Canad. Math. Bull. 2025, pp. 1–6
http://dx.doi.org/10.4153/S000843952510132X

A short proof of the Hilton–Milner Theorem

Denys Bulavka and Russ Woodroofe

Abstract. We give a short and relatively elementary proof of the Hilton–Milner Theorem.

The Hilton–Milner Theorem gives the maximum size of a uniform pairwise-intersecting family of sets that do not share a common element.

Theorem 1 (Hilton and Milner 1967 [9]) *Let $k \leq n/2$. If $\mathcal{F}$ is a family of pairwise-intersecting k-element subsets of $[n]$, where $\bigcap_{F \in \mathcal{F}} F = \varnothing$, then $|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-1-k}{k-1} + 1$.*

In the current article, we will show Theorem 1 to follow quickly from the following theorem, which we believe to be of some independent interest. Two set systems $\mathcal{A}$ and $\mathcal{B}$ are *cross-intersecting* if for every $A \in \mathcal{A}$, $B \in \mathcal{B}$, the intersection $A \cap B$ is nonempty. The *shadow* ∂B of a k-element set B consists of all the $(k-1)$-element subsets of B; the shadow of a uniform set family is the union of the shadows of its constituent sets, thus consists of all $(k-1)$-element subsets of constituent sets.

Theorem 2 *Let $2k - 1 \leq n$, let $\mathcal{A}$ be a family of $(k-1)$-element subsets of $[n]$, and let $\mathcal{B}$ be a family of k-element subsets of $[n]$. If $\mathcal{A}$ and $\mathcal{B}$ are cross-intersecting, $\mathcal{B}$ is nonempty, and $\partial\mathcal{B} \subseteq \mathcal{A}$, then*

$$|\mathcal{A}| + |\mathcal{B}| \leq \binom{n}{k-1} - \binom{n-k}{k-1} + 1.$$

Note that the bound of Theorem 1 is attained with a single k-element set B that does not contain 1, together with all the k-element sets that contain 1 and intersect B; the bound of Theorem 2 is attained with a single k-element set and all $(k-1)$-element sets that intersect it.

Our proof of Theorem 1 may be viewed as injective. Other recent proofs of Theorem 1 were given in [4, 10], but instead of relying on a simple cross-intersecting type theorem, both of these proofs rely on a certain "partial complement" operation. In somewhat older work [6] (see also [3]), Frankl and Tokushige gave a proof of Hilton–Milner from a different cross-intersection theorem, but the proof is less

Received by the editors July 1, 2025; accepted October 7, 2025.
Published online on Cambridge Core October 17, 2025.
Work of the first author is partially supported by the Israel Science Foundation grant ISF-2480/20. Work of the second author is supported in part by the Slovenian Research Agency (research program P1-0285 and research projects J1-9108, J1-2451, J1-3003, and J1-50000).
AMS subject classification: **05D05**.
Keywords: intersecting set-system, shifting, shadow.

elementary than that of Theorem 2, requiring the Schützenberger–Kruskal–Katona Theorem. A recent preprint of Wu, Li, Feng, Liu, and Yu [12] gives a proof based on still another cross-intersecting theorem, but the existing proofs of this underlying result also seem to be somewhat more difficult than our approach.

1 Shifting

We recall that a system $\mathcal{F}$ of subsets of $[n]$ is *shifted* if for each $i < j$, whenever $F \in \mathcal{F}$, $j \in F$, and $i \notin F$, then also $(F \backslash j) \cup i \in \mathcal{F}$. Here, we abuse notation to identify i, j with the singleton subsets $\{i\}, \{j\}$, where it causes no confusion. The *(combinatorial) shifting operation* $\mathrm{Shift}_{i\leftarrow j}$ is defined as

$$\begin{aligned}\mathrm{Shift}_{i\leftarrow j}\,\mathcal{F} = \{&F \in \mathcal{F} : j \notin F \text{ or } i \in F \text{ or } (F\backslash j) \cup i \in \mathcal{F}\}\\ &\cup \{(F\backslash j) \cup i : F \in \mathcal{F} \text{ is such that } j \in F, i \notin F\}.\end{aligned}$$

It is well-known that repeated applications of $\mathrm{Shift}_{i\leftarrow j}$ over $i < j$ will eventually reduce an arbitrary set system to a shifted set system, that the operation preserves the cross-intersecting property, and that $\partial\,\mathrm{Shift}_{i\leftarrow j}\,\mathcal{F} \subseteq \mathrm{Shift}_{i\leftarrow j}\,\partial\mathcal{F}$ [1, 2, 7, 8].

2 Proof of Theorem 2

We carry out a straightforward induction on n.

Proof If $n = 2k - 1$, then the upper bound is $\binom{n}{k-1}$, and the result follows by noticing that if a $(k-1)$-element set is in $\mathcal{A}$, then its complement cannot be in $\mathcal{B}$ (and vice-versa).

For the inductive step, we may assume that $\mathcal{A}$ and $\mathcal{B}$ are shifted; otherwise, shift. Let $\mathcal{A}(\neg n)$ and $\mathcal{B}(\neg n)$ consist of the subsets in $\mathcal{A}$ and $\mathcal{B}$ (respectively) that do not contain n. It is immediate that $\mathcal{A}(\neg n)$ and $\mathcal{B}(\neg n)$ are shifted, cross-intersecting, and satisfy the shadow condition. Let $\mathcal{A}(n)$ and $\mathcal{B}(n)$ be obtained by taking the families consisting of the subsets in $\mathcal{A}$ and $\mathcal{B}$ that contain n, then deleting n from each subset. It follows quickly from definitions that $\mathcal{A}(n)$ and $\mathcal{B}(n)$ are shifted, cross-intersecting, and satisfy the shadow condition.

As $\mathcal{A}$ and $\mathcal{B}$ are shifted, so $\mathcal{A}(\neg n)$ and $\mathcal{B}(\neg n)$ are nonempty, and hence by induction

$$|\mathcal{A}(\neg n)| + |\mathcal{B}(\neg n)| \le \binom{n-1}{k-1} - \binom{n-1-k}{k-1} + 1. \tag{1}$$

For $\mathcal{A}(n), \mathcal{B}(n)$, there are a few easy cases:

If $\mathcal{A}(n)$ is empty, then (by the shadow condition) also $\mathcal{B}(n)$ is empty.

If $\mathcal{B}(n)$ is empty, then since $\mathcal{B}$ is nonempty and shifted, we have $\{1, \dots, k\} \in \mathcal{B}$. Since every set in $\mathcal{A}(n)$ intersects with $\{1, \dots, k\}$, we get $|\mathcal{A}(n)| + |\mathcal{B}(n)| = |\mathcal{A}(n)| \le \binom{n-1}{k-2} - \binom{n-1-k}{k-2}$.

If $\mathcal{B}(n)$ is nonempty, then by induction it holds that

$$\begin{aligned}|\mathcal{A}(n)| + |\mathcal{B}(n)| &\le \binom{n-1}{k-2} - \binom{n-1-(k-1)}{k-2} + 1\\ &\le \binom{n-1}{k-2} - \binom{n-1-k}{k-2}.\end{aligned} \tag{2}$$

The result now follows from (1), the bound on $|\mathcal{A}(n)| + |\mathcal{B}(n)|$, and Pascal's Triangle identity. ■

3 Proof of Theorem 1

We will use the following lemma of Frankl and Füredi.

Lemma 3 (Essentially Frankl and Füredi [5]) *If $\mathcal{F}$ is a pairwise-intersecting family of k-element subsets of $[n]$ with $\bigcap_{F\in\mathcal{F}} F = \varnothing$, then there is a shifted family $\mathcal{F}'$ satisfying the same properties and with $|\mathcal{F}'| \geq |\mathcal{F}|$.*

Proof Given the lemma, the proof of Theorem 1 is nearly immediate. Let $\mathcal{F}$ be a shifted family satisfying the conditions of the theorem. Define

$$\begin{aligned} \mathcal{A} &= \{F \backslash 1 \ : F \in \mathcal{F} \text{ with } 1 \in \mathcal{F}\} \\ \mathcal{B} &= \{F \quad : F \in \mathcal{F} \text{ with } 1 \notin \mathcal{F}\}. \end{aligned}$$

Since $\mathcal{F}$ is shifted, if $F \in \mathcal{F}$ does not have 1, then $(F\backslash i) \cup 1 \in \mathcal{F}$ for each $i \in \mathcal{F}$. It follows that $\partial\mathcal{B} \subseteq \mathcal{A}$. Since $\mathcal{F}$ is intersecting, also $\mathcal{A}, \mathcal{B}$ are cross-intersecting systems of subsets of $\{2, \ldots, n\}$. Since $\mathcal{F}$ has empty intersection, both of $\mathcal{A}$ and $\mathcal{B}$ are nonempty. The desired bound is now immediate from Theorem 2. ■

Remark 4 This proof requires only the special case of Theorem 2 where the set systems are shifted.

4 Proof of Lemma 3

For completeness, we also prove the lemma.

Proof Given $\mathcal{F}$ as in Theorem 1, apply shifting operations $\mathrm{Shift}_{i\leftarrow j}$. Each such operation preserves the pairwise-intersecting property and cardinality, but may or may not result in a system with a common element of intersection.

If a sequence of shifting operations ends in a shifted system with empty intersection, then we are certainly done.

Otherwise, some $\mathrm{Shift}_{i_0\leftarrow j_0}$ results in a system where every set contains i_0. Thus, before this step, we have a system $\mathcal{F}$ where every set contains either i_0 or j_0. Relabel i_0 to 1 and j_0 to 2, and continue applying $\mathrm{Shift}_{i\leftarrow j}$ operations over all $3 \leq i < j$. Thus, after these additional shift operations, we have $\{1, 3, \ldots, k+1\}$ and $\{2, 3, \ldots, k+1\}$ in the system. Without loss of generality (since every set in $\mathcal{F}$ contains 1 or 2), we also have all k-element subsets containing $\{1, 2\}$; otherwise, add them. Thus, we have $\partial\{1, \ldots, k+1\}$ contained in our system. As $\partial\{1, \ldots, k+1\}$ has empty intersection and is preserved under all further shift operations (over $1 \leq i < j$), the result follows. ■

5 Discussion

In addition to being short and direct, our proof is relatively elementary, using only shifting theory. Indeed, we recover a completely elementary proof of the restriction of the Hilton–Milner Theorem to shifted systems.

A main difficulty in proofs of Hilton–Milner and/or Erdős–Ko–Rado type results is relating systems of $(k-1)$-element subsets to systems of k-element subsets. Our approach handles this with the shadow containment condition of Theorem 2.

Our motivation here comes partly from combinatorial algebraic topology. In particular, the simplicial complex generated by a shifted family of k-element sets has homology with generators in $\mathcal{B}$ (using notation as in the proof of Theorem 1). Thus, Lemma 3 transforms the combinatorial property of empty intersection into a homological property. Kalai comments on similar connections between intersection theorems and homology in [11, Section 6.4].

The approach also gives a unified proof of the well-known Erdős–Ko–Rado Theorem. More concretely, if we relax the hypothesis of Theorem 2 to allow $\mathcal{B}$ to be empty, then the corresponding bound is $|\mathcal{A}| + |\mathcal{B}| \leq \binom{n-1}{k-1}$. Erdős–Ko–Rado now follows from replacing Theorem 2 with the relaxed cross-intersection theorem in the proof of Theorem 1. The proof is similar to (and only slightly more complicated than) the standard inductive proof of Erdős–Ko–Rado for shifted systems.

The approach also recovers uniqueness of the largest family for Theorem 1 when $n/2 > k \geq 4$. Here, we strengthen the hypothesis of Theorem 2 to require $\mathcal{B}$ to have at least two elements. We discuss the details in the following section.

6 Uniqueness of the Hilton–Milner family

As mentioned in the discussion, the same techniques give uniqueness of the maximum family in Theorem 1. We prove the following.

Theorem 5 *In the situation of Theorem 1, if $4 \leq k < n/2$ and $|\mathcal{F}|$ achieves the upper bound, then there is some k-set B and $i \notin B$ so that $\mathcal{F}$ consists of B together with all k-sets that both contain i and intersect B.*

We require $k \geq 4$ in order to avoid some technicalities. In particular, there is another family achieving the bound for $k = 3$. See [10] for more details and a different argument.

As in the proof of Theorem 1, we reduce to a shifted family, and prove for a shifted family.

The proof for a shifted family requires a completely straightforward modification of Theorem 2. We obviously require $k \geq 4$. We also strengthen the hypothesis to require $|\mathcal{B}| \geq 2$, replacing the condition that $|\mathcal{B}| \geq 1$; with the strengthened hypothesis, the inequality is strict. Then in the proof, we may have $|\mathcal{B}(n)|$ empty or nonempty. If empty, then since $\mathcal{B}$ has at least two elements, so $\mathcal{A}(n)$ is strictly smaller than the given bound. If nonempty, then the bound in (2) is already strict so long as $k \geq 4$. In either case, the induction step yields a strict inequality.

Theorem 5 follows for shifted families by applying the variant of Theorem 2 with $|\mathcal{B}| \geq 2$ to the same families as in the proof of Theorem 1.

It remains only to reduce to shifted families. This reduction requires a bit of care. We did not find the following lemma in the literature, although we believe it to be known to experts in the field.

Lemma 6 *Let $\mathcal{F}$ be a family of pairwise-intersecting k-element subsets of $[n]$ with the additional property that for any $F_0 \in \mathcal{F}$, the intersection $\bigcap_{\mathcal{F}\setminus\{F_0\}} F$ is empty. Then there is a shifted family $\mathcal{F}'$ satisfying the same properties and with $|\mathcal{F}'| \geq |\mathcal{F}|$.*

Proof. By the *standard family*, we mean the shifted family with $A = \{2, \dots, k+1\}$, $A' = \{2, \dots, k, k+2\}$, and all k-element sets that both contain 1 and intersect A and A'. It is obvious that the standard family is at least as large as any family where all but two sets contain 1.

Given $\mathcal{F}$, we perform a sequence of shifts. If these terminate in a shifted family with the desired properties, then we are done. Otherwise, an operation results in a family without the additional property. Stopping just before this operation and relabeling elements, we have a family containing sets with 1 and not 2, with 2 and not 1, with both 1 and 2, and possibly the set $B = \{3, \dots, k+2\}$.

We may assume without loss of generality that we have all sets containing both 1 and 2 and intersecting with B. Since these sets do not have any common intersection other than 1 and 2, the operations $\mathrm{Shift}_{i \leftarrow j}$ over all $3 \leq i < j$ preserve the additional property.

After shifting over $3 \leq i < j$, if we have only one set with 1 and not 2, or only one set with 2 and not 1, then we replace with the standard family. Otherwise, we have in the family $\{a, 3, \dots, k+1\}$ and $\{a, 3, \dots, k, k+2\}$ for $a = 1, 2$, along with all sets containing $\{1, 2\}$ and intersecting B. In particular, the family contains as subfamilies both $\partial\{1, \dots, k+1\}$ and $\partial\{1, \dots, k, k+2\}$. Both subfamilies have empty intersection and are preserved under all shift operations, so we can now shift until the system stabilizes. ■

Acknowledgements We particularly thank Dániel Gerbner for several helpful comments about preprints of the article. We also thank Peter Frankl, Balázs Patkós, John Shareshian, and Tamás Szőnyi.

References

[1] P. Frankl, *The shifting technique in extremal set theory*. In: C. Whitehead (ed.), Surveys in combinatorics 1987 (new cross, 1987), London Mathematical Society Lecture Note Series, 123, Cambridge University Press, Cambridge, 1987, pp. 81–110.

[2] P. Frankl, *Shadows and shifting*. Graphs Comb. 7(1991), no. 1, 23–29.

[3] P. Frankl, *New inequalities for cross-intersecting families*. Mosc. J. Comb. Number Theory **6**(2016), no. 4, 27–32.

[4] P. Frankl, *A simple proof of the Hilton-Milner theorem*. Mosc. J. Comb. Number Theory **8**(2019), no. 2, 97–101.

[5] P. Frankl and Z. Füredi, *Nontrivial intersecting families*. J. Combin. Theory Ser. A **41**(1986), no. 1, 150–153.

[6] P. Frankl and N. Tokushige, *Some best possible inequalities concerning cross-intersecting families*. J. Combin. Theory Ser. A **61**(1992), no. 1, 87–97.
[7] D. Gerbner and B. Patkós, *Extremal finite set theory*. In: Discrete mathematics and its applications (Boca Raton), CRC Press, Boca Raton, FL, 2019.
[8] J. Herzog and T. Hibi, *Monomial ideals*, Graduate Texts in Mathematics, 260, Springer-Verlag, London, 2011.
[9] A. J. W. Hilton and E. C. Milner, *Some intersection theorems for systems of finite sets*. Q. J. Math. **18**(1967), 369–384.
[10] G. Hurlbert and V. Kamat, *New injective proofs of the Erdős-ko-Rado and Hilton-Milner theorems*. Discret. Math. **341**(2018), no. 6, 1749–1754.
[11] G. Kalai, *Algebraic shifting*. In: T. Hibi (ed.), Computational commutative algebra and combinatorics (Osaka, 1999), Advanced Studies in Pure Mathematics, 33, Mathematical Society of Japan, Tokyo, 2002, pp. 121–163.
[12] Y. Wu, Y. Li, L. Feng, J Liu, and G. Yu, *Maximal intersecting families revisited*. Discrete Math. **349**(2026), no. 1, 114654.

Department of Mathematics and Statistics, Dalhousie University, Halifax, Canada
e-mail: denys.bulavka@dal.ca
URL: https://kam.mff.cuni.cz/~dbulavka/

Faculty for Mathematics, Natural science, and Information Technology, University of Primorska, Slovenia
e-mail: russ.woodroofe@famnit.upr.si
URL: https://osebje.famnit.upr.si/~russ.woodroofe/