. Here they have been merged into one single equation.

[7] This formulation is not a literal translation of Newton's observation but it is totally equivalent.

[8] As we will see in Chapter 4, coupling the engine or applying the brakes to the wheels of a vehicle provokes changes in the wheel–ground interaction forces.

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of the calculations. The following reference frames are taken as Galilean ones in different fields of application.

Terrestrial reference frame (TRF): In short, it is the Earth reference frame. It can be considered as a Galilean one for the usual short-range applications of mechanical engineering: machines, ground vehicles. . . However, astronautical applications (such as the motion of artificial Earth satellites) and other medium-range phenomena reveal its non-Galilean characteristics: The rotation motion of cyclones and anticyclones, the rotation of the plane of oscillation of Foucault's pendulum, the eastward deviation of projectiles launched to the north/south from the north/south hemisphere, the eastward deviation of objects falling to the ground. . . Appendix 1B quantifies the non-Galilean characteristics of the TRF.

Conventional celestial reference frame (CCRF): It contains the Earth's center of mass and does not rotate with respect to the distant galaxies (or fixed stars). Relative to the CCRF, the Earth rotates about its axis with a constant angular velocity of one turn per sidereal day, which is the time needed by the Earth to complete one turn relative to the distant galaxies (Appendix 1B).

The CCRF is an acceptable Galilean reference frame in a first approximation for the astronautics of artificial Earth satellites and for the medium-range phenomena listed in the TRF comments, but not when studying tides and astronautics of flights to the Moon and the planets.

International celestial reference frame (ICRF): It is an astronomical reference frame. It contains the solar system barycenter and does not rotate with respect to the distant galaxies. It is the usual Galilean reference frame in astronomy and astronautics. With respect to the ICRF, the Earth rotates about its axis with a constant angular velocity of one turn per sidereal day, and its center of mass describes an elliptic orbit (slightly disturbed by the gravitational attraction of the Moon and the planets) with the Sun as one of its foci.

The International Astronomical Union (IAU) is in charge of the redefinition of the **standard astronomical reference frame** (SARF) according to modern high-precision measurements. As far as Newtonian mechanics is concerned, there is no need for a more precise definition: The accuracy of the results obtained under the hypothesis of the ICRF as a Galilean reference frame is comparable to that of the Newtonian approach.

1.7 Interaction Forces between Particles: Kinematic Dependency

The fundamental equation of dynamics (Newton's second law, Eq. (1.10) may be used to predict the acceleration of a particle **P** from the interaction forces $\overline{\mathbf{F}}_{\mathbf{Q}\rightarrow\mathbf{P}}$ exerted on it (**direct dynamics**), or to find out what forces would be required to obtain a prescribed motion for **P** (**inverse dynamics**). In the context of engineering, inverse problems in dynamics are associated with the use of actuators in mechanical systems. The forces that actuators may introduce between their endpoints to guarantee a given motion may be an unknown of the problem.

Among the other interaction forces, many of them may be formulated: their value can be obtained through a mathematical equation (formula) known before solving the problem. As the unknown in the fundamental equation is the acceleration, those forces may depend on the position and velocity of the interacting particles. However, that dependency has to be consistent with all principles and laws of Newtonian dynamics.

Any formulation we provide will have to include parameters: from the dimensional point of view, positions [length] and velocity [length/time] cannot yield the dimensions of the right-hand side of Eq. (1.10). The *uniformity* of time in Galilean reference frames restricts those parameters to constant ones. In other words: that principle forbids any time-dependent formulation of interaction forces.

Restrictions on the Position Dependency (Fig. 1.8)

- Because of the *homogeneity* of space in Galilean reference frames, the dependence of $\overline{\mathbf{F}}_{\mathbf{Q}\to\mathbf{P}}$ on the positions of **P** and **Q** cannot be through all the information contained in their position vectors $\overline{\mathbf{O}_{\mathrm{RGal}}\mathbf{P}}$ and $\overline{\mathbf{O}_{\mathrm{RGal}}\mathbf{Q}}$, but just through their difference $\overline{\mathbf{PQ}}$.
- *Isotropy* does not allow a dependency on a particular direction. Thus, only the distance $\rho = \left|\overline{\mathbf{PQ}}\right|$ is acceptable in the formulation of $\overline{\mathbf{F}}_{\mathbf{Q}\to\mathbf{P}}$.
- Because of the ***principle of determinacy***, the value of $\overline{\mathbf{F}}_{\mathbf{Q}\to\mathbf{P}}$ at any time instant may only depend on that of ρ at that same time instant.

Restrictions on the Velocity Dependency (Fig. 1.9)

- Because of ***Galileo's principle of relativity***, the dependence of $\overline{\mathbf{F}}_{\mathbf{Q}\to\mathbf{P}}$ on the velocities of **P** and **Q** cannot be through all the information contained in the vectors $\overline{\mathbf{v}}_{\mathrm{RGal}}(\mathbf{P})$ and $\overline{\mathbf{v}}_{\mathrm{RGal}}(\mathbf{Q})$ separately, but just through their difference:

$$\Delta\overline{\mathbf{v}}_{\mathrm{RGal}} = \overline{\mathbf{v}}_{\mathrm{RGal}}(\mathbf{P}) - \overline{\mathbf{v}}_{\mathrm{RGal}}(\mathbf{Q}) = \left.\frac{\mathrm{d}\overline{\mathbf{QP}}}{\mathrm{dt}}\right]_{\mathrm{RGal}}.$$

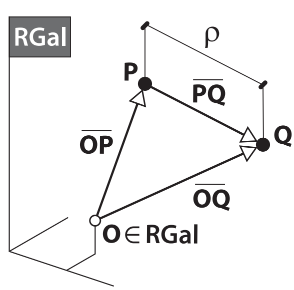

Fig. 1.8

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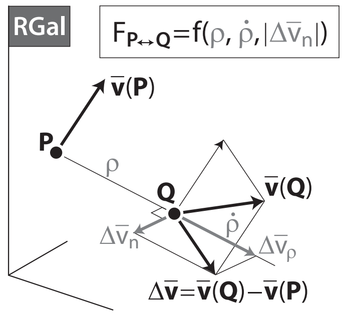

Fig. 1.9

- In general, that difference will have a component parallel to the line $\overline{\mathbf{PQ}}$ and another one perpendicular to that line: $\Delta\overline{\mathbf{v}} = \Delta\overline{\mathbf{v}}_{\|\overline{\mathbf{QP}}} + \Delta\overline{\mathbf{v}}_{\perp\overline{\mathbf{QP}}} \equiv \Delta\overline{\mathbf{v}}_\rho + \Delta\overline{\mathbf{v}}_\mathrm{n}$.
- *Isotropy* does not allow a dependency on a particular direction not associated to the existence of the two interacting particles. Isotropy restricts the type of dependence of $\overline{\mathbf{F}}_{\mathbf{Q}\to\mathbf{P}}$ on $\Delta\overline{\mathbf{v}}$. The interaction force may depend on the value of $\Delta\overline{\mathbf{v}}_\rho$ (that is, on $\dot{\rho}$, where $\dot{\rho} > 0$ and $\dot{\rho} < 0$ describe the separating and the approaching velocity of particles **P** and **Q**, respectively), and on just the magnitude of $|\Delta\overline{\mathbf{v}}_\mathrm{n}|$ (though usual formulations do not depend on that component).
- Because of the ***principle of determinacy***, the value of $\overline{\mathbf{F}}_{\mathbf{Q}\to\mathbf{P}}$ at any time instant may only depend on those of $\dot{\rho}$ and $|\Delta\overline{\mathbf{v}}_\mathrm{n}|$ at that same time instant.

Summarizing: $\mathrm{F}_{\mathbf{Q}\leftrightarrow\mathbf{P}} = \mathrm{f}(\rho, \dot{\rho}, |\Delta\overline{\mathbf{v}}_\mathrm{n}|)$. Note that the values $(\rho, \dot{\rho})$ are the same in all reference frames (whether Galilean or not), whereas $|\Delta\overline{\mathbf{v}}_\mathrm{n}|$ is the same only in Galilean reference frames but may change when moving to a non-Galilean one. Hence, the formulation of any interaction depending exclusively on ρ and $\dot{\rho}$ will be shared by all reference frames.

The usual interactions in mechanical systems do not depend on $|\Delta\overline{\mathbf{v}}_\mathrm{n}|$. Therefore, only formulations depending on ρ and $\dot{\rho}$ will be considered from now on.

1.8 Contact Forces between Particles and Extended Bodies

So far, we have been considering action–reaction forces between pairs of particles **P** and **Q**. However, the problems found in mechanical engineering do not deal with dimensionless particles but with extended bodies. The action–reaction forces have to be understood as interactions between particles belonging to different extended bodies, and this allows us to consider contact interactions. Those interactions are strictly impossible between a pair of dimensionless particles **P** and **Q**: if in contact, they would become one single particle.

Contact forces come from the local deformations of the objects in the vicinity of the contact points. Those deformations are accurately formulated in solid mechanics, but they cannot be dealt with when objects are modeled as rigid bodies: rigid bodies do not deform, they are impenetrable.

The modeling of those forces is not possible: consistency with the previous section leads to the conclusion that they cannot be formulated, and their value becomes an unknown of the dynamical problem.

Among the contact interactions, there is a subset whose value adapts in order to guarantee a kinematic restriction. They are the so-called **constraint forces**, and their value at every time instant is calculated through Newton's second law. The number of independent components (in the three directions of space), however, can be predicted beforehand from the kinematic restriction they are associated with. The description of those components is the **characterization** of the constraint force.

Constraint action–reaction pairs can also appear between particles at a finite distance ($\rho \neq 0$), provided there is an intermediate element with negligible mass between them guaranteeing a restriction on $\dot{\rho}$.

1.9 Formulation of Interaction Forces

The gravitational attraction was the first interaction to be formulated. Though not very intense, it is an infinite-range interaction, and it is the only one that matters when studying the motion of celestial objects (such as planets and stars).

Newton formulated the ***law of universal gravitation*** at the same time he proposed his three laws. The four laws were based on an important amount of empirical quantitative data (collected mainly by Tycho Brahe) on the motion of the planets and their synthetic kinematical description relative to a heliocentric reference frame given by Kepler's laws. In short: The planets' acceleration pointed toward the Sun and was inversely proportional to the distance squared between planet and Sun.

Gravitation has a particular status among the interactions considered in engineering mechanics. On the one hand, it is one of the so-called fundamental interactions in physics: It is not the result of any underlying phenomena which combine and yield that interaction; in other words, its formulation is not *phenomenological*.

On the other hand, it is the only interaction "at a distance" (that is, between particles separated in space without any intermediate element connecting them). All other interactions between particles that will be formulated in this section call for intermediate elements: elements connected to those particles and whose mass is significatively smaller than that of the particles. In a first approach, then, the mass of those elements can be neglected, and the force exerted by one particle at the endpoint of the element is equal, though with opposite sign, to that exerted by the other particle at the other endpoint. In other words, those two forces fulfill Newton's

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third law, and can be understood as an action–reaction pair of forces between the particles (Fig. 1.10).

The usual intermediate elements found in mechanical systems are springs, dampers, and linear actuators. Interactions between particles through those elements are the result of underlying phenomena whose overall effect on the particles is formulated at a phenomenological level through empirical models.

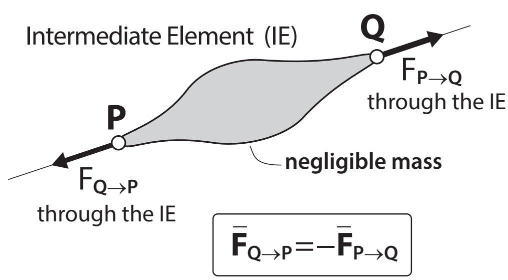

Fig 1.10

Gravitational Force

Newton's law of universal gravitation states that two particles **P** and **Q** attract each other with a force directly proportional to the product of their masses and inversely proportional to $|\overline{\mathbf{PQ}}|^2$ (Fig. 1.11):

$$\left|\overline{\mathbf{F}}^{\text{grav}}_{\mathbf{P}\leftrightarrow\mathbf{Q}}\right| = G_0 \frac{m_\mathbf{P} m_\mathbf{Q}}{|\overline{\mathbf{PQ}}|^2} = G_0 \frac{m_\mathbf{P} m_\mathbf{Q}}{\rho^2}. \tag{1.11}$$

Strictly speaking, $m_\mathbf{P}$ and $m_\mathbf{Q}$ in Eq. (1.11) are the ***gravitational masses*** of the particles. They are conceptually different from the inertial masses, but there is no empirical evidence so far that they should differ, so they will be treated as one same thing. The constant parameter G_0 in Eq. (1.11) is the ***universal constant of gravitation***

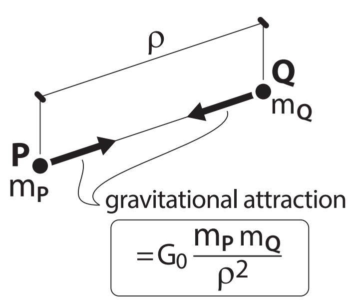

Fig. 1.11

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$G_0 = 6.67 \cdot 10^{-11} m^3/(kg \cdot s^2)$ in the International System of Units, SI), and is independent from the particles (that is why it is a *universal* constant).[9]

The vector $\overline{\mathbf{F}}^{\text{grav}}_{\mathbf{P}\to\mathbf{Q}}/m_{\mathbf{Q}}$ is called the **gravitational field** $\overline{g}_{\mathbf{P}\to\mathbf{Q}}$ created by **P** at **Q** location. Appendix 1.A presents the formulation of the Earth's gravitational field.

The universal law of gravitation has been widely discussed because of some very particular features. On the one hand, and as a result of the identification of inertial mass and gravitational mass, it predicts a same acceleration (relative to a Galilean reference frame) for any particle under the same gravitational field. On the other hand, Eq. (1.11) describes an *instantaneous* interaction. The direction and intensity of the gravitational attraction is adjusted instantaneously whatever the distance between the interacting particles might be: the gravitational field propagates with an infinite speed![10]

Interaction Force through Springs

The term **spring** is used to designate any element with negligible mass responsible for an interaction force depending exclusively on the distance between particles (called "spring length" from now on): $\left|\overline{\mathbf{F}}^{\text{spring}}_{\mathbf{P}\leftrightarrow\mathbf{Q}}\right| = f(\rho)$. Springs may introduce either an attraction or a repulsion between their endpoints.

The most usual case is the linear spring, where the force change associated with a change of spring length is proportional to that length change through a constant parameter called the **spring constant k**:

$$k \equiv \frac{\Delta \text{ attraction (repulsion) force between the spring ends}}{\text{increase (decrease) of spring length}}. \tag{1.12}$$

Springs constants are always positive. However, the term *spring* may be used in a more general sense for any physical phenomenon whose net force is a function of ρ (in this context, the gravitational force can be seen as a *spring*). In that case, we will talk of *equivalent spring*, and the constant may be negative.

A *real* spring (not an *equivalent* one) has a nonzero **slack length** ρ_{slack} (length for which the force between the spring endpoints is zero). From that state, the spring will generate an attraction force if the length is increased ($\Delta\rho = \rho - \rho_{\text{slack}} > 0$) and a repulsion force if it is decreased ($\Delta\rho < 0$).

The force associated with a spring within a mechanical system may evolve from attraction to repulsion with time. When solving the system dynamics, the spring force is drawn either as an attraction or a repulsion, and its value is formulated accordingly (Fig. 1.12).

If the spring force is formulated from a nonzero-tension state (a state where its length ρ_0 is higher or lower than ρ_{slack}, and the spring force is F_0), the choice of its description as an attraction or a repulsion is made according to the nature of the initial force F_0:

[9] The origin and the "constancy" of the fundamental constants are subjects still under discussion nowadays. Some scientists relate them to our limitations when it comes to measuring the external objective world, some others claim they might be slowly changing. . .

[10] This drawback is solved within Einstein's *Theory of General Relativity*.

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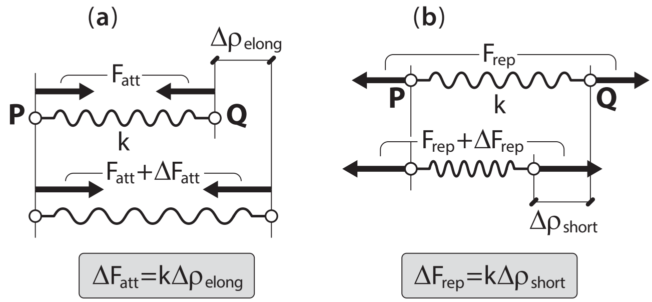

Fig. 1.12

$$\begin{aligned} F_0 &= \text{attraction} \Rightarrow F_{att}^{spring} = F_0 + k\Delta\rho_{elong} = F_0 - k\Delta\rho_{short}, \\ F_0 &= \text{repulsion} \Rightarrow F_{rep}^{spring} = F_0 + k\Delta\rho_{short} = F_0 - k\Delta\rho_{elong}. \end{aligned} \tag{1.13}$$

In the context of a particular problem, the spring length has to be expressed as a function of the generalized coordinates $(q_1, q_2, \ldots, q_n)$ describing the system configuration.

► **Example 1.2** The linear spring connecting the endpoints of two bars articulated at point **O** has a constant k and exerts an attraction force F_0 between **P** and **Q** when $\theta = \theta_0$ (Fig. 1.13). For any θ value, then, the spring force is formulated as an attraction force:

$$F_{att}^{spring} = F_0 + k\Delta\rho_{elong} = F_0 + 2Lk\left(\sin\frac{\theta}{2} - \sin\frac{\theta_0}{2}\right).$$

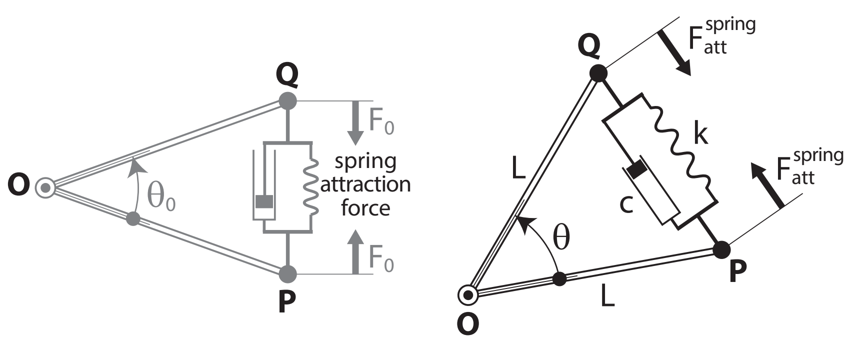

Fig. 1.13

◄

► **Example 1.3** The hydraulic cylinder prescribes a motion x(t), and the angular coordinate θ evolves dynamically. When $\theta = \theta_0$ and $x = x_0$, the spring is compressed and exerts a force F_0 between its endpoints (Fig. 1.14).

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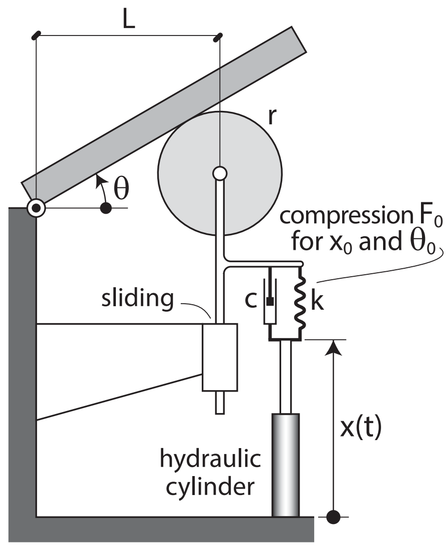

Fig. 1.14

As the initial force F_0 is a repulsion, the spring force for any configuration will be formulated as a repulsion force:

$$F_{rep}^{spring} = F_0 + k\Delta\rho_{short} = F_0 + k\left[(x - x_0) + \left(L\tan\theta_0 - \frac{r}{\cos\theta_0}\right) - \left(L\tan\theta - \frac{r}{\cos\theta}\right)\right]$$
$$= F_0 + k\left(x - x_0 + \frac{L\sin\theta_0 - r}{\cos\theta_0} - \frac{L\sin\theta - r}{\cos\theta}\right) .$$

◀

In a **nonlinear spring**, the force change is not proportional to the spring length change (Fig. 1.15).

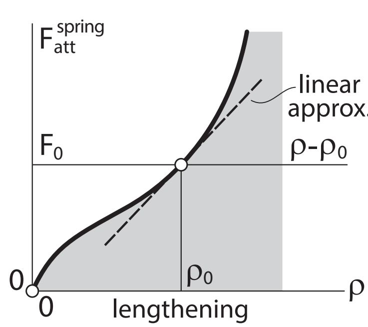

Fig. 1.15

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For small length changes around a given length ρ_0, the spring force may be approximated through a Taylor series up to the linear term:

$$F(\rho) \approx F(\rho_0) + \left.\frac{dF}{d\rho}\right]_{\rho_0} (\rho - \rho_0). \tag{1.14}$$

The derivative $dF/d\rho]_{\rho_0}$ can be understood as the constant of an equivalent linear spring for lengths close to ρ_0.

Interaction Force through Dampers

Dampers with negligible mass exert a force opposite to the approaching or separating velocity between their endpoints. The force value depends on that velocity, and sometimes also on the distance ρ between those points: $\left|\overline{\mathbf{F}}^{\text{damper}}_{\mathbf{P}\leftrightarrow\mathbf{Q}}\right| = f(\rho, \dot{\rho})$.

The most usual case is the **linear damper** (or **viscous friction damper**). In those dampers, the force is strictly proportional to $\dot{\rho}$ through the **damper constant**, c. As in springs, the damper force may be an attraction or a repulsion. When the damper is assembled in parallel with a spring (spring–damper system), the criterion to draw and formulate that force is the same as that chosen for the spring:

$$\begin{aligned} F^{\text{damp}}_{\text{att}} &= c \cdot v_{\text{separation}} \equiv c\dot{\rho} = -c \cdot v_{\text{approaching}}, \\ F^{\text{damp}}_{\text{rep}} &= c \cdot v_{\text{approaching}} \equiv -c\dot{\rho} = -c \cdot v_{\text{separation}}. \end{aligned} \tag{1.15}$$

► **Example 1.4** The attraction force exerted by the damper in Example 1.2 is

$$F^{\text{damp}}_{\text{att}} = c \cdot v_{\text{separation}} = c\dot{\rho} = c\frac{d}{dt}\left(2L\sin\frac{\theta}{2}\right) = cL\dot{\theta}\cos\frac{\theta}{2}.$$ ◄

► **Example 1.5** The repulsion force exerted by the damper in Example 1.3 is

$$F^{\text{damp}}_{\text{rep}} = -c \cdot v_{\text{separation}} = -c\dot{\rho} = -c\frac{d}{dt}\left(\frac{L\sin\theta - r}{\cos\theta} - x\right) = -c\frac{L - r\sin\theta}{\cos^2\theta}\dot{\theta} + c\dot{x}.$$

Actually, the distance between the damper endpoints is $\left(\frac{L\sin\theta - r}{\cos\theta} - x\right)$ plus a constant value, but that constant can be omitted because its time derivative is zero. ◄

Interaction Force through Actuators

An actuator (or driver) is an element able to produce a prescribed motion or a prescribed interaction according to a predefined time law. Its functioning requires a control system and an energy source.

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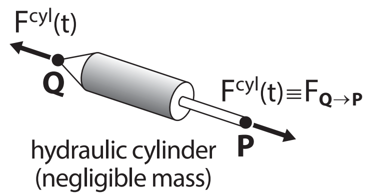

Fig. 1.16

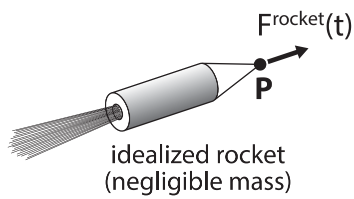

Fig.1.17

The most usual actuator between pairs of particles **P** and **Q** is the hydraulic cylinder. As with springs and dampers, the effect it has on **P** and **Q** is described as an action–reaction pair of forces when its mass is negligible (Fig. 1.16). The force $F_{cyl}(t)$ it introduces can either be a prescribed function or an unknown whose evolution has to be adjusted in order to guarantee a prescribed motion $\rho(t)$.[11]

A rocket with negligible mass can be considered as an idealized actuator. It is an interesting case as it seems to be using just one force of the action–reaction pair to control the motion of a particle **P** (Fig. 1.17). This is not so: the reaction force is applied to the high-speed gas particles ejected by the rocket in a direction opposite to $\overline{\mathbf{F}}^{rocket}(t)$.

1.10 Characterization and Limit Conditions of Constraint Forces: Friction Forces

The forces associated with the contact of a particle **P** and an extended rigid body are the consequence of two main features of rigid objects: impenetrability and roughness (that generates friction). They both may restrict the possible motion of **P** relative to the rigid body.

[11] Note that the formulation F(t) is not consistent with Newtonian axiomatics. Such contradiction is an oversimplification: The formulation of dynamics of the inner components of actuators is out of the scope of Newtonian mechanics. However, as these elements are found in most mechanical engineering problems, not including them in this textbook would be a strong limitation.

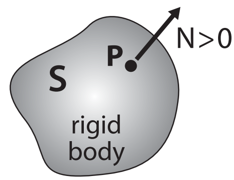

Fig. 1.18

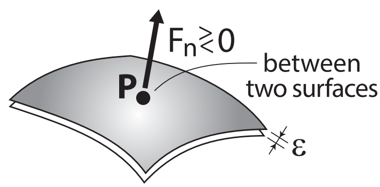

Fig. 1.19

Particle–Surface Interaction: Force Associated with Impenetrability

The impenetrability of a rigid body S forbids the normal approaching velocity of a particle **P** relative to it when they are in contact: $\overline{\mathbf{v}}_S(\mathbf{P})]_{n\ app} = 0$. It is a **unilateral** restriction, as the separating normal motion is not restricted.

The kinematic constraint $\overline{\mathbf{v}}_S(\mathbf{P})]_{n\ app} = 0$ translates into a repulsion normal force N on **P** that takes the value required to guarantee that constraint (Fig. 1.18). If that force did not exist, other interaction forces on the particle could provoke a penetration of **P** into the rigid body. As it is a unilateral constraint, the value of N has to be strictly positive.

The constraint normal force can be formulated as a function of the normal deformation of objects. When that deformation is neglected (hypothesis of rigid body), that force cannot be formulated, and it becomes an *unknown of the dynamical problem*.

In unilateral normal forces, a zero value $(N = 0)$ indicates that the contact is about to be lost. When the N value required to maintain that contact is negative $(N < 0)$, the contact is lost and a separating normal motion is actually taking place.

If the particle **P** motion is restricted by two close surfaces, we say that there is a **bilateral constraint** (Fig. 1.19). As an adimensional particle cannot be simultaneously in contact with two surfaces, we will assume an infinitesimal mutual separation $\varepsilon \to 0$ between them, and **P** will be either in contact with one surface or the other. In that case, the normal force on **P** may have either sign.

► **Example 1.6** The particle **P** is initially at rest relative to a curved ground (Fig. 1.20a). The normal force on **P** compensates the weight (otherwise **P** would penetrate the ground).

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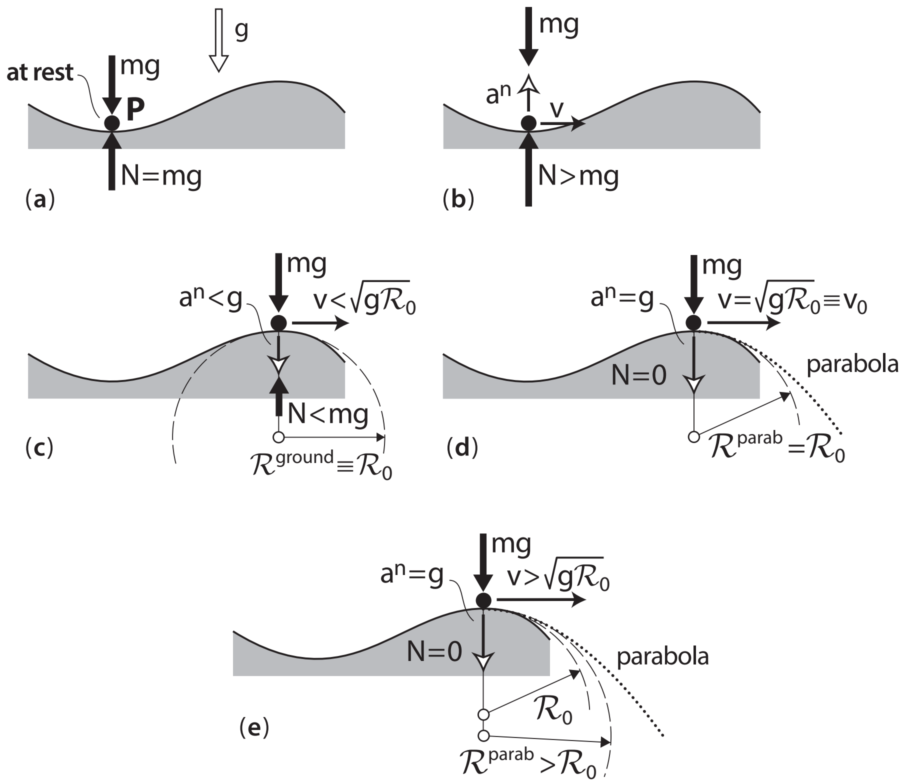

Fig. 1.20

If **P** slides on the ground with speed v (Fig. 1.20b), the value of the normal force is not the same. The normal acceleration of **P** relative to the ground is upward, thus N has to compensate the weight and provide that acceleration: $N > mg$.

When **P** reaches the highest position (Fig. 1.20c), its normal acceleration relative to the ground is downward, so $N < mg$. If the ground radius of curvature at that location is $\mathfrak{R}_0$ and the speed value is $v < \sqrt{g\mathfrak{R}_0}$, the normal acceleration is $a^n = v^2/\mathfrak{R}_0 < g$. As $mg - N = m(v^2/\mathfrak{R}_0)$ and $N > 0$, the weight compensates the normal force and provides that acceleration.

If **P** goes through that position with speed v_0 such that $v_0^2/\mathfrak{R}_0 = g$, the normal force is zero, and that indicates that the ground contact is about to be lost. The curvature radius of the **P** trajectory is not imposed by the ground curvature but by the speed value and the weight (Fig. 1.20e).

If $v > v_0$, the weight will be responsible for a parabolic trajectory with $\mathfrak{R}^{parab} > \mathfrak{R}_0$, and there will be no ground contact (Fig. 1.20e). ◀

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Particle–Surface Interaction: Force Associated with Friction

The roughness of a rigid surface S is responsible for a tangential force (or **friction force**) $\overline{\mathbf{F}}^{\text{fric}}$ on the particle **P** when they are in contact.[12] When there is sliding between **P** and S (that is, when $\overline{\mathbf{v}}_{\text{slide}}(\mathbf{P}) = \overline{\mathbf{v}}_{\text{S}}(\mathbf{P})]_{\text{tang}} \neq \overline{0}$, so there is no kinematic restriction in that direction), that force is called **dynamic** or **kinetic friction**,[13] it is opposed to the sliding velocity when the roughness is isotropic, and it may be formulated as a function of that velocity.

When there is no relative sliding, the tangential force is called **static friction**, and it is a constraint force: its value adapts in order to keep the kinematic restriction $\overline{\mathbf{v}}_{\text{S}}(\mathbf{P})]_{\text{tang}} = \overline{0}$. That force can be modeled as a function of the tangential deformation of objects. When that deformation is neglected (hypothesis of rigid body), that force cannot be modeled: it is an *unknown of the dynamical problem.*

When there is no lubricant between the particle and the surface, ***Coulomb's law of friction*** (or **dry friction** model) provides the simplest description for that phenomenon. If the roughness is isotropic:

- The static friction force $\overline{\mathbf{F}}^{\text{fric}}_{\text{st}}$ may have any direction and may take any value within the range $[0, \mu_s \text{N}]$ in order to guarantee the no-sliding restriction (Fig. 1.21a). The adimensional constant coefficient μ_s is called **static friction coefficient**. When the friction force required to maintain the kinematical constraint is $> \mu_s \text{N}$, the particle slides and $\overline{\mathbf{F}}^{\text{fric}}$ becomes a kinetic friction force.
- The kinetic friction force $\overline{\mathbf{F}}^{\text{fric}}_{\text{kin}}$ is opposed to the sliding velocity, and its value is $\left|\overline{\mathbf{F}}^{\text{fric}}_{\text{kin}}\right| = \mu_k \text{N}$ (Fig. 1.21b). The adimensional coefficient μ_k is the **kinetic friction coefficient**, and it is assumed to be constant.[14] Usually $\mu_k < \mu_s$.

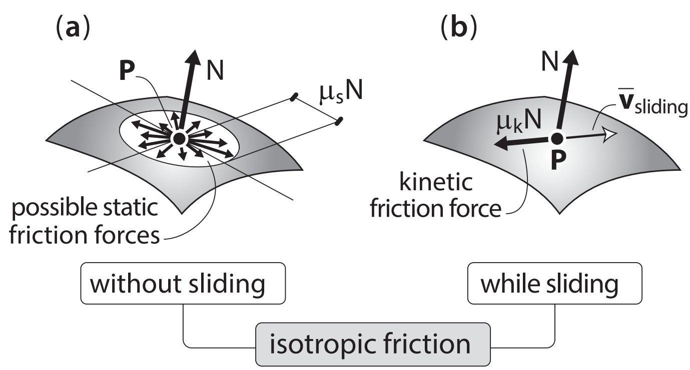

Fig. 1.21

[12] Friction is a phenomenon associated with the roughness of two rigid surfaces in contact. Hence, when dealing with friction between a rigid body S and a particle, the particle has to be pictured as a small object with a small flat surface in contact with S.

[13] Kinetic friction is also referred to as **sliding** friction.

[14] In a second approach, that coefficient depends on the sliding velocity.

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Friction in materials with anisotropic roughness may be modeled through direction-dependent friction coefficients. For the particular case of orthotropic surfaces (with different properties along two orthogonal directions), two different friction coefficients $\left(\mu_s^1, \mu_s^2\right)$ describe the phenomenon. The allowed range of values for $\left|\overline{\mathbf{F}}_{st}^{fric}\right|$ is $\left[0, \mu_s^{mean}N\right]$, where $\mu_s^{mean} = \sqrt{\mu_s^1\mu_s^2}$, and the direction of $\overline{\mathbf{F}}_{kin}^{fric}$ is not any more opposed to the sliding velocity.

When one of the two friction coefficients is much lower than the other $\left(\mu_s^1 \ll \mu_s^2\right)$, a first approximation with $\left(\mu_s^1 \approx 0, \mu_s^2 \equiv \mu_s\right)$ yields a simpler model.

When there is lubricant between the particle and the surface (so $|\overline{\mathbf{v}}_{slide}(\mathbf{P})| \neq 0$), the kinetic friction force can be formulated as a **viscous friction** proportional to the sliding velocity: $\left|\overline{\mathbf{F}}_{visc}^{fric}\right| = cv_{slide}$. The coefficient c may be a function of the normal constraint force.

Particle–Curve Interaction

The interaction between a rigid curve and a particle **P** moving along it can be visualized either as that of a small hollow sphere pierced by a thin wire (Fig. 1.22a), or a particle moving in a tube with an infinitesimal diameter $\varepsilon \to 0$ (Fig. 1.22b). In both cases, we will consider that **P** is in contact with just one side of the wire or the tube (as was assumed earlier in particles constrained by two close surfaces), hence the normal force **N** on **P** will have any direction on the plane perpendicular to the curve (Fig. 1.22c).

The friction force between curve and particle may be modeled through Coulomb's friction law (Fig. 1.22d,e).

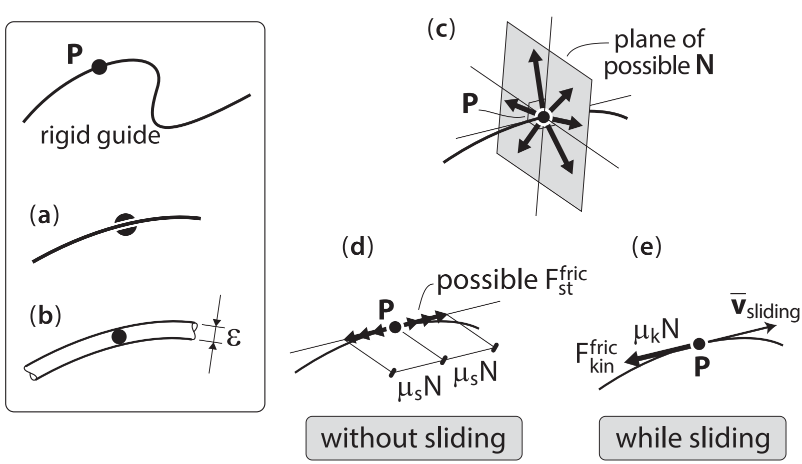

Fig. 1.22

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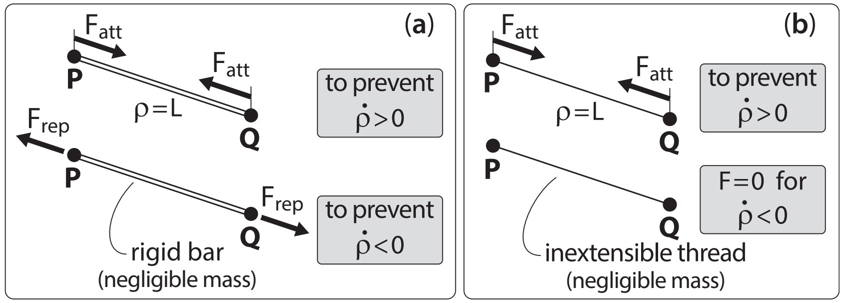

Fig. 1.23

Constraints between Particles through Intermediate Elements

Two different constraint forces have been introduced so far: the normal force and the static friction force associated with a single-point contact between a particle and a surface or a curve force. Following what has been done to introduce interactions "at a distance" between particles through springs, dampers, and actuators, we will consider now the possibility of constraints "at a distance" between particles through intermediate elements with negligible mass.

Constraints between two particles **P** and **Q** can be introduced through rigid thin bars and through inextensible threads. A rigid bar between **P** and **Q** generates a bilateral constraint preventing the particles to approach or separate: $\dot{\rho} = 0$ (Fig. 1.23a). That kinematical restriction translates into a pair of repulsion or attraction forces, respectively.

An inextensible thread under tension between **P** and **Q** generates a unilateral constraint preventing just the separation velocity between them, as the thread length cannot increase (Fig. 1.23b). Hence, a tensioned thread can only transmit attraction forces between **P** and **Q**.

Analytical Characterization of Constraint Forces

The previous description of the constraint forces of a rigid body S on a particle **P** relies on the description of the kinematical restrictions they are supposed to guarantee. Mathematically, this can be expressed as an orthogonality between the force and the particle allowed motion:

$$\overline{\mathbf{F}}_{\mathrm{S}\rightarrow\mathbf{P}} \cdot \overline{\mathbf{v}}_{\mathrm{S}}(\mathbf{P}) = 0. \tag{1.16}$$

The description of the velocity of **P** has to be done from the S reference frame. Eq. (1.16) is the equation for the *straightforward analytical characterization* of single-point constraint forces. Note that there is a one-to-one correspondence between the nonzero components in $\overline{\mathbf{F}}_{\mathrm{S}\rightarrow\mathbf{P}}$ and the zero components in $\overline{\mathbf{v}}_{\mathrm{S}}(\mathbf{P})$.

When the constraint force on **P** comes from another particle **Q** through an intermediate element, the kinematic description has to be done on any reference frame R where $\overline{\mathbf{v}}_{\mathrm{R}}(\mathbf{Q}) = \overline{0}$.

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Characterizing constraint forces and *calculating* them are two different things. The former consists on assessing the possibility of existence of those forces, and it *does not depend on the actual motion of the constraining elements* (the rigid body S or the particle **Q**). Calculating the constraint forces (finding out their values) requires the application of the laws of dynamics, and the result *does depend on the actual motion of the particle and the constraining elements.*

1.11 Dynamics in Non-Galilean Reference Frames: Inertial Forces

Non-Galilean reference frames move with an accelerated translational motion or a rotational motion relative to Galilean ones. The formulation of dynamics in those reference frames is more complicated than in Galilean ones because space and time do not fulfill some or all the three properties mentioned in Section 1.2 (homogeneity, isotropy, uniformity). Hence, besides the interaction forces between particles, that formulation includes terms (called **inertial forces**) which are location-, orientation- or time- dependent:

$$\mathrm{m}_{\mathbf{P}}\overline{\mathbf{a}}_{\mathrm{NGal}}(\mathbf{P}) = \sum_{\mathbf{Q}} \overline{\mathbf{F}}_{\mathbf{Q}\to\mathbf{P}} + \overline{\mathscr{F}}^{\,\mathrm{inertia}}_{\mathrm{NGal}\to\mathbf{P}}. \tag{1.17}$$

The acceleration $\overline{\mathbf{a}}_{\mathrm{NGal}}(\mathbf{P})$ can be obtained from $\overline{\mathbf{a}}_{\mathrm{Gal}}(\mathbf{P})$ through a composition of accelerations.[15] If the Galilean reference frame is the absolute (AB) frame, and the non-Galilean one is the relative (REL) frame:

$$\overline{\mathbf{a}}_{\mathrm{NGal}}(\mathbf{P}) = \overline{\mathbf{a}}_{\mathrm{Gal}}(\mathbf{P}) - \overline{\mathbf{a}}_{\mathrm{tr}}(\mathbf{P}) - \overline{\mathbf{a}}_{\mathrm{Cor}}(\mathbf{P}). \tag{1.18}$$

Combining Eq. (1.18) with Newton's second law (Eq. (1.10)) yields:

$$\mathrm{m}_{\mathbf{P}}\overline{\mathbf{a}}_{\mathrm{NGal}}(\mathbf{P}) = \sum_{\mathbf{Q}} \overline{\mathbf{F}}_{\mathbf{Q}\to\mathbf{P}} - \mathrm{m}_{\mathbf{P}}\overline{\mathbf{a}}_{\mathrm{tr}}(\mathbf{P}) - \mathrm{m}_{\mathbf{P}}\overline{\mathbf{a}}_{\mathrm{Cor}}(\mathbf{P}) \equiv \sum_{\mathbf{Q}} \overline{\mathbf{F}}_{\mathbf{Q}\to\mathbf{P}} + \overline{\mathscr{F}}^{\,\mathrm{tr}}_{\mathrm{NGal}\to\mathbf{P}} + \overline{\mathscr{F}}^{\,\mathrm{Cor}}_{\mathrm{NGal}\to\mathbf{P}}, \tag{1.19}$$

where $\overline{\mathscr{F}}^{\,\mathrm{tr}}_{\mathrm{NGal}\to\mathbf{P}}$ and $\overline{\mathscr{F}}^{\,\mathrm{Cor}}_{\mathrm{NGal}\to\mathbf{P}}$ are the **transportation inertial force** and the **Coriolis inertial force**, respectively.

The transportation force and the Coriolis force are proportional to the particle mass (as the gravitational force). The transportation force is location dependent, while the Coriolis force is velocity dependent. Both forces may depend on singular directions and may vary along time (different time instants are not equivalent). The lack of space homogeneity or isotropy, or of time uniformity, found in some of the reference frames presented in Example 1.1 comes from the transportation or the Coriolis forces that have to be taken into account in those reference frames.

Coriolis forces become zero in static situations (no motion relative to the non-Galilean reference frame), but transportation forces persist. For this reason, we develop

[15] Batlle, J.A. and Barjau Condomines, A. (2019) *Rigid Body Kinematics*, Cambridge University Press, chapter 2.

a certain intuitive knowledge concerning the latter. By way of an example, let's consider a vehicle accelerating (braking) on a straight road: the car passengers undergo the effect of the transportation forces (associated with the chassis reference frame) pushing them backward (forward). In cornering conditions, the passengers are pushed to the outer side of the road by the centrifugal transportation force.

That intuition may be misleading: it may seem that the transportation forces are actually associated with being at rest relative to the reference frame, and that the passenger motion tendency is actually the result of being "physically" transported by the reference frame.

In general, we are less aware of the Coriolis forces, as they add up to the transportation ones (when we are moving relative to the non-Galilean reference frame). As they are perpendicular to the our relative motion $\left(\text{such as } \overline{\mathbf{a}}_{\mathrm{Cor}}(\mathbf{P}) = 2\overline{\mathbf{\Omega}}_{\mathrm{RGal}}^{\mathrm{NGal}} \times \overline{\mathbf{v}}_{\mathrm{NGal}}(\mathbf{P})\right)$, they do not modify the relative speed.

Relationship between Inertial Forces Associated with Non-Galilean Reference Frames with a Relative Translational Motion

As for particles, the dynamics of rigid bodies may be formulated in Galilean or non-Galilean reference frames. However, once an initial choice of reference frame has been made, one may realize that the calculation of some magnitudes (such as angular momentum, as will be seen in Chapter 4) may be simpler in a frame with a translational motion (not necessarily rectilinear and uniform) relative to the initial frame. The formulation of the theorems in the new frame is not complicated, and overall that second choice may be advantageous. As that new frame does not rotate relative to the initial one, it will be described as the "**R**eference frame with a **T**ranslational motion and moving along with point **Q**": RTQ in short. Whenever that RTQ is not Galilean, the inertial forces on each point of the rigid body have to be taken into account.

If the *first reference frame is Galilean*, only transportation forces will have to be considered in the RTQ $\left(\text{as } \overline{\mathbf{\Omega}}_{\mathrm{RGal}}^{\mathrm{RTQ}} = \overline{0}\right)$. As all points in the RTQ have exactly the same motion relative to Gal, the field of inertial forces acting on a system of particles will be uniform and proportional to the particle masses (Fig. 1.24).

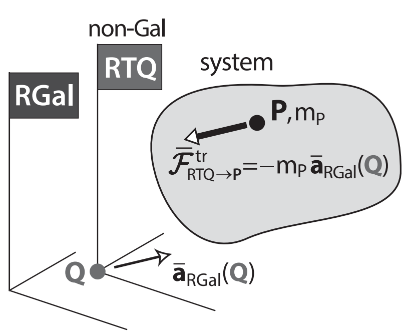

Fig. 1.24

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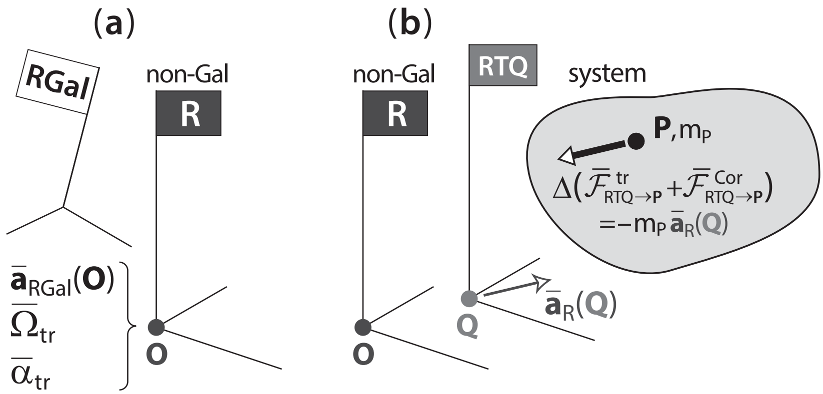

Fig. 1.25

If the *first reference frame is non-Galilean*, both the transportation and the Coriolis inertial forces $\overline{\mathscr{F}}^{\mathrm{tr}}_{\mathrm{NGal}\to\mathbf{P}}$ and $\overline{\mathscr{F}}^{\mathrm{Cor}}_{\mathrm{NGal}\to\mathbf{P}}$ have to be taken into account in general even if we do not move relative to the RTQ. They are associated with $\overline{\mathbf{a}}_{\mathrm{RGal}}(\mathbf{O}\in\mathrm{NGal}), \overline{\boldsymbol{\Omega}}_{\mathrm{tr}}\left(=\overline{\boldsymbol{\Omega}}^{\mathrm{NGal}}_{\mathrm{RGal}}\right)$ and $\overline{\boldsymbol{\alpha}}_{\mathrm{tr}}\left(=\overline{\boldsymbol{\alpha}}^{\mathrm{NGal}}_{\mathrm{RGal}}\right)$ (Fig. 1.25a).

If we move to the RTQ, the new inertial forces $\overline{\mathscr{F}}^{\mathrm{tr}}_{\mathrm{RTQ}\to\mathbf{P}}$ and $\overline{\mathscr{F}}^{\mathrm{Cor}}_{\mathrm{RTQ}\to\mathbf{P}}$ will be different. It is always possible to calculate them from $\overline{\mathbf{a}}_{\mathrm{RGal}}(\mathbf{Q}\in\mathrm{RTQ}), \overline{\boldsymbol{\Omega}}'_{\mathrm{tr}}\left(=\overline{\boldsymbol{\Omega}}^{\mathrm{RTQ}}_{\mathrm{RGal}}\right)$ and $\overline{\boldsymbol{\alpha}}'_{\mathrm{tr}}\left(=\overline{\boldsymbol{\alpha}}^{\mathrm{RTQ}}_{\mathrm{RGal}}\right)$, but they can be obtained from $\overline{\mathscr{F}}^{\mathrm{tr}}_{\mathrm{NGal}\to\mathbf{P}}$ and $\overline{\mathscr{F}}^{\mathrm{Cor}}_{\mathrm{NGal}\to\mathbf{P}}$ in a very straightforward way just by adding one term (Fig. 1.25b):

$$\left[\overline{\mathscr{F}}^{\mathrm{tr}}_{\mathrm{RTQ}\to\mathbf{P}}+\overline{\mathscr{F}}^{\mathrm{Cor}}_{\mathrm{RTQ}\to\mathbf{P}}\right]=\left[\overline{\mathscr{F}}^{\mathrm{tr}}_{\mathrm{NGal}\to\mathbf{P}}+\overline{\mathscr{F}}^{\mathrm{Cor}}_{\mathrm{NGal}\to\mathbf{P}}\right]-\mathrm{m}_{\mathbf{P}}\overline{\mathbf{a}}_{\mathrm{NGal}}(\mathbf{Q}). \tag{1.20}$$

The additional term that has to be added when moving to the RTQ corresponds to a uniform force field associated with the transportation force on **Q** that would have been required if R were Galilean.

♣ *Proof*

The inertial forces on **P** associated with the NGal and the RTQ reference frames are:

$$\overline{\mathscr{F}}^{\mathrm{tr}}_{\mathrm{NGal}\to\mathbf{P}}+\overline{\mathscr{F}}^{\mathrm{Cor}}_{\mathrm{NGal}\to\mathbf{P}}=-\mathrm{m}_{\mathbf{P}}\overline{\mathbf{a}}_{\mathrm{RGal}}(\mathbf{P}\in\mathrm{NGal})-2\mathrm{m}_{\mathbf{P}}\overline{\boldsymbol{\Omega}}^{\mathrm{NGal}}_{\mathrm{RGal}}\times\overline{\mathbf{v}}_{\mathrm{NGal}}(\mathbf{P}),$$
$$\overline{\mathscr{F}}^{\mathrm{tr}}_{\mathrm{RTQ}\to\mathbf{P}}+\overline{\mathscr{F}}^{\mathrm{Cor}}_{\mathrm{RTQ}\to\mathbf{P}}=-\mathrm{m}_{\mathbf{P}}\overline{\mathbf{a}}_{\mathrm{RGal}}(\mathbf{P}\in\mathrm{RTQ})-2\mathrm{m}_{\mathbf{P}}\overline{\boldsymbol{\Omega}}^{\mathrm{RTQ}}_{\mathrm{RGal}}\times\overline{\mathbf{v}}_{\mathrm{RTQ}}(\mathbf{P}).$$

Taking into account that the RTQ does not rotate relative to the NGal $\left(\text{so } \overline{\boldsymbol{\Omega}}^{\mathrm{RTQ}}_{\mathrm{RGal}}=\overline{\boldsymbol{\Omega}}^{\mathrm{NGal}}_{\mathrm{RGal}}\right)$, and that $\overline{\mathbf{v}}_{\mathrm{NGal}}(\mathbf{P})-\overline{\mathbf{v}}_{\mathrm{RTQ}}(\mathbf{P})=\overline{\mathbf{v}}_{\mathrm{NGal}}(\mathbf{Q})$,

$$\begin{aligned}\overline{\mathbf{a}}_{\mathrm{RGal}}(\mathbf{P}\in\mathrm{RTQ})&=\overline{\mathbf{a}}_{\mathrm{NGal}}(\mathbf{P}\in\mathrm{RTQ})+\overline{\mathbf{a}}_{\mathrm{RGal}}(\mathbf{P}\in\mathrm{NGal})+2\overline{\boldsymbol{\Omega}}^{\mathrm{NGal}}_{\mathrm{RGal}}\times\overline{\mathbf{v}}_{\mathrm{NGal}}(\mathbf{P}\in\mathrm{RTQ})\\&=\overline{\mathbf{a}}_{\mathrm{NGal}}(\mathbf{Q})+\overline{\mathbf{a}}_{\mathrm{RGal}}(\mathbf{P}\in\mathrm{NGal})+2\overline{\boldsymbol{\Omega}}^{\mathrm{NGal}}_{\mathrm{RGal}}\times\overline{\mathbf{v}}_{\mathrm{NGal}}(\mathbf{Q}),\end{aligned}$$

the difference between the inertial forces in those two reference frames becomes:

$$\left[\overline{\mathscr{F}}^{\,\mathrm{tr}}_{\mathrm{RTQ}\to\mathbf{P}} + \overline{\mathscr{F}}^{\,\mathrm{Cor}}_{\mathrm{RTQ}\to\mathbf{P}}\right] - \left[\overline{\mathscr{F}}^{\,\mathrm{tr}}_{\mathrm{NGal}\to\mathbf{P}} + \overline{\mathscr{F}}^{\,\mathrm{Cor}}_{\mathrm{NGal}\to\mathbf{P}}\right] = -\mathrm{m}_{\mathbf{P}}\overline{\mathbf{a}}_{\mathrm{NGal}}(\mathbf{Q}). \qquad \clubsuit$$

1.12 Examples

► **Example 1.7** A particle **P** with mass $\mathrm{m}_{\mathbf{P}}$ describes a circular trajectory with center **O**, radius R, and constant speed v_0 on smooth horizontal ground. **P** is attached to **O** through an inextensible thread.

The thread tension T can be calculated through Newton's second law formulated in the Galilean ground reference frame E (Fig. 1.26a). As the acceleration of **P** is centripetal with magnitude $|\overline{\mathbf{a}}_{\mathrm{E}}(\mathbf{P})| = \left(\mathrm{v}_0^2/\mathrm{r}\right)$, it requires a centripetal force with value $\mathrm{m}_{\mathbf{P}}\left(\mathrm{v}_0^2/\mathrm{r}\right)$ which is provided exclusively by the thread (as the ground is smooth).

The same result can be obtained with Newton's second law formulated in the non-Galilean reference frame R that rotates with angular velocity $\left|\overline{\mathbf{\Omega}}^{\mathrm{R}}_{\mathrm{E}}\right| = \mathrm{v}_0/\mathrm{r}$ relative to the ground (Fig. 1.26b).

The particle is permanently at rest in R, so the zero net force on **P** has to be zero:

$$\overline{\mathbf{T}} + \overline{\mathscr{F}}^{\,\mathrm{tr}}_{\mathrm{R}\to\mathbf{P}} + \overline{\mathscr{F}}^{\,\mathrm{Cor}}_{\mathrm{R}\to\mathbf{P}} = \overline{0}.$$

Here, $\overline{\mathscr{F}}^{\,\mathrm{Cor}}_{\mathrm{R}\to\mathbf{P}} = \overline{0}$ because $\overline{\mathbf{v}}_{\mathrm{R}}(\mathbf{P}) = \overline{0}$, and the transportation inertial force is $\overline{\mathscr{F}}^{\,\mathrm{tr}}_{\mathrm{R}\to\mathbf{P}} = -\mathrm{m}_{\mathbf{P}}\overline{\mathbf{\Omega}}_{\mathrm{tr}} \times \left(\overline{\mathbf{\Omega}}_{\mathrm{tr}} \times \overline{\mathbf{OP}}\right)$, centrifugal with value $\mathrm{m}_{\mathbf{P}}\Omega_{\mathrm{tr}}^2\mathrm{r} = \mathrm{m}_{\mathbf{P}}\left(\mathrm{v}_0^2/\mathrm{r}\right)$. Hence, the thread tension has to compensate $\overline{\mathscr{F}}^{\,\mathrm{tr}}_{\mathrm{R}\to\mathbf{P}}$, so it is centripetal with that same value.

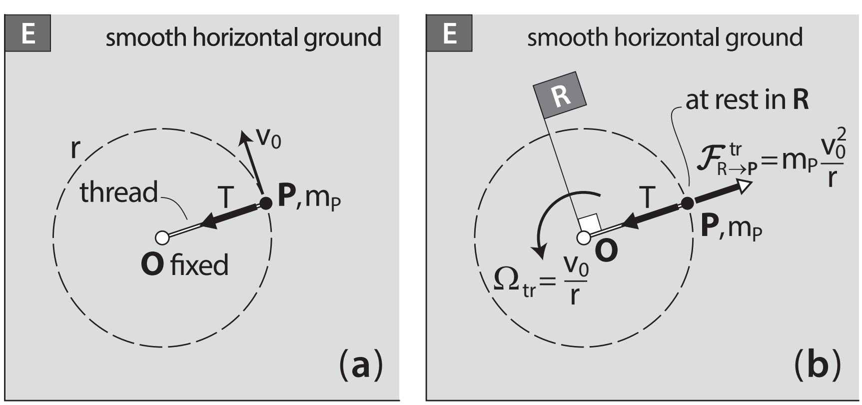

Fig. 1.26

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The comparison between the dynamical interpretations in these two reference frames is interesting:

- In the rotating frame R, there is static equilibrium, and the thread tension equilibrates the centrifugal force associated with the non-Galilean behaviour of R.
- In the ground reference frame E there is no static equilibrium, and the acceleration has to be provided by the thread tension. ◀

► **Example 1.8** A particle **P** with mass $m_{\mathbf{P}}$ slides along a smooth circular wire located on a vertical plane that rotates with constant angular velocity Ω_0 relative to the ground (Fig. 1.27a).

The equation of motion for the angular coordinate θ that gives the location of **P** on the guide and the two components of the constraint force between guide and particle can be obtained through Newton's second law formulated in the Galilean ground reference frame (E):

$$m_{\mathbf{P}}\bar{\mathbf{a}}_{E}(\mathbf{P}) = \sum \overline{F}_{\rightarrow \mathbf{P}} = m_{\mathbf{P}}\bar{\mathbf{g}} + \overline{\mathbf{F}}_{\text{guide}\rightarrow \mathbf{P}} \Rightarrow m_{\mathbf{P}}\begin{Bmatrix} R\ddot{\theta} - R\Omega_0^2 \sin\theta\cos\theta \\ R\dot{\theta}^2 + R\Omega_0^2 \sin^2\theta \\ -2R\Omega_0\dot{\theta}\cos\theta \end{Bmatrix}$$
$$= \begin{Bmatrix} -m_{\mathbf{P}}g\sin\theta \\ -m_{\mathbf{P}}g\cos\theta + F_2 \\ F_3 \end{Bmatrix}.$$

The first component is the equation of motion, and the other two yield the value of the constraint forces.

These results can be obtained through the application of Newton's second law in the guide reference frame. As it is a non-Galilean reference frame, the transportation and the Coriolis forces will have to be included (Fig. 1.27b,c):

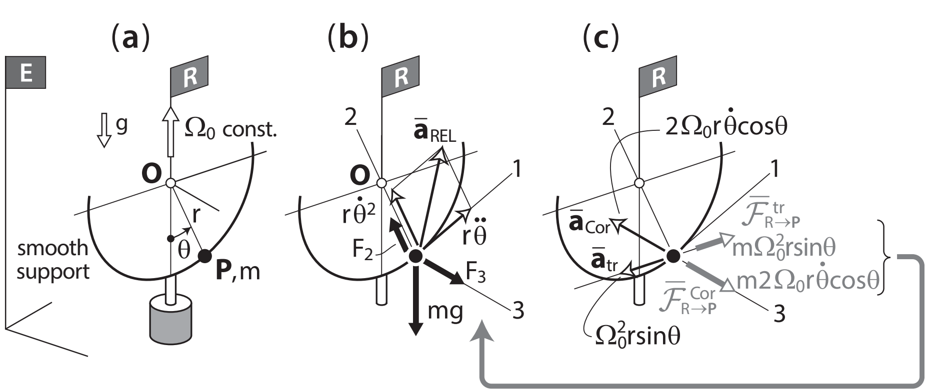

Fig. 1.27

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$$\{\bar{\mathbf{a}}_R(\mathbf{P})\} = \begin{Bmatrix} R\ddot{\theta} \\ R\dot{\theta}^2 \\ 0 \end{Bmatrix}, \quad \left\{\overline{\mathscr{F}}^{tr}_{R\to\mathbf{P}}\right\} = -m_{\mathbf{P}}\begin{Bmatrix} -R\Omega_0^2 \sin\theta\cos\theta \\ R\Omega_0^2 \sin^2\theta \\ 0 \end{Bmatrix},$$

$$\left\{\overline{\mathscr{F}}^{Cor}_{R\to\mathbf{P}}\right\} = -m_{\mathbf{P}}\begin{Bmatrix} 0 \\ 0 \\ -2R\Omega_0\dot{\theta}\cos\theta \end{Bmatrix}.$$

The equation $m_{\mathbf{P}}\bar{\mathbf{a}}_R(\mathbf{P}) = \sum \overline{F}_{\to\mathbf{P}} + \overline{\mathscr{F}}^{tr}_{R\to\mathbf{P}} + \overline{\mathscr{F}}^{Cor}_{R\to\mathbf{P}}$ yields the same results. ◄

► **Example 1.9** A particle **P** with mass $m_{\mathbf{P}}$ is initially at rest on a horizontal platform (*vibrating feeder*) which moves under the action of a hydraulic cylinder. That actuator prescribes the periodic speed $\dot{y}(t)$ relative to the ground E (Fig. 1.28). The friction coefficient between platform and particle is $\mu = 0.2$. Initially the particle is at rest relative to the platform.

The interaction forces on **P** are the weight and those associated with the platform contact: a normal constraint force and a horizontal friction force. If **P** does not slide on the platform, the horizontal force is a static friction whose value is within the range $[0, \mu_s N]$; if it slides, it is a kinetic friction with value $\mu_k N$. In the present case, $\mu_s = \mu_k = \mu = 0.2$.

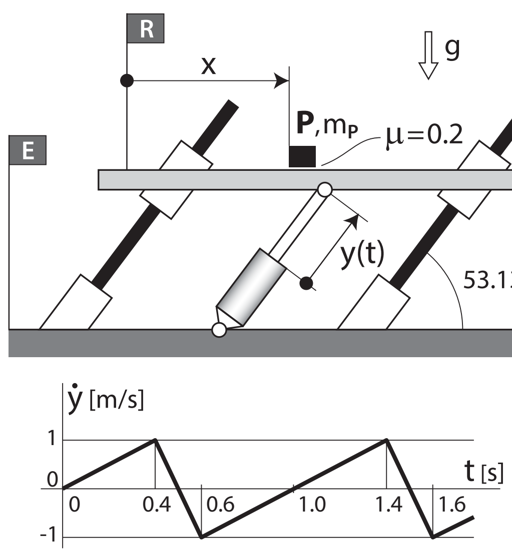

Fig. 1.28

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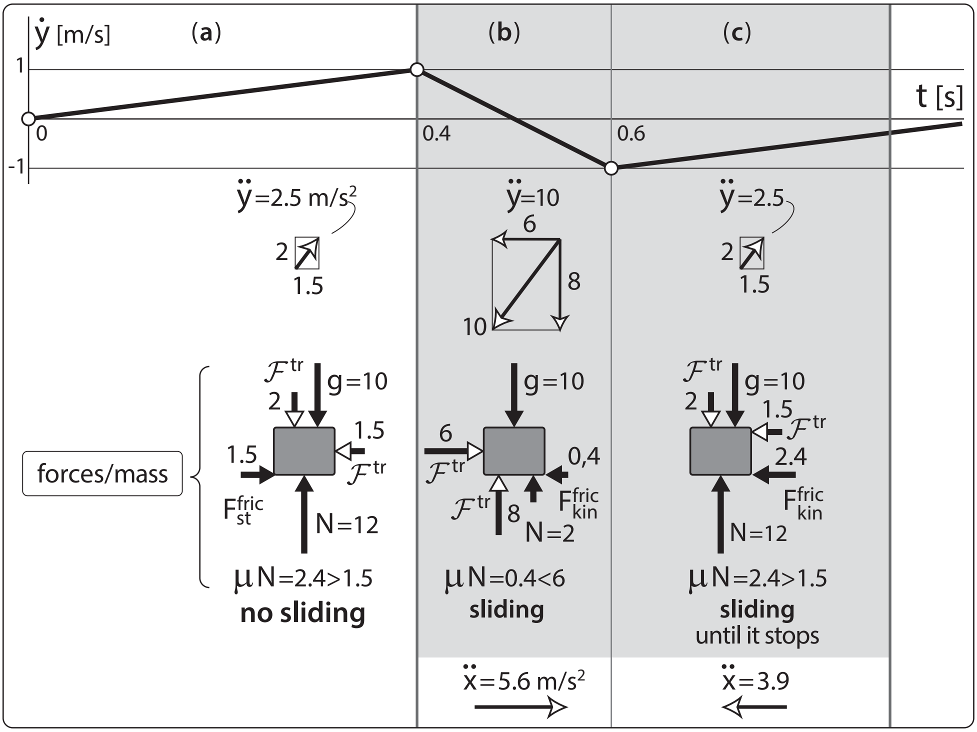

Fig. 1.29

If the dynamics of **P** is formulated in the platform reference frame R, the transportation force has to be added to the interaction forces: $\overline{\mathscr{F}}^{tr}_{R\to\mathbf{P}} = -m_{\mathbf{P}}\bar{\mathbf{a}}_{RGal}(\mathbf{P} \in R) = -m_{\mathbf{P}}\ddot{y}(t)$. As the platform motion is along an oblique direction, $\overline{\mathscr{F}}^{tr}_{R\to\mathbf{P}}$ will have two components.

As **P** is initially at rest on the platform ($\dot{x}(t=0)=0$), the friction force is a static one. Figure 1.29a shows the forces per unit mass that have to be considered during the first phase. The static friction compensates the horizontal component of $\overline{\mathscr{F}}^{tr}_{R\to\mathbf{P}}$, so $F^{fric}_{st}/m_{\mathbf{P}} = 1.5\ \mathrm{m/s^2}$. The normal force compensates the weight plus the vertical component of $\overline{\mathscr{F}}^{tr}_{R\to\mathbf{P}}$, and its value is $N/m_{\mathbf{P}} = 12\ \mathrm{m/s^2}$. As $F^{fric}_{st} < \mu N$, no sliding appears during the first phase.

During the second phase (Fig. 1.29b), the normal force is lower and the maximum value of the friction force cannot compensate the horizontal inertial force, so sliding starts ($\dot{x} > 0$).

In the third phase, $N/m_{\mathbf{P}} = 12\ \mathrm{m/s^2}$, so stiction can eventually happen. The two horizontal forces (inertial and dynamic friction) add up against the sliding (Fig. 1.29c). If they succeed in stopping it before the phase end, the nonsliding condition will be kept until the cycle $\dot{y}(t)$ restarts.

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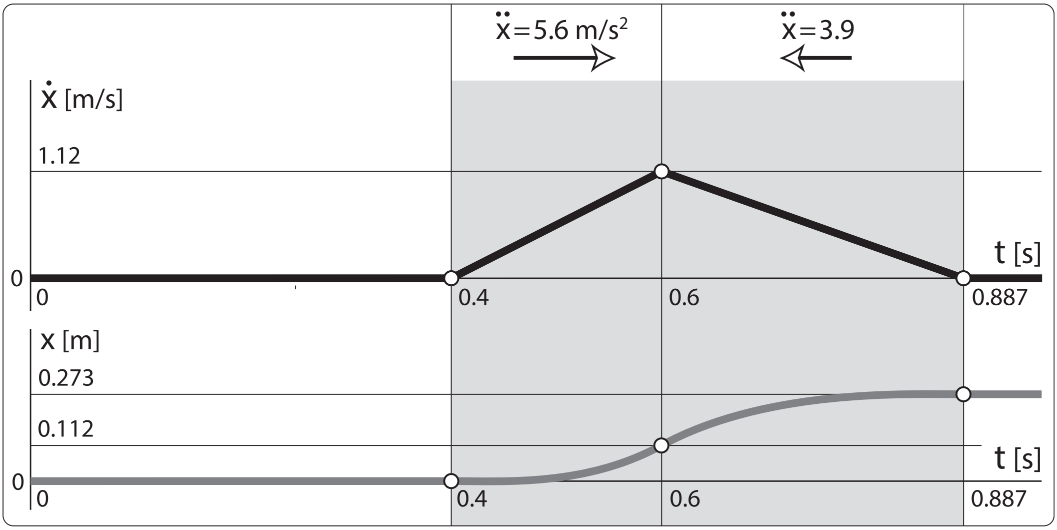

Fig. 1.30

The time evolution of the velocity $\dot{x}(t)$ and the position $x(t)$ of **P** on the platform is shown in Fig. 1.30. A straightforward time integration of $\dot{x}(t)$ shows that sliding does stop before the cycle is over. ◀

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Appendix 1A Gravitational Field Outside and Inside the Earth's Surface

Newton's formulation of the gravitational attraction (Eq. (1.11)) applies to pairs of interacting particles. The situation we are basically confronted with in mechanical engineering is that of an object undergoing the gravitational attraction of the Earth. The gravitational field experienced by that object is the result of the superposition of the fields associated with each particle in the Earth. The intensity of that resultant gravitational field depends on location. In particular, it is different over the Earth's surface (external gravitational field) or under it (internal gravitational field).

External Gravitational Field (Spherical Mass Distribution)

Assuming that the Earth has a spherical mass distribution (homogeneous concentrical layers) with radius R_E and mass M_E (Fig. 1A.1), the value of the external field at a distance r from the Earth's center, $g_E^{ext} = g_E(r > R_E)$, is equivalent to that created by the Earth's mass if it were concentrated at its center:

$$g_E^{ext} = g_E(r > R_E) = -G_0 \frac{M_E}{r^2} = -g_0 \left(\frac{R_E}{r}\right)^2, \tag{1A.1}$$

where g_0 is the filed intensity on the Earth's surface:

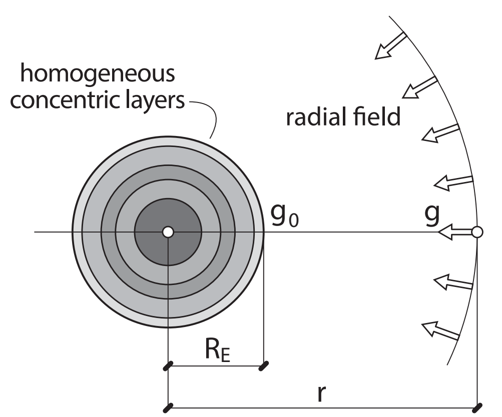

Fig. 1A.1

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$$g_0 = G_0 \frac{M_E}{R_E^2} = G_0 \frac{5.975 \cdot 10^{24}\ \mathrm{kg}}{\left(6365.10 \cdot 10^3\ \mathrm{m}\right)^2} \cong 9.814\ \mathrm{m/s^2}. \tag{1A.2}$$

♣ *Proof*

For symmetry reasons, $\mathbf{g}_E^{ext}$ is a *central field* whose center is the Earth's center (that is, its direction is radial toward the Earth's center and its value depends exclusively on the distance r to that center). Its intensity can be calculated through a 3D integration:

$$g_E^{ext}(r) = G_0 \int_{V_E} \frac{dm(\rho)}{\rho^2}. \tag{1A.3}$$

This integration may be rather cumbersome. An alternative is to calculate $g_E^{ext}(r)$ through the application of Gauss' theorem to a sphere with radius $r > R_E$ concentric with the Earth:

$$\int_V \overline{\nabla} \cdot \mathbf{g}_E^{ext} dV = \int_S \mathbf{g}_E^{ext} \cdot d\overline{\mathbf{S}} = -g_E^{ext} 4\pi r^2, \tag{1A.4}$$

since $\overline{\nabla} \cdot \mathbf{g}_E^{ext} dV$ is proportional to the mass contained in dV through the proportionality constant $4\pi G$:

$$\int_V \overline{\nabla} \cdot \mathbf{g}_E^{ext} dV = 4\pi G_0 M_E. \tag{1A.5}$$

The integration in Eq. (1A.3), or Eqs. (1A.4) and (1A.5), lead to Eq. (1A.1). ♣

Internal Gravitational Field (Uniform Mass Distribution)

Assuming that the Earth is a homogeneous sphere, the internal gravitational field is:

$$g_E^{int}(r < R_E) = -G_0 \frac{M_E}{R_E^3} r \equiv -g_0 \frac{r}{R_E}. \tag{1A.6}$$

This value is equivalent to the gravitational field at location r generated by the mass contained in the sphere with radius r if it were concentrated at its center. It is an attraction field that increases proportionally to the distance r.

♣ *Proof*

For symmetry reasons, it is also *central field*. Gauss' theorem applied to a sphere with radius $r < R_E$ concentric with the Earth (as in the previous proof) yields:

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$$-g_E^{int} 4\pi r^2 = 4\pi G_0 M_E \left(\frac{r}{R_E}\right)^3. \tag{1A.7}$$

The mass contained within the sphere is $M_E(r/R_E)^3$ instead of M_E (as before). Equation (1A.6) leads to Eq. (1A.6). ♣

Figure 1A.2 shows Earth's internal and external gravitational fields. In small domains close to the Earth's surface, the usual practice is to adopt the approximation of uniform field $|\overline{\mathbf{g}}| = g_0$

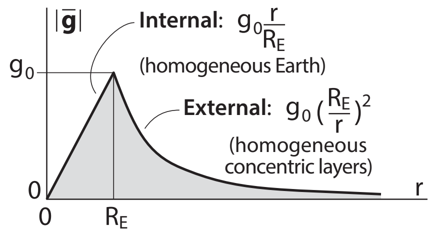

Fig. 1A.2

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Appendix 1B: Dynamics in the Terrestrial Reference Frame (TRF)

The formulation of dynamics has to include inertial forces whenever the reference frame is non-Galilean. The formulation of those forces depends on which reference frame is taken as Galilean. As explained in Section 1.5, that depends on the application field.

We present here two approximations for the inertial forces to be included in the TRF. They correspond to taking either the conventional celestial reference frame (CCRF) or the international celestial reference frame (ICRF) as the Galilean reference frame.

First Approximation

Assuming that the CCRF is Galilean, the non-Galilean behavior of the TRF is associated with the Earth spin $\Omega^{E}_{CCRF} = 1\,\frac{\text{tour}}{\text{sidereal day}} = \frac{2\pi}{86164}\frac{\text{rad}}{\text{s}}$ (see explanation below). The transportation and Coriolis forces on a particle $\mathbf{P}$ with mass $m_{\mathbf{P}}$ are:

$$\overline{\mathscr{F}}^{tr}_{\to\mathbf{P}} = -m_{\mathbf{P}}\overline{\Omega}^{E}_{CCRF} \times \left(\overline{\Omega}^{E}_{CCRF} \times \overline{\mathbf{CP}}\right), \quad \overline{\mathscr{F}}^{Cor}_{\to\mathbf{P}} = -2m_{\mathbf{P}}\overline{\Omega}^{E}_{CCRF} \times \overline{v}_{TRF}(\mathbf{P}), \qquad (1B.1)$$

where $\mathbf{C}$ is the center of the Earth.

As mentioned in Section 1.5, the Coriolis force is responsible for the rotation of the oscillation plane of Foucault's pendulum, of the spiral motion of cyclones and anticyclones, the deviation toward the east of objects falling to the Earth's surface and that of missiles launched to the north from the north hemisphere or to the south from the south hemisphere.

Orientative numerical values:

$\left|\overline{\Omega}^{E}_{CCRF} \times \left(\overline{\Omega}^{E}_{CCRF} \times \overline{\mathbf{CP}}\right)\right| = 3.450 \cdot 10^{-6} g_0$ at the Equator,

$\left|2\overline{\Omega}^{E}_{CCRF} \times \overline{v}_{TRF}(\mathbf{P})\right| = 413 \cdot 10^{-6} g_0$ in the parallels direction and for $v_{TRF}(\mathbf{P}) = 100$ km/h.

The duration of a sidereal day (measured from the observation of two consecutive crossings of a distant star through the meridian plane) corresponds to the time needed by the Earth to complete a whole turn about its axis relative to the distant stars.

The mean solar day (measured from two consecutive crossings of the Sun through the meridian plane) has a duration of $24 \times 24 \times 60 = 86{,}400$ seconds, and it is slightly longer than the sidereal day. The reason is simple: the Earth's center completes its trajectory around the Sun in one year, and the projection on the Earth axis of the angular

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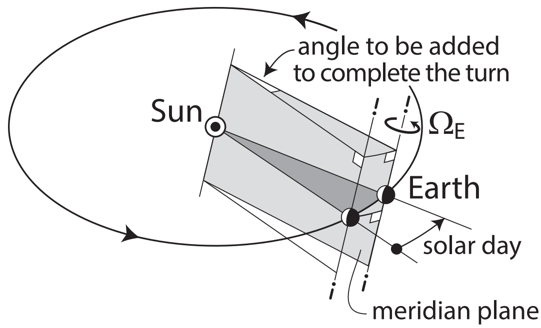


Fig. 1B.1

velocity associated to that motion has the same direction as the Earth's spin about its axis; hence, every solar day corresponds to a whole spin turn plus an additional angle (Fig. 1B.1). The number of solar and sidereal days in one year differs from one unit:

$$\text{sidereal day} = \frac{365.26}{365.26 + 1} \text{ mean solar day} = 86{,}164 \text{ seconds.}$$

Second Approximation

Assuming that the ICRF is Galilean and that the acceleration of the Earth's center **C** is exclusively associated with the gravitational attraction of the Sun and the Moon, the non-Galilean behavior of the TRF comes from the Earth's spin $\Omega^E_{ICRF}\left(= \Omega^E_{CCRF}\right)$ (as in the first approximation) and the Galilean acceleration of **C**: $\bar{a}_{ICRF}(\mathbf{C}) = \bar{a}_{ICRF}(\text{Sun} \to \mathbf{C}) + \bar{a}_{ICRF}(\text{Moon} \to \mathbf{C})$.

The inertial forces are:

$$\overline{\mathscr{F}}^{tr}_{TRF\to\mathbf{P}} = -m_\mathbf{P}[\bar{a}_{ICRF}(\text{Sun} \to \mathbf{C}) + \bar{a}_{ICRF}(\text{Moon} \to \mathbf{C})] - m_\mathbf{P}\overline{\Omega}^E_{ICRF} \times \left(\overline{\Omega}^E_{ICRF} \times \overline{\mathbf{CP}}\right),$$

$$\overline{\mathscr{F}}^{Cor}_{TRF\to\mathbf{P}} = -2m_\mathbf{P}\overline{\Omega}^E_{ICRF} \times \bar{v}_E(\mathbf{P}) \ . \tag{1B.2}$$

When formulating the dynamics of a particle **P** in the TRF, the interaction forces on **P** must include the gravitational attractions of both the Sun and the Moon (for consistency reasons). If $\sum \overline{\mathbf{F}}_{\to\mathbf{P}}$ stands for all the interaction forces on **P** but those two gravitational attractions, Newton's second law yields:

$$\begin{aligned}\bar{a}_E(\mathbf{P}) = &-m_\mathbf{P}^{-1}\sum \overline{\mathbf{F}}_{\to\mathbf{P}} - \overline{\Omega}^E_{ICRF} \times \left(\overline{\Omega}^E_{ICRF} \times \overline{\mathbf{CP}}\right) - 2\overline{\Omega}^E_{ICRF} \times \bar{v}_E(\mathbf{P}) \\ &+ [\bar{a}_{ICRF}(\text{Sun} \to \mathbf{P}) - \bar{a}_{ICRF}(\text{Sun} \to \mathbf{C})] \\ &+ [\bar{a}_{ICRF}(\text{Moon} \to \mathbf{P}) - \bar{a}_{ICRF}(\text{Moon} \to \mathbf{C})] \ .\end{aligned} \tag{1B.3}$$

The last two terms in brackets in Eq. (1B.3) are roughly equal to the difference between the gravitational fields of the Sun and the Moon at **P** and **C** locations, respectively. Those two terms are responsible for the sea tides.

Orientative numerical values:

$$
\begin{aligned}
&|\bar{a}_{\mathrm{ICRF}}(\mathrm{Sun} \to \mathbf{C})| = 624 \cdot 10^{-6} \mathrm{g}_0, \\
&|\bar{a}_{\mathrm{ICRF}}(\mathrm{Moon} \to \mathbf{C})| = 3.47 \cdot 10^{-6} \mathrm{g}_0, \\
&|\bar{a}_{\mathrm{ICRF}}(\mathrm{Sun} \to \mathbf{P}) - \bar{a}_{\mathrm{ICRF}}(\mathrm{Sun} \to \mathbf{C})| = 0.0542 \cdot 10^{-6} \mathrm{g}_0 \\
&\qquad \text{at the points closest to and furthest from the Sun,} \\
&|\bar{a}_{\mathrm{ICRF}}(\mathrm{Moon} \to \mathbf{P}) - \bar{a}_{\mathrm{ICRF}}(\mathrm{Moon} \to \mathbf{C})| = 0.116 \cdot 10^{-6} \mathrm{g}_0 \\
&\qquad \text{at the points closest to and furthest from the Moon.}
\end{aligned}
$$

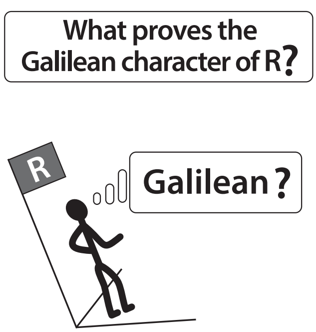
A surprising thing about the preceding values is that the influence of the Moon's gravitational field on the **P** dynamics (given by Eq. (1B.2)) is twice that of the Sun's, though the Moon's gravitational attraction is some 180 times weaker than that of the Sun. The reason is simply that the Moon is much closer to the Earth than the Sun.

Quiz Questions

For the sake of brevity, the following assumptions are made throughout the questions unless stated otherwise:

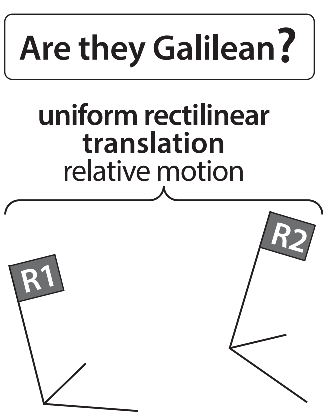
- Threads, ropes, and cables are inextensible and their mass is negligible.
- Bars have a negligible mass.
- Air friction and internal frictions in joints are negligible.
- Springs and dampers are linear.

All data are declared in the figures but not in the text. Mass values not declared are irrelevant.



1.1 A reference frame is Galilean when

A Its orientation does not change.
B It is permanently at rest.
C It has a constant orientation relative to the distant galaxies.
D Newton's second law is fulfilled within the degree of accuracy sought for in the problem being solved.
E It is fixed in the Earth reference frame or has a uniform rectilinear translation motion relative to it.



1.2 Two reference frames have a relative uniform translation motion. Are they Galilean reference frames?

A Not necessarily, but if both were non-Galilean, the transportation forces would be the same in all points in both reference frames..
B R1 is Galilean relative to R2 (or vice versa).
C Not necessarily, but if both were non-Galilean, the Coriolis forces would be zero.
D They are either both Galilean or both non-Galilean.
E Yes, they are both Galilean.

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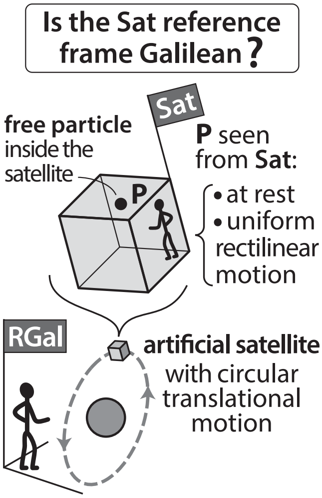
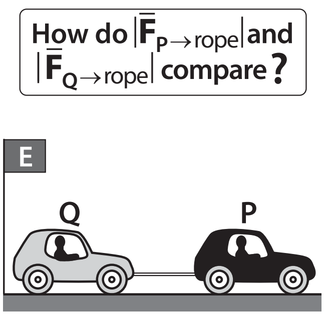


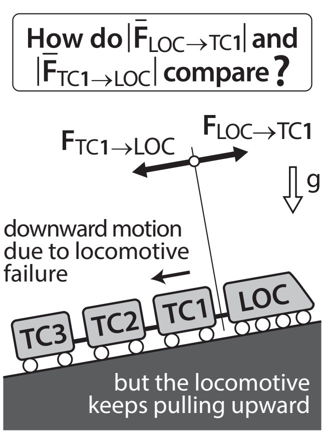
1.3 An artificial satellite has a circular translation motion at a distance of 1000 km from the Earth's surface, relative to a Galilean reference frame (RGal). Free particles in the satellite are seen either at rest or with a uniform rectilinear motion relative to the satellite. May we conclude from these observations that the satellite is a Galilean reference frame ?

A Yes, because the satellite has escaped from the Earth's gravitational field.
B No, because the particles are attracted by the Earth's gravitational field.
C Yes, because the law of inertia is fulfilled in the satellite reference frame.
D Not from a strict point of view, but it is a good approximation to a Galilean reference frame, even better than the Earth reference frame.
E Yes, because the satellite does not rotate relative to RGal.



1.4 The vehicle P drags the vehicle Q through a rope. How does the magnitude of the forces that the vehicles exert on the rope $\left(\left|\overline{\mathbf{F}}_{P\to rope}\right|, \left|\overline{\mathbf{F}}_{Q\to rope}\right|\right)$ compare?

A $\left|\overline{\mathbf{F}}_{P\to rope}\right| > \left|\overline{\mathbf{F}}_{Q\to rope}\right|$ if car P has a nonzero acceleration.
B $\left|\overline{\mathbf{F}}_{P\to rope}\right| > \left|\overline{\mathbf{F}}_{Q\to rope}\right|$ only if the rope is inextensible.
C $\left|\overline{\mathbf{F}}_{P\to rope}\right| > \left|\overline{\mathbf{F}}_{Q\to rope}\right|$ always.
D $\left|\overline{\mathbf{F}}_{P\to rope}\right| = \left|\overline{\mathbf{F}}_{Q\to rope}\right|$ only if the rope is inextensible.
E $\left|\overline{\mathbf{F}}_{P\to rope}\right| = \left|\overline{\mathbf{F}}_{Q\to rope}\right|$ always.



1.5 A train going up a slope starts moving downward because the locomotive has a sudden technical problem that prevents it from developing the force needed to drag the train up. How does the magnitude of the force that the locomotive exerts on the first train car $\left(\left|\overline{\mathbf{F}}_{LOC\to TC1}\right|\right)$ compare to that of the first train car on the locomotive $\left(\left|\overline{\mathbf{F}}_{TC1\to LOC}\right|\right)$?

A $\left|\overline{\mathbf{F}}_{TC1\to LOC}\right| > \left|\overline{\mathbf{F}}_{LOC\to TC1}\right|$
B $\left|\overline{\mathbf{F}}_{TC1\to LOC}\right| > \left|\overline{\mathbf{F}}_{LOC\to TC1}\right|$ only if the train acceleration when going down is nonzero.
C $\left|\overline{\mathbf{F}}_{TC1\to LOC}\right| = \left|\overline{\mathbf{F}}_{LOC\to TC1}\right|$ only if the train speed when going down is constant.
D $\left|\overline{\mathbf{F}}_{TC1\to LOC}\right| < \left|\overline{\mathbf{F}}_{LOC\to TC1}\right|$ $\left|\overline{\mathbf{F}}_{TC1\to LOC}\right| < \left|\overline{\mathbf{F}}_{LOC\to TC1}\right|$ only if the train speed when going down is constant.
E $\left|\overline{\mathbf{F}}_{TC1\to LOC}\right| = \left|\overline{\mathbf{F}}_{LOC\to TC1}\right|$ always.

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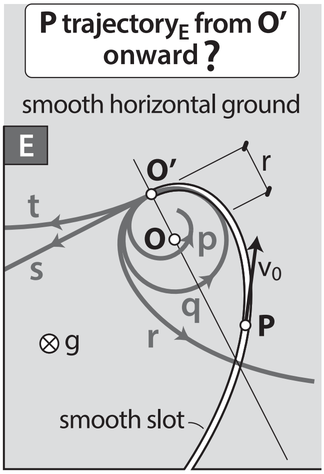
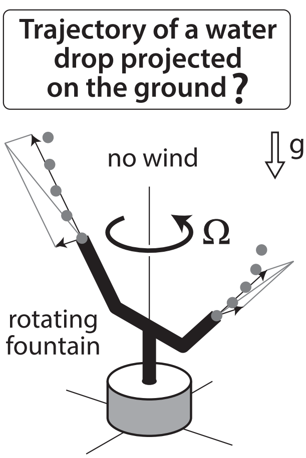


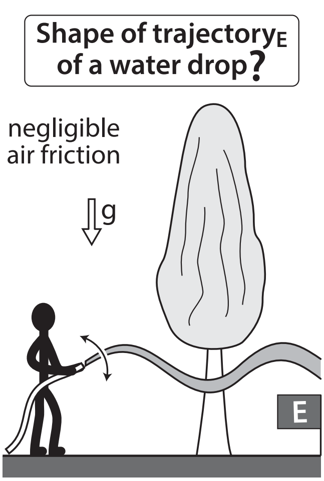
1.6 A particle **P** moves in a convergent spiral slot in a smooth horizontal ground (E). At the endpoint the slot has a curvature radius r and its curvature center is **O**. Which curve may correspond to the trajectory of **P** from **O**′ onward?

A p
B q
C r
D s
E t



1.7 What is the shape of the projection of a water drop on the ground if there is no wind?

A A convergent spiral.
B A divergent spiral.
C A circular line.
D A straight line.
E A conic curve.



1.8 A gardener waters a garden moving the water hose up and down. What is the shape of the trajectory of a water drop, relative to the ground, once it has left the hose if there is no wind and the air friction is negligible?

A A roughly sinusoidal one (as shown in the picture).
B A straight line.
C Parabolic.
D A nonparabolic shape with just one peak.
E Different from that on the picture but with the same number of maxima and minima.

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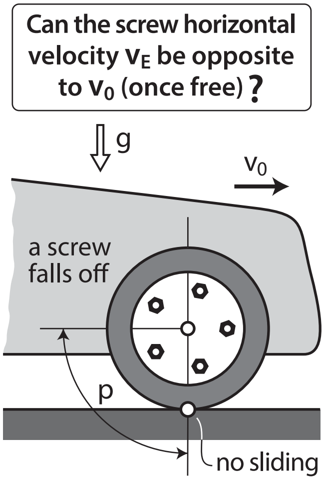
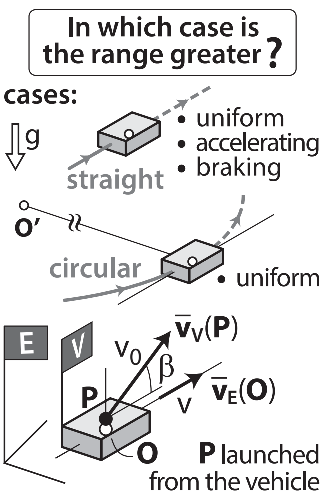


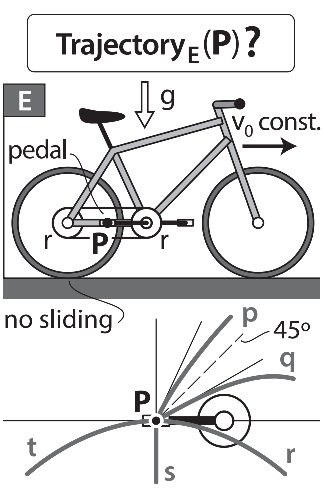
1.9 A vehicle moves on a straight road with constant speed v_0 relative to the ground. If a wheel's screw falls off suddenly and the wheels do not slide, is it possible that the horizontal component of the screw velocity relative to the ground be opposite to $\overline{\mathbf{v}}_0$?

A Only if the screw falls off when located under the horizontal diameter of the wheel.
B No, it is impossible.
C It is always the case.
D Only if the screw falls off when located in the p quadrant of the wheel.
E Only if the screw falls off when located at the left hand side of the wheel vertical diameter.



1.10 A particle **P** is launched forward from point **O** of a vehicle (V) moving on horizontal ground (E). At the time of launching, $|\overline{\mathbf{v}}_E(\mathbf{O})| = v$, and $\overline{\mathbf{v}}_v(\mathbf{P})$ points upward at an angle β to the horizontal and a value v_0. Under what conditions will the particle range be higher?

A When the vehicle motion is circular and uniform.
B When the vehicle accelerates on a straight road.
C When the vehicle brakes on a straight road.
D When the vehicle motion is rectilinear and uniform.
E The particle range will be the same in all cases.



1.11 A cyclist rides her bike with a constant speed v_0 relative to the ground. A particle **P** is released from the back pedal in its horizontal configuration. Which curve may correspond to the trajectory of **P** relative to the ground once released?

A p
B q
C r
D s
E t

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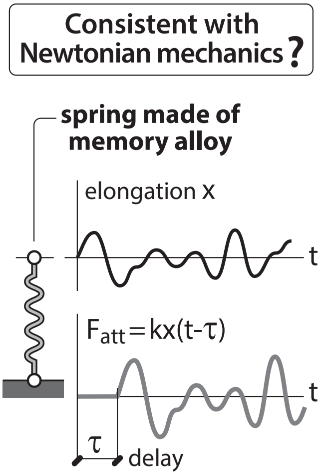
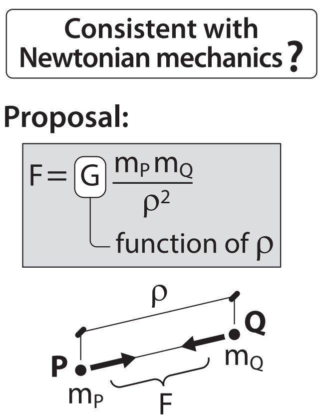


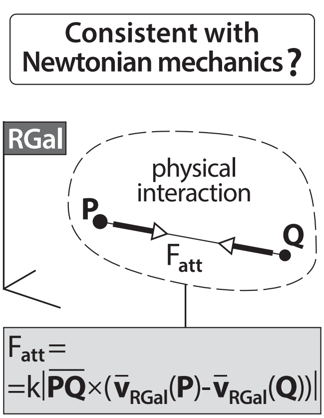
1.12 Someone claims to have discovered a memory alloy which allows a spring made of that material to develop an attraction force proportional to its deformation x, though with a time delay τ. Is that consistent with Newtonian mechanics?

A Yes, because it is an interaction force.
B No, because it does not fulfill the principle of action and reaction.
C No, because it does not fulfill Galileo's principle of relativity.
D No, because it does not fulfill the uniformity of time.
E No, because it does not fulfill the principle of determinacy.



1.13 Someone proposes changing the universal constant of gravitation G_0 for a function G changing with the separation ρ between the interacting particles. Is this formulation consistent with Newtonian mechanics?

A Yes.
B No, because it does not fulfill the principle of action and reaction.
C No, because it does not fulfill the principle of determinacy.
D No, because it is not consistent with Newton's absolute time.
E No, because it does not fulfill Galileo's principle of relativity.



1.14 A physical interaction between two particles **P** and **Q** results in an attraction force with value $\mathrm{F_{att}} = \mathrm{k}\left|\overline{\mathbf{PQ}}\times[\overline{\mathbf{v}}_{\mathrm{RGal}}(\mathbf{P}) - \overline{\mathbf{v}}_{\mathrm{RGal}}(\mathbf{Q})]\right|$. Is that interaction admissible in Newtonian mechanics?

A No, because it fulfills neither the principle of action and reaction nor Galileo's principle of relativity.
B Yes.
C No, because it does not fulfill the principle of action and reaction though it does fulfill Galileo's principle of relativity.
D No, because it does not fulfill Galileo's principle of relativity though it does fulfill the principle of action and reaction.
E No, because it does not fulfill the principle of determinacy.

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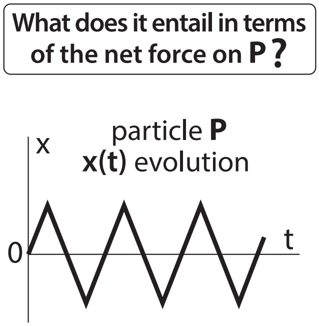
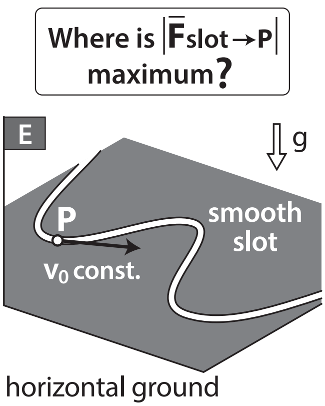


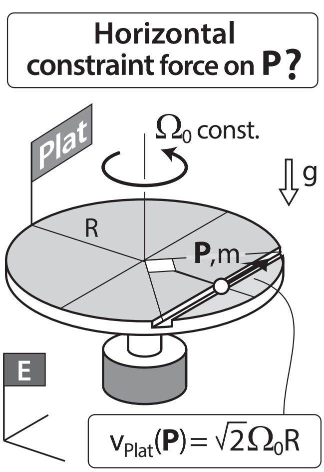
1.15 The location x of a particle in a straight line evolves according to a triangular wave. What does it entail in terms of the net force on the particle ?

A It is constant and $\neq 0$ between the vertices.
B Zero only when $x = 0$.
C Zero at the vertices.
D Discontinuous but bounded at the vertices.
E Its value is infinite at the vertices.



1.16 A particle **P** moves with constant speed in a smooth slot in a horizontal ground (E). At what location along the slot does the constraint force on **P** have a maximum value?

A At the location with minimum curvature radius.
B At the location with maximum curvature radius.
C Its value is the same at all locations.
D At the inflexion points.
E It depends on the v_0 value.



1.17 A platform rotates with constant angular velocity relative to the ground. A particle **P** moves in a straight slot on the platform with constant speed relative to it (under the action of an actuator). What is the magnitude of the horizontal constraint force on **P** when it goes through the slot midpoint?

A $(5/\sqrt{2})mR\Omega_0^2$
B $(4/\sqrt{2})mR\Omega_0^2$
C $(3/\sqrt{2})mR\Omega_0^2$
D $(2/\sqrt{2})mR\Omega_0^2$
E $(1/\sqrt{2})mR\Omega_0^2$

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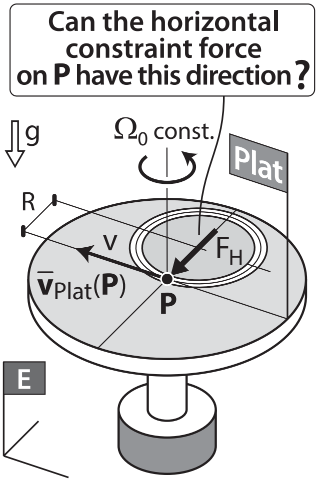
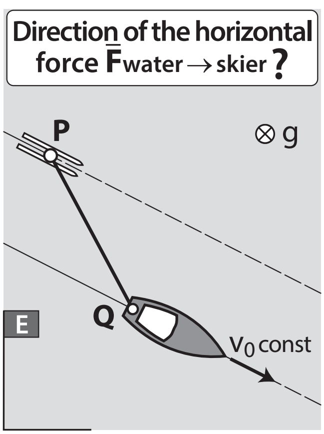


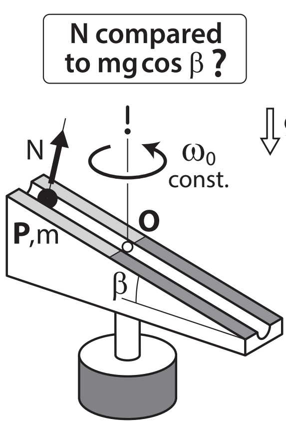
1.18 A platform rotates with constant angular velocity relative to the ground. A particle **P** moves in a circular slot on the platform If its speed when going through the platform midpoint is v (relative to Plat), are there any v values so that the horizontal constraint force on **P** has the shown direction?

A Any value fulfilling $v > (1/2)R\Omega_0$
B Any value fulfilling $v > 2R\Omega_0$
C Any value fulfilling $v < (1/2)R\Omega_0$
D Any value fulfilling $v < 2R\Omega_0$
E No



1.19 A canoe has a uniform rectilinear motion relative to the ground (E). A water skier **P** moves along a straight line parallel to the canoe trajectory. If the cable between canoe and skier is taut and horizontal at all times, what is the direction of the interaction force of the water on **P**?

A The canoe longitudinal direction.
B The canoe transverse direction.
C The cable direction.
D It lies between the canoe longitudinal direction and the cable direction.
E It lies between the canoe transverse direction and the cable direction.



1.20 A particle **P** moves along an inclined straight slot on a support which rotates with constant angular velocity relative to the ground. How does the magnitude of the normal constraint force N on **P** (contained in the vertical plane) compare to $mg\cos\beta$?

A $N = mg\cos\beta$ always.
B $N > mg\cos\beta$ always.
C $N < mg\cos\beta$ always.
D $N > mg\cos\beta$ when **P** has overpassed **O**.
E $N > mg\cos\beta$ before **P** overpasses **O**.

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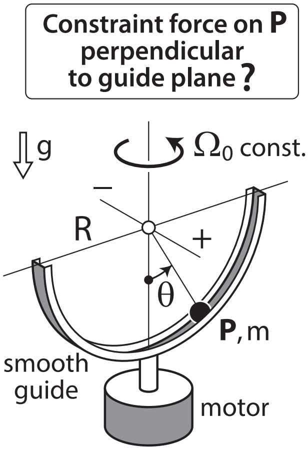
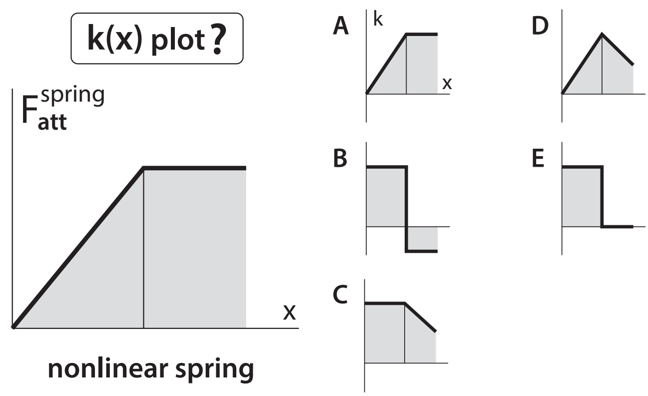
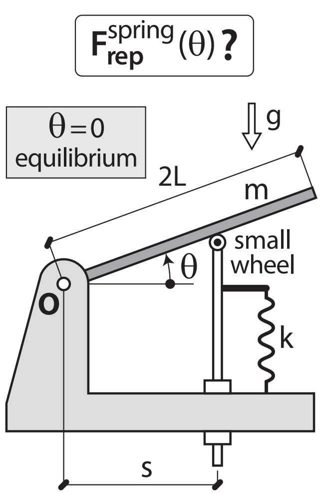


1.21 A particle **P** moves in a smooth circular straight slot with radius R which rotates with constant angular velocity relative to the ground. What is the value of the constraint force on **P** perpendicular to the guide plane?

A $-2mR\Omega_0\dot{\theta}\cos\theta$
B $+2mR\Omega_0\dot{\theta}\cos\theta$
C $-2mR\Omega_0\dot{\theta}\sin\theta$
D $+2mR\Omega_0\dot{\theta}\sin\theta$
E 0

1.22 The attraction force exerted by a nonlinear spring as a function of its deformation x follows the curve shown in the picture. What curve could represent k(x)?





1.23 The homogeneous rod is articulated to a support at point **O**, and keeps contact with a small wheel articulated to a vertical bar. Between the bar and the support there is a spring. If $\theta = 0$ corresponds to an equilibrium configuration, what is the formulation of the spring repulsion force as a function of θ?

A $(L/s)mg + sk\sin\theta$
B $(L/s)mg - sk\sin\theta$
C $(L/s)mg + sk\tan\theta$
D $(L/s)mg - sk\tan\theta$
E $-(L/s)mg - sk\sin\theta$

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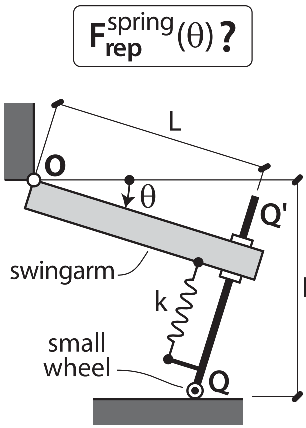
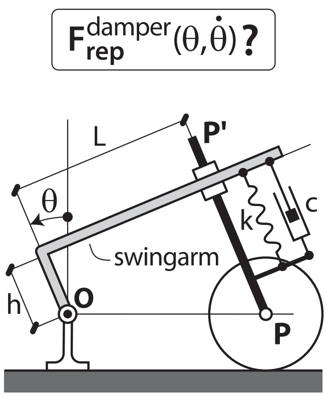


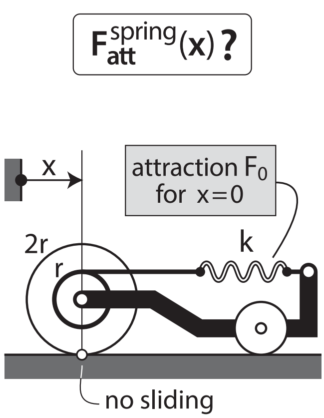
1.24 A swingarm is articulated to the ground at point **O**. The thin bar **QQ′** slides relative to the swingarm, and its endpoint **Q** is in contact with the ground through a small wheel. If the spring has no tension when $\theta = 0$, what is the formulation of the spring repulsion force as a function of θ?

A $kL(\cot\theta + \tan\theta - 1)$
B $kL(1 - \tan\theta)$
C $kL(\cos\theta + \sin\theta - 1)/\cos\theta$
D $kL\sin\theta$
E $kL(1 - \tan\theta + \cos\theta)$



1.25 A swingarm is articulated to the ground at point **O**. The thin bar **PP′** slides relative to the swingarm, and its endpoint **P** is articulated to the center of a wheel that moves on the ground. A (spring+damper) module acts between the swingarm and the bar. What is the formulation of the damper repulsion force as a function of θ and $\dot{\theta}$?

A $+cL\dot{\theta}$
B $-cL\dot{\theta}$
C $+cL\dot{\theta}\cos^2\theta$
D $-cL\dot{\theta}\cos^2\theta$
E $-cL\dot{\theta}/\cos^2\theta$



1.26 The wheels of the vehicle do not slide on the ground, and the wheel with radius 2r and the roller with radius r are mutually fixed. If the spring attraction force is F_0 when $x = 0$, what is the formulation of the general spring attraction force as a function of x?

A $F_0 - (1/2)kx$
B $F_0 + (1/2)kx$
C $F_0 - (3/2)kx$
D $F_0 + (3/2)kx$
E $F_0 - kx$

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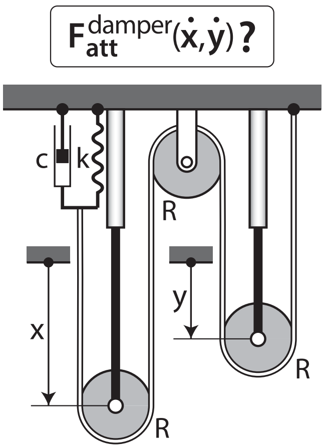
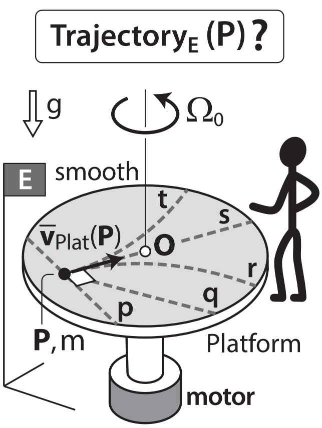


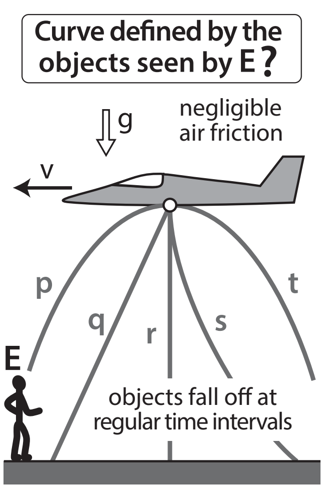
1.27 The three wheels are identical. Two of them are articulated at their centers to two hydraulic cylinders which control their vertical motion. The third one is articulated to the ceiling. The (spring+damper) module is attached to the ceiling and to a thread that does not slide on the wheels. What is the formulation of the damper attraction force as a function of $\dot{x}$ and $\dot{y}$?

A $c(2\dot{x} + 2\dot{y})$
B $c(2\dot{x} - 2\dot{y})$
C $c(\dot{x} + 2\dot{y})$
D $c(2\dot{x} + \dot{y})$
E $c2\dot{y}$



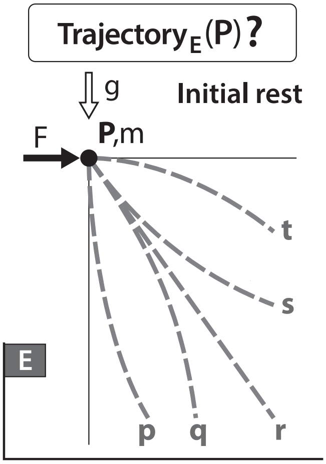
1.28 A particle **P** is launched tangentially to a platform which rotates with constant angular velocity relative to the ground. If its initial velocity relative to the platform is radial and points to the center **O**, which curve may correspond to the trajectory of **P** relative to the ground (E)?

A p
B q
C r
D s
E t



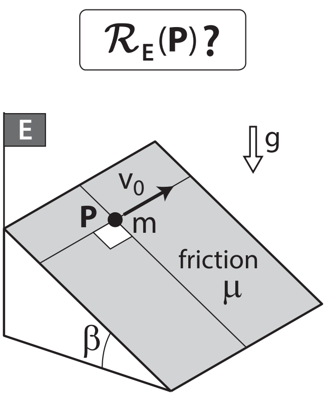
1.29 A plane moves with a uniform rectilinear motion relative to the ground. Some objects fall off the plane at regular time intervals. What curve may correspond to the line of falling objects relative to the ground (E)?

A p
B q
C r
D s
E t



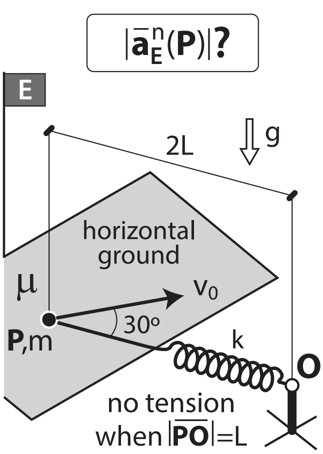
1.30 A particle **P** is initially at rest relative to the ground (E). If it moves under the action of the gravitational force and a constant horizontal force $\overline{\mathbf{F}}$, what curve may correspond to its trajectory relative to the ground (E)?

A p
B q
C r
D s
E t



1.31 A particle **P** moves on an inclined rough surface. At a certain time instant, its velocity relative to the ground has a value v_0 and is perpendicular to the line of maximum slope. What is the radius of curvature of the **P** trajectory relative to the ground at that time instant?

A $\left(v_0^2 - \mu g\right)/g\cos\beta$
B $v_0^2/(\mu g\cos\beta + g\sin\beta)$
C $v_0^2/g\sin\beta$
D $\mu v_0^2/g\sin\beta$
E $v_0^2/\sqrt{(\mu g\cos\beta)^2 + (g\sin\beta)^2}$



1.32 A particle **P** is attached to a spring and moves on rough horizontal ground. The other spring endpoint is attached to the ground at point **O**. What is the magnitude of the normal acceleration of **P** relative to the ground at the time instant shown in the figure?

A $kL/2m$
B kL/m
C μg
D $\left(\sqrt{3}kL/m\right) - \mu g$
E $\left(2\sqrt{3}kL/m\right) - \mu g$

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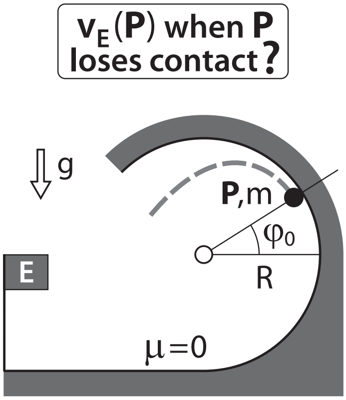
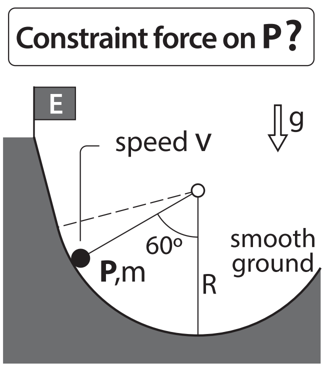


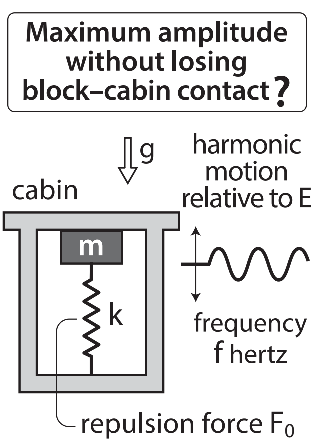
1.33 A particle **P** moves on a ground-fixed track with a straight span and a circular looping. When it reaches the shown position, it loses ground contact. What is its velocity relative to the ground at that time instant?

A 0
B $\sqrt{gR \sin \varphi_0}$
C $\sqrt{gR}$
D $\sqrt{2gR(1 - \sin \varphi_0)}$
E $\sqrt{4gR}$



1.34 A particle **P** moves on a ground-fixed circular track. What is the value of the constraint force on **P** at the time instant shown in the figure?

A $m(v^2/R) + mg$
B $m(v^2/R) - (1/2)mg$
C $m(v^2/R) + (1/2)mg$
D $(1/2)mg$
E $(\sqrt{3}/2)mg$



1.35 A block is held against the ceiling of a cabin by a compressed spring. If the cabin starts an up-and-down harmonic motion relative to the ground, what is the maximum amplitude of the oscillation without losing the contact between block and cabin?

A $(F_0 - mg)/(4\pi^2 f^2 m)$
B $(F_0 + mg)/(4\pi^2 f^2 m)$
C $(F_0 - mg)/k$
D $(F_0 + mg)/k$
E F_0/k

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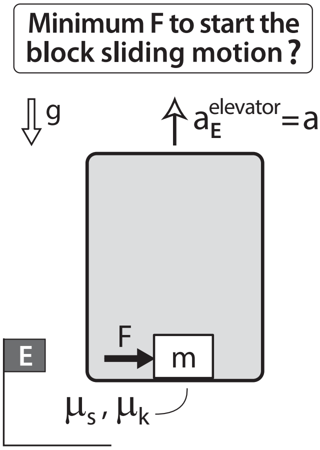
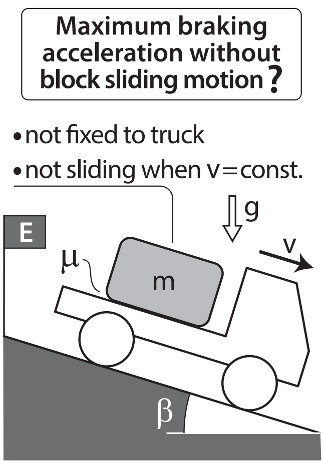


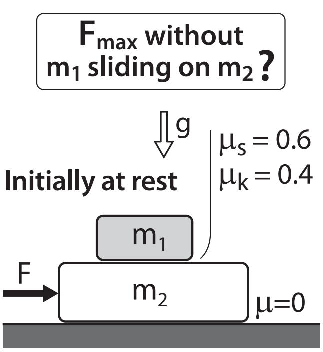
1.36 A block is at rest with respect to an elevator which has an accelerated vertical motion relative to the ground. What is the minimum horizontal force F required to start moving the block relative to the elevator?

A $\mu_s m(g + a)$
B $\mu_s m(g - a)$
C $\mu_k m(g + a)$
D $\mu_k mg$
E $\mu_s mg$



1.37 The block does not slide on the truck while it goes down the slope with constant speed relative to the ground. What is the maximum braking acceleration of the truck without block sliding motion ?

A $\mu g(\cos\beta - \sin\beta)$
B $g(\mu\cos\beta - \sin\beta)$
C $g(\mu\cos\beta + \sin\beta)$
D $g(\cos\beta - \mu\sin\beta)$
E $\mu g\cos\beta$



1.38 The two blocks are initially at rest relative to the ground. If the ground is smooth, what is the maximum horizontal force F that can be applied to block m_2 without sliding of m_1 on m_2?

A $0.4m_1g$
B $0.6m_2g$
C $0.4(m_1 + m_2)g$
D $0.6(m_1 + m_2)g$
E $(0.6m_1 + m_2)g$

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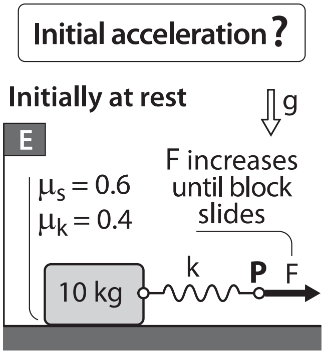
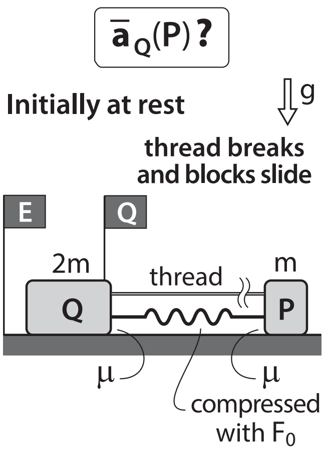


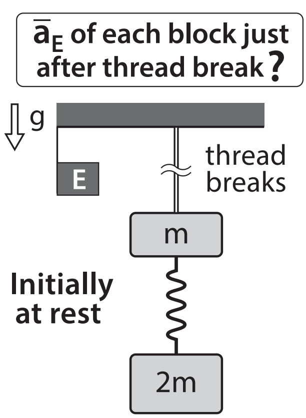
1.39 The block is initially at rest on the ground (E). If we apply an increasing horizontal force through a spring with no initial tension, what will be the initial acceleration of the block relative to the ground when it starts moving?

A 0.4g
B 0.5g
C 0.3g
D 0
E 0.2g



1.40 The two blocks Q and P are initially at rest relative to the ground (E) and attached together through a thread and a compressed spring. If the thread breaks, what is the initial acceleration of block P relative to block Q if they both slide on the ground?

A $[(3/2)(F_0/m) - 2\mu g] \rightarrow$
B $[(3/2)(F_0/m) - 2\mu g] \leftarrow$
C $[2(F_0/m) - 3\mu g] \rightarrow$
D $[(F_0/m) - \mu g] \rightarrow$
E $[(F_0/m) - \mu g] \leftarrow$



1.41 The two blocks are mutually attached through a spring, and hang from the ceiling by a thread. If they are initially at rest relative to the ground (E) and the thread breaks, what will be the initial acceleration of each of them relative to the ground?

A $\bar{\mathbf{a}}_E(m) = \downarrow g,\ \bar{\mathbf{a}}_E(2m) = \downarrow 2g$
B $\bar{\mathbf{a}}_E(m) = \downarrow 3g,\ \bar{\mathbf{a}}_E(2m) = \downarrow 3g$
C $\bar{\mathbf{a}}_E(m) = \downarrow 2g,\ \bar{\mathbf{a}}_E(2m) = \downarrow 2g$
D $\bar{\mathbf{a}}_E(m) = \downarrow g,\ \bar{\mathbf{a}}_E(2m) = \bar{0}$
E $\bar{\mathbf{a}}_E(m) = \downarrow 3g,\ \bar{\mathbf{a}}_E(2m) = \bar{0}$

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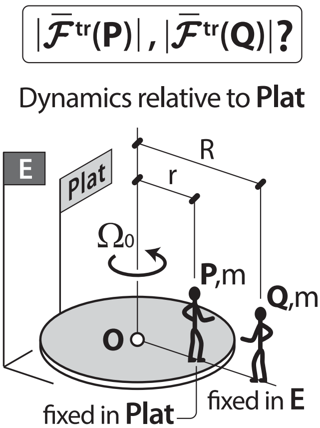


1.42 A person **P** is at rest on a platform which rotates with constant angular velocity relative to the ground (E). A person **Q** is fixed on the ground. If they are modeled as particles, what is the magnitude of the transportation force acting on them if we formulate their dynamics in the platform reference frame?

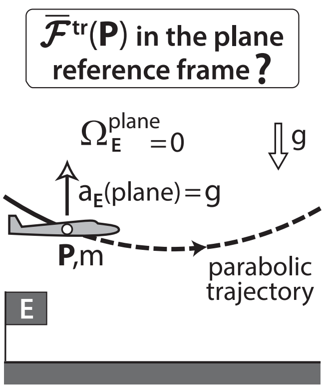
A $\left|\overline{\mathscr{F}}^{\mathrm{tr}}_{\mathrm{Plat}\to\mathbf{P}}\right| = m\Omega_0^2 r,\ \left|\overline{\mathscr{F}}^{\mathrm{tr}}_{\mathrm{Plat}\to\mathbf{Q}}\right| = 0$

B $\left|\overline{\mathscr{F}}^{\mathrm{tr}}_{\mathrm{Plat}\to\mathbf{P}}\right| = 0,\ \left|\overline{\mathscr{F}}^{\mathrm{tr}}_{\mathrm{Plat}\to\mathbf{Q}}\right| = m\Omega_0^2 R$

C $\left|\overline{\mathscr{F}}^{\mathrm{tr}}_{\mathrm{Plat}\to\mathbf{P}}\right| = 0,\ \left|\overline{\mathscr{F}}^{\mathrm{tr}}_{\mathrm{Plat}\to\mathbf{Q}}\right| = 0$

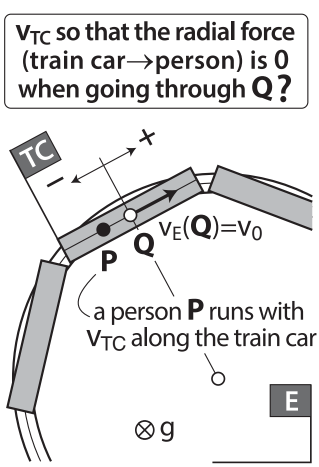
D $\left|\overline{\mathscr{F}}^{\mathrm{tr}}_{\mathrm{Plat}\to\mathbf{P}}\right| = m\Omega_0^2 r,\ \left|\overline{\mathscr{F}}^{\mathrm{tr}}_{\mathrm{Plat}\to\mathbf{Q}}\right| = m\Omega_0^2 R$

E $\left|\overline{\mathscr{F}}^{\mathrm{tr}}_{\mathrm{Plat}\to\mathbf{P}}\right| = m\Omega_0^2 r,\ \left|\overline{\mathscr{F}}^{\mathrm{tr}}_{\mathrm{Plat}\to\mathbf{Q}}\right| = m\Omega_0^2 r$



1.43 The plane moves according to a parabolic translation relative to the ground (E). At a certain time instant, its acceleration relative to the ground is ($\uparrow$ g). If we formulate the dynamics of a particle **P** in the plane reference frame, what is the transportation force on **P** at that particular time instant?

A $\downarrow$ mg
B $\downarrow$ 2mg
C 0
D $\uparrow$ mg
E $\uparrow$ 2mg



1.44 A train moves on a circular railroad, and the central point **Q** of its cars has a speed v_0 relative to the ground (E). A person, modeled as a particle **P**, runs along the train car with constant velocity v_{TC} relative to the train car. What should be the value of v_{TC} so that the total radial interaction force from the train car on **P** is zero when **P** goes through **Q**?

A $-v_0/2$
B $+v_0/2$
C It depends on the radius of the railroad.
D $-v_0$
E $+v_0$

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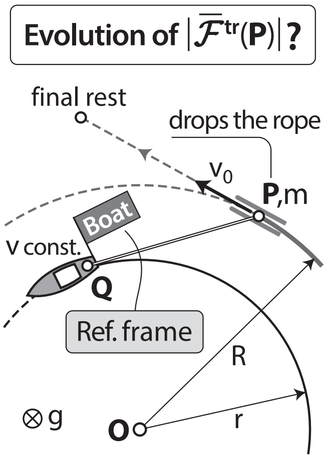
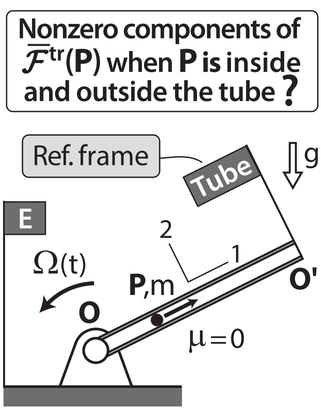


1.45 A water skier (modeled as a particle **P**) is dragged by a boat through a cable **QP**. Both the water skier and point **Q** of the boat describe a circular trajectory relative to the ground, both with constant speeds. At a certain time instant, the water skier drops the rope and keeps moving freely for a while along a straight line until reaching a final rest state relative to the ground. If the **P** dynamics are studied in the boat reference frame, how does the magnitude of the transportation force on **P** $\left(\left|\bar{\mathcal{F}}^{\text{tr}}_{\text{Boat}\to\mathbf{P}}\right|\right)$ evolves along the straight free path?

	$\left\lvert\bar{\mathcal{F}}^{\text{tr, free path}}_{\text{Boat}\to\mathbf{P}}\right\rvert$	$\left\lvert\bar{\mathcal{F}}^{\text{tr, final rest}}_{\text{Boat}\to\mathbf{P}}\right\rvert$
A	$\neq 0$ and $< m(r/R)\left(v_0^2/R\right)$	0
B	$> m\left(v_0^2/R\right)$	$> m\left(v_0^2/R\right)$
C	$\neq 0$ and $< m\left(v_0^2/R\right)$	0
D	$= m\left(v_0^2/R\right)$	$= m\left(v_0^2/R\right)$
E	0	0



1.46 A particle **P** slides in a smooth capless tube which rotates with variable angular velocity relative to the ground (E). If we formulate its dynamics in the tube reference frame, which components of the transportation force on **P** are nonzero when **P** is inside the tube and when it is outside?

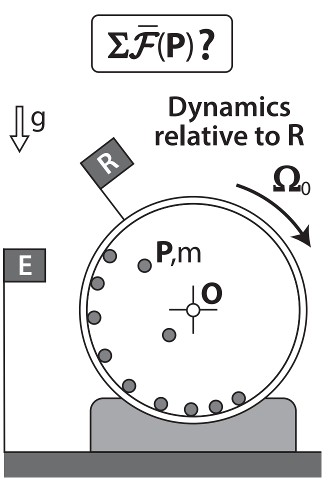
A $\left.\bar{\mathcal{F}}^{\text{tr in}}_{\text{Tube}\to\mathbf{P}}\right]_2$ and $\left.\bar{\mathcal{F}}^{\text{tr out}}_{\text{Tube}\to\mathbf{P}}\right]_1$

B $\left.\bar{\mathcal{F}}^{\text{tr in}}_{\text{Tube}\to\mathbf{P}}\right]_{1,2}$ but $\bar{\mathcal{F}}^{\text{tr out}}_{\text{Tube}\to\mathbf{P}} = \bar{0}$

C $\left.\bar{\mathcal{F}}^{\text{tr in}}_{\text{Tube}\to\mathbf{P}}\right]_{1,2}$ and $\left.\bar{\mathcal{F}}^{\text{tr out}}_{\text{Tube}\to\mathbf{P}}\right]_{1,2}$

D $\bar{\mathcal{F}}^{\text{tr in}}_{\text{Tube}\to\mathbf{P}} = \bar{0}$ and $\bar{\mathcal{F}}^{\text{tr out}}_{\text{Tube}\to\mathbf{P}} = \bar{0}$

E $\left.\bar{\mathcal{F}}^{\text{tr in}}_{\text{Tube}\to\mathbf{P}}\right]_2$ but $\bar{\mathcal{F}}^{\text{tr out}}_{\text{Tube}\to\mathbf{P}} = \bar{0}$



1.47 A few particles are inside a drum which rotates relative to the ground (E) with constant angular velocity about the ground-fixed point **O**. The particles may have a sliding or a nonsliding contact with the drum, or move freely without touching the drum. If the dynamics of any particle **P** is studied in the drum reference frame R, how is the resultant inertial force on **P**?

A It is always zero.
B It is always nonzero and its magnitude is constant.
C It is zero whenever **P** is in contact with the drum, and nonzero otherwise.
D It is always nonzero and its magnitude is not constant.
E It is only nonzero when **P** slides on the drum and is at rest relative to the ground.

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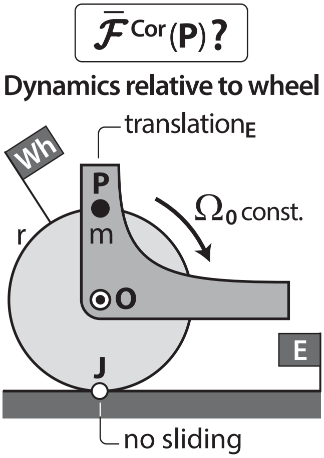


1.48 A particle **P** is glued to the arm, which is articulated to a wheel that rotates without sliding on the ground E with constant angular velocity. If the **P** dynamics is studied in the wheel reference frame (Wh), what is the Coriolis inertial force on **P**?

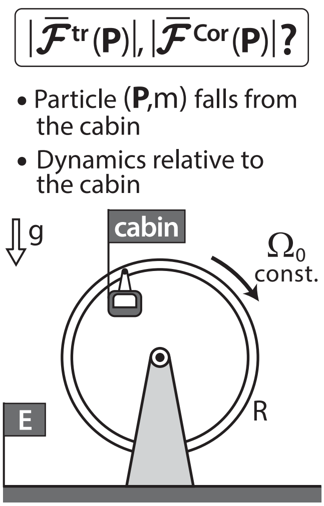
A 0

B $4mr\Omega_0^2 \downarrow$

C $4mr\Omega_0^2 \uparrow$

D $2mr\Omega_0^2 \uparrow$

E $2mr\Omega_0^2 \downarrow$



1.49 A particle **P** falls from the cabin of a Ferris wheel whose ring rotates with constant angular velocity relative to the ground (E). If its dynamics is studied in the cabin reference frame, and the cabin does no oscillate, how are the transportation and Coriolis inertial forces on **P**?

A There is no need to take into account any inertial forces as the cabin does not rotate relative to the ground and so is a Galilean reference frame.

B $\left|\overline{\mathscr{F}}^{tr}_{Cabin\to\mathbf{P}}\right| = mR\Omega_0^2$, $\left|\overline{\mathscr{F}}^{Cor}_{Cabin\to\mathbf{P}}\right| = 0$ all the time.

C Initially $\left|\overline{\mathscr{F}}^{tr}_{Cabin\to\mathbf{P}}\right| = mR\Omega_0^2$ and $\left|\overline{\mathscr{F}}^{Cor}_{Cabin\to\mathbf{P}}\right| = 0$, and then they both increase while **P** falls.

D Initially $\left|\overline{\mathscr{F}}^{tr}_{Cabin\to\mathbf{P}}\right| = mR\Omega_0^2$ and then it increases while **P** falls; $\left|\overline{\mathscr{F}}^{Cor}_{Cabin\to\mathbf{P}}\right| = 0$ all the time.

E $\left|\overline{\mathscr{F}}^{tr}_{Cabin\to\mathbf{P}}\right| = mR\Omega_0^2$ all the time; $\left|\overline{\mathscr{F}}^{Cor}_{Cabin\to\mathbf{P}}\right| = 0$ initially and then it increases while **P** falls.

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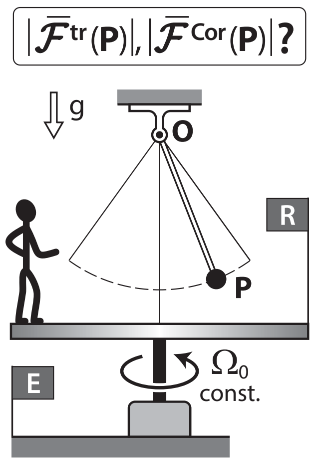
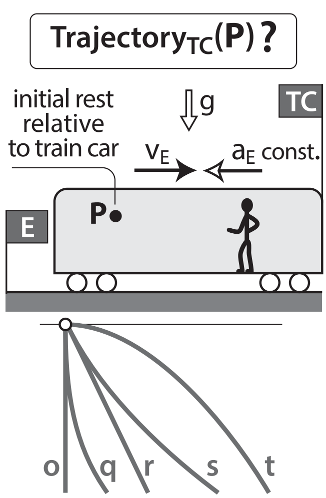


1.50 A pendulum articulated to the ground-fixed point **O** and whose mass is concentrated at its endpoint **P**, oscillates with an amplitude $< 90^o$. The platform rotates around the vertical axis through **O** with constant angular velocity relative to the ground (E). If the **P** dynamics is studied in the platform reference frame R, are the inertial transportation and Coriolis inertial forces on **P** zero or nonzero?

A Both are nonzero all the time.
B There is no need to take into account any inertial forces as the pendulum is articulated to the ground, not to the platform.
C $\left|\overline{\mathscr{F}}^{tr}_{R\to\mathbf{P}}\right| = 0$ only when the pendulum attains the extreme positions, and $\left|\overline{\mathscr{F}}^{Cor}_{R\to\mathbf{P}}\right| \neq 0$ all the time.
D $\left|\overline{\mathscr{F}}^{tr}_{R\to\mathbf{P}}\right| \neq 0$ all the time, and $\left|\overline{\mathscr{F}}^{Cor}_{R\to\mathbf{P}}\right| = 0$ only when the pendulum attains the extreme positions.
E $\left|\overline{\mathscr{F}}^{tr}_{R\to\mathbf{P}}\right| = 0$ only when the pendulum crosses the vertical through **O**, and $\left|\overline{\mathscr{F}}^{Cor}_{R\to\mathbf{P}}\right| \neq 0$ all the time.



1.51 A particle **P** falls from the ceiling of a train car which has a translation motion with constant braking acceleration relative to the ground (E). Which trajectory may correspond to the particle trajectory relative to the train car?

A o
B q
C r
D s
E t

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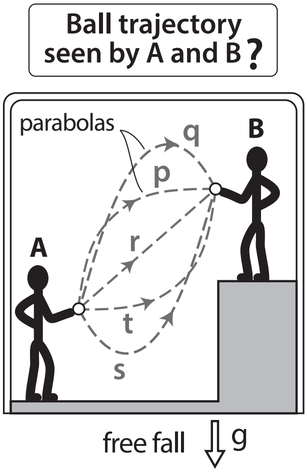
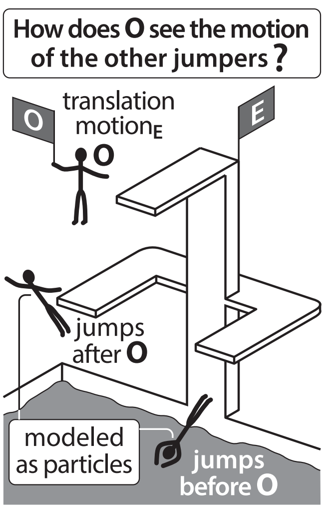


1.52 Two observers throw each other a ball in a freely falling elevator. If there is no air friction and the ball does not collide with the elevator between throws, which trajectory may correspond to the ball trajectory relative to the elevator?

A p
B q
C r
D s
E t



1.53 A person O jumps from a poll trampoline and acquires a translational motion relative to the ground (E). He observes other people (modeled as particles) who have also jumped from the trampoline and are in the air. How does O see their motion?

A He sees all of them at rest.
B He sees all of them either in rectilinear motion or at rest.
C He sees all of them either in parabolic motion or at rest.
D He only sees at rest those that have jumped simultaneously with him.
E He only sees rectilinear motion for those that have jumped simultaneously with him.

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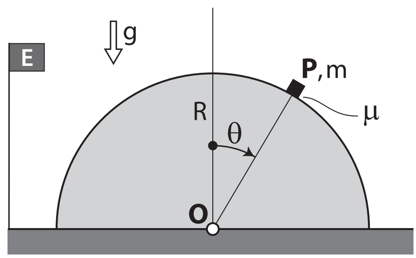
Exercises

1.1 A particle **P** with mass m is in contact with a cylindrical surface with radius R fixed in the ground (E). The friction coefficient between particle and surface is $\mu(= \mu_s = \mu_k)$. Consider just planar motion and find:

(a) The maximum value θ_{max} for which **P** may be at rest.
(b) The equation of movement for coordinate θ for the sliding downward motion (with increasing θ).
(c) The angle θ_{cont} for which **P** loses ground contact if $\mu = 0$ and **P** starts from rest at $\theta = 0$.

Hint: the change of variables $\ddot{\theta} = \dot{\theta}\frac{d\dot{\theta}}{d\theta}$ leads to a differential equation which may be integrated to obtain $\dot{\theta}(\theta)$.



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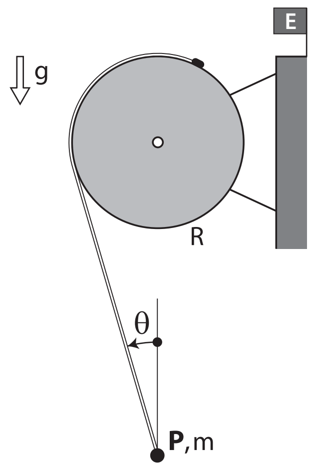
1.2 Circumference-envelope pendulum

A particle **P** with mass m is located at the endpoint of an inextensible thread partially rolled on a cylindrical support fixed in the ground. For $\theta = 0$, the length of the free portion of the thread is L. Consider just planar motion and find:

(a) The equation of movement for coordinate θ.
(b) The thread tension F as a function of θ and $\dot{\theta}$.



1.3 Cycloidal pendulum

Conventional pendulums have a drawback: their oscillation frequency depends on the oscillation amplitude (they are not strictly *isochronous*). The oscillations of a particle moving in a smooth cycloidal slot on a vertical plane are perfectly isochronous. Such a motion can be achieved through a simple pendulum consisting of a particle **P** with mass m attached to the endpoint of an inextensible thread suspended from a ground-fixed point **O** provided that the thread leans on a ground-fixed cycloidal surface. For this pendulum find:

(a) The equation of movement for coordinate θ.
(b) The thread tension F as a function of θ and $\dot{\theta}$.
(c) The equation of movement for coordinate $s = 4r\sin(\theta/2)$, which corresponds to the cycloid length from **O**′.

Hint: Solve again (from scratch) the problem with coordinate s (position of **P** along its cycloidal trajectory, with $s = 0$ when the thread is vertical). The expression of that coordinate as a function of the angle θ may be easily obtained from the **P** speed $\dot{s}(\theta, \dot{\theta})$ relative to the ground through integration.

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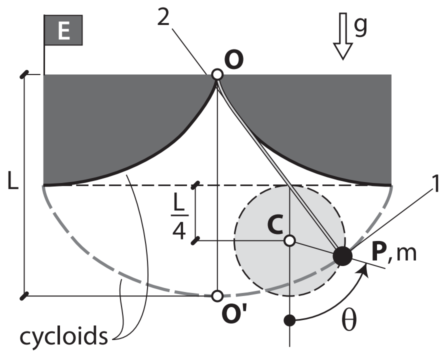
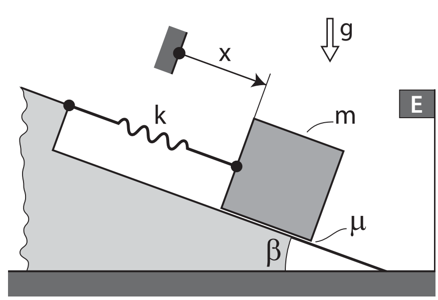
Note: Any point on the rim of a rigid wheel with a planar and nonsliding motion on the ground follows a cycloidal trajectory. Thus, the **P** trajectory can be seen as that of a point on the rim of the hypothetical wheel with center **C** and radiusr $= L/4$.



1.4 The block is on a slope and attached to the ground through a linear spring with constant k. The friction coefficient between ground and block is $\mu (= \mu_s = \mu_d)$. The block location is described through coordinate x, whose origin corresponds to a zero tension in the spring. Find:

(a) The range $[x_{min}, x_{max}]$ for which the block may be at rest (equilibrium interval).
(b) The equations of movement of the downward and the upward motions ($\dot{x} > 0$ and $\dot{x} < 0$, respectively).
(c) The spring constant k (as a function of the block weight mg) if its length increases by 10 cm when the block is hung from the ceiling by the spring.
(d) The permanent rest position of the block if it starts at rest from x = 21 cm, $\beta = 53.13°$, and $\mu = 0.5$.



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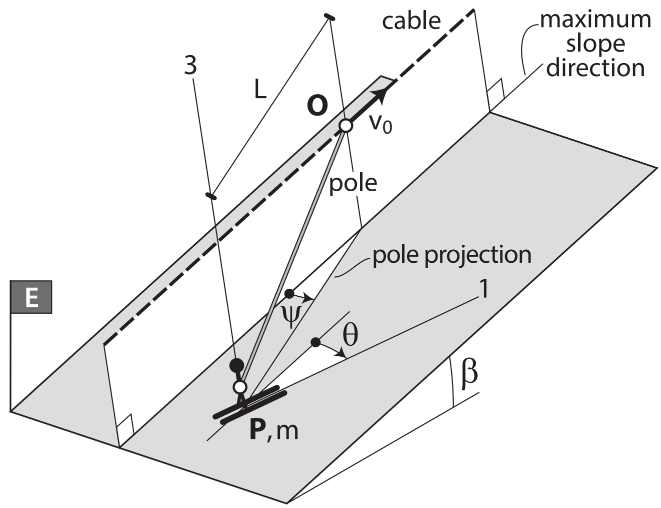
1.5 A ski lift drags a skier on a planar surface with slope β through a pole whose upper end **O** moves with constant speed v_0 relative to the ground (E) on a straight line parallel to the maximum slope direction. The pole has a negligible mass, length L, and its projection on the slope defines an angle ψ from the direction of the **O** motion. The skier controls the angle θ defined by the skis from that same direction. If the skier is modeled as a particle **P** with mass m, the skis do not skid and the friction coefficient between skis and ground is zero in the skis longitudinal direction, find:

(a) The skier's speed $v_E(\mathbf{P})$ relative to the ground, and the angular velocity $\dot{\psi}$ as a function of v_0 , ψ and θ.

(b) The radius of curvature of the skier's trajectory relative to the ground $(\mathfrak{R}_E(\mathbf{P}))$.



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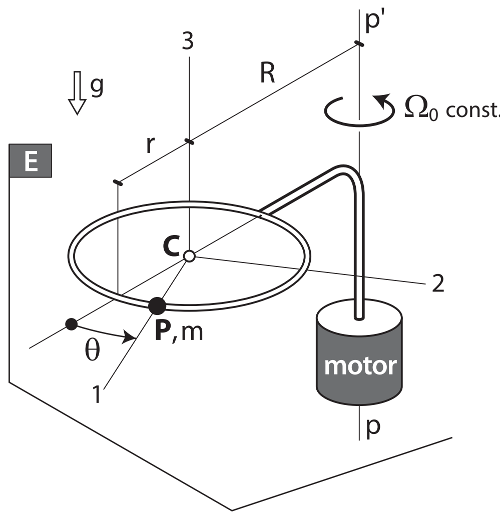
1.6 Centrifugal pendulum

A particle **P** with mass m slides in a circular smooth slot with radius r, which in turn rotates with constant angular velocity Ω_0 about the vertical axis p–p′ relative to the ground (E). Find:

(a) The equation of movement for coordinate θ.
(b) The magnitude of the horizontal and vertical components of the constraint force on **P** (F_2, F_3).
(c) The angular frequency ω of small-amplitude oscillations about the configuration $\theta = 0$ (which is an equilibrium configuration relative to the slot).



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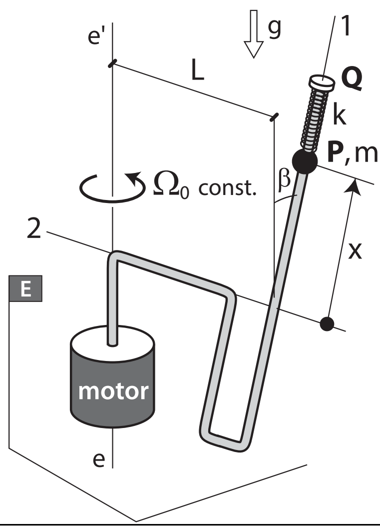
1.7 A particle **P** with mass m slides along an inclined straight smooth guide which in turn rotates with constant angular velocity Ω_0 about the vertical axis e–e′ relative to the ground (E). The particle is attached to a linear spring whose other endpoint is attached to the upper point **Q** of the guide. The coordinate x describes the particle location relative to the guide. When $\Omega_0 = 0$, the location $x = 0$ corresponds to an equilibrium position. Find:

(a) The equation of movement for coordinate x. What kind of motion does it correspond to?

(b) The components of the constraint force $\overline{\mathbf{F}}^{\text{const}}$ on **P**.



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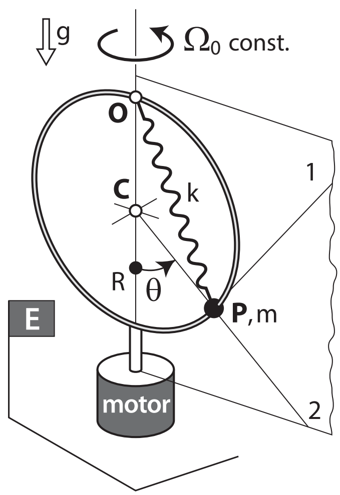
1.8 A particle **P** with mass m slides in a circular smooth slot with radius r, which in turn rotates with constant angular velocity Ω_0 about a vertical axis relative to the ground (E). A linear spring with constant k connects the particle to the upper point **O** of the guide, and introduces an attraction force between **P** and **O** strictly proportional to $|\overline{\mathbf{OP}}|$. Find:

(a) The equation of movement for coordinate θ.
(b) The components of the constraint force $\overline{\mathbf{F}}^{\text{const}}$ on **P**.
(c) The equilibrium configurations θ_{eq} relative to the guide (that is, the θ values which may remain constant), and the frequency of small-amplitude oscillations around $\theta_{eq} = 0$ and $\theta_{eq} = 180°$ when those configurations are stable.



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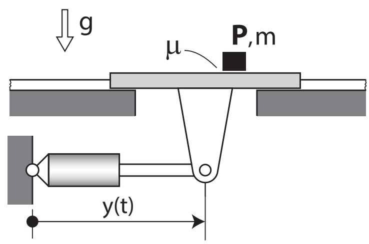
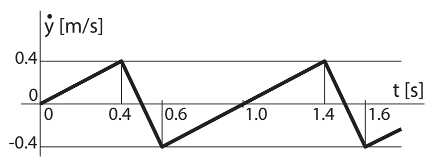
1.9 Vibrating feeder

A particle **P** with mass m is in contact with a horizontal platform (vibrating feeder) which moves under the action of a hydraulic cylinder. That actuator prescribes the periodic speed $\dot{y}(t)$ relative to the ground (E) plotted below. The friction coefficient between platform and particle is $\mu = 0.15$. Initially the particle is at rest relative to the platform. Assume that $g = 10\ \mathrm{m/s^2}$ and find:

(a) The time instant when **P** starts sliding on the platform, and that when the sliding stops.

(b) The maximum speed $v_{\mathrm{Plat}}(\mathbf{P})]_{\max}$ and displacement of **P** relative to the platform at each cycle $\left(\Delta d_{\mathrm{Plat}}(\mathbf{P})]_{\mathrm{cycle}}\right)$.





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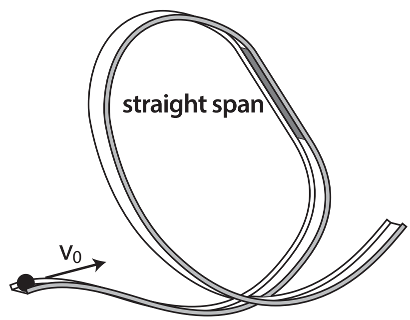
Puzzles

1.1 The loop

The loop is curved everywhere except for a short straight span. Between the loop and the particle **P** there is a unilateral constraint.

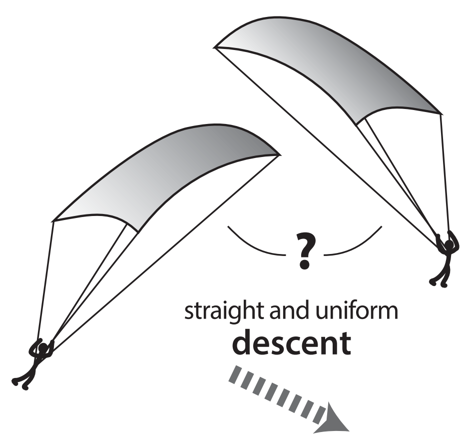
If the initial velocity of P is high enough, will it be able to go through the entire loop without losing contact?



1.2 Paragliding

A paraglider performs a rectilinear and uniform descent in a windless day.

Which image may represent the configuration of the system (paraglider+pilot)?

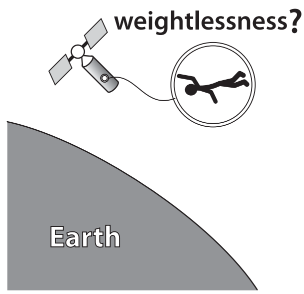


1.3 "Zero gravity" in the orbital station

Objects are said to be in "zero gravity" conditions inside an orbital station because they float in the air.

But at a distance of 300 km from the Earth's surface, the gravitational field is roughly 90 percent of that at the Earth's surface.

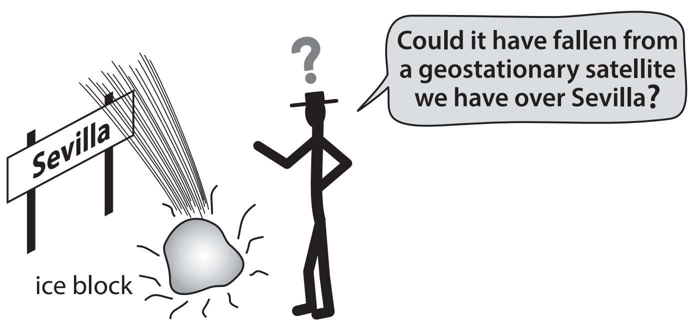
Can you explain this apparent inconsistency?



1.4 Ice blocks falling from the sky

This is what was said regarding the falling of some ice blocks of unknown origin on the Spanish city of Sevilla.

Are there any reasons why it could not be so?



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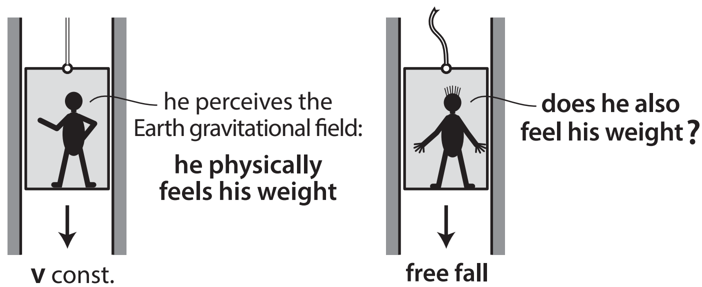
1.5 The falling elevator

When an elevator is either at rest or moving with constant speed relative to the ground, people inside the elevator physically feel the Earth's gravitational field.

If the elevator falls freely, **would they still feel it? Under what conditions do we physically feel that gravitational field?**



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Quiz Questions: Answers

	1	2	3	4	5	6	7	8	9	10
	D	D	B	E	E	D	D	C	B	E
+10	B	E	A	B	E	A	A	A	C	E
+20	A	E	D	C	E	A	A	B	C	C
+30	C	A	B	C	A	A	B	D	E	A
+40	E	D	A	A	B	C	D	E	B	E
+50	C	C	B							

Exercises: Results

1.1 (a) $\theta_{max} = \arctan \mu$

(b) $\ddot{\theta} = \frac{g}{R}(\sin\theta - \mu\cos\theta) + \mu\dot{\theta}^2$

(c) $\theta_{cont} = \arccos\left[\frac{1}{3}\left(2 + \frac{v_0^2}{gR}\right)\right]$ with $v_0 \leq \sqrt{gR}$; $\theta_{cont} = 0$ with $v_0 > \sqrt{gR}$

1.2 (a) $\ddot{\theta}(L - R\theta) = -g\sin\theta + R\dot{\theta}^2$

(b) $F = m\left[g\cos\theta + (L - r\theta)\dot{\theta}^2\right]$

1.3 (a) $\ddot{\theta} = -\frac{1}{2}\left(\frac{g}{r} - \dot{\theta}^2\right)\tan\frac{\theta}{2}$

(b) $F = m\left(g + r\dot{\theta}^2\right)\cos\frac{\theta}{2}$

(c) $\ddot{s} = -\frac{g}{L}s$

1.4 (a) $x_{min} \leq x \leq x_{max}$ with $x_{min} = \frac{mg}{k}(\sin\beta - \mu\cos\beta)$ and $x_{max} = \frac{mg}{k}(\sin\beta + \mu\cos\beta)$

(b) Downward motion $(\dot{x} > 0)$: $\ddot{x} + \frac{k}{m}x = g(\sin\beta - \mu\cos\beta)\left[= \frac{k}{m}x_{min}\right]$; sinusoidal oscillation with angular frequency $\omega = \sqrt{\frac{k}{m}}$ and equilibrium position x_{min}.

Upward motion $(\dot{x} < 0)$: $\ddot{x} + \frac{k}{m}x = g(\sin\beta + \mu\cos\beta)\left[= \frac{k}{m}x_{max}\right]$; sinusoidal oscillation with angular frequency $\omega = \sqrt{\frac{k}{m}}$ and equilibrium position x_{max}.

(c) $k = 10\,mg$ [N/m]

(d) $x = 9$ cm (it goes up to $x = 1$ cm and then down to $x = 9$ cm).

1.5 (a) $v_E(\mathbf{P}) = v_0\frac{\cos\psi}{\cos(\psi+\theta)} = v_0[\cos\theta + \sin\theta\tan(\psi+\theta)]$; $\dot{\psi} = \frac{v_0}{L}\frac{\sin\theta}{\cos(\psi+\theta)}$

(b) $\mathfrak{R}_E(\mathbf{P}) = \frac{v_0}{\dot{\theta}}\frac{\cos\psi}{\cos(\psi+\theta)}$

1.6 (a) $\ddot{\theta} = -\frac{R}{r}\Omega^2 \sin\theta$

(b) $F_2 = -m\left[r\left(\Omega + \dot{\theta}\right)^2 + R\Omega^2 \cos\theta\right]$; $F_3 = mg$

(c) $\omega = \Omega\sqrt{\frac{R}{R}}$

1.7 (a) $\ddot{x} = -\left(\omega^2 - \Omega^2 \sin\beta\right)x$ with $\omega^2 = \frac{k}{m}$

For $|\Omega \sin\beta| < \omega$, the motion is a sinusoidal oscillation with angular frequency $\sqrt{\omega^2 - \Omega^2 \sin\beta}$.

For $|\Omega \sin\beta| > \omega$, the equilibrium configuration $x = 0$ is unstable, hence there is no oscillation.

(b) $$\left\{\overline{\mathbf{F}}^{\text{const}}\right\} = \left\{\begin{array}{c} 0 \\ m\left(2\Omega\dot{x}\sin\beta + \Omega^2 R\right) \\ m\left(g + \Omega^2 x\cos\beta\right)\text{in}\beta \end{array}\right\}$$

1.8 (a) $\ddot{\theta} = \left(-\frac{g}{R} + \frac{k}{m} + \Omega^2 \cos\theta\right)\sin\theta$

(b) $$\left\{\overline{\mathbf{F}}^{\text{const}}\right\} = \left\{\begin{array}{c} 0 \\ -m\left(g\cos\theta + \Omega^2 R\sin^2\theta + \dot{\theta}^2 R\right) + kR(1 + \cos\theta) \\ 2m\Omega R\dot{\theta}\cos\theta \end{array}\right\}$$

(c) $\theta_{eq} = \pm 2n\pi$ with $n = 0, 1, 2\ldots$

If $\left(\frac{g}{R} - \frac{k}{m}\right) > \Omega^2, \theta_{eq}$ is a stable configuration and **P** oscillates with $\omega = \sqrt{\frac{g}{R} - \frac{k}{m} - \Omega^2}$.

$\theta_{eq} = \pm(1 + 2n)\pi$ with $n = 0, 1, 2\ldots$

If $\left(\frac{k}{m} - \frac{g}{R}\right) > \Omega^2, \theta_{eq}$ is a stable configuration and **P** oscillates with $\omega = \sqrt{\frac{k}{m} - \frac{g}{R} - \Omega^2}$.

$\theta_{eq} = \arccos\left(\frac{(g/R)-(k/m)}{\Omega^2}\right) + 2n\pi$, with $n = 0, 1, 2\ldots$ This family of equilibrium configurations exists only if $\left|\frac{k}{m} - \frac{g}{R}\right| < \Omega^2$, and its stability can be proved.

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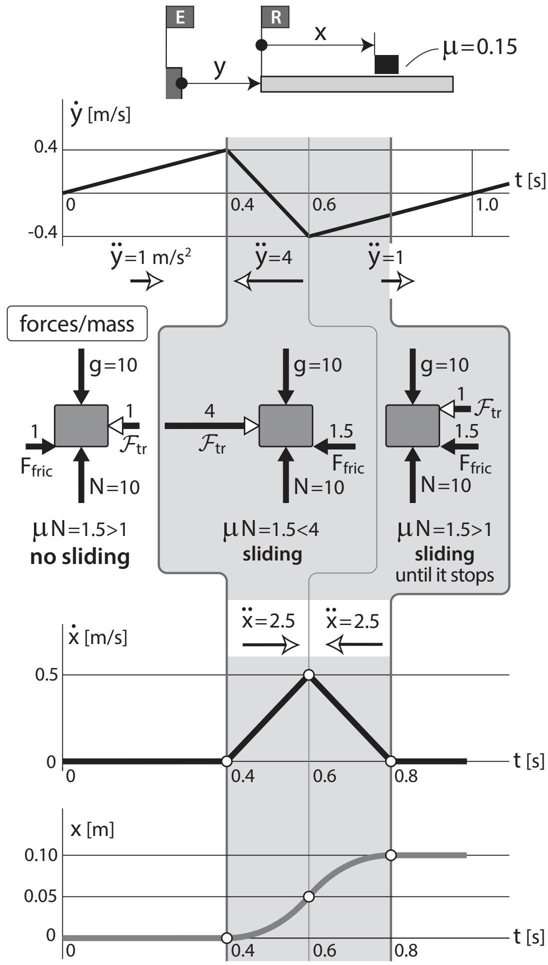
1.9 (a) It slides in the time interval $t \in [0.4 + n, 0.8 + n]$s with $n = 0, 1, 2 \ldots$

(b) $v_{Plat}(\mathbf{P})]_{max} = 0.5$ m/s; $\Delta d_{Plat}(\mathbf{P})]_{cycle} = 10$ cm



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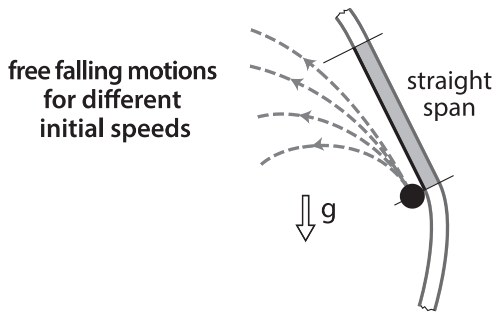
Puzzles: Solutions

1.1 The loop

P loses contact with the loop as soon as it enters the straight span. Following that span would imply a zero normal acceleration, hence a zero net force perpendicular to the span.

As the normal constraint force from the looping on **P** and the normal component of the weight have both the same sign, such zero net force is impossible.

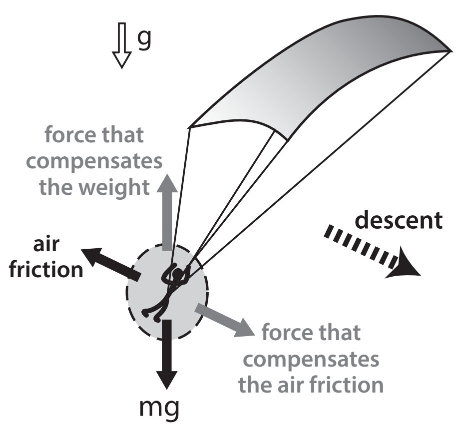


1.2 Paragliding

In a straight uniform descent in a windless day, the person flying the paraglider undergoes an air friction force opposite to her velocity relative to the ground.

As a straight uniform motion implies a zero acceleration, the net force on the person has to be zero: the friction force and the weight have to be compensated by the forces from the paraglider on the person.

Hence, the paraglider has to *pull* the person in the direction of the descent, and so it must be in an advanced position relative to the person.



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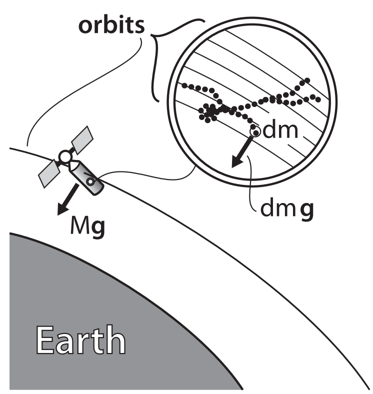
1.3 "Zero gravity" in the orbital station

As long as the station dimensions and the air friction are negligible, the station and all the objects in it are subjected exclusively to the Earth's gravitational attraction, which provokes the same acceleration $\bar{\mathbf{g}}$ to all of them (relative to the Earth). If the station does not modify its orientation relative to the distant galaxies and the object is initially at rest relative to the station, it will remain at rest. If its initial velocity relative to the station is nonzero, it will keep a rectilinear uniform motion relative to it. For this reason, we say that it "floats" inside the station.

An astronaut also "floats" inside the station and has the same weightlessness feeling as a person in an elevator falling freely (see puzzle 1.5) because every mass differential (dm) undergoes the amount of gravitational attraction (or weight) that justifies its Galilean acceleration. To be in orbit around the Earth is equivalent to be permanently falling without ever reaching the Earth's surface.



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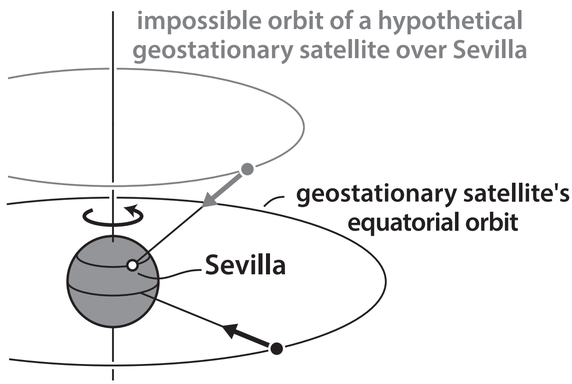
1.4 Ice blocks falling from the sky

There cannot be a geostationary satellite over Sevilla. A geostationary orbit can only be achieved on locations on the equatorial plane, because the Earth's gravitational field provides the centripetal force needed to generate the normal acceleration.

Objects cannot fall toward the Earth's surface from geostationary satellites. Objects inside those satellites are under geostationary conditions, therefore they float relative to the satellite. They are neither dragged nor supported by the satellite.



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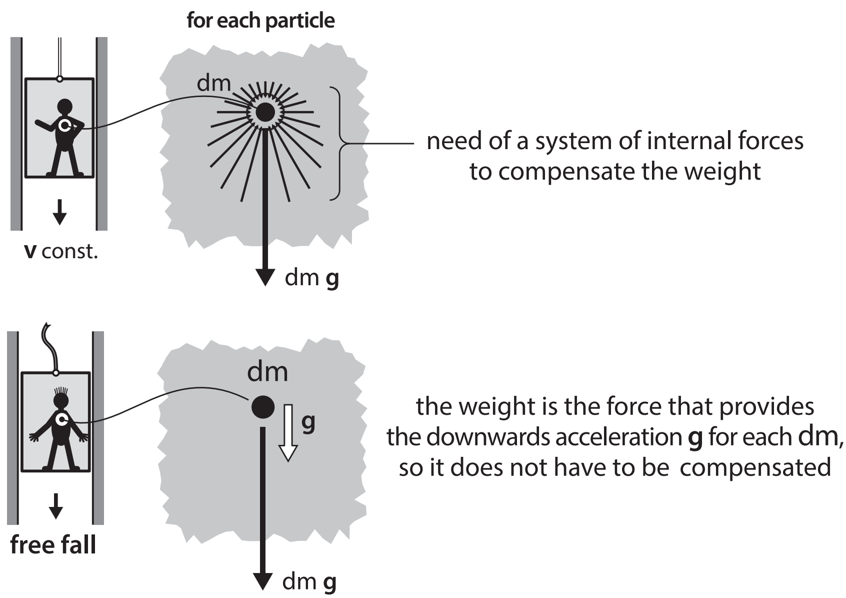
1.5 The falling elevator

When the elevator is at rest or moving with a constant velocity relative to a Galilean frame RGal, all the passenger's mass differentials dm have a zero acceleration relative to RGal. Hence, they have to be subjected to an interaction force with their surroundings equal to $-\mathrm{dm}\overline{\mathbf{g}}$ in order to compensate for their weight $\mathrm{dm}\overline{\mathbf{g}}$. In other words: there is a constant field of forces $-\mathrm{dm}\overline{\mathbf{g}}$, internal to the passenger, but we do not detect it because we are used to it.

When the elevator is in free fall, both it and the passenger have an acceleration $\overline{\mathbf{g}}$ relative to RGal, and every dm interacts with the Earth with a force $\mathrm{dm}\overline{\mathbf{g}}$ that provides that acceleration, so there is no need of an internal force field.

It is because we are so used to the internal field $(-\mathrm{dm}\overline{\mathbf{g}})$ that we "detect" its absence in a free fall as a feeling of "falling into the void".



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