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HYPERREFLEXIVITY CONSTANTS OF THE BOUNDED *N*-COCYCLE SPACES OF GROUP ALGEBRAS AND C*-ALGEBRAS

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Abstract

We introduce the concept of strong property ($\mathbb{B}$) with a constant for Banach algebras and, by applying a certain analysis on the Fourier algebra of the unit circle, we show that all C*-algebras and group algebras have the strong property ($\mathbb{B}$) with a constant given by $288\pi(1+\sqrt{2})$. We then use this result to find a concrete upper bound for the hyperreflexivity constant of $\mathcal{Z}^n(A, X)$, the space of bounded n-cocycles from A into X, where A is a C*-algebra or the group algebra of a group with an open subgroup of polynomial growth and X is a Banach A-bimodule for which $\mathcal{H}^{n+1}(A, X)$ is a Banach space. As another application, we show that for a locally compact amenable group G and $1 < p < \infty$, the space $CV_P(G)$ of convolution operators on $L^p(G)$ is hyperreflexive with a constant given by $384\pi^2(1+\sqrt{2})$. This is the generalization of a well-known result of Christensen ['Extensions of derivations. II', *Math. Scand.* **50**(1) (1982), 111–122] for $p = 2$.

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1. Introduction

The concept of hyperreflexivity is a strengthening of the well-known notion of reflexivity. The latter notion that was first defined in [23] has attracted much attention over the years. It has its origin in operator theory and at first was defined for the subspaces of $B(X)$. In [16], Larson generalized the concept of reflexivity to the subspaces of $B(X, Y)$, where X and Y are Banach spaces. One goal was to study the local behavior of derivations from a Banach algebra A to a Banach A-bimodule X.

Let A be a Banach algebra and X a Banach A-bimodule. One interesting question is under what conditions each local derivation from A into X is a derivation or, equivalently, when $Z^1(A, X)$ is algebraically reflexive. One could also study the continuous version of this question: when the space of bounded derivations from A

into X, namely $\mathcal{Z}^1(A, X)$, is reflexive. Johnson showed that $Z^1(A, X)$ is algebraically reflexive when A is a C*-algebra [14]. In [20], the first named author generalized the concept of local derivations to the higher cohomologies and defined the local n-cocycles. He showed that if A is a C*-algebra and X a Banach A-bimodule, then every bounded local n-cocycle from $A^{(n)}$ into X is an n-cocycle. In a subsequent paper, he introduced the concept of reflexivity for bounded n-linear maps [21]. He showed that if G is a locally compact group with an open subgroup of polynomial growth and X a Banach $L^1(G)$-bimodule, then $\mathcal{Z}^n(L^1(G), X)$, the space of bounded n-cocycles from A into X, is reflexive. More results related to these questions can be found in [5, 7, 9–13, 15–17, 19].

As was pointed out above, the concept of hyperreflexivity is a strengthening of reflexivity. This concept was first introduced by Arveson in [4] and proved to be powerful in operator theory. For instance, mainly due to the work of Christensen, it was shown that injective von Neumann algebras are hyperreflexive [6]. It is not known whether one can remove 'injectivity' from the preceding statement. In fact, this is equivalent with several open problems in operator algebras including Kadison's similarity problem [18]. The first attempt in studying the hyperreflexivity for the space of derivations was done by Shul'man in [24], where he showed that $\mathcal{Z}^1(A, A))$ is hyperreflexive for a C*-algebra A if $\mathcal{H}^2(A, A) = 0$. For group algebras, it was first shown in [2] that $\mathcal{Z}^1(L^1(G), L^1(G))$ is hyperreflexive for each amenable group with a small invariant neighborhood (SIN). In [21], the first named author extended the preceding result and showed that $\mathcal{Z}^1(L^1(G), X^*)$ is hyperreflexive if G is an amenable locally compact group with an open subgroup which is of polynomial growth and X is an essential Banach $L^1(G)$-bimodule. In particular, $\mathcal{Z}^1(L^1(G), L^1(G))$ is hyperreflexive for such a group. In [3], the latter result was extended further, so that one could drop the assumption of 'amenability'.

In the preceding work [22], we considered the extension of the concept of hyperreflexivity to the subspaces of bounded n-linear maps between Banach spaces, taking into account their multilinear structure. We mostly focused on $\mathcal{Z}^n(A, X)$, the space of bounded n-cocycles from a Banach algebra A into a Banach A-bimodule X, and found sufficient conditions under which $\mathcal{Z}^n(A, X)$ becomes hyperreflexive. We demonstrated that for a large class of Banach algebras, including nuclear C*-algebras and group algebras of groups with open subgroups of polynomial growth, these sufficient conditions hold, which gave evidence that our conditions are satisfactory. However, our approach in [22] did not give us any information about the hyperreflexivity constant of these spaces.

Our goal in this article is to fill out this gap by focusing on finding an upper bound for the hyperreflexivity constant. In this regard, we make use of the Fourier algebra of the unit circle $A(\mathbb{T})$ and give a characterization of bounded linear maps from $A(\mathbb{T})$ into its dual which almost preserve support (see Theorem 3.3). We then make an appropriate modification of our approach in [22], such as introducing the concept of the strong property $(\mathbb{B})$ with a constant for Banach algebras (Definition 3.1), and show that one could obtain hyperreflexivity of bounded n-cocycle spaces along with some information about the hyperreflexivity constant. We use our results to show that if A is a C*-algebra or the group algebra of a group with an open subgroup of polynomial growth and if X is a Banach A-bimodule for which $\mathcal{H}^{n+1}(A, X)$ is a Banach space, then the hyperreflexivity constant of $\mathcal{Z}^n(A, X)$, the space of bounded n-cocycles from A into X, is bounded by

$$C2^{n-1}(M^2 384\pi^2(1 + \sqrt{2}) + (M + 1)^2)^{n+1},$$

where M is a bound for the local units of A and C is a constant satisfying

$$\mathrm{dist}(T, Z^n(A, X)) \le C\|\delta^n(T)\| \quad (T \in B^n(Z, X)).$$

In the final section, we look at the space of convolution operators on $L^p(G)$ for a locally compact group G and $1 < p < \infty$. For $p = 2$, $CV_2(G)$ is nothing but the group von Neumann algebra of G generated by its left regular representation. It is well known that $CV_2(G)$ is an injective von Neumann algebra if G is an amenable group and so it is hyperreflexive by [6]. We extend this result to $CV_p(G)$ for all $1 < p < \infty$. For $p \neq 2$, $CV_p(G)$ is no longer a von Neumann algebra, so that we cannot use Christensen's result. However, we show that the methodology we developed to study the hyperreflexivity of n-cocycle spaces compensates the lack of the theory of von Neumann algebras in this situation. We apply successfully our method and show that for an amenable group G and $1 < p < \infty$, $CV_p(G)$ is hyperreflexive with a concrete upper bound estimate on the hyperreflexivity constant.

2. Preliminaries

Let X and Y be Banach spaces. For $n \in \mathbb{N}$, let $X^{(n)}$ be the Cartesian product of n copies of X and $B^n(X, Y)$ be the spaces of bounded n-linear maps from $X^{(n)}$ into Y. Let $\mathfrak{S}$ be a closed subspace of $B^n(X, Y)$. For every $T \in B^n(X, Y)$, we define

$$\mathrm{dist}(T, \mathfrak{S}) = \inf_{S \in \mathfrak{S}} \|T - S\|$$

and

$$\mathrm{dist}_r(T, \mathfrak{S}) = \sup_{\|x_i\| \le 1} \inf_{S \in \mathfrak{S}} \|T(x_1, \ldots, x_n) - S(x_1, \ldots, x_n)\|.$$

It is clear that for all $T \in B^n(X, Y)$,

$$\mathrm{dist}_r(T, \mathfrak{S}) \le \mathrm{dist}(T, \mathfrak{S}).$$

We define $\mathfrak{S}$ to be *reflexive* if for every $T \in B^n(X, Y)$, $\mathrm{dist}_r(T, \mathfrak{S}) = 0$ implies that $\mathrm{dist}(T, \mathfrak{S}) = 0$. We define $\mathfrak{S}$ to be *hyperreflexive* if there exists some $C > 0$ such that for all $T \in B^n(X, Y)$,

$$\mathrm{dist}(T, \mathfrak{S}) \le C\, \mathrm{dist}_r(T, \mathfrak{S}).$$

It is straightforward to verify that dist_r defines a seminorm on the quotient space $B^n(X, Y)/\mathfrak{S}$ given by

$$\|T + \mathfrak{S}\|_r = \mathrm{dist}_r(T, \mathfrak{S}).$$

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It follows easily from the definition that $\mathfrak{S}$ is reflexive if and only if $\|\cdot\|_r$ is a norm on $B^n(X,Y)/\mathfrak{S}$. On the other hand, $\mathfrak{S}$ is hyperreflexive if and only if $\|\cdot\|_r$ is equivalent to the dist norm on $B^n(X,Y)/\mathfrak{S}$, which is nothing but the quotient norm on $B^n(X,Y)/\mathfrak{S}$.

Let A be a Banach algebra and let X be a Banach A-bimodule. A bounded operator $D \in L(A,X)$ is a derivation if for all $a,b \in A$, $D(ab) = aD(b) + D(a)b$. For each $x \in X$, the operator $ad_x \in B(A,X)$ defined by $ad_x(a) = ax - xa$ is a bounded derivation, called an inner derivation. Let $\mathcal{Z}^1(A,X)$ be the linear spaces of bounded derivations from A into X. For $n \in \mathbb{N}$ and $T \in B^n(A,X)$, define

$$\begin{aligned}\delta^n T : (a_1,\dots,a_{n+1}) \mapsto\ & a_1 T(a_2,\dots,a_n)\\ &+ \sum_{j=1}^{n}(-1)^j T(a_1,\dots,a_j a_{j+1},\dots,a_{n+1})\\ &+ (-1)^{n+1} T(a_1,\dots,a_n)a_{n+1}.\end{aligned}$$

It is clear that δ^n is a linear map from $B^n(A,X)$ into $B^{n+1}(A,X)$; these maps are the *bounded connecting maps*. The elements of $\ker\delta^n$ are the *bounded n-cocycles*; we denote this linear space by $\mathcal{Z}^n(A,X)$. It is easy to check that $\mathcal{Z}^1(A,X)$ coincides with our previous definition of this notation.

Let A be a Banach algebra and let X be a Banach A-bimodule. By [8, Section 2.8], for $n \in \mathbb{N}$, the Banach space $B^n(A,X)$ turns into a Banach A-bimodule by the actions defined by

$$\begin{aligned}(a \star T)(a_1,\dots,a_n) &= aT(a_1,\dots,a_n),\\ (T \star a)(a_1,\dots,a_n) &= T(aa_1,\dots,a_n)\\ &\quad+ \sum_{j=1}^{n}(-1)^j T(a,a_1,\dots,a_j a_{j+1},\dots,a_n)\\ &\quad+ (-1)^{n+1} T(a,a_1,\dots,a_{n-1})a_n.\end{aligned}$$

In particular, when $n = 1$, $B(A,X)$ becomes a Banach A-bimodule with respect to the products

$$(a \star T)(b) = aT(b), \quad (T \star a)(b) = T(ab) - T(a)b.$$

Let $\Lambda_n : B^{n+1}(A,X) \to B^n(A,B(A,X))$ be the identification given by

$$(\Lambda_n(T)(a_1,\dots,a_n))(a_{n+1}) = T(a_1,\dots,a_{n+1}).$$

Then Λ_n is an A-bimodule isometric isomorphism. If we denote the connecting maps for the complex $B^n(A,(B(A,X),\star))$ by Δ^n, then it is shown in [8] that

$$\Lambda_{n+1} \circ \delta^{n+1} = \Delta^n \circ \Lambda_n.$$

3. A constant for the strong property ($\mathbb{B}$)

The concept of the strong property ($\mathbb{B}$) first appeared in [1] for C*-algebras and group algebras, where it was shown that they all possess this property. However, it was

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formally formulated and introduced in [22] for general Banach algebras and was used to obtain hyperreflexivity of bounded n-cocyle spaces from various Banach algebras. Since we are looking for further information such as a bound for the hyperreflexivity constant, we require a more refined version of the strong property $(\mathbb{B})$, that is, when its associated function is a line as described below.

DEFINITION 3.1. We say that a Banach algebra A has *the strong property* $(\mathbb{B})$ *with a constant* $R > 0$ if for each Banach space X and every bounded bilinear map $\varphi : A \times A \to X$ with the property that

$$a, b \in A, \quad ab = 0 \Rightarrow \|\varphi(a, b)\| \leq \alpha\|a\|\,\|b\|,$$

we can infer that

$$\|\varphi(ab, c) - \varphi(a, bc)\| \leq R\alpha\|a\|\,\|b\|\,\|c\| \quad (\forall a, b, c \in A).$$

In other words,

$$\|\varphi(ab, c) - \varphi(a, bc)\| \leq R\alpha(\varphi)\|a\|\,\|b\|\,\|c\| \quad (\forall a, b, c \in A),$$

where

$$\alpha(\varphi) = \sup\{\|\varphi(a, b)\| : a, b \in A, \|a\|, \|b\| \leq 1, ab = 0\}.$$

We note that by a simple application of the Hahn–Banach theorem, it suffices to check the preceding property for the case when $X = \mathbb{C}$. We will use this alternative definition when it is more convenient.

We will see later in Section 4 that existence of a constant for the strong property $(\mathbb{B})$ is fundamental in finding an upper bound for the hyperreflexivity constant of the bounded n-cocycle spaces.

3.1. Fourier algebra of the unit circle. As was mentioned above, in order to achieve our goal in finding an upper bound for the hyperreflexivity constant of the bounded n-cocycle spaces of C*-algebras and group algebras, we need to find a constant for the strong property $(\mathbb{B})$ for such Banach algebras. In the present section we aim to find such a constant for the Fourier algebra of the unit circle. Interestingly, we only need to study this case to find a constant for the strong property $(\mathbb{B})$ of C*-algebras and group algebras (see Theorem 3.4).

Let $\mathbb{T}$ denote the unit circle in $\mathbb{C}$, that is,

$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}.$$

Here we identify $\mathbb{T}$ with $\mathbb{R}/\mathbb{Z} \cong [-\pi, \pi]$. In this case $s = t$ if $s \equiv t (\text{mod}\, 2\pi\mathbb{Z})$. For every $f \in L^1(\mathbb{T})$, the Fourier transform on f, denoted by $\hat{f}$, is defined by

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-\text{int}}\,dt \quad (n \in \mathbb{Z}).$$

The Fourier algebra of the unit circle is defined as follows:

$$A(\mathbb{T}) = \Big\{ f \in L^1(\mathbb{T}) : \|f\|_{A(\mathbb{T})} = \sum_{n\in\mathbb{Z}} |\hat{f}(n)| < \infty \Big\}.$$

It is well known that $A(\mathbb{T}) \subseteq C(\mathbb{T})$, the space of continuous functions on $\mathbb{T}$. Also, $A(\mathbb{T})$ with pointwise addition and multiplication and the norm $\|\cdot\|_{A(\mathbb{T})}$ is a Banach algebra.

The following lemma is essential for us to get our result.

Lemma 3.2. *Let X be a Banach space and $F : A(\mathbb{T}) \to X$ a linear map with $\|F\| \le 1$. Suppose that $0 \le \alpha \le 1$ is such that for each $\varphi, \psi \in A(\mathbb{T})$ with $\operatorname{supp}\varphi \cap \operatorname{supp}\psi = \emptyset$,*

$$\|F(\varphi * \check{\psi})\| \le \alpha\|\varphi\|\,\|\psi\|,$$

where $\check{\psi}(x) = \psi(x^{-1})$. Let $f \in A(\mathbb{T})$ be given by $f(s) = e^{is} - 1$. Then

$$\|F(f)\| \le 8\pi\sqrt{3(1+\sqrt{2})}\,\sqrt{\alpha}.$$

Proof. Let $0 < \epsilon < 3$. Define

$$W_\epsilon = \{x \in \mathbb{T} : \|f - R_x f\|_{A(\mathbb{T})} < \epsilon\},$$

where $(R_x f)(s) = f(s + x)$. Note that for $s \in \mathbb{T}$,

$$(f - R_x f)(s) = e^{is}(1 - e^{ix}).$$

Hence, if we define $e_1(s) = e^{is}$, then

$$\|f - R_x f\|_{A(\mathbb{T})} = \|e_1\|\,|1 - e^{ix}| = |1 - e^{ix}|.$$

So

$$W_\epsilon = \{x \in \mathbb{T} : |1 - e^{ix}| < \epsilon\}.$$

We show that for each $0 < \delta < \epsilon$, $[-(\epsilon - \delta), (\epsilon - \delta)] \subseteq W_\epsilon$. Let $0 < x < \pi$. Applying the vector-valued mean value theorem to the function $f|_{[0,x]}$, we find $0 < c < x$ with

$$|f(x)| = |f(x) - f(0)| \le |f'(c)|\,|x| \le |x|.$$

If $-\pi < x < 0$, we use the same argument on the interval $[x, 0]$. For $x = 0$, the inequality trivially holds. So, for each $0 < \delta < \epsilon$ and for all $x \in [-(\epsilon - \delta), (\epsilon - \delta)]$,

$$|e^{ix} - 1| = |f(x)| \le |x| < \epsilon.$$

It means that $[-(\epsilon - \delta), (\epsilon - \delta)] \subseteq W_\epsilon$. Define

$$V_{\epsilon,\delta} = \Big[\frac{-(\epsilon-\delta)}{3}, \frac{(\epsilon-\delta)}{3}\Big], \qquad U_{\epsilon,\delta} = \Big[\frac{-(\epsilon-\delta)}{6}, \frac{(\epsilon-\delta)}{6}\Big].$$

Then $V_{\epsilon,\delta} + V_{\epsilon,\delta} + V_{\epsilon,\delta} \subseteq W_\epsilon$ and $U_{\epsilon,\delta} + U_{\epsilon,\delta} = V_{\epsilon,\delta}$. Now put

$$u = \frac{1}{\lambda(U_{\epsilon,\delta})^2} 1_{U_{\epsilon,\delta}} * 1_{U_{\epsilon,\delta}}$$

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and

$$v = f\Big(\frac{1}{\lambda(V_{\epsilon,\delta})} 1_{V_{\epsilon,\delta}+V_{\epsilon,\delta}} * 1_{V_{\epsilon,\delta}}\Big). \tag{3.1}$$

Obviously, $1_{U_{\epsilon,\delta}} \in L^2(\mathbb{T})$. Since $A(\mathbb{T}) = L^2(\mathbb{T}) * L^2(\mathbb{T})$, we have $u \in A(\mathbb{T}) \subseteq C(\mathbb{T}) \subseteq L^2(\mathbb{T}) \subseteq L^1(\mathbb{T})$. It is easy to check that $\|1_{U_{\epsilon,\delta}}\|_2 = \sqrt{\lambda(U_{\epsilon,\delta})}$. By definition of the Fourier norm,

$$\begin{aligned}\|u\|_{A(\mathbb{T})} &\le \frac{1}{\lambda(U_{\epsilon,\delta})^2}\|1_{U_{\epsilon,\delta}}\|_2\|1_{U_{\epsilon,\delta}}\|_2\\ &= \frac{1}{\lambda(U_{\epsilon,\delta})} = \frac{6\pi}{\epsilon-\delta}.\end{aligned} \tag{3.2}$$

Since $1_{U_{\epsilon,\delta}} \in L^2(\mathbb{T}) \subseteq L^1(\mathbb{T})$ and $L^2(\mathbb{T})$ is an $L^1(\mathbb{T})$-module with respect to the convolution,

$$\|u\|_2 \le \frac{1}{\lambda(U_{\epsilon,\delta})^2}\|1_{U_{\epsilon,\delta}}\|_1\|1_{U_{\epsilon,\delta}}\|_2. \tag{3.3}$$

It is easy to check that $\|1_{U_{\epsilon,\delta}}\|_1 = \lambda(U_{\epsilon,\delta})$. So, by (3.3),

$$\|u\|_2 \le \frac{\lambda(U_{\epsilon,\delta})^{3/2}}{\lambda(U_{\epsilon,\delta})^2} = \frac{1}{\lambda(U_{\epsilon,\delta})^{1/2}} = \sqrt{\frac{6\pi}{\epsilon-\delta}}. \tag{3.4}$$

We show that $\operatorname{supp} u \subseteq U_{\epsilon,\delta} + U_{\epsilon,\delta}$. Let $x \in [-\pi, \pi]$. Then

$$\begin{aligned}(1_{U_{\epsilon,\delta}} * 1_{U_{\epsilon,\delta}})(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} 1_{U_{\epsilon,\delta}}(y)1_{U_{\epsilon,\delta}}(x-y)\,dy\\ &= \frac{1}{2\pi}\int_{U_{\epsilon,\delta}} 1_{U_{\epsilon,\delta}}(x-y)\,dy.\end{aligned}$$

So, for x to be in $\operatorname{supp} u$, there should exist $y \in \operatorname{supp} 1_{U_{\epsilon,\delta}} = U_{\epsilon,\delta}$ such that $x - y \in \operatorname{supp} 1_{U_{\epsilon,\delta}} = U_{\epsilon,\delta}$. So, $x \in U_{\epsilon,\delta} + U_{\epsilon,\delta}$.

We also have

$$\|u\|_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(x)\,dx = \frac{1}{\lambda(U_{\epsilon,\delta})^2}\|1_{U_{\epsilon,\delta}}\|_1\|1_{U_{\epsilon,\delta}}\|_1 = 1. \tag{3.5}$$

Next we prove some properties related to v defined in (3.1).

First of all, note that $1_{V_{\epsilon,\delta}+V_{\epsilon,\delta}}, 1_{V_{\epsilon,\delta}} \in L^2(\mathbb{T})$. So, $1_{V_{\epsilon,\delta}+V_{\epsilon,\delta}} * 1_{V_{\epsilon,\delta}} \in A(\mathbb{T})$, which implies that $v \in A(\mathbb{T})$. Also,

$$\begin{aligned}\|v\|_{A(\mathbb{T})} &\le \|f\|_{A(\mathbb{T})}\Big\|\frac{1}{\lambda(V_{\epsilon,\delta})} 1_{V_{\epsilon,\delta}+V_{\epsilon,\delta}} * 1_{V_{\epsilon,\delta}}\Big\|_{A(\mathbb{T})}\\ &\le \frac{1}{\lambda(V_{\epsilon,\delta})}\|f\|_{A(\mathbb{T})}\|1_{V_{\epsilon,\delta}+V_{\epsilon,\delta}}\|_2\|1_{V_{\epsilon,\delta}}\|_2.\end{aligned}$$

Obviously, $\|1_{V_{\epsilon,\delta}+V_{\epsilon,\delta}}\|_2 = \sqrt{\lambda(V_{\epsilon,\delta}+V_{\epsilon,\delta})}$ and $\|1_{V_{\epsilon,\delta}}\|_2 = \sqrt{\lambda(V_{\epsilon,\delta})}$. So,

$$\begin{aligned}\|v\|_{A(\mathbb{T})} &\le \|f\|_{A(\mathbb{T})}\Big(\frac{\lambda(V_{\epsilon,\delta}+V_{\epsilon,\delta})}{\lambda(V_{\epsilon,\delta})}\Big)^{1/2}\\ &= 2\Bigg(\frac{\frac{4(\epsilon-\delta)}{6\pi}}{\frac{2(\epsilon-\delta)}{6\pi}}\Bigg)^{1/2} = 2\sqrt{2}.\end{aligned} \tag{3.6}$$

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Using (3.6), we can write

$$\begin{aligned}\|f - v\|_{A(\mathbb{T})} &\le \|f\|_{A(\mathbb{T})} + \|v\|_{A(\mathbb{T})}\\ &\le 2(1+\sqrt{2}).\end{aligned} \tag{3.7}$$

Similar to what we proved for u,

$$\operatorname{supp} v \subseteq \operatorname{supp}(1_{V_{\epsilon,\delta}+V_{\epsilon,\delta}} * 1_{V_{\epsilon,\delta}}) \subseteq V_{\epsilon,\delta} + V_{\epsilon,\delta} + V_{\epsilon,\delta} \subseteq W_\epsilon.$$

We now show that for each $x \in V_{\epsilon,\delta}$, $f(x) = v(x)$. To see this, take $x \in V_{\epsilon,\delta}$. Then

$$\begin{aligned}(1_{V_{\epsilon,\delta}+V_{\epsilon,\delta}} * 1_{V_{\epsilon,\delta}})(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} 1_{V_{\epsilon,\delta}+V_{\epsilon,\delta}}(x-w)1_{V_{\epsilon,\delta}}(w)\,dw\\ &= \frac{1}{2\pi}\int_{V_{\epsilon,\delta}} 1_{V_{\epsilon,\delta}+V_{\epsilon,\delta}}(x-w)\,dw\\ &= \lambda(V_{\epsilon,\delta}).\end{aligned}$$

Hence, $f(x) = v(x)$. This implies that

$$\operatorname{supp}(f - v) \subseteq V_{\epsilon,\delta}^c. \tag{3.8}$$

We show that $\|v\|_2 \le 2\epsilon\sqrt{(\epsilon-\delta)/6\pi}$. Let $x \in W_\epsilon$. Then

$$\begin{aligned}|f(x)| &= |f(0) - R_x f(0)|\\ &\le \|f - R_x f\|_\infty\\ &\le \|f - R_x f\|_{A(\mathbb{T})}\\ &< \epsilon.\end{aligned}$$

Since $\operatorname{supp} v \subseteq W_\epsilon$,

$$\begin{aligned}\|v\|_2^2 &= \frac{1}{2\pi}\int_{W_\epsilon} |f(t)|^2 \left|\frac{1}{\lambda(V_{\epsilon,\delta})} 1_{V_{\epsilon,\delta}+V_{\epsilon,\delta}} * 1_{V_{\epsilon,\delta}}(t)\right|^2 dt\\ &\le \epsilon^2 \frac{1}{\lambda(V_{\epsilon,\delta})^2}\|1_{V_{\epsilon,\delta}+V_{\epsilon,\delta}} * 1_{V_{\epsilon,\delta}}\|_2^2\\ &\le \epsilon^2 \frac{1}{\lambda(V_{\epsilon,\delta})^2}\|1_{V_{\epsilon,\delta}+V_{\epsilon,\delta}}\|_2^2\|1_{V_{\epsilon,\delta}}\|_1^2\\ &= \epsilon^2 \frac{1}{\lambda(V_{\epsilon,\delta})^2}\lambda(V_\epsilon + V_\epsilon)\lambda(V_{\epsilon,\delta})^2\\ &= \epsilon^2 \frac{4(\epsilon-\delta)}{6\pi}.\end{aligned}$$

This implies that

$$\|v\|_2 \le 2\epsilon\sqrt{\frac{\epsilon-\delta}{6\pi}}. \tag{3.9}$$

We now show that $\|f - f * \check{u}\|_{A(\mathbb{T})} \le \epsilon$. We can write $f * \check{u}$ as a Bochner integral

$$f * \check{u} = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(x) R_x f\,dx.$$

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By (3.5), $(1/2\pi)\int_{-\pi}^{\pi} u(x)\,dx = 1$. Therefore,

$$\begin{aligned}\|f - f * \check{u}\|_{A(\mathbb{T})} &= \frac{1}{2\pi}\Big\| \int_{-\pi}^{\pi} (f - R_x f) u(x)\,dx \Big\|_{A(\mathbb{T})} \\ &\leq \frac{1}{2\pi}\int_{U_{\epsilon,\delta}+U_{\epsilon,\delta}} \|(f - R_x f)\|_{A(\mathbb{T})} |u(x)|\,dx, \\ &< \epsilon,\end{aligned}$$

where the last inequality follows from the fact that $U_{\epsilon,\delta} + U_{\epsilon,\delta} \subseteq W_\epsilon$ and $\|u\|_1 = 1$. On the other hand, using (3.4) and (3.9),

$$\begin{aligned}\|v * \check{u}\|_{A(\mathbb{T})} &\leq \|u\|_2 \|v\|_2 \\ &\leq \sqrt{\frac{6\pi}{\epsilon - \delta}}\, 2\epsilon \sqrt{\frac{\epsilon - \delta}{6\pi}} = 2\epsilon.\end{aligned}$$

So, if we put $a = (f - v) * \check{u}$, then

$$\begin{aligned}\|f - a\|_{A(\mathbb{T})} &\leq \|f - f * \check{u}\|_{A(\mathbb{T})} + \|v * \check{u}\|_{A(\mathbb{T})} \\ &< \epsilon + 2\epsilon = 3\epsilon. \end{aligned} \tag{3.10}$$

Now we can write

$$\begin{aligned}\|F(f)\| &= \|F(f - a + a)\| \\ &\leq \|F(f - a)\| + \|F(a)\|.\end{aligned}$$

Since $a = (f - v) * \check{u}$ and by (3.8), $\operatorname{supp}(f - v) \cap \operatorname{supp}\check{u} \subseteq V^c_{\epsilon,\delta} \cap V_{\epsilon,\delta} = \emptyset$, we have (by hypothesis)

$$\|F(a)\| \leq \alpha \|f - v\|_{A(\mathbb{T})} \|u\|_{A(\mathbb{T})}.$$

Hence,

$$\|F(f)\| \leq \|f - a\|_{A(\mathbb{T})} + \alpha \|f - v\|_{A(\mathbb{T})} \|u\|_{A(\mathbb{T})}.$$

Using (3.2), (3.7) and (3.10),

$$\|F(f)\| \leq 3\epsilon + \alpha 2(1 + \sqrt{2}) \frac{6\pi}{\epsilon - \delta} \quad (0 < \epsilon < 3\ , 0 < \delta < \epsilon).$$

Letting $\delta \to 0$, $A = 4\pi$ and $B = 12\pi(1 + \sqrt{2})$,

$$\|F(f)\| \leq \inf\Big\{A\epsilon + \frac{\alpha B}{\epsilon}, 0 < \epsilon < 3\Big\}. \tag{3.11}$$

Define $k : (0, 3) \to R^+$ by $k(\epsilon) = A\epsilon + \alpha B/\epsilon$. Then

$$k'(\epsilon) = A - \frac{\alpha B}{\epsilon^2} = 0 \Rightarrow \epsilon = \sqrt{\frac{\alpha B}{A}}.$$

Note that for each $0 \leq \alpha \leq 1$, we have $\sqrt{\alpha B/A} \leq \sqrt{3(1 + \sqrt{2})} < 3$. So, by (3.11), we can write

$$\|F(f)\| \leq k\Big(\sqrt{\frac{\alpha B}{A}}\Big) = 2\sqrt{AB\alpha} = 8\pi\sqrt{3(1 + \sqrt{2})}\,\sqrt{\alpha}. \qquad \square$$

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We are now ready to prove the first main result of this section, which, in part, implies that $A(\mathbb{T})$ has the strong property $(\mathbb{B})$ with the constant $288\pi(1+\sqrt{2})$. The method of our proof was partly inspired by [1, Lemma 3.1] and its proof but it goes further to provide a concrete constant for the strong property $(\mathbb{B})$.

Theorem 3.3. *Let $\phi : A(\mathbb{T}) \times A(\mathbb{T}) \to \mathbb{C}$ be a continuous bilinear map satisfying the property*

$$f, g \in A(\mathbb{T}), \quad \operatorname{supp} f \cap \operatorname{supp} g = \emptyset \Rightarrow |\phi(f,g)| \le \alpha \|f\|\,\|g\| \tag{3.12}$$

for some $\alpha \ge 0$. Then

$$|\phi(fg,h) - \phi(f,gh)| \le 288\pi(1+\sqrt{2})\alpha\|f\|\,\|g\|\,\|h\|$$

for all $f, g, h \in A(\mathbb{T})$.

Proof. First assume that $0 \le \alpha < 1$ and $\|\phi\| \le 1$. The map ϕ gives rise to a continuous linear operator Φ on the projective tensor product $A(\mathbb{T})\hat{\otimes}A(\mathbb{T})(=A(\mathbb{T}\times\mathbb{T}))$ defined through

$$\Phi(f\otimes g) = \phi(f,g) \quad (f,g \in A(\mathbb{T})).$$

We define $N : A(\mathbb{T}) \to A(\mathbb{T}\times\mathbb{T})$ with

$$Nk(s,t) = k(s-t) \quad (k \in A(\mathbb{T}),\ s,t \in \mathbb{T}).$$

Pick $f, h \in A(\mathbb{T})$ with $\|f\|, \|h\| \le 1$ and define $N_{f,h} : A(\mathbb{T}) \to A(\mathbb{T}\times\mathbb{T})$ with

$$N_{f,h}k = Nk(f\otimes e_1 h),$$

where $e_1 \in A(\mathbb{T})$ is given by $e_1(s) = e^{is}$. Then it is easy to check that

$$N_{f,h}(e_1 - 1) = fe_1 \otimes h - f\otimes e_1 h. \tag{3.13}$$

Note that for $\psi, \varphi \in A(\mathbb{T})$, we have the Bochner integral equality

$$N(\varphi * \check{\psi}) = \int_{\mathbb{T}} R_x\varphi \otimes R_x\psi\, dx.$$

Hence,

$$N_{f,h}(\varphi * \check{\psi}) = \int_{\mathbb{T}} (R_x\varphi) f \otimes (R_x\psi)e_1 h\, dx. \tag{3.14}$$

If $\operatorname{supp}\varphi \cap \operatorname{supp}\psi = \emptyset$, then

$$\operatorname{supp}((R_x\varphi)f) \cap \operatorname{supp}((R_x\psi)e_1 h) = \emptyset.$$

Hence, using (3.14),

$$\begin{aligned} |\Phi\circ N_{f,h}(\varphi*\check{\psi})| &\le \int_{\mathbb{T}} \|\Phi((R_x\varphi)f\otimes(R_x\psi)e_1h)\|\,dx \\ &\le \int_{\mathbb{T}} \|\phi((R_x\varphi)f,(R_x\psi)e_1h)\|\,dx \quad \text{(by (3.12))} \\ &\le \int_{\mathbb{T}} \alpha\|\phi(R_x\varphi)f\|\,\|(R_x\psi)e_1\|\,dx \\ &\le \alpha\|\varphi\|\,\|\psi\|. \end{aligned}$$

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Hence, by Lemma 3.2,

$$|(\Phi \circ N_{f,h})(e_1 - 1)| \leq 8\pi \sqrt{3(1+\sqrt{2})}\sqrt{\alpha},$$

which, by (3.13), implies that

$$\begin{aligned} |\varphi(fe_1, h) - \varphi(f, e_1 h)| &= |\Phi(fe_1 \otimes h - f \otimes e_1 h)| \\ &\leq 8\pi \sqrt{3(1+\sqrt{2})}\sqrt{\alpha}. \end{aligned} \tag{3.15}$$

Now we show that

$$|\phi(fe_n, h) - \phi(f, e_n h)| \leq 8\pi \sqrt{3(1+\sqrt{2})}\sqrt{\alpha}\|f\|\,\|h\| \tag{3.16}$$

for all $f, h \in A(\mathbb{T})$, where e_n denotes the function in $A(\mathbb{T})$ defined by

$$e_n(s) = e^{\mathrm{ins}} \quad (s \in \mathbb{R}, n \in \mathbb{Z}).$$

For $a \in A(\mathbb{T})$, let $a_n \in A(\mathbb{T})$ be the function defined by

$$a_n(x) = a(nx).$$

Note that $e_n = (e_1)_n$. Define $\tau : A(\mathbb{T}) \times A(\mathbb{T}) \to \mathbb{C}$ by

$$\tau(a, b) = \phi(fa_n, hb_n) \quad (a, b \in A(\mathbb{T})).$$

Note that if $a \in A(\mathbb{T})$, then $a(s) = \sum_{k=-\infty}^{+\infty} \hat{a}(k)e^{\mathrm{iks}}$; hence, $a(ns) = \sum_{k=-\infty}^{+\infty} \hat{a}(k)e^{\mathrm{ikns}}$ and so $a_n \in A(\mathbb{T})$ with

$$\|a_n\| \leq \sum_{k=-\infty}^{+\infty} |\hat{a}(k)| = \|a\|.$$

Moreover, if $a, b \in A(\mathbb{T})$ are such that $\operatorname{supp} a \cap \operatorname{supp} b = \emptyset$, then it is easily seen that $\operatorname{supp} fa_n \cap \operatorname{supp} hb_n = \emptyset$. So,

$$\begin{aligned} |\tau(a, b)| &\leq \|\phi(fa_n, hb_n)\| \\ &\leq \alpha\|fa_n\|\,\|hb_n\| \\ &\leq \alpha\|a\|\,\|b\|. \end{aligned}$$

From (3.15),

$$|\tau(e_1, 1) - \tau(1, e_1)| \leq 8\pi \sqrt{3(1+\sqrt{2})}\sqrt{\alpha}. \tag{3.17}$$

On the other hand,

$$\tau(e_1, 1) = \phi(fe_n, h), \quad \tau(1, e_1) = \phi(f, e_n h),$$

which, together with (3.17), gives (3.16).

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Now let $g \in A(\mathbb{T})$. Since $g = \sum_{k=-\infty}^{+\infty} \hat{g}(k)e_k$, by applying (3.16),

$$\begin{aligned}|\phi(fg, h) - \phi(f, gh)| &= \left|\phi\Big(\sum_{k=-\infty}^{+\infty} \hat{g}(k)fe_k, h\Big) - \phi\Big(f, \sum_{k=-\infty}^{+\infty} \hat{g}(k)e_k h\Big)\right| \\ &\le \sum_{-\infty}^{+\infty} |\hat{g}(k)|\,|\phi(fe_k, h) - \phi(f, e_k h)| \\ &\le \sum_{-\infty}^{+\infty} |\hat{g}(k)| 8\pi\sqrt{3(1+\sqrt{2})}\sqrt{\alpha} \\ &= 8\pi\sqrt{3(1+\sqrt{2})}\sqrt{\alpha}\|g\|.\end{aligned}$$

Therefore, if $f, h \in A(\mathbb{T})$ are arbitrary elements,

$$|\phi(fg, h) - \phi(f, gh)| \le 8\pi\sqrt{3(1+\sqrt{2})}\sqrt{\alpha}\|f\|\,\|g\|\,\|h\|. \tag{3.18}$$

Next, let $m : A(\mathbb{T}\times\mathbb{T}) \to A(\mathbb{T})$ be the multiplication map which maps every elementary tensor $f \otimes g \in A(\mathbb{T}\times\mathbb{T})$ to $fg \in A(\mathbb{T})$. It follows from (3.18) that for $u = \sum_{i=1}^{\infty} f_i \otimes g_i \in A(\mathbb{T}\times\mathbb{T})$, we can write

$$\begin{aligned}|\Phi(u) - \phi(1, m(u))| &= \left|\Phi\Big(\sum_{i=1}^{\infty} f_i \otimes g_i - \sum_{i=1}^{\infty} 1 \otimes f_i g_i\Big)\right| \\ &\le 8\pi\sqrt{3(1+\sqrt{2})}\sqrt{\alpha}\sum_{i=1}^{\infty}\|f_i\|\,\|g_i\|.\end{aligned}$$

In particular, for every $u \in I := \ker m$,

$$|\Phi(u)| \le 8\pi\sqrt{3(1+\sqrt{2})}\sqrt{\alpha}\|u\|,$$

implying that

$$\|\Phi|_I\| \le 8\pi\sqrt{3(1+\sqrt{2})}\sqrt{\alpha}. \tag{3.19}$$

Now consider the general case. Let $\phi : A(\mathbb{T})\times A(\mathbb{T}) \to \mathbb{C}$ be a continuous bilinear map satisfying (3.12) for some $\alpha > 0$. Without loss of generality, we can assume that $\Phi|_I \neq 0$. Let $\Phi_0 \in I^*$ with $\Phi_0 = \Phi|_I/\|\Phi|_I\|$. Then $\|\Phi_0\| = 1$. By the Hahn–Banach theorem, Φ_0 can be extended to $\Psi \in A(\mathbb{T}\times\mathbb{T})^*$ with $\|\Psi\| = 1$. For $f, g \in A(\mathbb{T})$ with $\operatorname{supp} f \cap \operatorname{supp} g = \emptyset$,

$$\begin{aligned}|\Psi(f\otimes g)| &= |\Phi_0(f\otimes g)| \\ &= \frac{1}{\|\Phi|_I\|}|\Phi(f\otimes g)| \\ &\le \frac{\alpha}{\|\Phi|_I\|}\|f\|\,\|g\|.\end{aligned}$$

Put $\alpha_0 = \alpha/\|\Phi|_I\|$. Then $\|\Psi\| = 1$ and $0 \leq \alpha_0 \leq 1$ (we can assume that $\alpha \leq \|\Phi|_I\|$, otherwise the statement is trivial). By the first part and (3.19),

$$\begin{aligned} 1 = \|\Phi_0\| &= \|\Psi|_I\| \\ &\leq 8\pi\sqrt{3(1+\sqrt{2})}\sqrt{\alpha_0} \\ &= 8\pi\sqrt{3(1+\sqrt{2})}\sqrt{\frac{\alpha}{\|\Phi|_I\|}}. \end{aligned}$$

This implies that

$$\|\Phi|_I\| \leq 192\pi^2(1+\sqrt{2})\alpha.$$

In particular, for every $u \in I$,

$$|\Phi(u)| \leq 192\pi^2(1+\sqrt{2})\alpha\|u\|.$$

Finally, for $f, g, h \in A(\mathbb{T})$, it is clear that $fg \otimes h - f \otimes gh \in I$. So, we can write

$$\begin{aligned} |\phi(fg, h) - \phi(f, gh)| &= |\Phi(fg \otimes h - f \otimes gh)| \\ &\leq 192\pi^2(1+\sqrt{2})\alpha\|fg \otimes h - f \otimes gh\| \\ &\leq 384\pi^2(1+\sqrt{2})\alpha\|f\|\,\|g\|\,\|h\|. \end{aligned}$$ □

One important application of Theorem 3.3 is to obtain a constant for the strong property ($\mathbb{B}$) of C*-algebras and group algebras. The approach we need to use is the same as that provided in [1] with a slight modification using our result from the preceding subsection. Hence, we do not give a proof to the following theorem and we just refer to [1, Theorems 3.4 and 3.5] (see also [1, Lemma 3.2]). We highlight that the approach in [1] does not give a constant for the strong property ($\mathbb{B}$), whereas our modification does, as the following theorem shows.

THEOREM 3.4. *Let A be a C^*-algebra or a group algebra. Then A has the strong property ($\mathbb{B}$) with a constant given by* $384\pi^2(1+\sqrt{2})$.

4. A bound for the hyperreflexivity constant

In this section, we show how the existence of a constant for the strong property ($\mathbb{B}$) can help us to find an upper bound for the hyperreflexivity constant of the bounded n-cocycle spaces. We then apply our result to C*-algebras and group algebras. We achieve our goal by modifying the approach in [22] and its main result. We start by stating without proof the following theorem, which is a straightforward modification of [22, Theorem 3.6], taking into account our concept of the strong property ($\mathbb{B}$) with a constant.

THEOREM 4.1. *Let A be a unital Banach algebra with unit 1 having the strong property ($\mathbb{B}$) with a constant R. Suppose that X is a unital Banach A-bimodule, $n \in \mathbb{N}$, $T \in B^n(A, X)$ and let $\gamma \geq 0$ satisfying*

$$a_0a_1 = a_1a_2 = \cdots = a_na_{n+1} = 0 \Rightarrow \|a_0T(a_1, \ldots, a_n)a_{n+1}\| \leq \gamma\|a_0\| \cdots \|a_{n+1}\|.$$

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Also, $T(a_1, \ldots, a_n) = 0$ *if for some* $1 \le i \le n, a_i = 1$. *Then*

$$\|\delta^n(T)\| \le 2^{n-1} R^{n+1} \gamma.$$

The following theorem presents the platform for our approach toward computing a bound for the hyperreflexivity constant of bounded n-cocycle spaces. This is, again, the modification as well as an improvement of [22, Theorem 3.8] since it allows us to have a concrete bound for the hyperreflexivity constant. But, first, we need to recall the concept of bounded local unit for a Banach algebra defined in [22, Definition 5.1].

We say that a Banach algebra A has *bounded local units* or in brief *b.l.u.* if there are dense subsets A_l and A_r of A and $M > 0$ such that for every $a \in A_l$ (respectively $b \in A_r$), there is $c \in A$ (respectively $d \in A$) with $\|c\| \le M$ (respectively $\|d\| \le M$) satisfying

$$ca = a, \quad bd = b.$$

Remark 4.2. Suppose that A is a Banach algebra with b.l.u. It follows easily that A has a bounded approximate identity. On the other hand, if $\{e_i\}_{i \in I}$ is any bounded approximate identity for A, then it follows from the proof of [22, Proposition 5.4] that the bounded local units of A can be chosen such that they are bounded by $\sup_{i \in I}\{\|e_i\|\} + \epsilon$, where $\epsilon > 0$ is arbitrary.

Theorem 4.3. *Let A be a Banach algebra having b.l.u. and the strong property* ($\mathbb{B}$) *with a constant R. Let M be the least upper bound for a bounded approximate identity of A. Let $n \in \mathbb{N}$ and suppose that X is a Banach A-bimodule such that $\mathcal{H}^{n+1}(A, X)$ is a Banach space. Then, for each $T \in B^n(A, X)$,*

$$\operatorname{dist}(T, \mathcal{Z}^n(A, X)) \le C 2^{n-1} (M^2 R + (M+1)^2)^{n+1} \operatorname{dist}_r(T, \mathcal{Z}^n(A, X)),$$

where C is a constant satisfying

$$\operatorname{dist}(T, Z^n(A, X)) \le C \|\delta^n(T)\| \quad (T \in B^n(Z, X)). \tag{4.1}$$

Proof. Let $T \in B^n(A, X)$. By [22, Lemma 3.7], for every $a_i \in A^\sharp$, $i = 0, \ldots, n+1$, with $a_0 a_1 = \cdots = a_n a_{n+1} = 0$,

$$\|a_0 \sigma(T)(a_1, \ldots, a_n) a_{n+1}\| \le \operatorname{dist}_r(T, \mathcal{Z}^n(A, X)) \|a_1\| \cdots \|a_{n+1}\|,$$

where $\sigma(T) : A^\sharp \to X$ is defined by

$$\sigma(T)(b_1 + \lambda_1, \ldots, b_n + \lambda_n) = T(b_1, \ldots, b_n) \quad (b_i \in A, \ \lambda_i \in \mathbb{C}).$$

Moreover, if for some $1 \le i \le n$, $a_i = 1$, then

$$\sigma(T)(a_1, \ldots, a_n) = 0.$$

On the other hand, applying Remark 4.2 together with a close examination of [22, Theorem 5.3] and its proof shows that $A^\sharp$ has the strong property ($\mathbb{B}$) with a constant given by

$$(M+\epsilon)^2 R + (M + \epsilon + 1)^2,$$

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where $\epsilon > 0$ is arbitrary. Hence, we can use Theorem 4.1 to write

$$\|\delta^{\sharp n}(\sigma(T))\| \le 2^{n-1}((M+\epsilon)^2 R + (M+\epsilon+1)^2)^{n+1}\,\mathrm{dist}_r(T, \mathcal{Z}^n(A,X)). \tag{4.2}$$

Now, since $\mathcal{H}^{n+1}(A,X)$ is a Banach space, $\mathrm{Im}\delta^n$ is closed. Hence, by the open mapping theorem, there is a constant $C > 0$ such that for each $T \in B^n(A,X)$,

$$\mathrm{dist}(T, \mathcal{Z}^n(A,X)) \le C\|\delta^n(T)\|. \tag{4.3}$$

It is straightforward to check that

$$\|\delta^n(T)\| \le \|\delta^{\sharp n}(\sigma(T))\|. \tag{4.4}$$

Hence, putting (4.2), (4.3) and (4.4) together,

$$\mathrm{dist}(T, \mathcal{Z}^n(A,X)) \le C2^{n-1}((M+\epsilon)^2 R + (M+\epsilon+1)^2)^{n+1}\,\mathrm{dist}_r(T, \mathcal{Z}^n(A,X)).$$

By letting $\epsilon \to 0$, we get what we desired. □

We showed in Section 3.1 that every C*-algebra and group algebra has the strong property ($\mathbb{B}$) with the constant $384\pi^2(1+\sqrt{2})$. On account of Theorem 4.3, this enables us to obtain an upper bound for the hyperreflexivity constant of the bounded n-cocycle spaces of certain C*-algebras and group algebras.

Theorem 4.4. *Suppose that A is a C*-algebra or the group algebra of a group with an open subgroup of polynomial growth. Let $n \in \mathbb{N}$ and let X be a Banach A-bimodule such that $\mathcal{H}^{n+1}(A,X)$ is a Banach space. Then $\mathcal{Z}^n(A,X)$ is hyperreflexive with a constant bounded by*

$$C2^{n-1}(384\pi^2(1+\sqrt{2})+4)^{n+1}, \tag{4.5}$$

where C is the constant given in (4.1).

Proof. It is well known that C*-algebras and group algebras have contractive approximate identities. Hence, the statement of the theorem follows if we combine [22, Proposition 6.1], [22, Theorem 6.3(1)], Theorems 3.4 and 4.3. □

Remark 4.5. The preceding theorem can be applied to a large class of examples some of which we point out below. We refer the reader to [22, Section 6] for details:

(i) A is a nuclear C*-algebra and X is a dual Banach A-bimodule;
(ii) A is a von Neumann algebra of types I, II_∞ or III and $X = A$ or $X = B(\mathcal{H}) \supseteq A$ for a Hilbert space $\mathcal{H}$;
(iii) A is an injective von Neumann algebra and $X = A$ or $X = B(\mathcal{H}) \supseteq A$ for a Hilbert space $\mathcal{H}$;
(iv) $A = L^1(G)$ for G being an amenable locally compact group with open subgroup of polynomial growth and G being an amenable locally compact group with open subgroup of polynomial growth and X is a dual Banach A-bimodule;
(v) $A = L^1(G)$ for G being a locally compact group with open subgroup of polynomial growth and $X = L^1(G)^{(k)}$ for $k = 0$ and for each odd $k \in \mathbb{N}$, where $L^1(G)^{(k)}$ stands for the kth dual space of $L^1(G)$.

Moreover, for all cases in (i), (iii) and (iv), we can assume that the constant C in (4.5) is 1.

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5. Convolution operators

For a locally compact group G and $1 < p < \infty$, we recall that an operator $T \in B(L^p(G))$ is a *convolution operator* if for every $t \in G$ and $f \in L^p(G)$, $T(\delta_t * f) = \delta_t * T(f)$. The space of all convolution operators on $L^p(G)$ is denoted by $CV_p(G)$. It is straightforward to check that $CV_p(G)$ is a w^*-closed subalgebra of $B(L^p(G)) \cong (L^p(G) \widehat{\otimes} L^q(G))^*$, where q is the conjugate of p.

In this section, we discuss reflexivity and hyperreflexivity of $CV_p(G)$. For the latter case, our method once again relies on the concept of the strong property ($\mathbb{B}$) with a constant applied to group algebras. But, first, we need the following theorem, which is the generalization of a classical result of Christensen in [6] for C^*-algebras. We recall that for a Banach algebra A, a Banach A-bimodule X and $x \in X$, the inner derivation $\delta_x : A \to X$ is defined by $\delta_x(a) = a \cdot x - x \cdot a$.

Theorem 5.1. *Let A be a Banach algebra with an approximate identity bounded by K and having the strong property ($\mathbb{B}$) with a constant R, X a Banach space and $\pi : A \to B(X)$ a continuous nondegenerate representation. If $\mathcal{H}^1(A, B(X))$ is a Banach space, then $\pi(A)'$, the commutant of $\pi(A)$, is hyperreflexive and its hyperreflexivity constant is bounded by* $\mathrm{RCK}^2\|\pi\|^2$, *where C is a constant satisfying*

$$\mathrm{dist}(L, \pi(A)') \le C\|\delta_L\|, \quad L \in B(X).$$

Proof. Let $T \in B(X)$ and $\alpha = \mathrm{dist}_r(T, \pi(A)')$. Fix $x \in X$ and define

$$\varphi_{T,x} : A \times A \to X, \quad (a, b) \mapsto (\pi(a) \circ T \circ \pi(b))(x).$$

If $a, b \in A$ with $ab = 0$, then, for $S \in \pi(A)'$,

$$\begin{aligned}\|\varphi_{T,x}(a, b)\| &= \|\pi(a)T(\pi(b)(x)) - \pi(a)S(\pi(b)(x))\| \\ &= \|\pi(a)(T - S)(\pi(b)(x))\| \\ &\le \|\pi(a)\|\,\|(T - S)(\pi(b)(x))\|.\end{aligned}$$

Therefore,

$$\begin{aligned}\|\varphi_{T,x}(a, b)\| &\le \|\pi(a)\| \inf_{S\in A'} \|(T - S)(\pi(b)(x))\| \\ &\le \|\pi(a)\|\alpha\|\pi(b)(x)\| \\ &\le \alpha\|\pi\|^2\|x\|\,\|a\|\,\|b\|.\end{aligned}$$

Since A has the strong property ($\mathbb{B}$) with a constant R, we have that for every $a, b, c \in A$,

$$\|\varphi_{T,x}(ab, c) - \varphi_{T,x}(a, bc)\| \le R\alpha\|\pi\|^2\|x\|\,\|a\|\,\|b\|\,\|c\|. \tag{5.1}$$

However, if $\{e_i\}_{i\in I}$ is an approximate identity in A bounded by K, then it follows from the nondegeneracy of π that $\lim_{i\to\infty} \pi(e_i) = \mathrm{id}_X$, where the convergence happens in the

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strong operator topology of $B(X)$. Hence, if we put $a = c = e_i$ in (5.1) and let $i \to \infty$,

$$\begin{aligned} \|\delta_T(b)(x)\| &= \|(T \cdot b - b \cdot T)(x)\| \\ &= \|T(\pi(b)(x)) - \pi(b)T(x)\| \\ &= \lim_{i\to\infty} \|\pi(e_i b)T(\pi(e_i)(x)) - \pi(e_i)T(\pi(be_i)(x))\| \\ &= \lim_{i\to\infty} \|\varphi_{T,x}(e_i b, e_i) - \varphi_{T,x}(e_i, be_i)\| \\ &\leq RK^2\alpha\|\pi\|^2\|x\|\,\|b\|. \end{aligned}$$

Since $x \in X$ was arbitrary, we can conclude that for every $b \in A$,

$$\|\delta_T(b)\| \leq R\alpha\|\pi\|^2K^2\|b\|. \tag{5.2}$$

Now we define

$$\phi : B(X) \to \mathcal{Z}^1(A, B(X)), \quad L \mapsto \delta_L.$$

Using (5.2),

$$\|\delta_T\| \leq R\alpha\|\pi\|^2K^2. \tag{5.3}$$

But $\mathcal{H}^1(A, B(X)) = \mathcal{Z}^1(A, B(X))/\mathcal{N}^1(A, B(X))$ is a Banach space, so that $\mathcal{N}^1(A, B(X)) = \operatorname{Im}\phi$ is closed. Hence, applying the open mapping theorem to the map $i : B(X)/\ker\phi \to \mathcal{N}^1(A, B(X))$, we find $C > 0$ such that for all $L \in B(X)$,

$$\operatorname{dist}(L, \ker\phi) \leq C\|\delta_L\|,$$

which, together with (5.3), gives

$$\operatorname{dist}(T, \ker\phi) \leq \mathrm{RCK}^2\|\pi\|^2\alpha = \mathrm{RCK}^2\|\pi\|^2 \operatorname{dist}_r(T, \pi(A)').$$

The final result follows since $\ker\phi = \pi(A)'$. □

It is well known that all von Neumann algebras are reflexive and injective von Neumann algebras are hyperreflexive. The former is a direct and simple application of the double commutate theorem, whereas the latter is due to the beautiful work of Christensen [6]. In particular, for a locally compact group G, $CV_2(G)$, its group von Neumann algebra, is reflexive and, when G is amenable , it is hyperreflexive. In the following theorem, we extend this to all convolution operators on $L^p(G)$ for every $1 < p < \infty$.

Theorem 5.2. *Let G be a locally compact group and let $1 < p < \infty$. Then $CV_p(G) \subseteq B(L^p(G))$ is reflexive. If, in addition, G is amenable, then $CV_p(G)$ is hyperreflexive and its hyperreflexivity constant is bounded by* $384\pi^2(1 + \sqrt{2})$.

Proof. The reflexivity of $CV_p(G)$ follows from [19, Theorems 2.2 and 8.1]. Now, if G is amenable, then by the well-known result of Johnson, $L^1(G)$ is an amenable Banach algebra. Hence, if we let $A = L^1(G)$, $X = L^p(G)$ and $\pi = \rho$, the right regular representation of G on $L^p(G)$, then all the assumptions of Theorem 5.1 are satisfied

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with $C = K = \|\pi\| = 1$ (see Remark 4.5) and $R = 384\pi^2(1 + \sqrt{2})$ by Theorem 3.4. Therefore, $CV_p(G) = \pi(L^1(G))'$ is hyperreflexive and its hyperreflexivity constant is bounded by $384\pi^2(1 + \sqrt{2})$. □

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