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FUNCTORIAL PROPERTIES OF ALGEBRAIC CLOSURE AND SKOLEMIZATION

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Abstract

It is shown that no functor F exists from the category of sets with injections, to the category of algebraically closed fields of given characteristic, with monomorphisms, having the properties that for all sets A. $F(A)$ is an algebraically closed field having transcendence base A and for all injections f. $F(f)$ extends f. There does exist such a functor from the category of linearly-ordered sets with order monomorphisms.

An application to model-theory using the same methods is given showing that while the theory of algebraically closed fields is ω-stable, its Skolemization is not stable in any power.

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Introduction

A well-known theorem of Steinitz shows that, if A, B are transcendence bases for algebraically closed fields $\bar{A}$ and $\bar{B}$ of a given characteristic, then any injection f: $A \to B$ may be extended (in many ways) to a field monomorphism $F(f)$: $\bar{A} \to \bar{B}$. In this paper, we consider the extent to which F may be chosen to be a functor. We show using a theorem of Artin and Schreier that no such functor F exists from the full category of sets with injections. However, by an argument of the Ehrenfeucht–Mostowski sort, there does exist such a functor from the category of linearly-ordered sets with monomorphisms.

This investigation originated in the study of combinatorial functors, where the question arises whether the results of, for example, Crossley and Nerode (1974)

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for vector spaces can be applied to algebraically closed fields. It is thought that the application of a model-theoretic construction in our Proposition 3 is of independent interest.

A further connection with model-theory is obtained in Proposition 4 where both the Artin-Schreier Theorem and the Ehrenfeucht-Mostowski Theorem are used to show that the Skolemization of the ω-stable theory of algebraically closed fields is not stable in any power. This also may be viewed as a result concerning functors since Skolemization can be regarded as a functor on a suitable category of theories. Our result thus shows that this functor does not in any way preserve stability.

Preliminaries

We work in Zermelo–Fraenkel set theory with a universal choice operator. We thus identify cardinals with initial ordinals, and denote by card(A) the cardinal of the set A.

The notions of algebraic dependence, algebraically closed field, algebraic closure of a subset of a field, and transcendence base for an algebraically closed field are described in, for example, Jacobson (1964), p. 141 et seq.

S is the category of sets with injections, $\mathbf{ACF}_m$ the category of algebraically closed fields of characteristic m $(= 2, 3, \ldots, 0)$ with field monomorphisms and **L** that of linearly ordered sets with order monomorphisms.

Functors from S to $\mathbf{ACF}_m$

The following two lemmata are well known, see, for example, Jacobson (1964), pp. 142, 143.

LEMMA 1. *Each field F has an extension $\bar{F}$ which is an algebraically closed field in which the algebraic closure of F is $\bar{F}$. Thus, for every set A, there is an algebraically closed field of each characteristic for which A is a transcendence base.*

LEMMA 2. *If $\bar{A}, \bar{B} \in \mathbf{ACF}_m$ with transcendence bases A, B respectively, and f: $A \to B$ is an injection, then there is an extension f^+ of f to a monomorphism from $\bar{A}$ to $\bar{B}$. If f is a surjection then f^+ is an epimorphism.*

DEFINITION. By a *closure* functor we mean a functor from **S** to $\mathbf{ACF}_m$ whose object map assigns to each $A \in \mathbf{S}$ some $\bar{A} \in \mathbf{ACF}_m$ with transcendence base A.

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PROPOSITION 1. *There is a closure functor F such that if $f: A \to B$ then $F(f)$ maps $\bar{A}$ into $\bar{B}$.*

PROOF. First, for each set A, choose $\bar{A} \in \mathbf{ACF}_m$ arbitrarily with transcendence base A. Now choose monomorphisms $f_{\lambda\kappa}: \bar{\lambda} \to \bar{\kappa}$ for all cardinals λ, κ with $\lambda \leqslant \kappa$, such that $f_{\lambda k}$ maps λ into κ and if $\lambda \leqslant \mu \leqslant \kappa$ then $f_{\lambda\kappa} = f_{\mu\kappa} \circ f_{\lambda\mu}$, inductively, as follows. Let $f_{\kappa\kappa}$ be the identity on $\bar{\kappa}$. Suppose now that $f_{\lambda\mu}$ has been chosen for all λ, μ with $\lambda \leqslant \mu < \kappa$. If κ is a successor, $\kappa = \nu^+$, choose $f_{\nu\kappa}: \bar{\nu} \to \bar{\kappa}$ mapping ν into κ arbitrarily and define $f_{\lambda\kappa} = f_{\nu\kappa} \circ f_{\lambda\nu}$ for $\lambda < \kappa$. If κ is a limit cardinal, let F be a direct limit of the system, $\{\bar{\lambda}\}$, $\{f_{\lambda\mu}\}$ for $\lambda, \mu < \kappa$ with the usual monomorphisms $f_{\lambda\infty}: \bar{\lambda} \to F$. Now choose an isomorphism $g: F \to \bar{\kappa}$, and define $f_{\lambda\kappa} = g \circ f_{\lambda\infty}$.

For each set A, let b_A be a bijection from A to card(A). Let $\bar{b}_A$ extend b_A arbitrarily to an isomorphism from $\bar{A}$ to $\overline{\text{card}(A)}$. Now, if $f: A \to B$, define $F(f) = \bar{b}_B^{-1} \circ f_{\lambda\mu} \circ \bar{b}_A$ where $\lambda = \text{card}(A)$, $\mu = \text{card}(B)$.

REMARK. The proposition also follows quickly from Proposition 3 by first choosing a well-ordering of all sets.

Limitations on the possible properties of closure functors may, however, be derived from the following, see, for example, Jacobson (1964), p. 316.

THEOREM 1 (Artin–Schreier). *An automorphism of finite order of an algebraically closed field of characteristic m has order $\leqslant 2$ if $m = 0$, and is the identity if $m \neq 0$.*

PROPOSITION 2. *There is no closure functor F such that, for all f, $F(f)$ extends f.*

PROOF. Such a functor would have $f \neq g \Rightarrow F(f) \neq F(g)$. So taking $A \in \mathbf{S}$ with at least 3 elements, and a permutation f of A with order 3, $F(f): \bar{A} \to \bar{A}$ would have order 3, which is impossible by Theorem 1.

Ehrenfeucht-Mostowski sets and Skolemization

The following theorem is proved in Morley (1965).

THEOREM 2 (Ehrenfeucht–Mostowski). *Let T be a first-order theory with an infinite model. Then, for any linearly ordered set $(X, <)$ there is a model $\mathfrak{A}$ of T*

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whose base includes X and for which, if $x_1 < x_2 < \cdots < x_n$; $y_1 < y_2 < \cdots < y_n$ *where* $x_1, \ldots, x_n, y_1, \ldots, y_n \in X$ *then for each formula* φ, $\mathfrak{A} \models \varphi[x_1, \ldots, x_n] \leftrightarrow \mathfrak{A} \models \varphi[y_1, \ldots, y_n]$.

DEFINITION. Let T be a theory. A set Σ of formulae is called an *E.M. set for T* if, for every linearly-ordered set $(X, <)$ (equivalently for *some* infinite linearly-ordered set), there exists a model $\mathfrak{A}$ of T, whose base includes X, such that for all $x_1, \ldots, x_n \in X$ with $x_1 < x_2 < \cdots < x_n$,

$$\mathfrak{A} \models \varphi[x_1, x_2, \ldots, x_n] \leftrightarrow \varphi(v_1, v_2, \ldots, v_n) \in \Sigma.$$

Such an $\mathfrak{A}$ is then called an *E.M. model* for T, Σ and $(X, <)$. Theorem 2 thus states that every theory which has an infinite model has an E.M. set.

PROPOSITION 3. *If* **L** *is the category of linearly-ordered sets,* $\mathfrak{A} = (A, <)$ *with order monomorphisms, then there is a functor* F: $\mathbf{L} \to \mathbf{ACF}_m$ *for which* $F(A)$ *has transcendence base* A *and, for* $f \in \mathbf{L}$, $F(f)$ *extends* f.

PROOF. Let T be the theory of algebraically closed fields of characteristic m expressed in a language with function symbols for $+, \cdot, -, 0$. Let this language be augmented for each n by additional n-ary function symbols $\{f_p\}$ indexed by the $(n + 1)$-ary terms p of the original language (which correspond to integral polynomials). Let T^+ be the extension of T to this larger language obtained by adding the further axioms.

$$p(v_1, \ldots, v_n, f_p(v_1, \ldots, v_n)) = 0 \quad \text{for each } n$$

and each $(n + 1)$-ary p. Then T^+ has an infinite model, every model of T^+ is the expansion of some algebraically closed field and, in every such model, closure under the operations is algebraic closure.

Now, by Theorem 2, we may let Σ be an E.M. set for T^+. For each $\mathfrak{A} \in \mathbf{L}$, let $G(\mathfrak{A})$ be an E.M. model for T^+, Σ and $\mathfrak{A}$. Let $F(\mathfrak{A})$ be the submodel of $G(\mathfrak{A})$ generated by A. Now for $b \in F(\mathfrak{A})$ we have $b = t^{F(\mathfrak{A})}(A_1, \ldots, a_n)$ for some term t of T^+, and so if $\mathfrak{A}, \mathfrak{B} \in \mathbf{L}$, f: $A \to B$ we can define $F(f)$: $F(\mathfrak{A}) \to F(\mathfrak{B})$ by $t^{F(\mathfrak{A})}(a_1, \ldots, a_n) \mapsto t^{F(\mathfrak{B})}(f(a_1), \ldots, f(a_n))$.

DEFINITION. If T is a first-order theory, then by the *Skolemization* of T we mean the conservation extension, T^{Sk}, of T to a language with additional function letters $f_\varphi(v_1, \ldots, v_n)$ for each formula $\varphi(v_1, \ldots, v_n, v)$ and with the additional axioms $\varphi(v_1, \ldots, v_n, v) \to \varphi(v_1, \ldots, v_n, f_\varphi(v_1, \ldots, v_n))$.

The notion of *stability* of complete theories is given in Shelah (1971), where the following is shown, as in Morley (1965). An account of stability is also given in Keisler (1976).

LEMMA 3 (Morley-Shelah). *If a theory T is stable then no formula φ is both connected and antisymmetric over an infinite subset of a model of T.*

That is, for no formula $\varphi(v_1, \ldots, v_n)$ and no model $\mathfrak{A}$ of T with an infinite subset X do we have, for every n distinct elements $x_1, \ldots, x_n$ of X, permutations λ, μ of $\{1, 2, \ldots, n\}$ for which $\mathfrak{A} \vDash [x_{\lambda(1)}, \ldots, x_{\lambda(n)}]$ and $\mathfrak{A} \nvDash \varphi[x_{\mu(1)}, \ldots, x_{\mu(n)}]$.

Let T be the theory of algebraically closed fields of some characteristic. Then T is $\aleph_1$-categorical, therefore ω-stable. However:

PROPOSITION 4. *T^{Sk} has no complete extension which is stable.*

PROOF. Suppose, for a contradiction, that T' is a complete stableextension of T^{Sk}. Then, by Theorem 2 there is an E.M. set Σ for T'. Let X be any linearly ordered set and let $\mathfrak{A}$ be an E.M. model for T', Σ and X.

We observe that for any formula $\varphi(v_1, \ldots, v_n) \in \Sigma$, for no permutation λ of $\{1, 2, \ldots, n\}$ do we have $\varphi(v_{\lambda(1)}, v_{\lambda(2)}, \ldots, v_{\lambda(n)}) \in \Sigma$. For, if we did, then taking X to be infinite would give a model $\mathfrak{A}$ of T' in which φ was both connected and antisymmetric over the infinite set X, contradicting Theorem 3.

It follows that, if f is any permutation of X, then the map $t^{\mathfrak{A}}(x_1, \ldots, x_n) \mapsto t^{\mathfrak{A}}(f(x_1), \ldots, f(x_n))$ for $x_1, \ldots, x_n \in X$ is a well-defined automorphism f^+ of that submodel $\mathfrak{B}$ of $\mathfrak{A}$ generated by X. (Here t denotes a term of the language and $t^{\mathfrak{A}}$ its value in the model $\mathfrak{A}$.) By definition of T^{Sk}, $\mathfrak{B}$ is an expansion of an algebraically closed field. Now taking f to be a permutation of X of order 3, by definition of f^+, f^+ also has order 3. But this is a contradiction to Theorem 1.

Postscript

The referee has drawn our attention to Hodges (1974) which mentions a result similar to our Proposition 2. In that paper various algebraic constructions are treated uniformly as "word constructions". It is stated that formation of an algebraic closure of a field may *not* be construed in this way.

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