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S_3-Covers of Schemes

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Abstract. We analyze flat S_3-covers of schemes, attempting to create structures parallel to those found in the abelian and triple cover theories. We use an initial local analysis as a guide in finding a global description.

1 Introduction

Given a finite group G, a G-*cover* of a scheme X is a scheme Y together with a faithful G-action on Y and a finite G-equivariant morphism $\pi\colon Y \to X$ which identifies X with the geometric quotient Y/G. If we consider only schemes over a fixed algebraically closed field k of characteristic prime to the order of G, then to each G-cover $\pi\colon Y \to X$ there is a decomposition $\pi_*\mathcal{O}_Y = \bigoplus_{\rho\in G^\vee}\mathcal{F}_\rho$, where $\mathcal{F}_\rho$ is an $\mathcal{O}_X[G]$-module with G-action related to the irreducible representation ρ. Under suitable additional hypotheses (*e.g.*, X, Y integral and Noetherian, π flat), the sheaf $\mathcal{F}_\rho$ is locally free of rank equal to $(\dim\rho)^2$. Conversely, to construct a cover given an appropriate collection of locally free $\mathcal{O}_X[G]$-modules $\{\mathcal{F}_\rho\}_\rho$, we must define a commutative, associative $\mathcal{O}_X[G]$-algebra structure on $\mathcal{A} = \bigoplus_\rho \mathcal{F}_\rho$. We then obtain the G-cover $\pi\colon \mathbf{Spec}_X\mathcal{A} \to X$.

The theory for abelian groups was analyzed by Pardini in [19]. In that case, the decomposition runs over the irreducible characters of G, and the $\mathcal{O}_X[G]$-submodule $\mathcal{F}_\chi$ is the invertible χ-eigensheaf of $\pi_*\mathcal{O}_Y$, whose sections are those on which the group acts as multiplication by the character χ. The algebra structure on $\pi_*\mathcal{O}_Y$ is determined by a compatible collection of morphisms $\{\mathcal{F}_\chi \otimes \mathcal{F}_{\chi'} \to \mathcal{F}_{\chi\chi'}\}_{\chi,\chi'}$, or equivalently by a collection of global sections of $\{\mathcal{F}_\chi^{-1} \otimes \mathcal{F}_{\chi'}^{-1} \otimes \mathcal{F}_{\chi\chi'}\}_{\chi,\chi'}$. These sections are closely related to the branch divisor of the cover; given invertible sheaves $\{\mathcal{F}_\chi\}_\chi$, to construct a G-cover we may replace the explicit definition of the algebra structure with a specification of the branching data. As long as a "covering condition" is satisfied, we obtain a G-cover.

A key aspect of the abelian theory is that it allows us to understand geometry of the covering scheme in terms of geometry of the (usually simpler) base scheme. When X is a surface, we can use geometrically interesting configurations of curves in X to construct new surfaces whose intrinsic geometry reflects the geometry of the configuration. For example, a standard result in the theory of complex surfaces is the Bogomolov–Miyaoka–Yau inequality, which states that $K_X^2/\chi(\mathcal{O}_X) \leq 9$ for any smooth, complex surface X of general type [2], [17], [26]. This inequality is sharp, and Hirzebruch produced examples of equality by constructing abelian covers of $\mathbb{P}^2$

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branched over "extreme" configurations of lines [12]. The inequality is known to fail in positive characteristic [14], and infinite families of counterexamples can be produced using abelian covers branched over configurations of lines unique to positive characteristic [6]. A similar construction was employed in [8] to prove a higher-dimensional analogue of Belyi's theorem.

The situation in the nonabelian case is more complicated, but the permutation group S_3 is within reach. Indeed, complex dihedral covers have been studied by H. Tokunaga, who reduced the problem to a study of the Galois theory of the associated function fields [22], [23], [24], [25]. The S_3-covers we study here are a special case of Tokunaga's dihedral covers, but we pursue a different approach, one more closely paralleling the technique used to study triple covers in [16]. Our aim is to create structures parallel to those found in the abelian and triple cover cases, laying the groundwork for future applications. (For example, the triple cover results in [16] have recently been used in [3] to construct and study stacks of trigonal curves.) To realize our goal, we begin with a local analysis of S_3-covers, which we then use as a guide in a global analysis. The result of our analyses is the following description of S_3-covers.

1.1 Summary of Results

Let R be a domain in which 6 is invertible, and let $\tau \in S_3$ be a fixed transposition. If $\pi\colon Y \to X$ is a flat S_3-cover of integral, Noetherian R-schemes, then $Y \cong \mathbf{Spec}_X \pi_* \mathcal{O}_Y$ and we have a direct sum decomposition $\pi_*\mathcal{O}_Y \cong \mathcal{O}_X \oplus \mathcal{L} \oplus \mathcal{E} \oplus \tau\mathcal{E}$, with $\mathcal{L}$ the invertible subsheaf on which S_3 acts as the sign character and $\mathcal{E}$ a locally free sheaf of rank two (with S_3 acting on $\mathcal{E} \oplus \tau\mathcal{E}$ by its two-dimensional representation).

Conversely, suppose X is an integral, Noetherian R-scheme, and $\mathcal{L}$ and $\mathcal{E}$ are locally free sheaves of ranks one and two, respectively. Suppose S_3 acts on $\mathcal{L}$ by the sign character and on $\tilde{\mathcal{E}} = \mathcal{E} \oplus \tau\mathcal{E}$ by its two-dimensional representation. Letting $\mathcal{A} = \mathcal{O}_X \oplus \mathcal{L} \oplus \tilde{\mathcal{E}}$, the S_3-covers of the form $\mathbf{Spec}_X\mathcal{A} \to X$ are parameterized by the R-submodule $S_3\operatorname{Cov}_X(\mathcal{A}) \le \operatorname{Hom}_X(\operatorname{Sym}^2\mathcal{A}, \mathcal{A})$ of elements defining commutative, associative $\mathcal{O}_X$-algebra structures on $\mathcal{A}$ compatible with the given S_3-action. After identifying $S_3\operatorname{Cov}_X(\mathcal{A})$ with an R-submodule of $\operatorname{Hom}_X(\operatorname{Sym}^2\mathcal{L}, \mathcal{O}_X) \oplus \operatorname{Hom}_X(\mathcal{L} \otimes_{\mathcal{O}_X} \tilde{\mathcal{E}}, \tilde{\mathcal{E}}) \oplus \operatorname{Hom}_X(\operatorname{Sym}^2\tilde{\mathcal{E}}, \mathcal{A})$, a local analysis leads us to define the R-modules

$$
\begin{aligned}
\operatorname{Build}_X(\mathcal{A}) &= \operatorname{Hom}_X\big(\mathcal{L} \otimes \operatorname{Sym}^2\mathcal{E}, \textstyle\bigwedge^2\mathcal{E}\big)\\
&\qquad \oplus \operatorname{Hom}_X\big(\operatorname{Sym}^3\mathcal{E}, \textstyle\bigwedge^2\mathcal{E}\big) \oplus \operatorname{Hom}_X\big(\textstyle\bigwedge^2\mathcal{E}, \mathcal{L}\big),\\
\operatorname{Compat}_X(\mathcal{A}) &= \operatorname{Hom}_X\big(\mathcal{L} \otimes \big(\textstyle\bigwedge^2\mathcal{E}\big)^{\otimes 2} \otimes \mathcal{E}, \big(\textstyle\bigwedge^2\mathcal{E}\big)^{\otimes 2}\big)\\
&\qquad \oplus \operatorname{Hom}_X\big(\textstyle\bigwedge^2(\operatorname{Sym}^2\mathcal{E}), \textstyle\bigwedge^2\mathcal{E}\big),
\end{aligned}
$$

parameterizing the data required to build S_3-covers and the compatibility conditions on such data. After identifying $\operatorname{Build}_X(\mathcal{A})$ with an R-submodule of

$$
\operatorname{Hom}_X(\mathcal{L} \otimes \mathcal{E}, \mathcal{E}) \oplus \operatorname{Hom}_X(\operatorname{Sym}^2\mathcal{E}, \mathcal{E}) \oplus \operatorname{Hom}_X(\mathcal{E} \otimes \tau\mathcal{E}, \mathcal{L}),
$$

we construct R-module morphisms

$$\mathbf{P}\colon S_3\operatorname{Cov}_X(\mathcal{A}) \to \operatorname{Hom}_X(\mathcal{L}\otimes\mathcal{E},\mathcal{E}) \oplus \operatorname{Hom}_X(\operatorname{Sym}^2\mathcal{E},\mathcal{E}) \oplus \operatorname{Hom}_X(\mathcal{E}\otimes\tau\mathcal{E},\mathcal{L}),$$

$$\mathbf{C}\colon \operatorname{Build}_X(\mathcal{A}) \to \operatorname{Compat}_X(\mathcal{A}),$$

$$\mathbf{B}\colon \operatorname{Build}_X(\mathcal{A}) \to \operatorname{Hom}_X(\operatorname{Sym}^2\mathcal{L},\mathcal{O}_X) \oplus \operatorname{Hom}_X(\mathcal{L}\otimes\tilde{\mathcal{E}},\tilde{\mathcal{E}}) \oplus \operatorname{Hom}_X(\operatorname{Sym}^2\tilde{\mathcal{E}},\mathcal{A}),$$

such that $\mathbf{P}\big(S_3\operatorname{Cov}_X(\mathcal{A})^0\big) \subseteq \ker\mathbf{C}$ and $\mathbf{B}(\ker\mathbf{C}) \le S_3\operatorname{Cov}_X(\mathcal{A})$, where $S_3\operatorname{Cov}_X(\mathcal{A})^0$ is the subset of elements defining algebras that are domains. In other words, $\mathbf{P}$ extracts from those S_3-covers whose covering schemes are integral the building data of the covers, and such data lies in the kernel of $\mathbf{C}$; and conversely, given building data in the kernel of $\mathbf{C}$, the morphism $\mathbf{B}$ builds an S_3-cover (whose covering scheme might not be integral).

Remark 1.2 It is worth noting that S_3-covers are particular instances of degree six covers, and that our results, taken together with Pardini's abelian results [19], give a complete description of degree six Galois covers in terms of $\mathcal{O}_X$-algebra structures on $\pi_*\mathcal{O}_Y$. The problem of dealing with non-Galois covers, however, is wide open. This contrasts greatly with the situation for triple and quadruple covers, in which complete descriptions are known [10], [16], [20].

2 Preliminary Analysis

Fix a presentation, say $S_3 = \langle \sigma,\tau \mid \sigma^3 = \tau^2 = 1,\ \tau\sigma = \sigma^2\tau\rangle$, and let R be a domain in which 6 is invertible. As a free R-module, the group ring $R[S_3]$ decomposes as $R[S_3] = C_1 \oplus C_2 \oplus C_3$, where

$$\begin{aligned} C_1 &= \operatorname{span}_R\{1+\sigma+\sigma^2+\tau+\sigma\tau+\sigma^2\tau\} \\ &= \{v \in R[S_3] \mid \forall g \in S_3, g\cdot v = v\}, \\ C_2 &= \operatorname{span}_R\{1+\sigma+\sigma^2-\tau-\sigma\tau-\sigma^2\tau\} \\ &= \{v \in R[S_3] \mid \forall g \in S_3, g\cdot v = \operatorname{sgn}(g)v\}, \end{aligned}$$

and where a basis for C_3 is given by

$$\begin{aligned} u_{11} &= -1+\sigma+\tau-\sigma^2\tau, & u_{21} &= -1+\sigma^2+\tau-\sigma\tau, \\ u_{12} &= -\sigma+\sigma^2-\tau+\sigma\tau, & u_{22} &= 1-\sigma+\sigma\tau-\sigma^2\tau. \end{aligned}$$

Under this decomposition of $R[S_3]$, we have $1 = \epsilon_1+\epsilon_2+\epsilon_3$, where

$$\begin{aligned} \epsilon_1 &= \frac{1}{6}(1+\sigma+\sigma^2+\tau+\sigma\tau+\sigma^2\tau), \\ \epsilon_2 &= \frac{1}{6}(1+\sigma+\sigma^2-\tau-\sigma\tau-\sigma^2\tau), \\ \epsilon_3 &= \frac{1}{3}(2-\sigma-\sigma^2). \end{aligned}$$

Note that each ϵ_i is in the center of $R[S_3]$ and satisfies $\epsilon_i\epsilon_j = \delta_{ij}\epsilon_i$. We also have a non-equivariant decomposition $C_3 = C_{31} \oplus C_{32}$, where $C_{3i} = \operatorname{span}_R\{u_{i1}, u_{i2}\}$. Under this decomposition, ϵ_3 decomposes as $\epsilon_3 = \epsilon_{31} + \epsilon_{32}$, where

$$\epsilon_{31} = \frac{1}{3}(1 - \sigma + \sigma\tau - \sigma^2\tau), \quad \epsilon_{32} = \frac{1}{3}(1 - \sigma^2 - \sigma\tau + \sigma^2\tau),$$

which also satisfy $\epsilon_{3i}\epsilon_{3j} = \delta_{ij}\epsilon_{3i}$ (but are not central).

Suppose X is an R-scheme, and $\mathcal{F}$ is an $\mathcal{O}_X[S_3]$-module with action explicitly given by a group homomorphism $\mu\colon S_3 \to \operatorname{Aut}_{\mathcal{O}_X}(\mathcal{F})$. Extend this morphism R-linearly to a ring homomorphism $\mu\colon R[S_3] \to \operatorname{End}_{\mathcal{O}_X}(\mathcal{F})$. Recalling that for any $f \in \operatorname{End}_{\mathcal{O}_X}(\mathcal{F})$ satisfying $f \circ f = f$ the image presheaf of f is automatically a sheaf, let $\mathcal{F}_i \subseteq \mathcal{F}$ denote the image (pre)sheaf of $\mu(\epsilon_i)$. (In particular, $\mathcal{F}_1$ is the subsheaf of invariant sections of $\mathcal{F}$, and $\mathcal{F}_2$ is the subsheaf of sections on which S_3 acts as multiplication by the sign character.) Since $\mu(1) = \operatorname{Id}_{\mathcal{F}}$, and since 1 decomposes as $1 = \epsilon_1 + \epsilon_2 + \epsilon_3$ with elements ϵ_i satisfying the aforementioned properties, the morphisms $\mu(\epsilon_i)$ induce an $\mathcal{O}_X[S_3]$-module direct sum decomposition $\mathcal{F} \cong \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_3$. Using the decomposition $\epsilon_3 = \epsilon_{31} + \epsilon_{32}$, we can further obtain an $\mathcal{O}_X$-module direct sum decomposition $\mathcal{F}_3 \cong \mathcal{F}_{31} \oplus \mathcal{F}_{32}$. Note that the $\mathcal{F}_{3i}$ are not invariant under the group action: we have $\tau\epsilon_{31} = \epsilon_{32}\tau$, so $\mu(\tau)|_{\mathcal{F}_3}$ is an automorphism interchanging the summands, and hence $\mathcal{F}_3 \cong \mathcal{F}_{31} \oplus \tau\mathcal{F}_{31}$. This decomposition encodes the action of S_3 on $\mathcal{F}_3$: by a direct calculation we see that $\sigma\epsilon_{31} = -\tau\epsilon_{31}$ and $\sigma^2\epsilon_{31} = (-1+\tau)\epsilon_{31}$, so the S_3-action on $\mathcal{F}_{31} \oplus \tau\mathcal{F}_{31}$ is given by

$$\tau \cdot (s_1 + \tau s_2) = s_2 + \tau s_1, \quad \sigma \cdot (s_1 + \tau s_2) = s_2 + \tau(-s_1 - s_2), \tag{1}$$

for sections s_1, s_2 of $\mathcal{F}_{31}$. (For the second equality, note that we must have $\sigma \cdot (\tau s_2) = (\sigma\tau) \cdot s_2 = (\tau\sigma^2) \cdot s_2 = \tau \cdot (\sigma^2 \cdot s_2) = \tau \cdot (-s_2 + \tau s_2) = s_2 - \tau s_2$.)

Now suppose $\pi\colon Y \to X$ is a flat S_3-cover of R-schemes. Then $\pi_*\mathcal{O}_Y$ is a locally free $\mathcal{O}_X[S_3]$-module of rank six, with $(\pi_*\mathcal{O}_Y)^{S_3} \cong \mathcal{O}_X$. By the above, we have an induced $\mathcal{O}_X[S_3]$-module direct sum decomposition $\pi_*\mathcal{O}_Y \cong \mathcal{O}_X \oplus \mathcal{F}_2 \oplus \mathcal{F}_3$, as well as an $\mathcal{O}_X$-module decomposition $\mathcal{F}_3 \cong \mathcal{F}_{31} \oplus \tau\mathcal{F}_{31}$. If we assume X and Y are integral, Noetherian R-schemes, then $\mathcal{F}_2$ and $\mathcal{F}_{31}$ are locally free of ranks one and two, respectively. Moreover, since a finite morphism is affine, it follows that $Y \cong \mathbf{Spec}_X(\mathcal{O}_X \oplus \mathcal{F}_2 \oplus \mathcal{F}_{31} \oplus \tau\mathcal{F}_{31})$. Thus, to construct an S_3-cover of X we need the following data:

(i) an invertible $\mathcal{O}_X$-module $\mathcal{L}$, on which S_3 is to act via the sign character;

(ii) a locally free $\mathcal{O}_X$-module $\mathcal{E}$ of rank two; the group S_3 is then to act on $\mathcal{E} \oplus \tau\mathcal{E}$ as defined by (1);

(iii) a commutative, associative $\mathcal{O}_X[S_3]$-algebra structure on $\mathcal{O}_X \oplus \mathcal{L} \oplus \mathcal{E} \oplus \tau\mathcal{E}$. (As we will see, both associativity and compatibility with the group action impose strong conditions on the algebra structure.)

We aim to precisely describe data (iii), given data (i) and (ii). To that end, let us fix an integral, Noetherian R-scheme X, together with locally free $\mathcal{O}_X$-modules $\mathcal{L}$ and $\mathcal{E}$ of ranks one and two, respectively. For notational convenience, let us also define $\tilde{\mathcal{E}} = \mathcal{E} \oplus \tau\mathcal{E}$ and $\mathcal{A} = \mathcal{O}_X \oplus \mathcal{L} \oplus \tilde{\mathcal{E}}$. As commutative $\mathcal{O}_X$-algebra structures on $\mathcal{A}$ are given by the elements of $\operatorname{Hom}_X(\operatorname{Sym}^2 \mathcal{A}, \mathcal{A})$, we make the following definition.

Definition 2.1 Let $S_3 \operatorname{Cov}_X(\mathcal{A}) \leq \operatorname{Hom}_X(\operatorname{Sym}^2 \mathcal{A}, \mathcal{A})$ denote the R-submodule of elements that define commutative, associative $\mathcal{O}_X[S_3]$-algebra structures on $\mathcal{A}$.

Our goal, then, is to find an intrinsic description of $S_3 \operatorname{Cov}_X(\mathcal{A})$. We begin with the following preliminary observation.

Lemma 2.2 *Each commutative $\mathcal{O}_X[S_3]$-algebra structure on $\mathcal{A}$ is defined by an element of* $\operatorname{Hom}_X(\operatorname{Sym}^2 \mathcal{L}, \mathcal{O}_X) \oplus \operatorname{Hom}_X(\mathcal{L} \otimes_{\mathcal{O}_X} \tilde{\mathcal{E}}, \tilde{\mathcal{E}}) \oplus \operatorname{Hom}_X(\operatorname{Sym}^2 \tilde{\mathcal{E}}, \mathcal{A})$.

Proof *A priori*, an arbitrary $\mathcal{O}_X$-algebra structure on $\mathcal{A}$ is given by an $\mathcal{O}_X$-module morphism $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A} \to \mathcal{A}$, which is equivalent to an $\mathcal{O}_X$-module morphism

$$\begin{aligned}(\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X) \oplus (\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{L}) \oplus (\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_X) \oplus (\mathcal{O}_X \otimes_{\mathcal{O}_X} \tilde{\mathcal{E}}) \oplus (\tilde{\mathcal{E}} \otimes_{\mathcal{O}_X} \mathcal{O}_X) \\ \oplus (\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}) \oplus (\mathcal{L} \otimes_{\mathcal{O}_X} \tilde{\mathcal{E}}) \oplus (\tilde{\mathcal{E}} \otimes_{\mathcal{O}_X} \mathcal{L}) \oplus (\tilde{\mathcal{E}} \otimes_{\mathcal{O}_X} \tilde{\mathcal{E}}) \to \mathcal{A}.\end{aligned}$$

The first coordinate of this morphism must be given by the algebra structure on $\mathcal{O}_X$, and the following four coordinates must be given by the left and right $\mathcal{O}_X$-module structures on $\mathcal{L}$ and $\tilde{\mathcal{E}}$, respectively. If the algebra is to be commutative, then the sixth and ninth coordinates must factor through the canonical morphisms to the corresponding symmetric products, namely $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L} \to \operatorname{Sym}^2 \mathcal{L}$ and $\tilde{\mathcal{E}} \otimes_{\mathcal{O}_X} \tilde{\mathcal{E}} \to \operatorname{Sym}^2 \tilde{\mathcal{E}}$. Similarly, the seventh and eighth coordinates must agree after the canonical isomorphism $\mathcal{L} \otimes_{\mathcal{O}_X} \tilde{\mathcal{E}} \cong \tilde{\mathcal{E}} \otimes_{\mathcal{O}_X} \mathcal{L}$.

If the algebra is to be compatible with the S_3-action, then S_3 must act as $(\operatorname{sgn})^2 = \operatorname{Id}$ on the image of $\operatorname{Sym}^2 \mathcal{L} \to \mathcal{A}$, and so this morphism must factor through the embedding $\mathcal{O}_X \subseteq \mathcal{A}$ of invariant sections. Similarly, $\mathcal{L} \otimes_{\mathcal{O}_X} \tilde{\mathcal{E}} \to \mathcal{A}$ must factor through the embedding $\tilde{\mathcal{E}} \subseteq \mathcal{A}$. ■

We may thus regard $S_3 \operatorname{Cov}_X(\mathcal{A})$ as an R-submodule of $\operatorname{Hom}_X(\operatorname{Sym}^2 \mathcal{L}, \mathcal{O}_X) \oplus \operatorname{Hom}_X(\mathcal{L} \otimes_{\mathcal{O}_X} \tilde{\mathcal{E}}, \tilde{\mathcal{E}}) \oplus \operatorname{Hom}_X(\operatorname{Sym}^2 \tilde{\mathcal{E}}, \mathcal{A})$. Of course, not every element of the latter will satisfy all the necessary compatibilities with the S_3-action, nor will it necessarily define an associative algebra. To characterize those elements that do satisfy these conditions, and hence find a description of $S_3 \operatorname{Cov}_X(\mathcal{A})$, we first analyze the local situation.

3 Local Analysis

Let $U \subseteq X$ be an affine open such that $\mathcal{L}(U)$ and $\mathcal{E}(U)$ are free $\mathcal{O}_X(U)$-modules of ranks one and two, respectively. (In general, when we work with $\mathcal{L}$ and $\mathcal{E}$ locally, we will always mean over such an affine U.) Let t be a generator for $\mathcal{L}(U)$ and v_1, v_2 be generators for $\mathcal{E}(U)$. To understand the algebra structure on $\mathcal{A}(U)$, we need to analyze (the images of) the following products (with tensor symbols suppressed):

- $t^2 \in \mathcal{O}_X(U)$;
- $tv_i, t \cdot \tau v_i \in \tilde{\mathcal{E}}(U) = \mathcal{E}(U) \oplus \tau\mathcal{E}(U)$, for $i = 1, 2$;
- $v_i v_j, v_i \cdot \tau v_j, \tau v_i \cdot \tau v_j \in \mathcal{A}(U)$, for $i, j = 1, 2$.

Momentarily leaving aside associativity concerns, we find the following lemma.

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Lemma 3.1 *With respect to the basis $\{1, t, v_1, v_2, \tau v_1, \tau v_2\}$ for $\mathcal{A}(U)$, the commutative $\mathcal{O}_X(U)[S_3]$-algebra structures on $\mathcal{A}(U)$ are precisely those of the form*

$$\begin{aligned}
t^2 &= a\\
tv_1 &= b_1v_1 + b_2v_2 - 2b_1\tau v_1 - 2b_2\tau v_2\\
tv_2 &= c_1v_1 + c_2v_2 - 2c_1\tau v_1 - 2c_2\tau v_2\\
t\cdot\tau v_1 &= 2b_1v_1 + 2b_2v_2 - b_1\tau v_1 - b_2\tau v_2\\
t\cdot\tau v_2 &= 2c_1v_1 + 2c_2v_2 - c_1\tau v_1 - c_2\tau v_2\\
v_1^2 &= d_1 + d_3v_1 + d_4v_2 - 2d_3\tau v_1 - 2d_4\tau v_2\\
v_1v_2 &= e_1 + e_3v_1 + e_4v_2 - 2e_3\tau v_1 - 2e_4\tau v_2\\
v_2^2 &= f_1 + f_3v_1 + f_4v_2 - 2f_3\tau v_1 - 2f_4\tau v_2\\
v_1\cdot\tau v_1 &= \frac{1}{2}d_1 - d_3v_1 - d_4v_2 - d_3\tau v_1 - d_4\tau v_2\\
v_1\cdot\tau v_2 &= \frac{1}{2}e_1 + h_2t + h_3v_1 + h_4v_2 - e_3\tau v_1 - e_4\tau v_2\\
v_2\cdot\tau v_1 &= \frac{1}{2}e_1 - h_2t - e_3v_1 - e_4v_2 + h_3\tau v_1 + h_4\tau v_2\\
v_2\cdot\tau v_2 &= \frac{1}{2}f_1 - f_3v_1 - f_4v_2 - f_3\tau v_1 - f_4\tau v_2\\
(\tau v_1)^2 &= d_1 - 2d_3v_1 - 2d_4v_2 + d_3\tau v_1 + d_4\tau v_2\\
(\tau v_1)(\tau v_2) &= e_1 - 2e_3v_1 - 2e_4v_2 + e_3\tau v_1 + e_4\tau v_2\\
(\tau v_2)^2 &= f_1 - 2f_3v_1 - 2f_4v_2 + f_3\tau v_1 + f_4\tau v_2,
\end{aligned}$$

for some $a, b_i, c_i, d_i, e_i, f_i, h_i \in \mathcal{O}_X(U)$.

Remark 3.2 In particular, every element of $S_3\,\mathrm{Cov}_X(\mathcal{A})$ is locally of the above form.

Proof Compatibility with the action of τ requires $\tau\cdot(tv_i) = (\tau\cdot t)(\tau v_i) = -t\cdot\tau v_i$, and so $t\cdot\tau v_i$ is determined by tv_i. Similarly, $\tau v_i\cdot\tau v_j$ is determined by v_iv_j, and $v_2\cdot\tau v_1$ is determined by $v_1\cdot\tau v_2$. Suppose

$$\begin{aligned}
t^2 &= a\\
tv_1 &= b_1v_1 + b_2v_2 + b_3\tau v_1 + b_4\tau v_2\\
tv_2 &= c_1v_1 + c_2v_2 + c_3\tau v_1 + c_4\tau v_2\\
v_1^2 &= d_1 + d_2t + d_3v_1 + d_4v_2 + d_5\tau v_1 + d_6\tau v_2\\
v_1v_2 &= e_1 + e_2t + e_3v_1 + e_4v_2 + e_5\tau v_1 + e_6\tau v_2
\end{aligned}$$

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$$\begin{aligned} v_2^2 &= f_1 + f_2 t + f_3 v_1 + f_4 v_2 + f_5 \tau v_1 + f_6 \tau v_2 \\ v_1 \cdot \tau v_1 &= g_1 + g_2 t + g_3 v_1 + g_4 v_2 + g_5 \tau v_1 + g_6 \tau v_2 \\ v_1 \cdot \tau v_2 &= h_1 + h_2 t + h_3 v_1 + h_4 v_2 + h_5 \tau v_1 + h_6 \tau v_2 \\ v_2 \cdot \tau v_2 &= i_1 + i_2 t + i_3 v_1 + i_4 v_2 + i_5 \tau v_1 + i_6 \tau v_2, \end{aligned}$$

for some $a, b_j, c_j, d_j, e_j, f_j, g_j, h_j, i_j \in \mathcal{O}_X(U)$. By our initial remark, we then must have

$$\begin{aligned} t \cdot \tau v_1 &= -b_3 v_1 - b_4 v_2 - b_1 \tau v_1 - b_2 \tau v_2 \\ t \cdot \tau v_2 &= -c_3 v_1 - c_4 v_2 - c_1 \tau v_1 - c_2 \tau v_2 \\ (\tau v_1)^2 &= d_1 - d_2 t + d_5 v_1 + d_6 v_2 + d_3 \tau v_1 + d_4 \tau v_2 \\ (\tau v_1)(\tau v_2) &= e_1 - e_2 t + e_5 v_1 + e_6 v_2 + e_3 \tau v_1 + e_4 \tau v_2 \\ (\tau v_2)^2 &= f_1 - f_2 t + f_5 v_1 + f_6 v_2 + f_3 \tau v_1 + f_4 \tau v_2 \\ v_2 \cdot \tau v_1 &= h_1 - h_2 t + h_5 v_1 + h_6 v_2 + h_3 \tau v_1 + h_4 \tau v_2. \end{aligned}$$

Such an algebra structure is now ensured to be compatible with the action of τ, and so compatibility with the full S_3-action will follow from compatibility with the action of σ. Using the equations above, together with equation (1), we compute

$$\sigma \cdot (t v_1) = b_3 v_1 + b_4 v_2 + (-b_1 - b_3) \tau v_1 + (-b_2 - b_4) \tau v_2,$$

and

$$\begin{aligned} (\sigma \cdot t)(\sigma \cdot v_1) = t(\sigma \cdot v_1) &= t(-\tau v_1) = -t(\tau v_1) \\ &= b_3 v_1 + b_4 v_2 + b_1 \tau v_1 + b_2 \tau v_2. \end{aligned}$$

So, compatibility with σ requires $b_3 = -2b_1$ and $b_4 = -2b_2$. The corresponding computation for the relation $\sigma \cdot (t v_2) = (\sigma \cdot t)(\sigma \cdot v_2)$ requires $c_3 = -2c_1$ and $c_4 = -2c_2$.

Computing

$$\sigma \cdot (v_1^2) = d_1 + d_2 t + d_5 v_1 + d_6 v_2 + (-d_3 - d_5) \tau v_1 + (-d_4 - d_6) \tau v_2,$$

and

$$\begin{aligned} (\sigma \cdot v_1)^2 &= (-\tau v_1)^2 = (\tau v_1)^2 \\ &= d_1 - d_2 t + d_5 v_1 + d_6 v_2 + d_3 \tau v_1 + d_4 \tau v_2, \end{aligned}$$

we must have $d_2 = 0$, $d_5 = -2d_3$, and $d_6 = -2d_4$. Similarly, the relation $\sigma \cdot (v_1 v_2) = (\sigma \cdot v_1)(\sigma \cdot v_2)$ requires $e_2 = 0$, $e_5 = -2e_3$, and $e_6 = -2e_4$. The relation $\sigma \cdot (v_2)^2 = (\sigma \cdot v_2)^2$ requires $f_2 = 0$, $f_5 = -2f_3$, and $f_6 = -2f_4$.

Lastly, we compute

$$\sigma \cdot (v_1 \cdot \tau v_1) = g_1 + g_2 t + g_5 v_1 + g_6 v_2 + (-g_3 - g_5)\tau v_1 + (-g_4 - g_6)\tau v_2,$$

and

$$\begin{aligned}(\sigma \cdot v_1)(\sigma \cdot \tau v_1) &= (-\tau v_1)(v_1 - \tau v_1) = -(v_1 \cdot \tau v_1) + (\tau v_1)^2 \\ &= (-g_1 + d_1) + (-g_2 - d_2)t + (-g_3 + d_5)v_1 + (-g_4 + d_6)v_2 \\ &\quad + (-g_5 + d_3)\tau v_1 + (-g_6 + d_4)\tau v_2.\end{aligned}$$

From this (and the previously obtained relations), we deduce $g_1 = \frac{1}{2}d_1$, $g_2 = 0$, $g_3 = g_5 = -d_3$, and $g_4 = g_6 = -d_4$. The relation $\sigma \cdot (v_1 \cdot \tau v_2) = (\sigma \cdot v_1)(\sigma \cdot \tau v_2)$ requires $h_1 = \frac{1}{2}e_1$, $h_5 = -e_3$, and $h_6 = -e_4$. The relation $\sigma \cdot (v_2 \cdot \tau v_2) = (\sigma \cdot v_2)(\sigma \cdot \tau v_2)$ requires $i_1 = \frac{1}{2}f_1$, $i_2 = 0$, $i_3 = i_5 = -f_3$, and $i_4 = i_6 = -f_4$. ■

The previous lemma specifically omitted any mention of associativity. Indeed, associativity imposes many additional constraints.

Proposition 3.3 *With respect to the basis $\{1, t, v_1, v_2, \tau v_1, \tau v_2\}$ for $\mathcal{A}(U)$, every commutative, associative $\mathcal{O}_X(U)[S_3]$-algebra structure on $\mathcal{A}(U)$ that defines a domain is of the form*

$$\begin{aligned}
t^2 &= -3b_1^2 - 3b_2c_1 \\
tv_1 &= b_1v_1 + b_2v_2 - 2b_1\tau v_1 - 2b_2\tau v_2 \\
tv_2 &= c_1v_1 - b_1v_2 - 2c_1\tau v_1 + 2b_1\tau v_2 \\
t \cdot \tau v_1 &= 2b_1v_1 + 2b_2v_2 - b_1\tau v_1 - b_2\tau v_2 \\
t \cdot \tau v_2 &= 2c_1v_1 - 2b_1v_2 - c_1\tau v_1 + b_1\tau v_2 \\
v_1^2 &= 6(d_3^2 - d_4f_4) + d_3v_1 + d_4v_2 - 2d_3\tau v_1 - 2d_4\tau v_2 \\
v_1v_2 &= 3(d_4f_3 - d_3f_4) - f_4v_1 - d_3v_2 + 2f_4\tau v_1 + 2d_3\tau v_2 \\
v_2^2 &= 6(f_4^2 - d_3f_3) + f_3v_1 + f_4v_2 - 2f_3\tau v_1 - 2f_4\tau v_2 \\
v_1 \cdot \tau v_1 &= 3(d_3^2 - d_4f_4) - d_3v_1 - d_4v_2 - d_3\tau v_1 - d_4\tau v_2 \\
v_1 \cdot \tau v_2 &= \frac{3}{2}(d_4f_3 - d_3f_4) + h_2t + f_4v_1 + d_3v_2 + f_4\tau v_1 + d_3\tau v_2 \\
v_2 \cdot \tau v_1 &= \frac{3}{2}(d_4f_3 - d_3f_4) - h_2t + f_4v_1 + d_3v_2 + f_4\tau v_1 + d_3\tau v_2 \\
v_2 \cdot \tau v_2 &= 3(f_4^2 - d_3f_3) - f_3v_1 - f_4v_2 - f_3\tau v_1 - f_4\tau v_2 \\
(\tau v_1)^2 &= 6(d_3^2 - d_4f_4) - 2d_3v_1 - 2d_4v_2 + d_3\tau v_1 + d_4\tau v_2 \\
(\tau v_1)(\tau v_2) &= 3(d_4f_3 - d_3f_4) + 2f_4v_1 + 2d_3v_2 - f_4\tau v_1 - d_3\tau v_2 \\
(\tau v_2)^2 &= 6(f_4^2 - d_3f_3) - 2f_3v_1 - 2f_4v_2 + f_3\tau v_1 + f_4\tau v_2,
\end{aligned}$$

for some $b_1, b_2, c_1, d_3, d_4, f_3, f_4, h_2 \in \mathcal{O}_X(U)$ *satisfying*

$$\text{(i) } 2b_1d_3 - b_2f_4 + c_1d_4 = 0; \qquad \text{(iii) } b_2h_2 + 3(d_3^2 - d_4f_4) = 0;$$

$$\text{(ii) } 2b_1f_4 - b_2f_3 + c_1d_3 = 0; \qquad \text{(iv) } 2b_1h_2 + 3(d_3f_4 - d_4f_3) = 0;$$

$$\text{(v) } c_1h_2 + 3(d_3f_3 - f_4^2) = 0.$$

Conversely, any multiplicative structure on $\mathcal{A}(U)$ *of the above form defines a commutative, associative* $\mathcal{O}_X(U)[S_3]$*-algebra structure on* $\mathcal{A}(U)$*, although possibly not a domain.*

Remark 3.4 Consequently, if we define $S_3\operatorname{Cov}_X(\mathcal{A})^0 \subset S_3\operatorname{Cov}_X(\mathcal{A})$ as the subset of elements defining domains, then we see that every element of $S_3\operatorname{Cov}_X(\mathcal{A})^0$ is locally of the form described above. Conversely, every element of $\operatorname{Hom}_X(\operatorname{Sym}^2\mathcal{L}, \mathcal{O}_X) \oplus \operatorname{Hom}_X(\mathcal{L} \otimes_{\mathcal{O}_X} \tilde{\mathcal{E}}, \tilde{\mathcal{E}}) \oplus \operatorname{Hom}_X(\operatorname{Sym}^2\tilde{\mathcal{E}}, \mathcal{A})$ that is locally of the above form lies in $S_3\operatorname{Cov}_X(\mathcal{A})$.

Proof The proof consists of systematically imposing the third-order associativity conditions. As the calculations are straightforward but tedious, we only give full details for the first few. So as to maintain symmetry between v_1 and v_2, we endeavor to express e_i and h_i in terms of d_i and f_i whenever possible. Using Lemma 3.1, we compute

$$(t^2)v_1 = av_1$$

$$t(tv_1) = (-3b_1^2 - 3b_2c_1)v_1 + (-3b_1b_2 - 3b_2c_2)v_2.$$

Equating coefficients gives

$$a = -3b_1^2 - 3b_2c_1 \tag{2}$$

$$0 = b_2(b_1 + c_2). \tag{3}$$

Similarly, we compute

$$(t^2)v_2 = av_2$$

$$t(tv_2) = (-3b_1c_1 - 3c_1c_2)v_1 + (-3b_2c_1 - 3c_2^2)v_2,$$

and so must have

$$0 = c_1(b_1 + c_2) \tag{4}$$

$$a = -3b_2c_1 - 3c_2^2. \tag{5}$$

Note that the relations $(t^2)\tau v_1 = t(t\tau v_1)$ and $(t^2)\tau v_2 = t(t\tau v_2)$ immediately follow from the above relations and the compatibility with τ. Indeed, we have $(t^2)\tau v_1 = (-t)^2\tau v_1 = (\tau \cdot t)^2\tau v_1 = \tau \cdot \big((t^2)v_1\big) = \tau \cdot \big(t(tv_1)\big) = -t(-t \cdot \tau v_1) = t(t\tau v_1)$, and similarly for $(t^2)\tau v_2$.

We next compute

$$t(v_1^2) = d_1 t + (-3b_1d_3 - 3c_1d_4)v_1 + (-3b_2d_3 - 3c_2d_4)v_2$$
$$(tv_1)v_1 = (-2b_2h_2)t + (3b_1d_3 + b_2e_3 - 2b_2h_3)v_1 + (3b_1d_4 + b_2e_4 - 2b_2h_4)v_2,$$

which implies

$$d_1 = -2b_2h_2 \tag{6}$$
$$6b_1d_3 + b_2e_3 - 2b_2h_3 + 3c_1d_4 = 0 \tag{7}$$
$$3b_1d_4 + 3b_2d_3 + b_2e_4 - 2b_2h_4 + 3c_2d_4 = 0. \tag{8}$$

We claim $c_2 = -b_1$. Indeed, suppose $c_2 \neq -b_1$. Recalling that we have assumed X is integral, and hence $\mathcal{O}_X(U)$ is a domain, equations (3) and (4) imply $b_2 = c_1 = 0$. Equations (2) and (5) then become $-3c_2^2 = a = -3b_1^2$. By hypothesis, the algebra $\mathcal{A}(U)$ is a domain, so $t^2 = a$ is nonzero, and hence it follows that b_1 and c_2 are both nonzero. Since their squares are equal and $c_2 \neq -b_1$, we must have $c_2 = b_1$. But then equations (6)–(8) imply $d_1 = d_3 = d_4 = 0$, and hence $v_1^2 = 0$, which again violates the assumption $\mathcal{A}(U)$ is a domain.

We must therefore have $c_2 = -b_1$, and equations (2)–(8) reduce to

$$a = -3b_1^2 - 3b_2c_1 \tag{9}$$
$$d_1 = -2b_2h_2 \tag{10}$$
$$6b_1d_3 + b_2e_3 - 2b_2h_3 + 3c_1d_4 = 0 \tag{11}$$
$$3b_2d_3 + b_2e_4 - 2b_2h_4 = 0. \tag{12}$$

Similar computations for the relations $t(v_1v_2) = (tv_1)v_2$ and $t(v_2^2) = (tv_2)v_2$ yield

$$h_3 = -e_3 = f_4 \tag{13}$$
$$h_4 = -e_4 \tag{14}$$
$$e_1 = 2b_1h_2 \tag{15}$$
$$f_1 = 2c_1h_2 \tag{16}$$
$$2b_1f_4 - b_2f_3 - c_1e_4 = 0. \tag{17}$$

Imposing the conditions so far collected, a calculation reveals the relations $t(v_1\tau v_1) = (tv_1)\tau v_1$, $t(v_1\tau v_2) = (tv_1)\tau v_2$, and $t(v_2\tau v_2) = (tv_2)\tau v_2$ all now hold. Compatibility with the action of τ then implies that the relations $t(\tau v_1)^2 = (t\cdot\tau v_1)\tau v_1$, $t(\tau v_1\cdot\tau v_2) = (t\cdot\tau v_1)\tau v_2$, $t(\tau v_2)^2 = (t\cdot\tau v_2)\tau v_2$, and $t(v_2\cdot\tau v_1) = (tv_2)\tau v_1$ also hold.

Similar calculations for the relations $(v_1^2)v_2 = v_1(v_1v_2)$, $v_1(v_2^2) = (v_1v_2)v_2$, and $v_1^2(\tau v_1) = v_1(v_1\tau v_1)$ yield

$$e_4 = -d_3 \tag{18}$$
$$2b_1h_2 - 3d_4f_3 + 3d_3f_4 = 0 \tag{19}$$
$$b_2h_2 + 3d_3^3 - 3d_4f_4 = 0 \tag{20}$$
$$c_1h_2 + 3d_3f_3 - 3f_4^2 = 0. \tag{21}$$

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At this point, we've reached the statement of the proposition: equation (9) gives a in terms of b_1, b_2, c_1; equations (10) and (20), (15) and (19), and (16) and (21) give d_1, e_1, and f_1 in terms of d_3, d_4, f_3, f_4, respectively; equations (11) and (13) give condition (i); equations (17) and (18) give condition (ii); and equations (19)–(21) gives conditions (iii)–(v). A calculation verifies that all remaining associativity relations are now satisfied. ■

Remark 3.5 Examining the coefficients in Proposition 3.3, it is evident that elements of

$$\operatorname{Hom}\big(\operatorname{Sym}^2 \mathcal{L}(U), \mathcal{O}_X(U)\big) \oplus \operatorname{Hom}\big(\mathcal{L}(U) \otimes_{\mathcal{O}_X(U)} \tilde{\mathcal{E}}(U), \tilde{\mathcal{E}}(U)\big) \\ \oplus \operatorname{Hom}\big(\operatorname{Sym}^2 \tilde{\mathcal{E}}(U), \mathcal{A}(U)\big)$$

of the given form are in bijection with elements

$$(\phi_U, \psi_U, \xi_U) \in \operatorname{Hom}\big(\mathcal{L}(U) \otimes \mathcal{E}(U), \mathcal{E}(U)\big) \oplus \operatorname{Hom}\big(\operatorname{Sym}^2 \mathcal{E}(U), \mathcal{E}(U)\big) \\ \oplus \operatorname{Hom}\big(\mathcal{E}(U) \otimes \tau\mathcal{E}(U), \mathcal{L}(U)\big)$$

of the form

$$\begin{aligned} &\phi_U(t \otimes v_1) = av_1 + bv_2 && \psi_U(v_1^2) = dv_1 + ev_2 && \xi_U(v_1 \otimes \tau v_1) = 0 \\ &\phi_U(t \otimes v_2) = cv_1 - av_2 && \psi_U(v_1 v_2) = -gv_1 - dv_2 && \xi_U(v_1 \otimes \tau v_2) = ht \\ & && \psi_U(v_2^2) = fv_1 + gv_2 && \xi_U(v_2 \otimes \tau v_1) = -ht \\ & && && \xi_U(v_2 \otimes \tau v_2) = 0. \end{aligned}$$

for $a, b, c, d, e, f, g, h \in \mathcal{O}_X(U)$ satisfying the relations

$$\begin{aligned} &\text{(i)} \ -bg + 2ad + ce = 0; && \text{(iii)}\ bh + 3(d^2 - eg) = 0; \\ &\text{(ii)} \ -bf + 2ag + cd = 0; && \text{(iv)}\ 2ah + 3(dg - ef) = 0; \\ & && \text{(v)}\ ch + 3(df - g^2) = 0. \end{aligned}$$

Remark 3.6 In Miranda's study of triple covers, morphisms $\psi \in \operatorname{Hom}_X(\operatorname{Sym}^2 \mathcal{E}, \mathcal{E})$ locally of the form in Remark 3.5 are called *triple cover homomorphisms* [16, Definition 3.1]. They are shown to induce commutative, associative $\mathcal{O}_X$-algebra structures on $\mathcal{O}_X \oplus \mathcal{E}$, and hence define triple covers $\mathbf{Spec}_X(\mathcal{O}_X \oplus \mathcal{E}) \to X$ [16, Theorem 2.7]. This connection with triple covers is not unexpected. Indeed, if $\mathbf{Spec}_X\mathcal{A} \to X$ is an S_3-cover and we let $\mathcal{A}^\tau \subset \mathcal{A}$ denote the submodule of τ-invariant sections, then it is straightforward to verify (using Proposition 3.3 and [16, Theorem 2.7]) that $\mathbf{Spec}_X\mathcal{A}^\tau \to X$ is a triple cover. Similarly, $\mathbf{Spec}_X\mathcal{A}^{\sigma\tau} \to X$ and $\mathbf{Spec}_X\mathcal{A}^{\sigma^2\tau} \to X$ are triple covers (and all three of these triple covers are conjugate).

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Remark 3.7 Proposition 3.3 gives a full description of affine S_3-covers, as well as a simple method for generating them, namely by choosing solutions to equations (i)–(v), above. (For instance, $b_1 = b_2 = c_1 = d_3 = f_4 = 1$, $d_4 = -1$, $f_3 = 3$, $h_2 = -6$ and $b_1 = b_2 = c_1 = d_3 = f_3 = 1$, $d_4 = -2$, $f_4 = 0$, $h_2 = -3$ are both solutions.) The description of the multiplicative structure of $\mathcal{A}(U)$ defines an $\mathcal{O}_X(U)$-algebra isomorphism $\mathcal{A}(U) \cong \mathcal{O}_X(U)[1, t, v_1, v_2, \tau v_1, \tau v_2]/I$, where I is the ideal of relations generated by the fifteen equations of Proposition 3.3. One consequence of this explicit description is that we also have a description of the ramification locus of $\pi|_{\pi^{-1}(U)}\colon \operatorname{Spec}\mathcal{A}(U) \to U$, as the zero locus of the ideal generated by all 5 x 5 minors of the matrix

$$\begin{bmatrix} 2t & 0 & 0 & 0 & 0 \\ v_1 & t-b_1 & -b_2 & 2b_1 & 2b_2 \\ v_2 & -c_1 & t+b_1 & 2c_1 & -2b_1 \\ \tau v_1 & -2b_1 & -2b_2 & t+b_1 & b_2 \\ \tau v_2 & -2c_1 & 2b_1 & c_1 & t-b_1 \\ 0 & 2v_1-d_3 & -d_4 & 2d_3 & 2d_4 \\ 0 & v_2+f_4 & v_1+d_3 & -2f_4 & -2d_3 \\ 0 & -f_3 & 2v_2-f_4 & 2f_3 & 2f_4 \\ 0 & \tau v_1+d_3 & d_4 & v_1+d_3 & d_4 \\ -h_2 & \tau v_2-f_4 & -d_3 & -f_4 & v_1-d_3 \\ h_2 & -f_4 & \tau v_1-d_3 & v_2-f_4 & -d_3 \\ 0 & f_3 & \tau v_2+f_4 & f_3 & v_2+f_4 \\ 0 & 2d_3 & 2d_4 & 2\tau v_1-d_3 & -d_4 \\ 0 & -2f_4 & -2d_3 & \tau v_2+f_4 & \tau v_1+d_3 \\ 0 & 2f_3 & 2f_4 & -f_3 & 2\tau v_2-f_4 \end{bmatrix}.$$

4 Global Analysis

We now use our local description of S_3-covers to obtain a global description. In light of Proposition 3.3 and Remark 3.5, we expect to characterize such covers by an R-submodule of $\operatorname{Hom}_X(\mathcal{L}\otimes\mathcal{E},\mathcal{E}) \oplus \operatorname{Hom}_X(\operatorname{Sym}^2\mathcal{E},\mathcal{E}) \oplus \operatorname{Hom}_X(\mathcal{E}\otimes\tau\mathcal{E},\mathcal{L})$. We need a global restatement of Remark 3.5.

Definition 4.1 Let $M_1 \leq \operatorname{Hom}_X(\mathcal{L}\otimes\mathcal{E},\mathcal{E})$, $M_2 \leq \operatorname{Hom}_X(\operatorname{Sym}^2\mathcal{E},\mathcal{E})$, and $M_3 \leq \operatorname{Hom}_X(\mathcal{E}\otimes\tau\mathcal{E},\mathcal{L})$ denote the R-submodules consisting of elements locally of the forms

$$\begin{aligned} &\phi_U(t\otimes v_1) = av_1 + bv_2 && \psi_U(v_1^2) = dv_1 + ev_2 && \xi_U(v_1\otimes\tau v_1) = 0 \\ &\phi_U(t\otimes v_2) = cv_1 - av_2 && \psi_U(v_1v_2) = -gv_1 - dv_2 && \xi_U(v_1\otimes\tau v_2) = ht \\ & && \psi_U(v_2^2) = fv_1 + gv_2 && \xi_U(v_2\otimes\tau v_1) = -ht \\ & && && \xi_U(v_2\otimes\tau v_2) = 0, \end{aligned}$$

respectively, for each affine open $U \subseteq X$ (over which $\mathcal{L}(U)$ and $\mathcal{E}(U)$ are free) with generators t for $\mathcal{L}(U)$ and v_1, v_2 for $\mathcal{E}(U)$.

Lemma 4.2 *M_1, M_2, and M_3 are well defined,* i.e., *do not depend on the choices of local generators for $\mathcal{L}$ and $\mathcal{E}$.*

Proof Let $U \subseteq X$ be an open affine over which $\mathcal{L}(U)$ and $\mathcal{E}(U)$ are free, and suppose $(\phi,\psi,\xi) \in \mathrm{Hom}_X(\mathcal{L}\otimes\mathcal{E},\mathcal{E}) \oplus \mathrm{Hom}_X(\mathrm{Sym}^2\mathcal{E},\mathcal{E}) \oplus \mathrm{Hom}_X(\mathcal{E}\otimes\tau\mathcal{E},\mathcal{L})$ are locally of the above form with respect to generators t for $\mathcal{L}(U)$ and v_1, v_2 for $\mathcal{E}(U)$. Let $s = \eta t$ be another generator for $\mathcal{L}(U)$, and $\{w_1, w_2\}$ be another basis for $\mathcal{E}(U)$, with change of basis matrix $C = \left[\begin{smallmatrix}\lambda_1 & \mu_1\\ \lambda_2 & \mu_2\end{smallmatrix}\right]$. A straightforward calculation reveals

$$\begin{aligned}
&\phi(s\otimes w_1) = a'w_1 + b'w_2 && \psi(w_1^2) = d'w_1 + e'w_2 && \xi(w_1\otimes\tau w_1) = 0\\
&\phi(s\otimes w_2) = c'w_1 - a'w_2 && \psi(w_1w_2) = -g'w_1 - d'w_2 && \xi(w_1\otimes\tau w_2) = h's\\
& && \psi(w_2^2) = f'w_1 + g'w_2 && \xi(w_2\otimes\tau w_1) = -h's\\
& && && \xi(w_2\otimes\tau w_2) = 0,
\end{aligned}$$

where

$$\begin{aligned}
a' &= \frac{\eta}{\det(C)}(-\lambda_1\lambda_2 b + \lambda_1\mu_2 a + \lambda_2\mu_1 a + \mu_1\mu_2 c)\\
b' &= \frac{\eta}{\det(C)}(\lambda_1^2 b - 2\lambda_1\mu_1 a - \mu_1^2 c)\\
c' &= \frac{\eta}{\det(C)}(-\lambda_2^2 b + 2\lambda_2\mu_2 a + \mu_2^2 c)\\
d' &= \frac{1}{\det(C)}(-\lambda_1^2\lambda_2 e + \lambda_1^2\mu_2 d + 2\lambda_1\lambda_2\mu_1 d - 2\lambda_1\mu_1\mu_2 g - \lambda_2\mu_1^2 g + \mu_1^2\mu_2 f)\\
e' &= \frac{1}{\det(C)}(\lambda_1^3 e - 3\lambda_1^2\mu_1 d + 3\lambda_1\mu_1^2 g - \mu_1^3 f)\\
f' &= \frac{1}{\det(C)}(-\lambda_2^3 e + 3\lambda_2^2\mu_2 d - 3\lambda_2\mu_2^2 g + \mu_2^3 f)\\
g' &= \frac{1}{\det(C)}(\lambda_1\lambda_2^2 e - 2\lambda_1\lambda_2\mu_2 d + \lambda_1\mu_2^2 g - \lambda_2^2\mu_1 d + 2\lambda_2\mu_1\mu_2 g - \mu_1\mu_2^2 f)\\
h' &= \frac{\det(C)}{\eta}h. \qquad \blacksquare
\end{aligned}$$

The following lemma gives intrinsic descriptions of the R-modules M_1, M_2, and M_3.

Lemma 4.3 *There exist R-module isomorphisms*

(i) $\mathbf{G}_1\colon \mathrm{Hom}_X(\mathcal{L}\otimes\mathrm{Sym}^2\mathcal{E}, \bigwedge^2\mathcal{E}) \xrightarrow{\sim} M_1$,

(ii) $\mathbf{G}_2\colon \mathrm{Hom}_X(\mathrm{Sym}^3\mathcal{E}, \bigwedge^2\mathcal{E}) \xrightarrow{\sim} M_2$,

(iii) $\mathbf{G}_3\colon \mathrm{Hom}_X(\bigwedge^2\mathcal{E}, \mathcal{L}) \xrightarrow{\sim} M_3$.

Proof The proof of (ii) is detailed in [16, Proposition 3.3]. The proofs of (i) and (iii) are similar and are given here. The method of proof will be used repeatedly. We begin by proving (iii).

Define $\mathbf{G}_3\colon \operatorname{Hom}_X(\bigwedge^2\mathcal{E},\mathcal{L}) \to \operatorname{Hom}_X(\mathcal{E}\otimes\tau\mathcal{E},\mathcal{L})$ by precomposing elements of $\operatorname{Hom}_X(\bigwedge^2\mathcal{E},\mathcal{L})$ with the morphism $\mathcal{E}\otimes\tau\mathcal{E} \xrightarrow{1\otimes\tau} \mathcal{E}\otimes\mathcal{E} \xrightarrow{\text{can}} \bigwedge^2\mathcal{E}$. We claim $\mathbf{G}_3$ maps $\operatorname{Hom}_X(\bigwedge^2\mathcal{E},\mathcal{L})$ isomorphically onto M_3. To see this, suppose $\Xi \in \operatorname{Hom}_X(\bigwedge^2\mathcal{E},\mathcal{L})$ is locally given by

$$v_1\wedge v_2 \longmapsto ht.$$

Then $\mathbf{G}_3(\Xi)$ is locally given by

$$\begin{aligned}
v_1\otimes\tau v_1 &\longmapsto v_1\otimes v_1 \longmapsto v_1\wedge v_1 = 0 \longmapsto 0\\
v_1\otimes\tau v_2 &\longmapsto v_1\otimes v_2 \longmapsto v_1\wedge v_2 \longmapsto ht\\
v_2\otimes\tau v_1 &\longmapsto v_2\otimes v_1 \longmapsto v_2\wedge v_1 \longmapsto -ht\\
v_2\otimes\tau v_2 &\longmapsto v_2\otimes v_2 \longmapsto v_2\wedge v_2 = 0 \longmapsto 0.
\end{aligned}$$

It is then clear that $\mathbf{G}_3$ is an isomorphism onto M_3. Indeed, the inverse morphism can be described locally as follows. Suppose $\xi \in M_3$ is locally of the form of Definition 4.1. The morphism $\mathcal{E}(U)\otimes\mathcal{E}(U) \xrightarrow{1\otimes\tau} \mathcal{E}(U)\otimes\mathcal{E}(U) \xrightarrow{\xi_U} \mathcal{L}(U)$ is then locally given by

$$\begin{aligned}
v_1\otimes v_1 &\longmapsto v_1\otimes\tau v_1 \longmapsto 0\\
v_1\otimes v_2 &\longmapsto v_1\otimes\tau v_2 \longmapsto ht\\
v_2\otimes v_1 &\longmapsto v_2\otimes\tau v_1 \longmapsto -ht\\
v_2\otimes v_2 &\longmapsto v_2\otimes\tau v_2 \longmapsto 0,
\end{aligned}$$

and hence factors through the canonical morphism $\mathcal{E}(U)\otimes\mathcal{E}(U) \to \bigwedge^2\mathcal{E}(U)$. The induced morphism is $\mathbf{G}_3^{-1}(\xi)_U\colon \bigwedge^2\mathcal{E}(U) \to \mathcal{L}(U)$. Letting U vary, this defines the sheaf morphism $\mathbf{G}_3^{-1}(\xi)\colon \bigwedge^2\mathcal{E} \to \mathcal{L}$.

We now prove (i). Observe that we have isomorphisms

$$\begin{aligned}
&\operatorname{Hom}_R\bigl(\operatorname{Hom}_X\bigl(\mathcal{L}\otimes\operatorname{Sym}^2\mathcal{E},\textstyle\bigwedge^2\mathcal{E}\bigr),\operatorname{Hom}_X(\mathcal{L}\otimes\mathcal{E},\mathcal{E})\bigr)\\
&\qquad\cong \operatorname{Hom}_X\bigl(\mathcal{L}^*\otimes(\operatorname{Sym}^2\mathcal{E})^*\otimes\textstyle\bigwedge^2\mathcal{E},\mathcal{L}^*\otimes\mathcal{E}^*\otimes\mathcal{E}\bigr)\\
&\qquad\cong \operatorname{Hom}_X\bigl(\mathcal{L}\otimes\textstyle\bigwedge^2\mathcal{E}\otimes\mathcal{E},\mathcal{L}\otimes\mathcal{E}\otimes\operatorname{Sym}^2\mathcal{E}\bigr).
\end{aligned}$$

An element of this final group is the morphism $\mathbf{G}_1$ locally defined by

$$l\otimes(e_1\wedge e_2)\otimes e_3 \longmapsto l\otimes e_1\otimes e_2e_3 - l\otimes e_2\otimes e_1e_3,$$

for $l \in \mathcal{L}(U), e_i \in \mathcal{E}(U)$. We claim $\mathbf{G}_1$ (considered as an element of the first group) maps $\operatorname{Hom}_X(\mathcal{L}\otimes\operatorname{Sym}^2\mathcal{E},\bigwedge^2\mathcal{E})$ isomorphically onto M_1. To see this, suppose $\Phi \in \operatorname{Hom}_X(\mathcal{L}\otimes\operatorname{Sym}^2\mathcal{E},\bigwedge^2\mathcal{E})$ is locally given by

$$\begin{aligned}
t\otimes v_1^2 &\longmapsto -b\, v_1\wedge v_2\\
t\otimes v_1v_2 &\longmapsto a\, v_1\wedge v_2\\
t\otimes v_2^2 &\longmapsto c\, v_1\wedge v_2.
\end{aligned}$$

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Considered as a section of $\mathcal{L}^* \otimes (\operatorname{Sym}^2 \mathcal{E})^* \otimes \bigwedge^2 \mathcal{E}$, Φ corresponds to the element locally given by

$$\big(-bt^* \otimes (v_1^2)^* + at^* \otimes (v_1 v_2)^* + ct^* \otimes (v_2^2)^*\big) \otimes (v_1 \wedge v_2).$$

Now let us chase $\mathbf{G}_1$ backwards through the chain of isomorphisms. Locally, $\mathbf{G}_1$ is given by

$$\begin{aligned} t \otimes (v_1 \wedge v_2) \otimes v_1 &\longmapsto t \otimes v_1 \otimes v_1 v_2 - t \otimes v_2 \otimes v_1^2 \\ t \otimes (v_1 \wedge v_2) \otimes v_2 &\longmapsto t \otimes v_1 \otimes v_2^2 - t \otimes v_2 \otimes v_1 v_2. \end{aligned}$$

As an element of $\operatorname{Hom}_X\big(\mathcal{L}^* \otimes (\operatorname{Sym}^2 \mathcal{E})^* \otimes \bigwedge^2 \mathcal{E}, \mathcal{L}^* \otimes \mathcal{E}^* \otimes \mathcal{E}\big)$, this corresponds to the morphism locally given by

$$\begin{aligned} t^* \otimes (v_1^2)^* \otimes (v_1 \wedge v_2) &\longmapsto -t^* \otimes v_1^* \otimes v_2 \\ t^* \otimes (v_1 v_2)^* \otimes (v_1 \wedge v_2) &\longmapsto t^* \otimes v_1^* \otimes v_1 - t^* \otimes v_2^* \otimes v_2 \\ t^* \otimes (v_2^2)^* \otimes (v_1 \wedge v_2) &\longmapsto t^* \otimes v_2^* \otimes v_1. \end{aligned}$$

Hence, the image of Φ under $\mathbf{G}_1$ is the section locally given by

$$bt^* \otimes v_1^* \otimes v_2 + a(t^* \otimes v_1^* \otimes v_1 - t^* \otimes v_2^* \otimes v_2) + ct^* \otimes v_2^* \otimes v_1,$$

which corresponds to the map locally given by

$$\begin{aligned} t \otimes v_1 &\longmapsto av_1 + bv_2 \\ t \otimes v_2 &\longmapsto cv_1 - av_2. \end{aligned}$$

It is now clear that $\mathbf{G}_1$ does indeed map isomorphically onto M_1. Indeed, the inverse morphism can also be described locally, as follows. Suppose $\phi \in M_1$ is locally of the form of Definition 4.1. The morphism

$$\mathcal{L}(U) \otimes \mathcal{E}(U) \otimes \mathcal{E}(U) \xrightarrow{\phi_U \otimes 1} \mathcal{E}(U) \otimes \mathcal{E}(U) \xrightarrow{\text{can}} \textstyle\bigwedge^2 \mathcal{E}(U)$$

maps

$$\begin{aligned} t \otimes v_1 \otimes v_1 &\longmapsto (av_1 + bv_2) \otimes v_1 \longmapsto -b\, v_1 \wedge v_2 \\ t \otimes v_1 \otimes v_2 &\longmapsto (av_1 + bv_2) \otimes v_2 \longmapsto a\, v_1 \wedge v_2 \\ t \otimes v_2 \otimes v_1 &\longmapsto (cv_1 - av_2) \otimes v_1 \longmapsto a\, v_1 \wedge v_2 \\ t \otimes v_2 \otimes v_2 &\longmapsto (cv_1 - av_2) \otimes v_2 \longmapsto c\, v_1 \wedge v_2, \end{aligned}$$

and hence factors through the canonical morphism $\mathcal{L}(U) \otimes \mathcal{E}(U) \otimes \mathcal{E}(U) \to \mathcal{L}(U) \otimes \operatorname{Sym}^2 \mathcal{E}(U)$. The induced morphism is $\mathbf{G}_1^{-1}(\phi)_U \colon \mathcal{L}(U) \otimes \operatorname{Sym}^2 \mathcal{E}(U) \to \bigwedge^2 \mathcal{E}(U)$. Letting U vary, this defines the sheaf morphism $\mathbf{G}_1^{-1}(\phi) \colon \mathcal{L} \otimes \operatorname{Sym}^2 \mathcal{E} \to \bigwedge^2 \mathcal{E}$. ■

Definition 4.4 Define $\mathrm{Build}_X(\mathcal{A})$ to be

$$\mathrm{Hom}_X\big(\mathcal{L} \otimes \mathrm{Sym}^2 \mathcal{E}, \textstyle\bigwedge^2 \mathcal{E}\big) \oplus \mathrm{Hom}_X\big(\mathrm{Sym}^3 \mathcal{E}, \textstyle\bigwedge^2 \mathcal{E}\big) \oplus \mathrm{Hom}_X\big(\textstyle\bigwedge^2 \mathcal{E}, \mathcal{L}\big).$$

Recall that, by Lemma 2.1, we are viewing $S_3\,\mathrm{Cov}_X(\mathcal{A})$ as an R-submodule of $\mathrm{Hom}_X(\mathrm{Sym}^2 \mathcal{L}, \mathcal{O}_X) \oplus \mathrm{Hom}_X(\mathcal{L} \otimes_{\mathcal{O}_X} \tilde{\mathcal{E}}, \tilde{\mathcal{E}}) \oplus \mathrm{Hom}_X(\mathrm{Sym}^2 \tilde{\mathcal{E}}, \mathcal{A})$ (where $\tilde{\mathcal{E}} = \mathcal{E} \oplus \tau\mathcal{E}$). Let $\mathbf{P}$ denote the projection from this latter module to

$$\mathrm{Hom}_X(\mathcal{L} \otimes \mathcal{E}, \mathcal{E}) \oplus \mathrm{Hom}_X(\mathrm{Sym}^2 \mathcal{E}, \mathcal{E}) \oplus \mathrm{Hom}_X(\mathcal{E} \otimes \tau\mathcal{E}, \mathcal{L}).$$

Using the isomorphisms of Lemma 4.3, we may regard $\mathrm{Build}_X(\mathcal{A})$ as an R-submodule of this latter R-module. Recalling that $S_3\,\mathrm{Cov}_X(\mathcal{A})^0 \subset S_3\,\mathrm{Cov}_X(\mathcal{A})$ denotes the subset of those elements that define domains, we have the following corollary.

Corollary 4.5 $\mathbf{P}\big(S_3\,\mathrm{Cov}_X(\mathcal{A})^0\big) \subseteq \mathrm{Build}_X(\mathcal{A})$.

Proof This follows from Proposition 3.3, Remark 3.5, Lemma 4.2, and Lemma 4.3. ∎

Intuitively, we can think of the projection map $\mathbf{P}$ as extracting from an S_3-cover the "building data" necessary to construct the cover.

We next need a global restatement of conditions (i)–(v) of Remark 3.5. We first translate those conditions (using the isomorphisms of Lemma 4.3) into local conditions on the elements of $\mathrm{Build}_X(\mathcal{A})$. Note that if $(\Phi, \Psi, \Xi) \in \mathrm{Build}_X(\mathcal{A})$ are locally of the form

$$\begin{aligned}
\Phi_U(t \otimes v_1^2) &= A(v_1 \wedge v_2) & \Psi_U(v_1^3) &= D(v_1 \wedge v_2) & \Xi_U(v_1 \wedge v_2) &= ht\\
\Phi_U(t \otimes v_1 v_2) &= B(v_1 \wedge v_2) & \Psi_U(v_1^2 v_2) &= E(v_1 \wedge v_2) & &\\
\Phi_U(t \otimes v_2^2) &= C(v_1 \wedge v_2) & \Psi_U(v_1 v_2^2) &= F(v_1 \wedge v_2) & &\\
& & \Psi_U(v_2^3) &= G(v_1 \wedge v_2), & &
\end{aligned}$$

then the corresponding elements $(\psi, \phi, \xi) \in M_1 \oplus M_2 \oplus M_3$ are locally of the form

$$\begin{aligned}
\phi_U(t \otimes v_1) &= Bv_1 - Av_2 & \psi_U(v_1^2) &= Ev_1 - Dv_2 & \xi_U(v_1 \otimes \tau v_2) &= h\\
\phi_U(t \otimes v_2) &= Cv_1 - Bv_2 & \psi_U(v_1 v_2) &= Fv_1 - Ev_2 & &\\
\psi_U(v_2^2) &= Gv_1 - Fv_2. & & & &
\end{aligned}$$

The correspondence with our earlier notation is thus $A = -b, B = a, C = c, D = -e$, $E = d$, $F = -g$, $G = f$. Using this dictionary, the five conditions of Remark 3.5 become the following five local conditions on $(\Phi, \Psi, \Xi) \in \mathrm{Build}_X(\mathcal{A})$:

$$\begin{aligned}
&(\mathrm{i})'\ AF - 2BE + CD = 0; & &(\mathrm{iii})'\ Ah + 3(DF - E^2) = 0;\\
&(\mathrm{ii})'\ AG - 2BF + CE = 0; & &(\mathrm{iv})'\ 2Bh + 3(DG - EF) = 0;\\
& & &(\mathrm{v})'\ Ch + 3(EG - F^2) = 0.
\end{aligned}$$

Lemma 4.6 *There exists an R-module morphism*

$$\mathbf{C}_1\colon\ \mathrm{Build}_X(\mathcal{A}) \to \mathrm{Hom}_X\big(\mathcal{L}\otimes\big(\textstyle\bigwedge^2\mathcal{E}\big)^{\otimes 2}\otimes\mathcal{E}, \big(\textstyle\bigwedge^2\mathcal{E}\big)^{\otimes 2}\big)$$

whose kernel consists of precisely those elements that locally satisfy conditions (i)′ *and* (ii)′.

Proof First, consider the morphism $\Theta_1\colon (\bigwedge^2\mathcal{E})^{\otimes 2}\to(\mathrm{Sym}^2\mathcal{E})^{\otimes 2}$ locally defined by

$$(e_1\wedge e_2)\otimes(e_3\wedge e_4)\longmapsto e_1e_3\otimes e_2e_4 - e_1e_4\otimes e_2e_3 - e_2e_3\otimes e_1e_4 + e_2e_4\otimes e_1e_3,$$

and the canonical morphism $\Theta_2\colon\ \mathrm{Sym}^2\mathcal{E}\otimes\mathcal{E}\to\mathrm{Sym}^3\mathcal{E}$. Given any pair $(\Phi,\Psi)\in \mathrm{Hom}_X(\mathcal{L}\otimes\mathrm{Sym}^2\mathcal{E},\bigwedge^2\mathcal{E})\oplus\mathrm{Hom}_X(\mathrm{Sym}^3\mathcal{E},\bigwedge^2\mathcal{E})$, we have an induced morphism

$$\begin{aligned}\mathcal{L}\otimes\big(\textstyle\bigwedge^2\mathcal{E}\big)^{\otimes 2}\otimes\mathcal{E} &\xrightarrow{1\otimes\Theta_1\otimes 1}\mathcal{L}\otimes(\mathrm{Sym}^2\mathcal{E})^{\otimes 2}\otimes\mathcal{E}\\ &\xrightarrow{1\otimes 1\otimes\Theta_2}\mathcal{L}\otimes\mathrm{Sym}^2\mathcal{E}\otimes\mathrm{Sym}^3\mathcal{E}\xrightarrow{\Phi\otimes\Psi}\big(\textstyle\bigwedge^2\mathcal{E}\big)^{\otimes 2}.\end{aligned}$$

Let $\mathbf{C}_1(\Phi,\Psi)$ denote this morphism. If (Φ,Ψ) are locally of the form described above, then this morphism is locally given by

$$\begin{aligned} t\otimes(v_1\wedge v_2)\otimes(v_1\wedge v_2)\otimes v_1 &\longmapsto t\otimes\big(v_1^2\otimes v_2^2 - 2(v_1v_2\otimes v_1v_2)+v_2^2\otimes v_1^2\big)\otimes v_1\\ &\longmapsto t\otimes\big(v_1^2\otimes v_1v_2^2 - 2(v_1v_2\otimes v_1^2v_2)+v_2^2\otimes v_1^3\big)\\ &\longmapsto (AF-2BE+CD)(v_1\wedge v_2)^{\otimes 2}\\ t\otimes(v_1\wedge v_2)\otimes(v_1\wedge v_2)\otimes v_2 &\longmapsto t\otimes\big(v_1^2\otimes v_2^2 - 2(v_1v_2\otimes v_1v_2)+v_2^2\otimes v_1^2\big)\otimes v_2\\ &\longmapsto t\otimes\big(v_1^2\otimes v_2^3 - 2(v_1v_2\otimes v_1v_2^2)+v_2^2\otimes v_1^2v_2\big)\\ &\longmapsto (AG-2BF+CE)(v_1\wedge v_2)^{\otimes 2}\end{aligned}$$

Thus, conditions (i)′ and (ii)′ together are equivalent to the morphism $\mathbf{C}_1(\Phi,\Psi)$ being the zero map. (Extend $\mathbf{C}_1$ to all of $\mathrm{Build}_X(\mathcal{A})$ in the trivial way.) ■

Lemma 4.7 *There exists an R-module morphism*

$$\mathbf{C}_2\colon\ \mathrm{Build}_X(\mathcal{A})\to\mathrm{Hom}_X\big(\textstyle\bigwedge^2(\mathrm{Sym}^2\mathcal{E}),\bigwedge^2\mathcal{E}\big)$$

whose kernel consists of precisely those elements satisfying conditions (iii)′–(v)′.

Proof Define a morphism $\Theta\colon \bigwedge^2(\mathrm{Sym}^2\mathcal{E})\to\bigwedge^2\mathcal{E}\otimes\mathrm{Sym}^2\mathcal{E}$ locally by

$$e_1e_2\wedge e_3e_4\longmapsto\frac{1}{2}\big((e_1\wedge e_2)\otimes e_2e_4+(e_1\wedge e_4)\otimes e_2e_3+(e_2\wedge e_3)\otimes e_1e_4+(e_2\wedge e_4)\otimes e_1e_3\big).$$

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With respect to our usual basis, this morphism is locally given by

$$\begin{aligned} v_1^2 \wedge v_1 v_2 &\longmapsto (v_1 \wedge v_2) \otimes v_1^2 \\ v_1^2 \wedge v_2^2 &\longmapsto 2(v_1 \wedge v_2) \otimes v_1 v_2 \\ v_1 v_2 \wedge v_2^2 &\longmapsto (v_1 \wedge v_2) \otimes v_2^2. \end{aligned}$$

The composition $\bigwedge^2(\operatorname{Sym}^2 \mathcal{E}) \xrightarrow{\Theta} \bigwedge^2 \mathcal{E} \otimes \operatorname{Sym}^2 \mathcal{E} \xrightarrow{\Xi \otimes 1} \mathcal{L} \otimes \operatorname{Sym}^2 \mathcal{E} \xrightarrow{\Phi} \bigwedge^2 \mathcal{E}$ is then locally given by

$$\begin{aligned} v_1^2 \wedge v_1 v_2 &\longmapsto Ah(v_1 \wedge v_2) \\ v_1^2 \wedge v_2^2 &\longmapsto 2Bh(v_1 \wedge v_2) \\ v_1 v_2 \wedge v_2^2 &\longmapsto Ch(v_1 \wedge v_2). \end{aligned}$$

Letting $\psi = \mathbf{G}_2(\Psi)$, we also have the map $\bigwedge^2(\psi)$, locally given by

$$\begin{aligned} v_1^2 \wedge v_1 v_2 &\longmapsto (DF - E^2)(v_1 \wedge v_2) \\ v_1^2 \wedge v_2^2 &\longmapsto (DG - EF)(v_1 \wedge v_2) \\ v_1 v_2 \wedge v_2^2 &\longmapsto (EG - F^2)(v_1 \wedge v_2). \end{aligned}$$

So, define $\mathbf{C}_2(\Phi, \Psi, \Xi) = \big(\Phi \circ (\Xi \otimes 1) \circ \Theta\big) + 3 \bigwedge^2 \circ \mathbf{G}_2(\Psi)$. Then conditions (iii)$'$–(v)$'$ are equivalent to the morphism $\mathbf{C}_2(\Phi, \Psi, \Xi)$ being the zero map. ■

Definition 4.8 Define

$$\operatorname{Compat}_X(\mathcal{A}) = \operatorname{Hom}_X\big(\mathcal{L} \otimes \big(\textstyle\bigwedge^2 \mathcal{E}\big)^{\otimes 2} \otimes \mathcal{E}, \big(\bigwedge^2 \mathcal{E}\big)^{\otimes 2}\big) \oplus \operatorname{Hom}_X\big(\bigwedge^2(\operatorname{Sym}^2 \mathcal{E}), \bigwedge^2 \mathcal{E}\big).$$

Corollary 4.9 *There exists an R-module morphism*

$$\mathbf{C}\colon \operatorname{Build}_X(\mathcal{A}) \to \operatorname{Compat}_X(\mathcal{A})$$

with $\mathbf{P}\big(S_3 \operatorname{Cov}_X(\mathcal{A})^0\big) \subseteq \ker \mathbf{C}$.

Proof Take $\mathbf{C} = \mathbf{C}_1 \oplus \mathbf{C}_2$, and apply Proposition 3.3, Remark 3.5, and Lemmas 4.6 and 4.7. ■

By construction, the morphism $\mathbf{C}$ tests building data for the compatibility conditions necessary for the data to induce an S_3-cover.

Lastly, we construct from any building data the element of $\operatorname{Hom}_X(\operatorname{Sym}^2 \mathcal{L}, \mathcal{O}_X) \oplus \operatorname{Hom}_X(\mathcal{L} \otimes \tilde{\mathcal{E}}, \tilde{\mathcal{E}}) \oplus \operatorname{Hom}_X(\operatorname{Sym}^2 \tilde{\mathcal{E}}, \mathcal{A})$ that defines the corresponding algebra structure.

Lemma 4.10 *There exists an R-module morphism*

$$\mathbf{B}_1\colon \operatorname{Hom}_X\big(\mathcal{L}\otimes \operatorname{Sym}^2\mathcal{E}, \textstyle\bigwedge^2\mathcal{E}\big) \to \operatorname{Hom}_X(\operatorname{Sym}^2\mathcal{L}, \mathcal{O}_X)$$

taking an element Φ *locally of the form*

$$\Phi_U(t\otimes v_1^2) = A(v_1\wedge v_2)$$
$$\Phi_U(t\otimes v_1v_2) = B(v_1\wedge v_2)$$
$$\Phi_U(t\otimes v_2^2) = C(v_1\wedge v_2)$$

to an element α *locally of the form*

$$\alpha_U(t^2) = -3(B^2 - AC).$$

Proof Observe that we have isomorphisms

$$\begin{aligned}&\operatorname{Hom}_R\Big(\operatorname{Hom}_X\big(\textstyle\bigwedge^2(\mathcal{L}\otimes\mathcal{E}), \bigwedge^2\mathcal{E}\big), \operatorname{Hom}_X(\operatorname{Sym}^2\mathcal{L}, \mathcal{O}_X)\Big)\\ &\qquad\cong \operatorname{Hom}_X\Big(\big(\textstyle\bigwedge^2(\mathcal{L}\otimes\mathcal{E})\big)^*\otimes \bigwedge^2\mathcal{E}, (\operatorname{Sym}^2\mathcal{L})^*\Big)\\ &\qquad\cong \operatorname{Hom}_X\big(\textstyle\bigwedge^2\mathcal{E}\otimes\operatorname{Sym}^2\mathcal{L}, \bigwedge^2(\mathcal{L}\otimes\mathcal{E})\big).\end{aligned}$$

An element of this last group is the morphism $\tilde{\mathbf{B}}_1$ locally defined by

$$(e_1\wedge e_2)\otimes l_1l_2 \longmapsto \frac{3}{2}\big((l_1\otimes e_1)\wedge(l_2\otimes e_2) - (l_1\otimes e_2)\wedge(l_2\otimes e_1)\big).$$

In terms of our usual basis, this morphism is locally given by

$$(v_1\wedge v_2)\otimes t^2 \longmapsto \frac{3}{2}\big((t\otimes v_1)\wedge(t\otimes v_2) - (t\otimes v_2)\wedge(t\otimes v_1)\big) = 3(t\otimes v_1)\wedge(t\otimes v_2).$$

This morphism, considered as an element of

$$\operatorname{Hom}_X\Big(\big(\textstyle\bigwedge^2(\mathcal{L}\otimes\mathcal{E})\big)^*\otimes\bigwedge^2\mathcal{E}, (\operatorname{Sym}^2\mathcal{L})^*\Big),$$

is locally given by

$$\big((t\otimes v_1)\wedge(t\otimes v_2)\big)^*\otimes(v_1\wedge v_2) \longmapsto 3(t^2)^*.$$

Letting $\psi = \mathbf{G}_2(\Psi)$, the morphism $\bigwedge^2(\phi)$, when considered as a section of $\big(\bigwedge^2(\mathcal{L}\otimes\mathcal{E})\big)^*\otimes\bigwedge^2\mathcal{E}$, is locally

$$-(B^2 - AC)\big((t\otimes v_1)\wedge(t\otimes v_2)\big)^*\otimes(v_1\wedge v_2).$$

Thus, $\tilde{\mathbf{B}}_1$ maps $\bigwedge^2(\phi)$ to the section locally given by

$$-3(B^2 - AC)(t^2)^*,$$

which corresponds to the map

$$t^2 \longmapsto -3(B^2 - AC).$$

The composition $\mathbf{B}_1 = \tilde{\mathbf{B}}_1\circ\bigwedge^2\circ\mathbf{G}_1$ is therefore the desired morphism. ■

Lemma 4.11 *There exists an R-module morphism*

$$\mathbf{B}_2\colon \operatorname{Hom}_X\big(\mathcal{L}\otimes \operatorname{Sym}^2\mathcal{E}, \textstyle\bigwedge^2\mathcal{E}\big) \to \operatorname{Hom}_X(\mathcal{L}\otimes\tilde{\mathcal{E}}, \tilde{\mathcal{E}})$$

taking an element Φ *locally of the form*

$$\Phi_U(t\otimes v_1^2) = A(v_1\wedge v_2)$$
$$\Phi_U(t\otimes v_1v_2) = B(v_1\wedge v_2)$$
$$\Phi_U(t\otimes v_2^2) = C(v_1\wedge v_2)$$

to an element β *locally of the form*

$$\beta_U(t\otimes v_1) = Bv_1 - Av_2 - 2B\tau v_1 + 2A\tau v_2$$
$$\beta_U(t\otimes v_2) = Cv_1 - Bv_2 - 2C\tau v_1 + 2B\tau v_2$$
$$\beta_U(t\otimes \tau v_1) = 2Bv_1 - 2Av_2 - B\tau v_1 + A\tau v_2$$
$$\beta_U(t\otimes \tau v_2) = 2Cv_1 - 2Bv_2 - C\tau v_1 + B\tau v_2.$$

Proof Observe that $\phi = \mathbf{G}_1(\Phi)$ induces a morphism $\mathcal{L}\otimes\mathcal{E}\xrightarrow{\phi}\mathcal{E}\xrightarrow{\tau}\tau\mathcal{E}\subset\tilde{\mathcal{E}}$, and hence two morphisms:

$$\beta_1\colon \mathcal{L}\otimes\mathcal{E}\xrightarrow{(\phi,-2(\tau\circ\phi))}\mathcal{E}\oplus\tau\mathcal{E} = \tilde{\mathcal{E}},$$

and

$$\beta_2\colon \mathcal{L}\otimes\tau\mathcal{E}\xrightarrow{\tau\otimes\tau}\mathcal{L}\otimes\mathcal{E}\xrightarrow{(\phi,-2(\tau\circ\phi))}\mathcal{E}\oplus\tau\mathcal{E} = \tilde{\mathcal{E}}\xrightarrow{\tau}\tilde{\mathcal{E}}.$$

Let $\mathbf{B}_2(\Phi) = \langle\beta_1,\beta_2\rangle\colon \mathcal{L}\otimes\tilde{\mathcal{E}}\cong(\mathcal{L}\otimes\mathcal{E})\oplus(\mathcal{L}\otimes\tau\mathcal{E})\to\tilde{\mathcal{E}}$. In terms of our usual basis, this morphism is locally given by

$$t\otimes v_1 \longmapsto Bv_1 - Av_2 - 2B\tau v_1 + 2A\tau v_2$$
$$t\otimes v_2 \longmapsto Cv_1 - Bv_2 - 2C\tau v_1 + 2B\tau v_2$$
$$t\otimes \tau v_1 \longmapsto 2Bv_1 - 2Av_2 - B\tau v_1 + A\tau v_2$$
$$t\otimes \tau v_2 \longmapsto 2Cv_1 - 2Bv_2 - C\tau v_1 + B\tau v_2. \qquad \blacksquare$$

Lemma 4.12 *There exists an R-module morphism*

$$\mathbf{B}_3\colon \operatorname{Hom}_X\big(\operatorname{Sym}^3\mathcal{E}, \textstyle\bigwedge^2\mathcal{E}\big)\oplus\operatorname{Hom}_X\big(\textstyle\bigwedge^2\mathcal{E},\mathcal{L}\big)\to\operatorname{Hom}_X(\operatorname{Sym}^2\tilde{\mathcal{E}},\mathcal{A})$$

taking a pair (Ψ,Ξ) *locally of the form*

$$\Psi_U(v_1^3) = D(v_1\wedge v_2)$$
$$\Psi_U(v_1^2v_2) = E(v_1\wedge v_2)$$
$$\Psi_U(v_1v_2^2) = F(v_1\wedge v_2)$$
$$\Psi_U(v_2^3) = G(v_1\wedge v_2)$$
$$\Xi_U(v_1\wedge v_2) = ht$$

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to an element γ *locally of the form*

$$\begin{aligned}
\gamma_U(v_1^2) &= 6(E^2 - DF) + Ev_1 - Dv_2 - 2E\tau v_1 + 2D\tau v_2\\
\gamma_U(v_1v_2) &= 3(EF - DG) + Fv_1 - Ev_2 - 2F\tau v_1 + 2E\tau v_2\\
\gamma_U(v_2^2) &= 6(F^2 - EG) + Gv_1 - Fv_2 - 2G\tau v_1 + 2F\tau v_2\\
\gamma_U(v_1 \cdot \tau v_1) &= 3(E^2 - DF) - Ev_1 + Dv_2 - E\tau v_1 + D\tau v_2\\
\gamma_U(v_1 \cdot \tau v_2) &= \frac{3}{2}(EF - DG) + ht - Fv_1 + Ev_2 - F\tau v_1 + E\tau v_2\\
\gamma_U(v_2 \cdot \tau v_1) &= \frac{3}{2}(EF - DG) - ht - Fv_1 + Ev_2 - F\tau v_1 + E\tau v_2\\
\gamma_U(v_2 \cdot \tau v_2) &= 3(F^2 - EG) - Gv_1 + Fv_2 - G\tau v_1 + F\tau v_2\\
\gamma_U\big((\tau v_1)^2\big) &= 6(E^2 - DF) - 2Ev_1 + 2Dv_2 + E\tau v_1 - D\tau v_2\\
\gamma_U\big((\tau v_1)(\tau v_2)\big) &= 3(EF - DG) - 2Fv_1 + 2Ev_2 + F\tau v_1 - E\tau v_2\\
\gamma_U\big((\tau v_2)^2\big) &= 6(F^2 - EG) - 2Gv_1 + 2Fv_2 + G\tau v_1 - F\tau v_2.
\end{aligned}$$

Proof Observe that we have isomorphisms

$$\begin{aligned}
&\operatorname{Hom}_R\Big(\operatorname{Hom}_X\big(\textstyle\bigwedge^2(\operatorname{Sym}^2 \mathcal{E}), \bigwedge^2\mathcal{E}\big), \operatorname{Hom}_X(\operatorname{Sym}^2 \mathcal{E}, \mathcal{O}_X)\Big)\\
&\quad\cong \operatorname{Hom}_X\Big(\big(\textstyle\bigwedge^2(\operatorname{Sym}^2 \mathcal{E})\big)^* \otimes \bigwedge^2\mathcal{E}, (\operatorname{Sym}^2 \mathcal{E})^*\Big)\\
&\quad\cong \operatorname{Hom}_X\Big(\big(\textstyle\bigwedge^2\mathcal{E}\big) \otimes \operatorname{Sym}^2 \mathcal{E}, \bigwedge^2(\operatorname{Sym}^2 \mathcal{E})\Big).
\end{aligned}$$

An element of this last group is the morphism $\tilde{\mathbf{B}}_{3,1}$ defined locally by

$$(e_1 \wedge e_2) \otimes e_3e_4 \longmapsto -3(e_1e_3 \wedge e_2e_4 + e_1e_4 \wedge e_2e_3).$$

In terms of our usual basis, this morphism is locally given by

$$\begin{aligned}
(v_1 \wedge v_2) \otimes v_1^2 &\longmapsto -3(v_1^2 \wedge v_1v_2 + v_1^2 \wedge v_1v_2) = -6(v_1^2 \wedge v_1v_2)\\
(v_1 \wedge v_2) \otimes v_1v_2 &\longmapsto -3(v_1^2 \wedge v_2^2 + v_1v_2 \wedge v_1v_2) = -3(v_1^2 \wedge v_2^2)\\
(v_1 \wedge v_2) \otimes v_2^2 &\longmapsto -3(v_1v_2 \wedge v_2^2 + v_1v_2 \wedge v_2^2) = -6(v_1v_2 \wedge v_2^2).
\end{aligned}$$

As an element of $\operatorname{Hom}_X\Big(\big(\bigwedge^2(\operatorname{Sym}^2 \mathcal{E})\big)^* \otimes \bigwedge^2\mathcal{E}, (\operatorname{Sym}^2 \mathcal{E})^*\Big)$, this corresponds locally to the map

$$\begin{aligned}
(v_1^2 \wedge v_1v_2)^* \otimes (v_1 \wedge v_2) &\longmapsto -6(v_1^2)^*\\
(v_1^2 \wedge v_2^2)^* \otimes (v_1 \wedge v_2) &\longmapsto -3(v_1v_2)^*\\
(v_1v_2 \wedge v_2^2)^* \otimes (v_1 \wedge v_2) &\longmapsto -6(v_2^2)^*.
\end{aligned}$$

Taking $\psi = \mathbf{G}_2(\Psi)$, the morphism $\bigwedge^2(\psi)$, as a section of $\left(\bigwedge^2(\operatorname{Sym}^2 \mathcal{E})\right)^* \otimes \bigwedge^2 \mathcal{E}$, is locally given by

$$(DF - E^2)(v_1^2 \wedge v_1 v_2)^* \otimes (v_1 \wedge v_2) + (DG - EF)(v_1^2 \wedge v_2^2)^* \\ \otimes (v_1 \wedge v_2) + (EG - F^2)(v_1 v_2 \wedge v_2^2)^* \otimes (v_1 \wedge v_2).$$

Thus, $\tilde{\mathbf{B}}_{3,1}$ maps $\bigwedge^2(\psi)$ to the section locally given by

$$6(E^2 - DF)(v_1^2)^* + 3(EF - DG)(v_1 v_2)^* + 6(F^2 - EG)(v_2^2)^*,$$

which corresponds to the morphism $\tilde{\gamma}_1$ locally defined by

$$\begin{aligned} v_1^2 &\longmapsto 6(E^2 - DF) \\ v_1 v_2 &\longmapsto 3(EF - DG) \\ v_2^2 &\longmapsto 6(F^2 - EG). \end{aligned}$$

This induces morphisms

$$\gamma'_{1,1} \colon \mathcal{E} \otimes \mathcal{E} \xrightarrow{\quad \text{can} \quad} \operatorname{Sym}^2 \mathcal{E} \xrightarrow{\tilde{\gamma}_1} \mathcal{O}_X$$

$$\gamma'_{1,2} : \mathcal{E} \otimes \tau\mathcal{E} \xrightarrow{1 \otimes \tau} \mathcal{E} \otimes \mathcal{E} \xrightarrow{\text{can}} \operatorname{Sym}^2 \mathcal{E} \xrightarrow{\frac{1}{2}\tilde{\gamma}_1} \mathcal{O}_X$$

$$\gamma'_{1,3} : \tau\mathcal{E} \otimes \mathcal{E} \xrightarrow{\tau \otimes 1} \mathcal{E} \otimes \mathcal{E} \xrightarrow{\text{can}} \operatorname{Sym}^2 \mathcal{E} \xrightarrow{\frac{1}{2}\tilde{\gamma}_1} \mathcal{O}_X$$

$$\gamma'_{1,4} : \tau\mathcal{E} \otimes \tau\mathcal{E} \xrightarrow{\tau \otimes \tau} \mathcal{E} \otimes \mathcal{E} \xrightarrow{\text{can}} \operatorname{Sym}^2 \mathcal{E} \xrightarrow{\tilde{\gamma}_1} \mathcal{O}_X,$$

which together define a morphism

$$\gamma'_1 \colon \tilde{\mathcal{E}} \otimes \tilde{\mathcal{E}} \cong (\mathcal{E} \otimes \mathcal{E}) \oplus (\mathcal{E} \otimes \tau\mathcal{E}) \oplus (\tau\mathcal{E} \otimes \mathcal{E}) \oplus (\tau\mathcal{E} \otimes \tau\mathcal{E}) \to \mathcal{O}_X.$$

By construction, this morphism factors through the canonical morphism $\tilde{\mathcal{E}} \otimes \tilde{\mathcal{E}} \to \operatorname{Sym}^2 \tilde{\mathcal{E}}$, and gives the first-coordinate morphism $\gamma_1 \colon \operatorname{Sym}^2 \tilde{\mathcal{E}} \to \mathcal{O}_X$. Let $\mathbf{B}_{3,1}(\Psi, \Xi) = \gamma_1$.

Next, observe that $\xi = \mathbf{G}_3(\Xi)$ induces a morphism $\xi' \colon \tau\mathcal{E} \otimes \mathcal{E} \xrightarrow{\text{can}} \mathcal{E} \otimes \tau\mathcal{E} \xrightarrow{\xi} \mathcal{L}$. Define $\gamma'_2 \colon \tilde{\mathcal{E}} \otimes \tilde{\mathcal{E}} \to \mathcal{L}$ by $\gamma'_2 = \langle 0, \xi, \xi', 0 \rangle$. By construction, this morphism factors through the canonical morphism $\tilde{\mathcal{E}} \otimes \tilde{\mathcal{E}} \to \operatorname{Sym}^2 \tilde{\mathcal{E}}$, and gives the second-coordinate morphism $\gamma_2 \colon \operatorname{Sym}^2 \tilde{\mathcal{E}} \to \mathcal{L}$. Let $\mathbf{B}_{3,2}(\Psi, \Xi) = \gamma_2$.

Lastly, observe that the morphism $\psi = \mathbf{G}_2(\Psi)$ induces morphisms

$$\gamma'_{3,1}\colon \mathcal{E}\otimes\mathcal{E} \xrightarrow{\quad\text{can}\quad} \operatorname{Sym}^2\mathcal{E} \xrightarrow{\psi-2(\tau\circ\psi)} \mathcal{E}\oplus\tau\mathcal{E}$$

$$\gamma'_{3,2}\colon \mathcal{E}\otimes\tau\mathcal{E} \xrightarrow{1\otimes\tau} \mathcal{E}\otimes\mathcal{E} \xrightarrow{\text{can}} \operatorname{Sym}^2\mathcal{E} \xrightarrow{-\psi-(\tau\circ\psi)} \mathcal{E}\oplus\tau\mathcal{E}$$

$$\gamma'_{3,3}\colon \tau\mathcal{E}\otimes\mathcal{E} \xrightarrow{\tau\otimes 1} \mathcal{E}\otimes\mathcal{E} \xrightarrow{\text{can}} \operatorname{Sym}^2\mathcal{E} \xrightarrow{-\psi-(\tau\circ\psi)} \mathcal{E}\oplus\tau\mathcal{E}$$

$$\gamma'_{3,4}\colon \tau\mathcal{E}\otimes\tau\mathcal{E} \xrightarrow{\tau\otimes\tau} \mathcal{E}\otimes\mathcal{E} \xrightarrow{\text{can}} \operatorname{Sym}^2\mathcal{E} \xrightarrow{-2\psi+(\tau\circ\psi)} \mathcal{E}\oplus\tau\mathcal{E}.$$

These together define a morphism $\gamma'_3\colon \tilde{\mathcal{E}}\otimes\tilde{\mathcal{E}}\to\tilde{\mathcal{E}}$ by $\gamma'_3 = \langle\gamma'_{3,1},\gamma'_{3,2},\gamma'_{3,3},\gamma'_{3,4}\rangle$. By construction, this factors through the canonical morphism $\tilde{\mathcal{E}}\otimes\tilde{\mathcal{E}}\to\operatorname{Sym}^2\tilde{\mathcal{E}}$, and gives the third-coordinate morphism $\gamma_3\colon \operatorname{Sym}^2\tilde{\mathcal{E}}\to\mathcal{L}$. Let $\mathbf{B}_{3,3}(\Psi,\Xi) = \gamma_3$.

The desired morphism is then given by $\mathbf{B}_3 = \mathbf{B}_{3,1}\oplus\mathbf{B}_{3,2}\oplus\mathbf{B}_{3,3}$. ∎

Corollary 4.13 *There is an R-module morphism*

$$\mathbf{B}\colon \operatorname{Build}_X(\mathcal{A})\to\operatorname{Hom}_X(\operatorname{Sym}^2\mathcal{L},\mathcal{O}_X)\oplus\operatorname{Hom}_X(\mathcal{L}\otimes\tilde{\mathcal{E}},\tilde{\mathcal{E}})\oplus\operatorname{Hom}_X(\operatorname{Sym}^2\tilde{\mathcal{E}},\mathcal{A})$$

with $\mathbf{B}(\ker\mathbf{C})\le S_3\operatorname{Cov}_X(\mathcal{A})$.

Proof Take $\mathbf{B} = \mathbf{B}_1\oplus\mathbf{B}_2\oplus\mathbf{B}_3$, and apply Proposition 3.3, Remark 3.5, Corollary 4.9, and Lemmas 4.10, 4.11, and 4.12. ∎

4.14 Summary

We fixed an integral, Noetherian scheme X over a domain R in which 6 was invertible, together with locally free sheaves $\mathcal{L}$ and $\mathcal{E}$ of ranks one and two, respectively. We let S_3 act on $\mathcal{L}$ by the sign character and on $\tilde{\mathcal{E}} = \mathcal{E}\oplus\tau\mathcal{E}$ by its two-dimensional representation. Letting $\mathcal{A} = \mathcal{O}_X\oplus\mathcal{L}\oplus\tilde{\mathcal{E}}$, our goal was to understand $S_3\operatorname{Cov}_X(\mathcal{A})\le\operatorname{Hom}_X(\operatorname{Sym}^2\mathcal{A},\mathcal{A})$, the R-submodule of elements defining commutative, associative $\mathcal{O}_X[S_3]$-algebra structures on $\mathcal{A}$ (and hence inducing S_3-covers of the form $\mathbf{Spec}_X\mathcal{A}\to X$). After identifying $S_3\operatorname{Cov}_X(\mathcal{A})$ with an R-submodule of $\operatorname{Hom}_X(\operatorname{Sym}^2\mathcal{L},\mathcal{O}_X)\oplus\operatorname{Hom}_X(\mathcal{L}\otimes_{\mathcal{O}_X}\tilde{\mathcal{E}},\tilde{\mathcal{E}})\oplus\operatorname{Hom}_X(\operatorname{Sym}^2\tilde{\mathcal{E}},\mathcal{A})$, a local analysis led us to define R-modules

$$\begin{aligned}\operatorname{Build}_X(\mathcal{A}) &= \operatorname{Hom}_X\big(\mathcal{L}\otimes\operatorname{Sym}^2\mathcal{E},\textstyle\bigwedge^2\mathcal{E}\big)\oplus\operatorname{Hom}_X\big(\operatorname{Sym}^3\mathcal{E},\textstyle\bigwedge^2\mathcal{E}\big)\\ &\qquad\oplus\operatorname{Hom}_X\big(\textstyle\bigwedge^2\mathcal{E},\mathcal{L}\big),\\ \operatorname{Compat}_X(\mathcal{A}) &= \operatorname{Hom}_X\big(\mathcal{L}\otimes\big(\textstyle\bigwedge^2\mathcal{E}\big)^{\otimes 2}\otimes\mathcal{E},\big(\textstyle\bigwedge^2\mathcal{E}\big)^{\otimes 2}\big)\\ &\qquad\oplus\operatorname{Hom}_X\big(\textstyle\bigwedge^2(\operatorname{Sym}^2\mathcal{E}),\textstyle\bigwedge^2\mathcal{E}\big),\end{aligned}$$

parameterizing the data required to build S_3-covers and the compatibility conditions on such data. After identifying $\mathrm{Build}_X(\mathcal{A})$ with an R-submodule of $\mathrm{Hom}_X(\mathcal{L}\otimes\mathcal{E},\mathcal{E})\oplus \mathrm{Hom}_X(\mathrm{Sym}^2\,\mathcal{E},\mathcal{E})\oplus\mathrm{Hom}_X(\mathcal{E}\otimes\tau\mathcal{E},\mathcal{L})$, we constructed R-module morphisms

$$\mathbf{P}\colon S_3\,\mathrm{Cov}_X(\mathcal{A})\to\mathrm{Hom}_X(\mathcal{L}\otimes\mathcal{E},\mathcal{E})\oplus\mathrm{Hom}_X(\mathrm{Sym}^2\,\mathcal{E},\mathcal{E})\oplus\mathrm{Hom}_X(\mathcal{E}\otimes\tau\mathcal{E},\mathcal{L}),$$

$$\mathbf{C}\colon \mathrm{Build}_X(\mathcal{A})\to\mathrm{Compat}_X(\mathcal{A}),$$

$$\mathbf{B}\colon \mathrm{Build}_X(\mathcal{A})\to\mathrm{Hom}_X(\mathrm{Sym}^2\,\mathcal{L},\mathcal{O}_X)\oplus\mathrm{Hom}_X(\mathcal{L}\otimes\tilde{\mathcal{E}},\tilde{\mathcal{E}})\oplus\mathrm{Hom}_X(\mathrm{Sym}^2\,\tilde{\mathcal{E}},\mathcal{A}),$$

such that $\mathbf{P}\big(S_3\,\mathrm{Cov}_X(\mathcal{A})^0\big)\subseteq\ker\mathbf{C}$ and $\mathbf{B}(\ker\mathbf{C})\leq S_3\,\mathrm{Cov}_X(\mathcal{A})$, where $S_3\,\mathrm{Cov}_X(\mathcal{A})^0$ denoted the subset of elements defining algebras that were domains. In other words, **P** extracted from those S_3-covers whose covering schemes were integral the building data of the covers, and such data lay in the kernel of **C**; and conversely, given building data in the kernel of **C**, the morphism **B** built an S_3-cover (whose covering scheme might not be integral).

4.15 Closing Remarks

In analogy with the abelian case, it would be useful to know when building data uniquely determine a cover, as well as find a global description of the branch locus, the location and nature of singularities of the covering scheme, and the relationship between the (cohomological) invariants of the base scheme and those of the covering scheme.

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