6

Causal Variational Principles

The causal action principle as introduced in Section 5.6 has quite a rich structure and is rather involved. Therefore, it is difficult to analyze it in full generality in one step. It is preferable to begin with special cases and simplified situations and to proceed from there step by step. In fact, doing so leads to a whole class of variational principles, referred to as *causal variational principles*. These different variational principles capture different features and aspects of the causal action principle. Proceeding in this way also gives a better understanding of the physical meaning of the different structures of a causal fermion system and of the interaction as described by the causal action principle. We now give an overview of the different settings considered so far. This has the advantage that in the later chapters of this book, we can always work in the setting that is most suitable for the particular question in mind. Moreover, for pedagogical reasons, in this book, we shall sometimes idealize the setting, for example, by assuming for technical simplicity that the Lagrangian is smooth.

6.1 The Causal Variational Principle on the Sphere

Clearly, the trace constraint (5.38) and the boundedness constraint (5.39) complicate the analysis. Therefore, it might be a good idea to consider a simplified setting where these constraints are not needed. This can be accomplished most easily by *prescribing the eigenvalues* of the operators in $\mathcal{F}$. This method was first proposed in [43, Section 2] in a slightly different formulation. We now explain the method in a way that best fits to our setting. Given $n \in \mathbb{N}$, we choose real numbers $\nu_1, \ldots, \nu_{2n}$ with

$$\nu_1 \leq \cdots \leq \nu_n \;\leq 0 \;\leq\; \nu_{n+1} \leq \ldots \leq \nu_{2n} \,. \tag{6.1}$$

We let $\mathcal{F}$ be the set of all symmetric operators on $\mathcal{H}$ of rank $2n$ whose eigenvalues (counted with multiplicities) coincide with $\nu_1, \ldots, \nu_{2n}$. If $\mathcal{H}$ is finite-dimensional, the set $\mathcal{F}$ is compact. This is the reason why it is sensible to minimize the causal action (5.36) keeping only the volume constraint (5.37), which for simplicity we implement by restricting attention to normalized measures,

$$\rho(\mathcal{F}) = 1 \,. \tag{6.2}$$

Note that, since $\mathcal{F}$ is compact and the Lagrangian $\mathcal{L}$ is continuous on $\mathcal{F} \times \mathcal{F}$, also the action $S(\rho)$ is finite for any normalized measure ρ.

The simplest interesting case is obtained by choosing the spin dimension $n = 1$ and the Hilbert space $\mathcal{H} = \mathbb{C}^2$. In this case, according to (6.1), we have one nonnegative and one nonpositive eigenvalue. If these eigenvalues have the same absolute value, all the operators have trace zero. This case is not of interest because there are trivial minimizers (for details, see Example 12.4.1 in Section 12.4). With this in mind, it suffices to consider the case that $|\nu_1| \neq |\nu_2|$. Since scaling all the eigenvalues in (6.1) by a real constant does not change the essence of the variational principle, it is no loss of generality to assume that the two eigenvalues ν_1, ν_2 satisfy the relation $\nu_1 + \nu_2 = 2$, making it possible to parametrize the eigenvalues by

$$\nu_{1/2} = 1 \mp \tau \qquad \text{with} \qquad \tau \geq 1 \,. \tag{6.3}$$

Then, $\mathcal{F}$ consists of all Hermitian 2×2-matrices F that have eigenvalues ν_1 and ν_2. These matrices can be represented using the Pauli matrices by

$$\mathcal{F} = \left\{ F = \tau\, \vec{x}\vec{\sigma} + \mathbb{1} \quad \text{with} \quad \vec{x} \in S^2 \subset \mathbb{R}^3 \right\}. \tag{6.4}$$

Thus, the set $\mathcal{F}$ can be identified with the unit sphere S^2.

The volume constraint (5.37) can be implemented most easily by restricting attention to *normalized* regular Borel measures on $\mathcal{F}$ (i.e., measures with $\rho(\mathcal{F}) = 1$). Such a measure can be both continuous, discrete or a mixture. Examples of *continuous measures* are obtained by multiplying the Lebesgue measure on the sphere by a nonnegative smooth function. By a *discrete measure*, on the other hand, we here mean a *weighted counting measure*, that is, a measure obtained by inserting weight factors into (5.22),

$$\rho = \sum_{i=1}^{L} c_i\, \delta_{x_i} \qquad \text{with} \qquad x_i \in \mathcal{F}\,, \quad c_i \geq 0 \quad \text{and} \quad \sum_{i=1}^{L} c_i = 1\,. \tag{6.5}$$

A straightforward computation yields for the Lagrangian (5.35) (see Exercise 6.1)

$$\begin{aligned} &\mathcal{L}(x,y) = \max\big(0, \mathcal{D}(x,y)\big) \qquad \text{with} \\ &\mathcal{D}(x,y) = 2\tau^2\, \big(1 + \langle x,y \rangle\big) \Big(2 - \tau^2\, \big(1 - \langle x,y \rangle\big)\Big)\,, \end{aligned} \tag{6.6}$$

and $\langle x,y \rangle$ denotes the Euclidean scalar product of two unit vectors $x, y \in S^2 \subset \mathbb{R}^3$ (thus $\langle x,y \rangle = \cos\vartheta$, where ϑ is the angle between x and y).

The resulting so-called *causal variational principle on the sphere* was introduced in [43, Chapter 1] and analyzed in [74, Sections 2 and 5] and more recently in [10]. We now explain a few results from these papers.

First of all, the causal variational principle on the sphere is well posed, meaning that the minimum is attained in the class of all normalized regular Borel measures (the proof of this statement will be given in Chapter 12 using measure-theoretic methods to be developed later in this book). Minimizing numerically in the class of weighted counting measures for increasing number L of points and different values of the parameter τ, the resulting minimal value of the action has an interesting non-smooth structure shown in Figure 6.1. In particular, one finds that the minimizing measure is not unique; indeed, there are typically many minimizers. From the mathematical perspective, this nonuniqueness can be understood from the fact

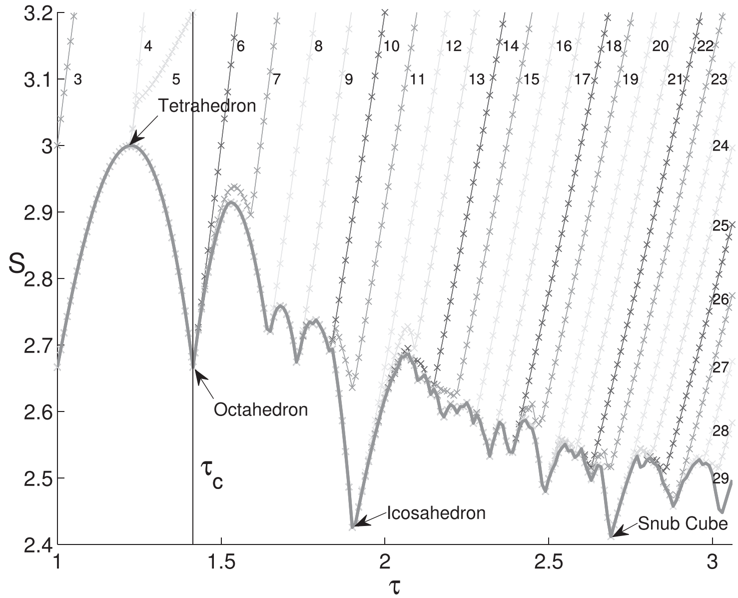

Figure 6.1 Numerical minima for the weighted counting measure on the sphere. From [80]. Reproduced with permission from Springer Nature.

that the causal action principle is a *non-convex* variational principle where one cannot expect uniqueness. To give a concrete example for the nonuniqueness, we note that in the case $\tau = \tau_c = \sqrt{2}$, there is a minimizing measure that is supported on an octahedron (for details, see [74, Section 2]). This minimizing measure is not unique because every measure obtained from it by a rotation in SO(3) is again a minimizer. But the nonuniqueness goes even further in the sense that there are pairs of minimizing measures that cannot be obtained from each other by a rotation or reflection. For example, again in the case $\tau = \sqrt{2}$, the normalized Lebesgue measure on the sphere is also a minimizer.

Moreover, the study in [74, Section 2] gives the following

numerical result: If $\tau > \sqrt{2}$, every minimizing measure is a weighted counting measure (6.5).

Thus, although we minimize over all regular Borel measures (i.e., measures that can have discrete and continuous components), a minimizing measure always describes a discrete spacetime consisting of a finite number of spacetime points. This result can be interpreted physically as an indication that the causal action principle should give rise to discrete spacetime structures. More details on the numerical findings and the physical interpretation can be found in the review [80, Section 4]. A more advanced numerical study of the causal action principle in low dimensions can be found in [83].

The above-mentioned numerical findings can be underpinned by analytic results. We finally mention some of these results, although they will not be needed later on, and the methods of proof will not be covered in this book. First, it was proven in [74, Section 5.1] that the support has an empty interior:

Theorem 6.1.1 *If $\tau > \sqrt{2}$, the support of any minimizing measure does not contain an open subset of S^2.*

Intuitively speaking, this result shows that the spacetime points are a subset of the sphere of dimension strictly smaller than two. More recently, it was shown in [10] that the dimension of the support is even strictly smaller than one:

Theorem 6.1.2 *In the case $\tau > \sqrt{6}$, the support of any minimizing measure is totally disconnected and has a Hausdorff dimension at most $6/7$.*

The proof of these theorems uses techniques that will not be covered in this book. Therefore, we refer the reader interested in more details to the papers cited earlier.

6.2 Causal Variational Principles in the Compact and Smooth Settings

Generalizing the causal variational principle on the sphere, one can replace $\mathcal{F}$ by a smooth compact manifold of dimension $m \geq 1$.

Definition 6.2.1 *Let $\mathcal{F}$ be a smooth compact manifold of dimension $m \geq 1$ and $\mathcal{D} \in C^\infty(\mathcal{F} \times \mathcal{F}, \mathbb{R})$. Define the Lagrangian $\mathcal{L} \in C(\mathcal{F} \times \mathcal{F}, \mathbb{R}_0^+)$ by*

$$\mathcal{L}(x, y) := \max\big(0, \mathcal{D}(x, y)\big), \tag{6.7}$$

and assume that $\mathcal{L}$ has the following properties:

(i) *$\mathcal{L}$ is symmetric: $\mathcal{L}(x, y) = \mathcal{L}(y, x)$ for all $x, y \in \mathcal{F}$.*
(ii) *$\mathcal{L}$ is strictly positive on the diagonal: $\mathcal{L}(x, x) > 0$ for all $x \in \mathcal{F}$.*

The **causal variational principle in the compact setting** *is to minimize the causal action*

$$\mathcal{S} = \int_{\mathcal{F}} d\rho(x) \int_{\mathcal{F}} d\rho(y)\, \mathcal{L}(x, y), \tag{6.8}$$

under variations of measures ρ in the class of all regular Borel measures on $\mathcal{F}$ that are normalized, that is,

$$\rho(\mathcal{F}) = 1\,. \tag{6.9}$$

This setting was introduced in [74, Section 1.2]. It is the preferable choice for studying phenomena for which the detailed form of the Lagrangian as well as the constraints of the causal action principle are irrelevant. Note also that in the compact setting the action $S(\rho)$ is finite for any normalized measure ρ because $\mathcal{F}$ is compact and $\mathcal{L}$ is continuous.

Given a minimizing measure ρ, the Lagrangian induces on spacetime $M := \operatorname{supp} \rho$ a causal structure. Namely, two spacetime points $x, y \in M$ are said to be *timelike* and *spacelike* separated if $\mathcal{L}(x, y) > 0$ and $\mathcal{L}(x, y) = 0$, respectively. But, of course, compared to the causal action principle for causal fermion systems, spin spaces and physical wave functions (as defined in Section 5.7) are missing in this setting.

We point out that in (6.7), we merely assumed that the function $\mathcal{D}$ is smooth. The Lagrangian, however, is only Lipschitz continuous. It is in general non-differentiable on the boundary of the light cone as defined by the level set $\mathcal{D}(x,y)=0$. In order to avoid differentiability issues, it is sometimes useful to simplify the setting even further by assuming that the Lagrangian itself is smooth,

$$\mathcal{L} \in C^\infty(\mathcal{F} \times \mathcal{F}, \mathbb{R}^+_0) . \tag{6.10}$$

This is the so-called **smooth setting**. We point out that the Lagrangian of the causal action (5.35) is not smooth if some of the eigenvalues vanish or are degenerate (more precisely, the causal Lagrangian is only Hölder continuous, as is worked out in detail in [67, Section 5]). Indeed, this non-smoothness yields interesting effects like the results on the singular support in [74, 10]. In view of these results, the smoothness assumption (6.10) is a mathematical simplification that, depending on the application in mind, may or may not be justified. In this book, we choose the smooth setting mainly for pedagogical reasons, keeping in mind that the generalizations to non-smooth Lagrangians are rather straightforward. The reader who is interested in or needs these generalizations will find the details in the research papers.

Before going on, we point out that the assumptions that $\mathcal{F}$ is a smooth manifold and that the function $\mathcal{D}$ in (6.7) is smooth are convenient and avoid certain technicalities. But these assumptions are much more than what is needed for the analysis. More generally, one can choose $\mathcal{L}$ as a nonnegative continuous function,

$$\mathcal{L} \in C^0(\mathcal{F} \times \mathcal{F}, \mathbb{R}^+_0) . \tag{6.11}$$

Going one step further, one may relax the continuity of the Lagrangian by the condition that $\mathcal{L}$ should be *lower semicontinuous*, that is, for all sequences $x_n \to x$ and $y_{n'} \to y$,

$$\mathcal{L}(x,y) \leq \liminf_{n,n' \to \infty} \mathcal{L}(x_n, y_{n'}) . \tag{6.12}$$

Since the Lagrangian of the causal action (5.35) is continuous, lower semicontinuity is an unphysical generalization. Nevertheless, this setting is useful for two reasons: First, from the point of view of the calculus of variations, it is a natural generalization to which most methods still apply. And second, lower semicontinuous Lagrangians are convenient for formulating explicit examples (like the lattice model in [62, Section 5]).

We finally note also that the assumption of $\mathcal{F}$ being a smooth manifold can be relaxed. From the point of view of the calculus of the variations, the right setting is to assume that $\mathcal{F}$ is a compact topological Hausdorff space.

In this book, for pedagogical reasons, we do not aim for the highest generality and minimal smoothness and regularity assumptions. An introduction to a more general and more abstract setting can be found in [66, Section 3].

6.3 Causal Variational Principles in the Non-compact Setting

In the compact setting, spacetime M is a compact subset of $\mathcal{F}$. This is not suitable for describing situations when spacetime has an asymptotic future or past or when spacetime has singularities (like at the big bang or inside a black hole). For studying such situations, it is preferable to work in the so-called *non-compact setting* introduced in [62, Section 2.1], where $\mathcal{F}$ is chosen to be a non-compact manifold.

In the non-compact setting, it is not sensible to restrict attention to normalized measures. Instead, the total volume $\rho(\mathcal{F})$ is typically infinite. In this situation, the causal action (6.8) could also be infinite. Therefore, we need to define in which sense a measure is a minimizer of the action.

Definition 6.3.1 (Causal variational principles in the non-compact setting) *Let $\mathcal{F}$ be a non-compact smooth manifold of dimension $m \geq 1$, and let $\mathcal{D} \in C^\infty(\mathcal{F} \times \mathcal{F}, \mathbb{R})$ be such that the Lagrangian $\mathcal{L} \in C^0(\mathcal{F} \times \mathcal{F}; \mathbb{R}^+_0)$ defined by* (6.7) *has the properties* (i) *and* (ii) *in Definition 6.2.1. Given a regular Borel measure ρ on $\mathcal{F}$, another regular Borel measure $\tilde{\rho}$ on $\mathcal{F}$ is a* **variation** *of ρ* **of finite volume** *if it satisfies the conditions*

$$\big|\tilde{\rho} - \rho\big|(\mathcal{F}) < \infty \qquad \text{and} \qquad \big(\tilde{\rho} - \rho\big)(\mathcal{F}) = 0\,, \tag{6.13}$$

where $|\tilde{\rho} - \rho|$ is the total variation measure (see Definition 2.3.5 in Section 2.3). For such a variation of finite volume, we consider the (formal) difference of the actions defined by

$$\begin{aligned}\big(\mathcal{S}(\tilde{\rho}) - \mathcal{S}(\rho)\big) := &\int_{\mathcal{F}} d(\tilde{\rho} - \rho)(x) \int_{\mathcal{F}} d\rho(y)\, \mathcal{L}(x,y) \\ &+ \int_{\mathcal{F}} d\rho(x) \int_{\mathcal{F}} d(\tilde{\rho} - \rho)(y)\, \mathcal{L}(x,y) \\ &+ \int_{\mathcal{F}} d(\tilde{\rho} - \rho)(x) \int_{\mathcal{F}} d(\tilde{\rho} - \rho)(y)\, \mathcal{L}(x,y)\,.\end{aligned} \tag{6.14}$$

The measure ρ is said to be a **minimizer** *of the causal action with respect to variations of finite volume if this difference is nonnegative for all $\tilde{\rho}$ satisfying* (6.13),

$$\big(\mathcal{S}(\tilde{\rho}) - \mathcal{S}(\rho)\big) \geq 0\,. \tag{6.15}$$

We note for clarity that integrals with respect to $\tilde{\rho} - \rho$ are defined by

$$\int_{\mathcal{F}} f(x)\, d(\tilde{\rho} - \rho)(x) := \int_{\mathcal{F}} f\ d\mu^+ - \int_{\mathcal{F}} f\ d\mu^- \tag{6.16}$$

with the finite measures $\mu_\pm$ as given in Definition 2.3.5. In particular, $(\tilde{\rho} - \rho)(\mathcal{F}) := \mu^+(\mathcal{F}) - \mu^-(\mathcal{F})$.

Exactly as mentioned at the end of the previous section, the assumption that $\mathcal{F}$ is a smooth manifold could be weakened. From the point of view of calculus of variations, the right setting is to assume that $\mathcal{F}$ is a σ-locally compact topological Hausdorff space (for details, see again [66, Section 3]).

6.4 The Local Trace Is Constant

Causal variational principles as introduced in the previous sections are of interest in their own right as a novel class of nonlinear variational principles. Nevertheless, since we are primarily interested in causal fermion systems, it is important to get a concise mathematical connection to the causal action principle. In preparation, we now analyze the trace constraint and derive a first general result on minimizing measures of the causal action principle. We present this result at such an early stage of this book because this result can be used to simplify the setup of causal fermion systems, getting the desired connection to causal variational principles (see Section 6.5). The following result was first obtained in [13] (albeit with a different method); see also [45, Proposition 1.4.1]. For technical simplicity, we restrict attention to the finite-dimensional setting (in the infinite-dimensional case, this problem has not yet been studied). Then, the total volume of spacetime as well as the minimal action are finite.

Proposition 6.4.1 *Consider the causal action principle in the finite-dimensional setting* $\dim \mathcal{H} < \infty$. *Let* ρ *be a minimizer of finite total volume,* $\rho(\mathcal{F}) < \infty$. *Then, there is a real constant* c *such that*

$$\mathrm{tr}(x) = c \qquad \textit{for all } x \in M := \operatorname{supp} \rho \,. \tag{6.17}$$

We often refer to $\mathrm{tr}(x)$ as the *local trace* at the point x.

Proof of Proposition 6.4.1. We will prove the theorem by contradiction and therefore assume that the local trace is *not* constant. The idea is to use this assumption to construct a suitable variation

$$(\rho_\tau)_{\tau \in (-\delta, \delta)} \qquad \text{with} \qquad \rho_0 = \rho, \tag{6.18}$$

which satisfies the constraints, but makes the action smaller, in contradiction to ρ being a minimizer.

For the construction of the variation, we combine two different general methods. One method is to multiply the measure ρ by a positive measurable function $f_\tau \,:\, M \to \mathbb{R}^+$,

$$\rho_\tau = f_\tau \, \rho, \tag{6.19}$$

(alternatively, one can also write this relation as $\mathrm{d}\rho_\tau(x) = f_\tau(x)\, \mathrm{d}\rho(x)$). Clearly, such a variation does not change the support of the measure. In order to change the support, our second method is to consider a measurable function $F_\tau \,:\, M \to \mathcal{F}$ and take the push-forward measure,

$$\rho_\tau = (F_\tau)_* \rho \,. \tag{6.20}$$

Combining these two methods, we are led to considering variations of the form

$$\rho_\tau = (F_\tau)_* \big(f_\tau \, \rho \big) \,. \tag{6.21}$$

The condition $\rho_0 = \rho$ gives rise to the conditions

$$f_0 \equiv 1 \qquad \text{and} \qquad F_0 \equiv \mathbb{1} \,. \tag{6.22}$$

Finally, we assume that the functions f_τ and F_τ are smooth in τ. The ansatz (6.21) is particularly convenient for computations. Namely, by definition of the push-forward measure,

$$\int_{\mathcal{F}} \mathcal{L}(x,y)\, d\rho_\tau(y) = \int_M \mathcal{L}\big(x, F_\tau(y)\big)\, f_\tau(y)\, d\rho(y)\,, \tag{6.23}$$

and similarly for all other integrals.

Next, for arbitrarily given f_τ, we want to choose F_τ in such a way that the last integral becomes independent of τ. To this end, we choose

$$F_\tau(x) := \frac{x}{\sqrt{f_\tau(x)}}\,. \tag{6.24}$$

Using that the causal Lagrangian $\mathcal{L}(x,y)$ is homogeneous of degree two in y (as is obvious from (5.35) and the fact that the eigenvalues λ_i^{xy} are homogeneous of degree one in y), it follows that

$$\int_M \mathcal{L}\big(x, F_\tau(y)\big)\, f_\tau(y)\, d\rho(y) = \int_M \mathcal{L}\bigg(x, \frac{y}{\sqrt{f_\tau(y)}}\bigg)\, f_\tau(y)\, d\rho(y) \tag{6.25}$$

$$= \int_M \mathcal{L}(x,y)\frac{1}{f_\tau(y)}\, f_\tau(y)\, d\rho(y) = \int_M \mathcal{L}(x,y)\, d\rho(y)\,. \tag{6.26}$$

Arguing similarly in the variable x, one sees that our variation does not change the action. Using that the integrand $|\lambda_j^{xy}|^2$ of the boundedness constraint (5.39) is again homogeneous of degree two in x and y, the above-mentioned argument applies similarly to the functional $\mathcal{T}$, showing that the boundedness constraint is respected by our variations.

Let us analyze the volume and trace constraints. In order to satisfy the volume constraint, we make the ansatz

$$f_\tau = 1 + \tau g\,, \tag{6.27}$$

where g is a bounded function with zero mean,

$$\int_M g(x)\, d\rho(x) = 0\,. \tag{6.28}$$

This ensures that the volume constraint is satisfied. We finally consider the variation of the trace constraint,

$$\begin{aligned}
&\int_{\mathcal{F}} \mathrm{tr}(x)\, d\rho_\tau(x) - \int_{\mathcal{F}} \mathrm{tr}(x)\, d\rho(x) \\
&= \int_M \mathrm{tr}\,\big(F_\tau(x)\big)\, f_\tau(x)\, d\rho(x) - \int_M \mathrm{tr}(x)\, d\rho(x) \\
&= \int_M \mathrm{tr}\,\bigg(\frac{x}{\sqrt{f_\tau(x)}}\bigg)\, f_\tau(x)\, d\rho(x) - \int_M \mathrm{tr}(x)\, d\rho(x) \\
&= \int_M \mathrm{tr}(x)\,\Big(\sqrt{f_\tau(x)} - 1\Big)\, d\rho(x)\,.
\end{aligned} \tag{6.29}$$

Employing again the ansatz (6.27) and differentiating with respect to τ, we obtain for the first variation

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \int_{\mathcal{F}} \mathrm{tr}(x)\, \mathrm{d}\rho_\tau \Big|_{\tau=0} = \frac{1}{2} \int_M \mathrm{tr}(x)\, g(x)\, \mathrm{d}\rho(x), \tag{6.30}$$

(here we may differentiate the integrand using Lebesgue's dominated convergence theorem). Now we use of our assumption that the local trace is *not* constant on M, making it possible to choose $x_1, x_2 \in M$ with $\mathrm{tr}(x_1) > \mathrm{tr}(x_2)$. Moreover, we choose a function g that supported in a small neighborhood of x_1 and x_2, has zero mean (6.28) and is positive at x_1 and negative at x_2. In this way, we can arrange that the right-hand side of (6.30) is strictly positive. Hence, using (6.28), it follows that

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \int_{\mathcal{F}} \mathrm{tr}(x)\, \mathrm{d}\big(\rho_\tau - \rho\big)(x) \Big|_{\tau=0} > 0\,. \tag{6.31}$$

To summarize, we have found a variation that respects the boundedness and the volume constraint and preserves the causal action but increases the integral of the trace (6.31).

The final step is to modify the variation in such a way that the trace and volume constraints are respected, whereas the action and the boundedness constraints become smaller. To this end, we transform the measures according to

$$\rho_\tau \to (G_\tau)_* (\rho_\tau) \tag{6.32}$$

with

$$G_\tau(x) = x \left(\int_M \mathrm{tr}(x)\, \mathrm{d}\rho \right) \Big/ \left(\int_M \mathrm{tr}(x)\, \mathrm{d}\rho_t \right). \tag{6.33}$$

A short computation shows that the trace constraint is respected and so is the volume constraint. Moreover, in view of (6.31), for small positive τ, the scaling factor in (6.33) is strictly smaller than one, implying that the first variations of the action and of the boundedness constraint are both strictly negative (here we use again the homogeneity of the Lagrangian and of the integrand of the boundedness constraint). This is a contradiction to the fact that ρ is a minimizer (here we make essential use of the fact that the boundedness constraint (5.39) is an *inequality* constraint, so that decreasing $\mathcal{T}$ in the variation is allowed). We conclude that the local trace must be constant. □

6.5 How the Causal Action Principle Fits into the Non-compact Setting

Under mild technical assumptions on the minimizing measure, the causal action principle for causal fermion systems is a special case of the causal variational principle in the non-compact setting, as we now explain.

Since for minimizers of the causal action principle, all operators in M have the same trace (see Proposition 6.4.1), we can simplify the setting by restricting attention to linear operators in $\mathcal{F}$ that all have the same trace. Then, the trace constraint can be disregarded, as it follows from the volume constraint. We now implement this simplification by modifying our setting. At the same time, we implement the boundedness constraint by a *Lagrange multiplier* term. Here, we apply this method naively by modifying the Lagrangian to

$$\mathcal{L}_\kappa(x,y) := \frac{1}{4n} \sum_{i,j=1}^{2n} \Big(\big|\lambda_i^{xy}\big| - \big|\lambda_j^{xy}\big| \Big)^2 + \kappa \left(\sum_{i=1}^{2n} \big|\lambda_i^{xy}\big| \right)^2 , \tag{6.34}$$

where $\kappa > 0$ is the Lagrange multiplier. We refer to $\mathcal{L}_\kappa$ as the *κ-Lagrangian*. The justification for this procedure as given in [13] is a bit subtle and, for brevity, we shall not enter these constructions here. It is important to note that, in contrast to the usual Lagrange multiplier, where a minimizer under constraints in general merely is a critical point of the Lagrangian including the Lagrange multipliers, here we obtain again a *minimizer* of the effective action (for details, see [13, Theorem 3.13]).

Finally, we make a mild technical simplification. A spacetime point $x \in M$ is said to be *regular* if x has the maximal possible rank $2n$. Otherwise, the spacetime point is *singular*. In typical physical applications, all spacetime points are regular, except maybe at singularities like the center of black holes. For example, the construction of a causal fermion system in Minkowski space from Dirac wave functions in Section 5.5 gives regular spacetime points if $\mathcal{H}$ is chosen sufficiently large (in particular for all negative-frequency solutions). More generally, the interacting systems considered in [45, Chapters 3–5] all have regular spacetime points. The same is true for the similar construction in globally hyperbolic spacetimes (for details, see [47]). With this in mind, in this book, we usually assume that the *causal fermion system* is *regular* in the sense that all spacetime points are regular. This assumption has the advantage that the set of all regular points of $\mathcal{F}$ is a smooth manifold (see Proposition 3.1.3 in Section 3.1). We remark that, in the case that $\mathcal{H}$ is infinite-dimensional, the set of regular points of $\mathcal{F}$ can be endowed with the structure of a Banach manifold (for details, see [67, Section 3]). These considerations lead us to the following setting:

Definition 6.5.1 *Let $\mathcal{H}$ be a complex Hilbert space. Moreover, suppose we are given parameters $n \in \mathbb{N}$ (the spin dimension), $c > 0$ (the constraint for the local trace) and $\kappa > 0$ (the Lagrange multiplier of the boundedness constraint). We then let $\mathcal{F}^{\text{reg}} \subset \mathrm{L}(\mathcal{H})$ be the set of all symmetric operators F on $\mathcal{H}$ with the following properties:*

(a) *F has finite rank and (counting multiplicities) has exactly n positive and n negative eigenvalues.*

(b) *The local trace is equal to c, that is,*

$$\mathrm{tr}(F) = c\,. \tag{6.35}$$

On $\mathcal{F}^{\text{reg}}$, we again consider the topology induced by the sup-norm on $\mathrm{L}(\mathcal{H})$. The **reduced causal action principle for regular systems** *is to minimize the reduced causal action*

$$\mathcal{S}_\kappa(\rho) = \iint_{\mathcal{F}\times\mathcal{F}} \mathcal{L}_\kappa(x,y)\, \mathrm{d}\rho(x)\, \mathrm{d}\rho(y), \tag{6.36}$$

over all regular Borel measures under variations that preserve the total volume.

In this way, the causal action principle fits into the framework of causal variational principles in the non-compact setting as introduced in Section 6.3. In agreement with (6.11), the causal Lagrangian is continuous (in fact, it is even locally Hölder continuous; for details, see [67, Section 5.1]). Moreover, it has the desired properties (i) and (ii) in Definition 6.2.1 (it is strictly positive because the Lagrangian can be estimated from below in terms of the local trace; see Exercise 5.4).

In order to avoid misunderstandings, we point out that the above-mentioned description of causal fermion systems by measures on $\mathcal{F}^{\text{reg}}$ is not a suitable setting for the existence theory (as will be developed in Chapter 12). The reason is that $\mathcal{F}^{\text{reg}}$ is not closed in $\mathcal{F}$ because the boundary points in $\mathcal{F}$ are missing (in a converging sequence, some of the eigenvalues could tend to zero in the limit). As a consequence, considering a minimizing sequence $(\rho_n)_{n\in\mathbb{N}}$ of measures in $\mathcal{F}^{\text{reg}}$, the limiting measure might well be supported also on $\mathcal{F} \setminus \mathcal{F}^{\text{reg}}$. For this reason, there is no existence theory for measures on $\mathcal{F}^{\text{reg}}$. But if a minimizing measure is given, it seems sensible to assume that the resulting causal fermion system is regular. Under this assumption, the analysis of the causal fermion system can be carried out exclusively in $\mathcal{F}^{\text{reg}}$, whereas $\mathcal{F}$ is no longer needed. For a convenient and compact notation, in such situations, we shall even omit the superscript "reg," so that $\mathcal{F}$ denotes the set of all symmetric operators on $\mathcal{H}$ with the above-mentioned properties (a) and (b). Moreover, we shall omit the subscript κ. Thus, with a slight abuse of notation, we shall denote the Lagrangian including the Lagrange multiplier term (6.34) again by $\mathcal{L}$.

In this way, assuming that the causal fermion systems under consideration are regular, we have recovered the causal action principle as a specific causal variational principle. The connection is summarized schematically as follows:

$$\begin{array}{c}
\boxed{\text{causal action principle for causal fermion systems}} \\
\Big\downarrow \;\; \begin{array}{l}\text{implement boundedness constraint}\\ \text{by a Lagrange multiplier}\end{array} \\
\boxed{\text{reduced causal action principle}} \\
\Big\downarrow \;\; \begin{array}{l}\text{build in trace constraint,}\\ \text{restrict attention to regular}\\ \qquad\quad \text{causal fermion systems}\end{array} \\
\boxed{\text{reduced causal action principle for regular systems}} \\
\big\downarrow \;\; \text{generalize} \\
\boxed{\text{causal variational principles}}
\end{array}$$

Whenever the specific form of the causal Lagrangian (6.34) is not needed, we will work in the more general setting of causal variational principles. Apart from the sake of greater generality, this has the advantage that it becomes clearer which structures are needed for which results. Moreover, it is often more convenient not to specify the form of the Lagrangian. Generally speaking, we can work with causal variational principles unless the physical wave functions and their induced geometric and analytic structures are invoked.

6.6 Exercises

Exercise 6.1 *(Derivation of the causal variational principle on the sphere)* We consider the causal fermion systems in the case $n = 1$ and $f = 2$. For a given parameter $\tau > 1$, we introduce the mapping $F : S^2 \subset \mathbb{R}^3 \to \mathcal{F}$ by

$$F(\vec{x}) = \tau\, \vec{x}\vec{\sigma} + \mathbb{1}\,. \tag{6.37}$$

(a) Compute the eigenvalues of the matrix $F(\vec{x})$ and verify that it has one positive and one negative eigenvalue.

(b) Use the identities between Pauli matrices

$$\sigma^i \sigma^j = \delta^{ij} + \mathrm{i}\epsilon^{ijk}\, \sigma^k\,, \tag{6.38}$$

in order to compute the matrix product,

$$F(\vec{x})\, F(\vec{y}) = \big(1 + \tau^2\, \vec{x}\vec{y}\big)\, \mathbb{1} + \tau\, (\vec{x} + \vec{y})\vec{\sigma} + \mathrm{i}\tau^2\, (\vec{x} \wedge \vec{y})\vec{\sigma}. \tag{6.39}$$

(c) Compute the eigenvalues of this matrix product to obtain

$$\lambda_{1/2} = 1 + \tau^2 \cos\vartheta \pm \tau\sqrt{1 + \cos\vartheta}\, \sqrt{2 - \tau^2\, (1 - \cos\vartheta)}\,, \tag{6.40}$$

where ϑ denotes the angle ϑ between $\vec{x}$ and $\vec{y}$.

(d) Verify that if $\vartheta \leq \vartheta_{\max}$ with

$$\vartheta_{\max} := \arccos\Big(1 - \frac{2}{\tau^2}\Big)\,, \tag{6.41}$$

then the eigenvalues $\lambda_{1/2}$ are both real. Conversely, if $\vartheta > \vartheta_{\max}$, then the eigenvalues form a complex conjugate pair.

(e) Use the formula

$$\begin{aligned} \lambda_1 \lambda_2 &= \det(F(\vec{x})F(\vec{y})) \\ &= \det(F(\vec{x}))\, \det(F(\vec{y})) = (1+\tau)^2(1-\tau)^2 > 0, \end{aligned} \tag{6.42}$$

to conclude that if the eigenvalues $\lambda_{1/2}$ are both real, then they have the same sign.

(f) Combine the findings of (a)–(e) to conclude that the causal Lagrangian in (5.35) can be simplified to (6.6).

Exercise 6.2 *(The action and boundedness constraint of the Lebesgue measure on the sphere)* We consider the causal variational principle on the sphere as introduced in Section 6.1. We let $\mathrm{d}\mu$ be the surface area measure, normalized such that $\mu(S^2) = 1$.

(a) Use the formula for the causal Lagrangian on the sphere (6.6) to compute the causal action (5.36). Verify that

$$\mathcal{S}[F] = \frac{1}{2} \int_0^{\vartheta_{\max}} \mathcal{L}(\cos\vartheta)\, \sin\vartheta\; \mathrm{d}\vartheta = 4 - \frac{4}{3\tau^2}\,. \tag{6.43}$$

(b) Show that the functional $\mathcal{T}$ is given by

$$\mathcal{T}[F] = 4\tau^2(\tau^2 - 2) + 12 - \frac{8}{3\tau^2}\,. \tag{6.44}$$

Hence, the causal action (6.43) is bounded uniformly in τ, although the function F, (6.37), and the functional $\mathcal{T}$, (6.44), diverge as $\tau \to \infty$.

Exercise 6.3 *(A minimizer with singular support)* We again consider the causal variational principle on the sphere as introduced in Section 6.1. Verify by direct computation that in the case $\tau = \sqrt{2}$, the causal action of than the normalized counting measure supported on the *octahedron* is smaller than the action of μ. *Hint:* For $\tau = \sqrt{2}$, the opening angle of the light cone is given by $\vartheta = 90°$, so that all distinct spacetime points are spacelike separated. Moreover, the causal action of the normalized Lebesgue measure is given in Exercise 6.2(a).

It turns out that the normalized counting measure supported on the octahedron is indeed a minimizer of the causal action. More details and related considerations can be found in [43, 74, 80].

Exercise 6.4 *(A causal variational principle on $\mathbb{R}$)* We let $\mathcal{F} = \mathbb{R}$ and consider the Lagrangians

$$\mathcal{L}_2(x,y) = (1+x^2)(1+y^2) \qquad \text{and} \qquad \mathcal{L}_4(x,y) = (1+x^4)(1+y^4)\,. \tag{6.45}$$

We minimize the corresponding causal actions (6.8) within the class of all normalized regular Borel measures on $\mathbb{R}$. Show with a direct estimate that the Dirac measure δ supported at the origin is the unique minimizer of these causal variational principles.

Exercise 6.5 *(A causal variational principle on S^1)* We let $\mathcal{F} = S^1$ be the unit circle parametrized as $\mathrm{e}^{\mathrm{i}\varphi}$ with $\varphi \in \mathbb{R} \bmod 2\pi$ and consider the Lagrangian

$$\mathcal{L}(\varphi, \varphi') = 1 + \sin^2\big(\varphi - \varphi' \bmod 2\pi\big)\,. \tag{6.46}$$

We minimize the corresponding causal action (6.8) within the class of all normalized regular Borel measures on S^1. Show by direct computation and estimates that every minimizer is of the form

$$\rho = \tau\, \delta\big(\varphi - \varphi' - \varphi_0 \bmod 2\pi\big) + (1-\tau)\, \delta\big(\varphi - \varphi' - \varphi_0 + \pi \bmod 2\pi\big) \tag{6.47}$$

for suitable values of the parameters $\tau \in [0,1]$ and $\varphi_0 \in \mathbb{R} \bmod 2\pi$.