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ON MULTIPLIERS WITH UNCONDITIONALLY CONVERGING FOURIER SERIES

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Let G be a compact abelian group with dual group Γ. For $1 \leqq p < \infty$, $1 \leqq q < \infty$, let $M_p^q(\Gamma)$ denote the Banach space of complex-valued functions on Γ which are multipliers of type (p, q) and $m_p^q(\Gamma)$ the subspace of compact multipliers.

Grothendieck [**10**; **11**] has proven that a function in $L^p(G)$, $1 \leqq p < 2$, has an unconditionally converging Fourier series in $L^p(G)$ if and only if it is in $L^2(G)$, and Helgason [**12**] has proven that the derived algebra of $L^p(G)$, $1 \leqq p < 2$, is $L^2(G)$. Using these results we show in § 2 that a multiplier of type (p, q), $1 \leqq p \leqq 2$, $1 \leqq q \leqq 2$, has an unconditionally converging Fourier series in $M_p^q(\Gamma)$ if and only if it is in $m_p^2(\Gamma)$ (Theorem 2.1), and that, for $1 \leqq p \leqq q \leqq 2$, the derived algebra of $M_p^q(\Gamma)$ is $M_p^2(\Gamma)$ (Theorem 2.2). Statements equivalent to the above are also given. Thus, by a result of the first author and John Gilbert [**2**] the derived algebra of $M_p^q(\Gamma)$ is the double dual of the (p, q) multipliers with unconditionally converging Fourier series. This last result is valid for $1 \leqq p \leqq q < \infty$ (Remark 2.5) and is in contrast to the situation for $L^p(G)$, where the unconditionally converging Fourier series coincide with the derived algebra [**1**] and form a reflexive Banach space [**2**].

Figà-Talamanca and Gaudry [**7**] have given an example of an element of $C_0(\mathbf{Z})$ (i.e., a function on the integers vanishing at infinity) which is also in $M_p^2(\mathbf{Z})$ but not in $m_p^p(\mathbf{Z})$, where $1 < p < 2$. In § 3 we show that the absolute value of this function gives that

$$m_p^2(\mathbf{Z}) \subsetneqq M_p^2(\mathbf{Z}) \cap m_p^q(\mathbf{Z}), \quad 1 < p \leqq q < 2,$$

and that

$$m_p^2(\mathbf{Z}) \not\supset L^s(T)^\wedge \cap M_p^2(\mathbf{Z}), \; 1 < p < 2, \; 1 \leqq s < 2p/(3p-2)$$

The first inequality is due to Haskell Rosenthal.

1. Preliminaries. We use freely the notation and basic results in Rudin's book [**17**]. We use without reference fundamental facts about multipliers as presented in Edward's book [**6**]. The fact that results are stated there only for $G = \mathbf{T}$ (the circle group) should cause the reader no difficulty.

Let $1 \leqq p < \infty$, $1 \leqq q < \infty$. A complex-valued function φ defined on Γ is said to be a *multiplier* of type (p, q) if it determines an operator from $L^p(G)$ to

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$L^q(G)$, T_φ, given by

$$(T_\varphi f)^\wedge = \varphi\hat{f} \quad (f \in L^p(G)).$$

Let $M_p{}^q(\Gamma)$ denote the set of multipliers of type (p, q). Then $M_p{}^q(\Gamma)$ is a Banach space where the norm $||\cdot||_{(p,q)}$ of the multiplier φ is defined to be the norm of the multiplier operator T_φ. We denote by $m_p{}^q(\Gamma)$ the set of *compact multipliers* of type (p, q), that is, the set of $\varphi \in M_p{}^q(\Gamma)$ for which the corresponding operator T_φ is compact. The set $m_p{}^q(\Gamma)$ is a closed subspace of $M_p{}^q(\Gamma)$.

We have:

1.1 LEMMA. *Let* $1 \leqq p < \infty$, $1 \leqq q < \infty$. *Then*

(i) $m_p{}^q(\Gamma)$ *is the closure of* $C_c(\Gamma)$ *in* $M_p{}^q(\Gamma)$. *If* $\varphi \in m_p{}^q(\Gamma)$ *then* φ *can be approximated in* $M_p{}^q(\Gamma)$ *by functions in* $C_c(\Gamma)$ *with supports contained in that of* φ.

(ii) (cf. Hormander [**14**]). *If* $p \leqq q$, *then* $M_p{}^q(\Gamma)$ *is a commutative semi-simple Banach algebra whose maximal ideal space contains* Γ. *The set* $m_p{}^q(\Gamma)$ *is a closed ideal in* $M_p{}^q(\Gamma)$ *whose maximal ideal space equals* Γ.

Proof. (i) See [**2**, Theorem 3.1] or [**8**, Theorem 4.2.2]. We note that $\varphi \in C_c(\Gamma)$ if and only if T_φ is given by convolution with a trigonometric polynomial.

(ii) If $p \leqq q$, then $L^q(G) \subset L^p(G)$ and $||\cdot||_q \geqq ||\cdot||_p$. From this it follows that $M_p{}^q(\Gamma)$ is a commutative Banach algebra under pointwise multiplication. Its maximal ideal space contains Γ, since $\varphi \to \varphi(\gamma)$ is a multiplicative linear functional. In particular, $M_p{}^q(\Gamma)$ is semi-simple. Since $C(G)^\wedge \subset m_p{}^q(\Gamma) \subset C_0(\Gamma)$, it follows that the maximal ideal space of $m_p{}^q(\Gamma)$ is Γ. Since $M_p{}^q(\Gamma) \subset M_p{}^p(\Gamma)$, if follows from operator theory that $m_p{}^q(\Gamma)$ is an ideal in $M_p{}^q(\Gamma)$.

We next discuss the notions of unconditional convergence and the derived algebra.

1.2 *Definition.* Let $\{J\}$ denote the collection of finite subsets of Γ, directed by inclusion, and let $\chi_J \in C_c(\Gamma)$ denote the characteristic function of J. If $\varphi \in M_p{}^q(\Gamma)$, we say that φ has an *unconditionally converging Fourier series* if

$$\lim_{\{J\}} ||\varphi\chi_J - \varphi||_{(p,q)} = 0.$$

The motivation for this terminology is as follows: If $\varphi = \hat{f}$ for some $f \in L^1(G)$ then $T_\varphi(g) = f*g$, and the operator corresponding to $\varphi\chi_J$ is convolution by the trigonometric polynomial, $\sum_{\gamma\in J} \hat{f}(\gamma)\gamma$. Thus φ has an unconditionally converging Fourier series, in our terminology, if and only if the Fourier series for f converges unconditionally to f in the (p,q)- multiplier norm; that is,

$$\lim_{\{J\}} \sup_{||g||_p \leqq 1} \left\| \sum_J \hat{f}(\gamma)\hat{g}(\gamma)\gamma - f*g \right\|_q = 0.$$

For basic facts on unconditional convergence, the reader is referred to Day's book [**4**]. It is straightforward to verify that the set of elements of $M_p{}^q(\Gamma)$ or

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$m_p{}^q(\Gamma)$ with an unconditionally converging Fourier series is a Banach space with the norm given by

$$||\varphi||_S = \sup_J ||\varphi\chi_J||_{(p,q)}$$

and that, for such φ,

$$\lim_{\{J\}} ||\varphi - \varphi\chi_J||_S = 0.$$

1.3 *Definition* (Helgason [**12**]). If A is a commutative semi-simple Banach algebra with maximal ideal space $\mathscr{M}$, define the *derived algebra*, A_0, to be the the set of $x \in A$ such that

$$\sup_{y \in A} \frac{||xy||}{||\tilde{y}||_\infty} \equiv ||x||_0 < \infty,$$

where $\tilde{y}$ denotes the Gelfand transform of y, so that

$$||\tilde{y}||_\infty = \sup\{|\tilde{y}(M)| : M \in \mathscr{M}\}.$$

If $q \leqq p$, and $A = M_p{}^q(\Gamma)$ or $m_p{}^q(\Gamma)$, one verifies that A_0 is a Banach algebra and that $||\cdot||_0 \geqq ||\cdot||_{(p,q)}$.

Let $S^p(G)$ denote the set of functions in $L^p(G)$ with unconditionally converging Fourier series in $L^p(G)$. In § 2 we will make use of the following results.

1.4 Theorem. *Let* $1 \leqq p < 2$.

(i) (Helgason [**12**]) $L^p(G)_0 = L^2(G)$.

(ii) (Grothendieck [**10**; **11**]) $S^p(G) = L^2(G)$.

(iii) (Grothendieck [**11**]) *If* φ *is a complex-valued function on* Γ *such that* $\epsilon\varphi \in M(G)$ *for all* ϵ *with* $\epsilon(\gamma) = \pm 1$, *then* $\varphi \in l^2(\Gamma)$.

Part (iii) is a generalization of a theorem of Littlewood. For related results, see also [**12**, Theorem 10; **18**, V(8.13); **5**].

2. Multipliers which have an unconditionally converging Fourier series or are in the derived algebra. We first give several equivalent conditions for a multiplier to have an unconditionally converging Fourier series.

2.1 Theorem. *Let* $1 \leqq p \leqq 2$, $1 \leqq q \leqq 2$, *and let* φ *be a complex-valued function on* Γ. *Then the following statements are equivalent.*

(i) $\varphi \in M_p{}^q(\Gamma)$ *and has an unconditionally converging Fourier series.*

(ii) $\varphi \in m_p{}^2(\Gamma)$.

(iii) $a\varphi \in m_p{}^q(\Gamma)$ *for all* $a \in l^\infty(\Gamma)$.

(iv) $\epsilon\varphi \in m_p{}^q(\Gamma)$ *for all* ϵ *with* $\epsilon(\gamma) = \pm 1$.

Proof. (i) implies (ii). Let S denote the set of elements in $M_p{}^q(\Gamma)$ with an unconditionally converging Fourier series, and let $R \subset M_p{}^q(\Gamma)$ denote the set of compact multipliers from $L^p(G)$ to $S^q(G)$, with norm $||\cdot||_R$. Since $S^q(G) = L^2(G)$, $R = m_p{}^2(\Gamma)$. We will show that $R = S$, and hence (ii) follows.

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If $\psi \in C_c(\Gamma)$, then

$$\begin{aligned}||\psi||_R &= \sup_{||f||_p \leqq 1} ||T_\psi f||_{S^q} \\ &= \sup_{||f||_p \leqq 1} \sup_J \left\| \sum_J \psi(\gamma)\hat{f}(\gamma)\gamma \right\|_q \\ &= \sup_J ||\psi \chi_J||_{(p,q)} \\ &= ||\psi||_S.\end{aligned}$$

Since $C_c(\Gamma)$ is dense in each of the spaces R and S, $R = S$.

(ii) implies (iii). If $a \in l^\infty(\Gamma) = M_2^2(\Gamma)$, then

$$a\varphi \in m_p^2(\Gamma)M_2^2(\Gamma) \subset m_p^2(\Gamma) \subset m_p^q(\Gamma).$$

(iii) implies (iv) is immediate.

(iv) implies (i). Choosing $\epsilon(\gamma) = 1$ for all γ, we have that $\varphi \in m_p^q(\Gamma)$. Let Γ_1 denote the support of φ and let

$$B = \{\psi \in m_p^q(\Gamma): \psi(\gamma) = 0, \gamma \notin \Gamma_1\}.$$

Then B is a Banach space. It follows from Lemma 1.1 (i) that Γ_1 is countable and that B is separable.

Let $\Gamma_1 = (\gamma_n)$ and define $(b_n) \subset B$, $(\beta_n) \subset B^*$ by

$$b_n(\gamma) = \begin{cases} 1, & \gamma = \gamma_n \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\beta_n(\psi) = \psi(\gamma_n), \quad \psi \in B, \ n = 1, 2, \ldots .$$

Then (b_n, β_n) is a biorthogonal sequence in B, and (β_n) is total. Condition (iv) implies that, given a sequence (a_n), with $a_n = 0$ or 1, there exists $\psi \in B$ such that $\beta_n(\psi) = a_n\beta_n(\varphi)$. Thus by [**3**, Theorem 1], $\Sigma_n \beta_n(\varphi)b_n$ converges unconditionally to φ in B. But this is precisely the statement that

$$\lim_{\{J\}} ||\varphi\chi_J - \varphi||_{(p,q)} = 0.$$

We now give conditions equivalent to a multiplier being in the derived algebra.

2.2 THEOREM. *Let $1 \leqq p \leqq 2$, $1 \leqq q \leqq 2$, and let φ be a complex-valued function on Γ. Then the following statements are equivalent.*

(i) *$\varphi \in M_p^2(\Gamma)$.*

(ii) *$a\varphi \in m_p^q(\Gamma)$ for all $a \in C_0(\Gamma)$.*

(iii) *$a\varphi \in M_p^q(\Gamma)$ for all $a \in C_0(\Gamma)$.*

(iv) *$a\varphi \in M_p^q(\Gamma)$ for all $a \in l^\infty(\Gamma)$.*

(v) *$\epsilon\varphi \in M_p^q(\Gamma)$ for all ϵ with $\epsilon(\gamma) = \pm 1$.*

If $p \leqq q$, then the above are equivalent to

(vi) *φ is in the derived algebra of $M_p^q(\Gamma)$.*

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Proof. "(i) implies (ii)" and "(i) implies (iv)" both follow in a manner similar to "(ii) implies (iii)" of Theorem 2.1.

"(ii) implies (iii)" and "(iv) implies (v)" are immediate.

(iii) implies (i). If $f \in L^p(G)$, then $a\varphi\hat{f} \in L^q(G)^\wedge$ for all $a \in C_0(\Gamma)$. Thus, by [**12**, Theorem 2],

$$\varphi\hat{f} \in (L^q(G)_0)^\wedge = l^2(\Gamma),$$

so $\varphi \in M_p{}^2(\Gamma)$.

"(v) implies (i)" follows as above, using Theorem 1.4 (iii).

(i) implies (vi). Let $\psi \in M_p{}^q(\Gamma)$ and let $\tilde{\psi}$ denote the Gelfand transform of ψ. Then $||\psi||_\infty \leqq ||\tilde{\psi}||_\infty$.

If $f \in L^p(G)$, then

$$\begin{aligned} ||T_{\varphi\psi}f||_q &\leqq ||\varphi\psi\hat{f}||_2 \\ &\leqq ||\psi||_\infty \, ||\varphi\hat{f}||_2 \\ &\leqq ||\tilde{\psi}||_\infty \, ||\varphi||_{(p,2)} \, ||f||_p, \end{aligned}$$

so $||\varphi\psi||_{(p,q)} \leqq ||\varphi||_{(p,2)} \, ||\tilde{\psi}||_\infty$. Thus $\varphi \in M_p{}^q(\Gamma)_0$.

(vi) implies (iii). If $a \in C_c(\Gamma)$, then $||\tilde{a}||_\infty = ||a||_\infty$ so

$$||a\varphi||_{(p,q)} \leqq ||\varphi||_0 \, ||\tilde{a}||_\infty = ||\varphi||_0 \, ||a||_\infty.$$

Since $C_c(\Gamma)$ is dense in $C_0(\Gamma)$, this implies that $a \to a\varphi$ is a bounded operator from $C_0(\Gamma)$ to $M_p{}^q(\Gamma)$. Thus (iii) holds.

From Theorems 2.1, 2.2, and [**12**, Theorem 2] the following corollary is immediate:

2.3 COROLLARY. *Let* $1 \leqq p \leqq 2$, $1 \leqq q \leqq 2$. *Then*:

(i) *An element* $\varphi \in m_p{}^q(\Gamma)$ *has an unconditionally converging Fourier series if and only if* $\varphi \in m_p{}^2(\Gamma)$.

(ii) *If* $p \leqq q$, *then the derived algebra of* $m_p{}^q(\Gamma)$ *is* $M_p{}^2(\Gamma) \cap m_p{}^q(\Gamma)$.

2.4 *Remark.* For $1 \leqq p \leqq \infty$, $q > 2$, let $M(p, S^q)$ denote the set of (p, q) multipliers φ for which $T_\varphi(L^p) \subset S^q$, and let $m(p, S^q)$ denote the subspace for which T_φ is compact as an operator into S^q. Then the results of this section all hold, with $M_p{}^2(\Gamma)$ replaced by $M(p, S^q)$ and $m_p{}^2(\Gamma)$ replaced by $m(p, S^q)$. The proofs are identical, since all properties of $L^2(G)$ used above are valid for $S^q(G)$ as well. (See [**1**] and [**2**] for details about S^q.)

2.5 *Remark.* Let $1 \leqq p \leqq q < \infty$. Since $L^2(G)$, $L^p(G)$, and $S^q(G)$ are reflexive homogeneous Banach spaces, by [**2**, Theorem 3.8] $m_p{}^2(\Gamma)^{**} = M_p{}^2(\Gamma)$ and $m(p, S^q)^{**} = M(p, S^q)$. In view of Theorems 2.1, 2.2, and the above Remark, this means that, in every case, the derived algebra of $M_p{}^q(\Gamma)$ is the double dual of the (p, q) multipliers with unconditionally converging Fourier series.

Let $1 < p \leqq q < 2$, $1/p + 1/p' = 1$. Now $M_p{}^2(\Gamma) \not\subset C_0(\Gamma)$, since the characteristic function of a $\Lambda_{p'}$ set is in $M_p{}^2(\Gamma)$ [**13**, Theorem 37.9]. In addition, $m_p{}^q(\Gamma) \subset C_0(\Gamma)$. Thus

$$m_p{}^q(\Gamma) \cap M_p{}^2(\Gamma) \neq M_p{}^2(\Gamma) = m_p{}^2(\Gamma)^{**},$$

so Corollary 2.3 shows that the derived algebra of $m_p{}^q(\Gamma)$ is not the double dual of the compact (p, q) multipliers with unconditionally converging Fourier series. The example of the next section shows that for $G = \mathbf{T}$, the derived algebra does not coincide with the unconditionally converging compact (p, q) multipliers either, that is,

$$m_p{}^2(\mathbf{Z}) \underset{\neq}{\subset} m_p{}^q(\mathbf{Z}) \cap M_p{}^2(\mathbf{Z}), \quad 1 < p \leqq q < 2.$$

3. An example. We now give an example of a multiplier on $\mathbf{Z}$ which helps clarify the relationship between some of the spaces mentioned in the previous section. Throughout we assume that $1 < p < 2$ and that $r = 2p/(2-p)$.

For $n = 0, 1, \ldots$ define ψ_n on $\mathbf{Z}$ by

$$\psi_n(k) = \begin{cases} \dfrac{1}{2^{n/r}}, & k = 2^n, 2^n + 1, \ldots, 2^{n+1} - 1 \\ 0, & \text{otherwise,} \end{cases}$$

and let

$$\psi(k) = \sum_{n=0}^{\infty} \psi_n(k), \quad k \in \mathbf{Z}.$$

The following proposition is due to Haskell Rosenthal.

3.1 Proposition. *The function ψ is in $M_p{}^2(\mathbf{Z}) \cap m_p{}^q(\mathbf{Z})$, $p \leqq q < 2$, but not in $m_p{}^2(\mathbf{Z})$.*

Proof. Let φ be the example constructed in [**7**, Theorem *B*]. Then $\varphi \in C_0(\mathbf{Z}) \cap M_p{}^p(\mathbf{Z})$ but $\varphi \notin m_p{}^p(\mathbf{Z})$. Thus $\varphi \notin m_p{}^2(\mathbf{Z})$ The proof of Theorem *B* shows that φ is actually in $M_p{}^2(\mathbf{Z})$ and that $\psi = |\varphi|$. Since $\psi = a\varphi$ and $\varphi = b\psi$, where a and b are both sequences of absolute value one, it is clear that $\psi \in M_p{}^2(\mathbf{Z})$ and that $\psi \notin m_p{}^2(\mathbf{Z})$.

It remains to show that $\psi \in m_p{}^q(\mathbf{Z})$, $p \leqq q < 2$. By Interpolation Theory (see e.g. [**15**, p. 36]) it is enough to show that $\psi \in m_p{}^p(\mathbf{Z})$. Let μ_n denote the characteristic function of $\{2^n, \ldots, 2^{n+1} - 1\}$, $n = 0, 1, \ldots$. Since $p > 1$, the M. Riesz and Littlewood-Paley Theorems [**17**, p. 217; **18**, p. 224] imply that (μ_n) is a uniformly bounded sequence in $M_p{}^p(\mathbf{Z})$. Thus

$$\sum_{n=0}^{\infty} \frac{1}{2^{n/r}} ||\mu_n||_{(p,p)} < \infty.$$

Now

$$\psi_n = \frac{1}{2^{n/r}} \mu_n,$$

so $\Sigma_{n=0}^{\infty} \psi_n$ converges to ψ in $M_p{}^p(\mathbf{Z})$. Since each $\psi_n \in C_c(\mathbf{Z})$, $\psi \in m_p{}^p(\mathbf{Z})$.

Let $1/s = 1 + 1/q - 1/p$. Then Young's Inequality states that $L^s * L^p \subset L^q(\mathbf{T})$. Hence $L^s(\mathbf{T})^\wedge \subset M_p{}^q(\mathbf{Z})$, and since the trigonometric polynomials are dense in $L^s(\mathbf{T})$, $L^s(\mathbf{T})^\wedge \subset m_p{}^q(\mathbf{Z})$. In particular, if $s = r' =$

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$2p/(3p-2)$, then $q = 2$, so that $L^s(\mathbf{T})^\wedge \subset m_p^2(\mathbf{Z})$. Hence $\psi \notin L^s(\mathbf{T})^\wedge$. However, we do have:

3.2 PROPOSITION. *If* $1 \leqq s < 2p/(3p-2)$, *then* $\psi \in L^s(\mathbf{T})^\wedge$, *and hence* $M_p^2(\mathbf{Z}) \cap L^s(\mathbf{T})^\wedge \not\subset m_p^2(\mathbf{Z})$.

Proof. Let

$$f_n(x) = \sum_{k=2^n}^{2^{n+1}-1} \frac{1}{2^{n/r}} e^{ikx}, \quad n = 0, 1, \ldots .$$

We will show that $\Sigma_{n=0}^{\infty} ||f_n||_s < \infty$. Hence $\Sigma_{n=0}^{\infty} f_n$ converges in $L^s(\mathbf{T})$ to (say) f, and $\hat{f} = \psi$. Whence the conclusion follows.

Now

$$f_n(x) = \frac{1}{2^{n/r}} \{e^{i2^n x} D_{2^n}(x) - e^{i2^{n+1}x}\}$$

where $D_N(x)$ denotes the N-th Dirichlet kernel. Since $p < 2$ we may assume $s > 1$. Thus $||D_n||_s = O(N^{1/s'})$, and hence

$$||f_n||_s = O\left(\frac{(2^n)^{1/s'}}{2^{n/r}}\right) = O(2^{n(1/s'-1/r)}).$$

Since $s < 2p(3p-2) = r'$, $s' > r$, so

$$\sum_{n=0}^{\infty} 2^{n(1/s'-1/r)} < \infty.$$

Thus $\Sigma_{n=0}^{\infty} ||f_n||_s < \infty$.

3.3 *Remark.* Results analogous to those of this section hold when Γ is an infinite discrete torsion group of bounded order (see [**7**, Theorem D; **9**, p. 92; **16**]).

For Γ a discrete abelian group, Γ_1 a subgroup of Γ, and $1 \leqq p \leqq q$, let

$$i(\varphi)(\gamma) = \begin{cases} \varphi(\gamma), & \gamma \in \Gamma_1 \\ 0, & \gamma \notin \Gamma_1 \end{cases} \quad (\varphi \in M_p^q(\Gamma_1)),$$

and let $r(\varphi) = \varphi|\Gamma_1$, $\varphi \in M_p^q(\Gamma)$. Then i maps $M_p^q(\Gamma_1)$ into $M_p^q(\Gamma)$ and r maps $M_p^q(\Gamma)$ into $M_p^q(\Gamma_1)$ [**9**, Lemma 4.6]. Since $i(C_c(\Gamma_1)) \subset C_c(\Gamma)$, $r(C_c(\Gamma)) \subset C_c(\Gamma_1)$, and i and r are continuous, we see that $i(m_p^q(\Gamma_1)) \subset m_p^q(\Gamma)$ and $r(m_p^q(\Gamma)) \subset m_p^q(\Gamma_1)$. Since ri is the identity on $M_p^q(\Gamma_1)$, this means that $\varphi \in m_p^q(\Gamma_1)$ if and only if $i\varphi \in m_p^q(\Gamma)$. Thus if Γ contains $\mathbf{Z}$ or an infinite torsion group of bounded order, then results analogous to those of this section also hold for Γ.

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