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Alexandrov sphere theorems for $W^{2,n}$-hypersurfaces

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We prove that the proximal unit normal bundle of the subgraph of a $W^{2,n}$-function carries a natural structure of Legendrian *cycle*. This result is used to obtain an Alexandrov-type sphere theorem for hypersurfaces in $\mathbf{R}^{n+1}$, which are locally graphs of *arbitrary* $W^{2,n}$-functions. We also extend the classical umbilicality theorem to $W^{2,1}$-graphs, under the Lusin (N) condition for the graph map.

keywords: higher-order mean curvatures; Legendrian cycles; $W^{2,n}$-functions; Lusin (N) property; sphere theorem

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1. Introduction

1.1. Background and motivation

It is important and natural to understand if classical results of smooth differential geometry still hold if one weakens the regularity hypothesis. Since most of the classical differential-geometric techniques rely on some smoothness assumption, such a question often calls for substantial generalizations of the existing methods.

Our starting point in this paper is the following general version of the sphere theorem by Alexandrov [1].

Theorem (Alexandrov). A bounded and connected C^2-domain $\Omega \subseteq \mathbf{R}^{n+1}$ must be a round ball, provided there exist a C^1-function $\varphi : \mathbf{R}^n \to \mathbf{R}$ and $\lambda \in \mathbf{R}$ such that

$$\varphi(\chi_{\Omega,1}(p), \dots, \chi_{\Omega,n}(p)) = \lambda$$

and

$$\frac{\partial \varphi}{\partial t_i}(\chi_{\Omega,1}(p), \dots, \chi_{\Omega,n}(p)) > 0 \quad \text{for } i = 1, \dots, n, \tag{1.1}$$

for every $p \in \partial\Omega$. Here $\chi_{\Omega,1} \leq \dots \leq \chi_{\Omega,n}$ are the principal curvatures of $\partial\Omega$.

The simplest case of this result is the famous rigidity result for hypersurfaces with constant mean curvature. More generally, choosing $\varphi = \sigma_k$, where σ_k is the k-th elementary symmetric function (see definition 5.11), one can deduce (see [36, Appendix]) that if $\Omega \subseteq \mathbf{R}^{n+1}$ is a bounded C^2-domain such that $H_{\Omega,k} = \sigma_k(\chi_{\Omega,1}, \dots, \chi_{\Omega,n})$ is constant for some k, then Ω is a round ball. This result was proved by Alexandrov using the moving plane method. A completely different approach to treat the special case $\varphi = \sigma_k$ and based on integral identities that was found by Ros in [35] and Montiel–Ros [26].

It is natural to ask about generalization of the sphere theorem beyond the classical smooth regime. This problem was addressed by Alexandrov in [2], where he generalized the sphere theorem to bounded domains whose boundary can be locally represented as graph of C^1-functions with second-order distributional derivatives in L^n, and under the *uniform ellipticity condition*

$$0 < \mu_1 \leq \frac{\partial \varphi}{\partial t_i}(\chi_{\Omega,1}(p), \dots, \chi_{\Omega,n}(p)) \leq \mu_2 < \infty \quad \text{for } i = 1, \dots, n, \tag{1.2}$$

for $\mathcal{H}^n$ a.e. $p \in \partial\Omega$. Here $\chi_{\Omega,1} \leq \dots \leq \chi_{\Omega,n}$ are the weak principal curvatures of $\partial\Omega$. Obviously, (1.2) reduces to (1.1) for C^2-domains. The proof of this result is based on the generalization of the moving plane method by means of maximum principles for $W^{2,n}$-solutions of uniformly elliptic partial differential equations. Both the hypothesis of C^1-regularity and the *uniform ellipticity condition* (1.2) are important for the applicability of this method. On the other hand, it is natural to ask if these hypotheses are convenient conditions rather than necessary restrictions. Additionally, arbitrary $W^{2,n}$-functions exhibit very different behaviours than C^1-functions. For instance, T. Toro in [43] constructs a $W^{2,n}$-function with a (countable) dense subset of singular points, and J. Fu in [17] points out the existence of $W^{2,n}$-functions on $\mathbf{R}^n$ whose gradient has a dense graph in $\mathbf{R}^n \times \mathbf{R}^n$.

With these motivations in mind, in this paper, we generalize Alexandrov sphere theorem to *arbitrary $W^{2,n}$-domains* (i.e. open sets which are locally subgraphs of $W^{2,n}$-functions) when φ is a symmetric function of the weak principal curvatures. Moreover, we prove our result under a more general hypothesis than uniform ellipticity, namely the *degenerate ellipticity condition* (1.3), cf. theorem C. Instead of using a moving plane method, we extend the Montiel–Ros integral-geometric approach to prove our result. In recent years, Montiel–Ros argument has been generalized to some classes of non-smooth geometric sets, namely sets of finite perimeter with *bounded* distributional mean curvature (see [8] and [7]) and sets of positive reach (cf. [14]). On the other hand, the aforementioned examples show that $W^{2,n}$-domains exhibit some very different behaviour than the sets treated in [8], [7], or in [14]. As explained below, this requires a substantially novel approach.

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1.2. Legendrian cycles and sphere theorem

Here we discuss the generalization of the Montiel–Ros method (see [26]) to $W^{2,n}$-domains. In the smooth setting, this method is based on a clever combination of the Heintze–Karcher inequality (see [12] and [26]) with the variational formulae for the higher-order mean curvature integrals of a C^2-domain (cf. [13] and [31]). As noted by Fu (cf. [16]), the variational formulae are strictly related to the structure of Legendrian cycle carried by the unit-normal bundle of a smooth submanifold. We recall that an integer multiplicity locally rectifiable n-current T of $\mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$ with compact support in $\mathbf{R}^{n+1} \times \mathbf{S}^n$ is called a *Legendrian cycle* of $\mathbf{R}^{n+1}$ if and only if $\partial T = 0$ and $T \llcorner \alpha = 0$, where α is the contact form in $\mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$ (see §2). Indeed, it is a simple exercise to prove that the unit-normal bundle of a C^2-domain carries a natural structure of Legendrian cycle. It is also well known that if f is a $W^{2,n}$-function on an open subset U of $\mathbf{R}^{n+1}$ then there exists an integral current $\mathbb{D}(f)$ (also denoted by $[df]$) of $U \times \mathbf{R}^n$ with zero distributional boundary (i.e. an integral cycle), which serves as a substitute for the graph of the differential of f (see [15], [32, (4.1) p. 332], and [17]). This current is called *differential cycle of f*. The construction of this integral cycle can be naturally extended to construct a Legendrian cycle associated with the graph of *f*. However, this information alone is not sufficient to extend the Montiel–Ros method, as such extension seems to require a crucial geometric property for the carrier W of this Legendrian cycle: namely that the segments $a + tu$ with $t \geq 0$, at least for 'many' points $(a, u) \in W$, must be distance-minimizing segments near 0. This observation leads us to consider the proximal unit-normal bundle, which is defined for an arbitrary set $C \subseteq \mathbf{R}^{n+1}$ as

$$\mathrm{nor}(C) = \{(x, u) \in \overline{C} \times \mathbf{S}^n : \mathrm{dist}(x + su, C) = s \text{ for some } s > 0\}.$$

It is always true that $\mathrm{nor}(C)$ is a *Legendrian rectifiable set* of $\mathbf{R}^{n+1} \times \mathbf{S}^n$ (see definition 2.1), namely it can be $\mathcal{H}^n$ almost everywhere covered by a countable union of n-dimensional C^1-submanifold of $\mathbf{R}^{n+1} \times \mathbf{S}^n$ and the contact form α vanishes on the approximate tangent plane of $\mathrm{nor}(C)$ at $\mathcal{H}^n$ almost all points; cf. lemma 2.11. On the other hand, it is *not* always true that $\mathcal{H}^n \llcorner \mathrm{nor}(C)$ is a Radon measure over $\mathbf{R}^{n+1} \times \mathbf{S}^n$ (for instance even when C coincides with the closure of a smooth submanifold with bounded mean curvature, cf. lemma A.3), henceforth, $\mathrm{nor}(C)$ cannot always carry an integer-multiplicity rectifiable current.

Our first and central result asserts that the proximal unit normal bundle of a $W^{2,n}$-domain carries a natural structure of Legendrian cycle. More precisely, denoting by $\pi_0 : \mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ the projection onto the first factor, cf. (2.2), and by $E^{'}$ a volume form of $\mathbf{R}^{n+1}$, cf. (2.5), we prove the following result.

Theorem A (cf. theorem 3.9 and theorem 5.7). *If $\Omega \subseteq \mathbf{R}^{n+1}$ is a bounded $W^{2,n}$-domain then $\mathcal{H}^n(\mathrm{nor}(\Omega)) < \infty$ and there exists a unique n-dimensional Legendrian cycle T such that*

$$T = (\mathcal{H}^n \llcorner \mathrm{nor}(\Omega)) \wedge \eta,$$

where η is a $\mathcal{H}^n \llcorner \mathrm{nor}(\Omega)$ measurable n-vectorfield such that

$$|\eta(x, u)| = 1, \quad \eta(x, u) \text{ is simple},$$

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$$\mathrm{Tan}^n(\mathcal{H}^n \llcorner \mathrm{nor}(\Omega), (x,u)) \text{ is associated with } \eta(x,u)$$

and

$$\langle [\textstyle\bigwedge_n \pi_0](\eta(x,u)) \wedge u, E' \rangle > 0$$

for $\mathcal{H}^n$ a.e. $(x,u) \in \mathrm{nor}(\Omega)$. We write $T = N_\Omega$.

A simple functional-analytic reformulation of theorem A, which can be proved along the same lines as theorem 3.9, states what follows.

THEOREM B. *Suppose $f \in W^{2,n}(U)$. Then $\mathbb{D}(f)$ is a unit-density integral cycle carried over $\{(x, \nabla f(x)) : x \in \mathcal{S}(f)\}$, where $\mathcal{S}(f)$ is the set of pointwise* twice-*differentiabily points of f (cf.* definition 2.15*), and ∇f is the pointwise gradient of f.*

In this direction, we recall that $\mathcal{L}^n(U \setminus \mathcal{S}(f)) = 0$ by lemma 2.16, and the subtlety of theorem B becomes more transparent if we observe that $\mathcal{S}(f)$ cannot be replaced by the set of pointwise differentiability points $\mathrm{Diff}(f)$ of f: indeed, there exists $f \in C^1\big([-1,1]^n\big) \cap W^{2,n}\big((-1,1)^n\big)$ such that

$$[-1,1]^n \subseteq \nabla f([-1,1] \times \{0\}^{n-1}),$$

cf. [37], and denoting by $\overline{\nabla f}$ the graph map of the gradient of f, we use lemma 3.5 to conclude that $\mathcal{H}^n\big(\overline{\nabla f}(U) \setminus \overline{\nabla f}(\mathcal{S}(f))\big) > 0$.

Combining theorem A with the variational formulae for Legendrian cycles in [16], we can extend Reilly variational formulae (see [31]) to $W^{2,n}$-domains, whence we deduce the Minkowski–Hsiung formulae in our setting (see theorems 5.15 and 5.17). Moreover, the Heintze–Karcher inequality for $W^{2,n}$-domains (see theorem 6.1) can be deduced from the general inequality [14, Theorem 3.20] employing some of the structural properties of the proximal unit-normal bundle (see theorem 5.7(2)–(3)), that already play a role in theorem A.

Combining these results we can eventually prove our generalization of Alexandrov sphere theorem.

THEOREM C (cf. theorem 6.2 and remark 6.3). *A bounded and connected $W^{2,n}$-domain $\Omega \subseteq \mathbf{R}^{n+1}$ must be a round ball, provided there exist $k \in \{2, \dots, n\}$ and $\lambda \in \mathbf{R}$ such that*

$$\sigma_k(\chi_{\Omega,1}(p), \dots, \chi_{\Omega,n}(p)) = \lambda$$

and

$$\frac{\partial \sigma_k}{\partial t_i}(\chi_{\Omega,1}(p), \dots, \chi_{\Omega,n}(p)) \geq 0 \quad \text{for } i = 1, \dots, n \tag{1.3}$$

for $\mathcal{H}^n$ a.e. $p \in \partial\Omega$.

If $k = 1$ the result would reduce to the smooth Alexandrov's sphere theorem for constant mean curvature hypersurfaces, since the condition $H_{\Omega,1}(z) = \lambda$ for $\mathcal{H}^n$ a.e. $z \in \partial\Omega$ implies that $\partial\Omega$ is smooth by Allard's regularity theorem (notice that by theorem 5.15 the function $H_{\Omega,1}$ is the generalized mean curvature of $\partial\Omega$ in the sense of varifolds, see [45]). No analogous regularity result is available when $k \geq 2$.

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1.3. The support of Legendrian cycles

Theorem A finds natural application in other problems, beyond the rigidity questions that we have considered so far. In §4, we employ it to answer a question implicit in [29]. In [29, Remark 2.3], the authors asked if there exist n-dimensional Legendrian cycles in $\mathbf{R}^{n+1}$ whose support is not locally $\mathcal{H}^n$-rectifiable or even has positive $\mathcal{H}^{n+1}$-measure. Combining theorem A with an observation by J. Fu in [17] about the existence of $W^{2,n}$-functions whose differential has a graph dense in $\mathbf{R}^n \times \mathbf{R}^n$, we prove the following result.

THEOREM D. *There exist n-dimensional Legendrian cycles of* $\mathbf{R}^{n+1}$ *whose support has positive* $\mathcal{H}^{2n}$*-measure.*

1.4. The Nabelpunktsatz

In the final section of this paper, we study the problem of extending the classical umbilicality theorem (or Nabelpunktsatz). The classical proof of this theorem works for hypersurfaces that are at least C^3-regular. A proof for C^2-hypersurfaces is given in [42] (see also [28] and [22]). Considering more general hypersurfaces with curvatures defined only almost everywhere, the question about the validity of the Nabelpunktsatz goes back to the classical paper of Busemann and Feller [4], where they also pointed out the existence of non-spherical convex C^1-hypersurfaces which are umbilical at almost every point; see also remark 7.8. In [7], the Nabelpunksatz is extended to $C^{1,1}$-hypersurfaces. Here we obtain the following far-reaching generalization of this result.

THEOREM E (cf. theorem 7.6). *The Nabelpunktsatz holds for almost everywhere umbilical* $W^{2,1}$*-graphs, provided the Lusin condition (N) holds for the graph function* (*cf.* definition 7.5).

The Lusin condition (N) is necessary in order to guarantee the existence of weak curvatures on the graphs (i.e. second-order rectifiability) and it is automatically verified for graphs of $W^{2,p}$-functions, with $p > \frac{n}{2}$; cf. remark 7.7.

2. Notation and background

Given a set of parameters $\{p_1, p_2, \dots p_n\}$, we denote a *generic* positive constant depending only $p_1, \dots, p_n$ by $c(p_1, \dots, p_n)$.

If $f : S \to T$ is a function we define

$$\overline{f} : S \to S \times T, \qquad \overline{f}(x) = (x, f(x)). \tag{2.1}$$

The characteristic function of a set X is $\mathbf{1_X}$ and the Grassmannian of m-dimensional subspaces of $\mathbf{R}^k$ is $\mathbf{G}(k, m)$. Moreover, we often use the following projection maps

$$\pi_0 : \mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \to \mathbf{R}^{n+1} \quad \pi_1 : \mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \to \mathbf{R}^{n+1} \tag{2.2}$$

defined as $\pi_0(x, u) = x$ and $\pi_1(x, u) = u$.

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In this paper, we use the symbol $\bullet$ to denote *scalar products*. In particular, we fix a scalar product $\bullet$ on $\mathbf{R}^{n+1}$,

$$\text{an orthonormal basis } e_1, \ldots, e_{n+1} \text{ of } \mathbf{R}^{n+1} \text{ and its dual basis } e_1', \ldots, e_{n+1}'. \tag{2.3}$$

For a subset S of an Euclidean space, $\overline{S}$ is the closure of S. We use the symbols D and ∇ for the classical differential and the gradient. If $f : U \to \mathbf{R}$ is a continuous function defined on an open set U, we denote the set of $x \in U$, where f is pointwise differentiable by

$$\mathrm{Diff}(f).$$

2.1. Basic notions from geometric measure theory

In this paper, we use standard notation from geometric measure theory, for which we refer to [10]. For the reader's convenience, we recall some basic notions here.

For a subset $X \subseteq \mathbf{R}^m$ and a positive integer μ, we define $\mathrm{Tan}^\mu(\mathcal{H}^\mu \llcorner X, a)$ to be the set of all $v \in \mathbf{R}^m$ such that

$$\Theta^{*\mu}\big(\mathcal{H}^\mu \llcorner X \cap \{x : |r(x-a) - v| < \epsilon \text{ for some } r > 0\}, a\big) > 0$$

for every $\epsilon > 0$. This is a cone with vertex at 0 and we set

$$\mathrm{Nor}^\mu(\mathcal{H}^\mu \llcorner X, a) = \{v \in \mathbf{R}^m : v \bullet u \leq 0 \text{ for } u \in \mathrm{Tan}^\mu(\mathcal{H}^\mu \llcorner X, a)\}.$$

Suppose $X \subseteq \mathbf{R}^m$ and f maps a subset of $\mathbf{R}^m$ into $\mathbf{R}^k$. Given a positive integer μ and $a \in \mathbf{R}^m$, we say that f is $\mathcal{H}^\mu \llcorner X$ *approximately differentiable at* a (cf. [10, 3.2.16]) if and only if there exists a map $g : \mathbf{R}^m \to \mathbf{R}^k$ pointwise differentiable at a such that

$$\Theta^\mu(\mathcal{H}^\mu \llcorner X \cap \{b : f(b) \neq g(b)\}, a) = 0.$$

In this case, (see [10, 3.2.16]) f determines the restriction of $Dg(a)$ on the approximate tangent cone $\mathrm{Tan}^\mu(\mathcal{H}^\mu \llcorner X, a)$ and we define

$$\mathrm{ap}\, Df(a) = Dg(a)|\, \mathrm{Tan}^\mu(\mathcal{H}^\mu \llcorner X, a).$$

Suppose $X \subseteq \mathbf{R}^m$ and μ is a positive integer. We say that X is *countably $\mathcal{H}^\mu$-rectifiable* if there exist countably many μ-dimensional C^1-submanifolds Σ_i of $\mathbb{R}^m$ such that

$$\mathcal{H}^\mu\big(X \setminus \bigcup_{i=1}^{\infty} \Sigma_i\big) = 0.$$

It is well known that *if X is countably $\mathcal{H}^\mu$-rectifiable with $\mathcal{H}^\mu(X) < \infty$, then $\mathrm{Tan}^\mu(\mathcal{H}^\mu \llcorner X, a)$ is a μ-dimensional plane at $\mathcal{H}^\mu$ a.e. $a \in X$, and every Lipschitz function $f : X \to \mathbf{R}^k$ has an $\mathcal{H}^\mu \llcorner X$-approximate differential* $\mathrm{ap}\, Df(a)$: $\mathrm{Tan}^\mu(\mathcal{H}^\mu \llcorner X, a) \to \mathbb{R}^k$ *at $\mathcal{H}^\mu$ a.e. $a \in X$*. At such points a we define for $h \in \{1, \ldots, k\}$ the h-dimensional approximate Jabobian of f

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$$J_h^X f(a) = \sup\left\{\left|[\textstyle\bigwedge_h \operatorname{ap} Df(a)](\xi)\right| : \xi \in \textstyle\bigwedge_h \operatorname{Tan}^\mu(\mathcal{H}^\mu \llcorner X, a),\ |\xi| = 1\right\}$$

(see (2.4) for the definition of $\bigwedge_h \operatorname{ap} Df(a)$). The approximate Jacobian naturally appears in area and coarea formula for f; see [10, 3.2.20, 3.2.22].

2.2. Differential forms and currents

Let V be a vector space. We denote by $v_1 \wedge \cdots \wedge v_m$ the simple m-vector obtained by the exterior multiplication of vectors $v_1, \ldots, v_m$ in V and $\bigwedge_m V$ is the vector space generated by all simple m vectors of V. Each linear map $f : V \to V'$ can be uniquely extended to a linear map

$$\textstyle\bigwedge_m f : \bigwedge_m V \to \bigwedge_m V' \tag{2.4}$$

such that $\bigwedge_m f(v_1 \wedge \cdots \wedge v_m) = f(v_1) \wedge \cdots \wedge f(v_m)$ for every $v_1, \ldots, v_m \in V$ (cf. [10, 1.3.1–1.3.3]).

The vector space of all alternating m-linear functions $f : V^m \to \mathbf{R}$ (i.e. $f(v_1, \ldots, v_m) = 0$ whenever $v_1, \ldots v_m \in V$ and $v_i = v_j$ for some $i \neq j$) is denoted by $\bigwedge^m V$. There is a natural isomorphism between $\bigwedge^m V$ and the space of all linear $\mathbf{R}$-valued maps on $\bigwedge_m V$ (cf. [10, 1.4.1–1.4.3]). It is often convenient to use the following customary notation (see [10]):

$$\langle \xi, h \rangle = h(\xi) \quad \text{whenever } \xi \in \textstyle\bigwedge_m V \text{ and } h \in \textstyle\bigwedge^m V.$$

Using the notation introduced in (2.3), we define

$$E = e_1 \wedge \cdots \wedge e_{n+1} \in \textstyle\bigwedge_{n+1} \mathbf{R}^{n+1} \quad \text{and} \quad E' = e_1' \wedge \cdots \wedge e_{n+1}' \in \textstyle\bigwedge^{n+1} \mathbf{R}^{n+1}. \tag{2.5}$$

If V is an inner product space, then both $\bigwedge_m V$ and $\bigwedge^m V$ can be endowed with natural scalar products, whose *associated norms* are denoted by $|\cdot|$; see [10, 1.7.5].

Suppose $U \subseteq \mathbf{R}^p$ is open and $k \geq 0$. A k-form is a smooth map $\phi : U \to \bigwedge^k \mathbf{R}^p$ (if $k = 0$ we set $\bigwedge^0 \mathbf{R}^p = \mathbf{R}$). Following [10, 4.1.1, 4.1.7], we denote by $\mathcal{E}^k(U)$ the space of all smooth k-forms on U and we denote by $\mathcal{D}^k(U)$ the space of all k-forms with compact support in U. If $\phi \in \mathcal{E}^k(U)$, we denote by $d\phi$ the *exterior derivative* of ϕ (cf. [10, 4.1.6]). Moreover, if f is a smooth function mapping U into $\mathbf{R}^q$ and ψ is a k-form defined on an open subset V of $\mathbf{R}^q$ with $f(U) \subseteq V$, then we define the k-form $f^\# \psi$ on U by the formula

$$\langle v_1 \wedge \cdots \wedge v_k, f^\# \psi(x) \rangle = \langle \textstyle\bigwedge_k Df(x)(v_1 \wedge \cdots \wedge v_k), \psi(f(x)) \rangle$$

for $x \in U$ and $v_1, \ldots, v_k \in \mathbf{R}^p$. We refer to [10, 4.1.6] for the basic properties of $f^\#$. Functions mapping a subset of U into $\bigwedge_k(\mathbf{R}^p)$ are called *k-vectorfields*.

Suppose $U \subseteq \mathbf{R}^p$ is open and $k \geq 0$. A k-current is a continuous $\mathbf{R}$-valued linear map T on $\mathcal{D}^k(U)$, with respect to the standard topology (cf. [10, 4.1.1]) and we denote the space of all k-currents on U by $\mathcal{D}_k(U)$. We say that a sequence

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$T_\ell \in \mathcal{D}_k(U)$ *weakly converges* to $T \in \mathcal{D}_k(U)$ if and only if

$$T_\ell(\phi) \to T(\phi) \qquad \text{for all } \phi \in \mathcal{D}^k(U).$$

If T is a k-current on U, then *the boundary of* T is the $(k-1)$-current $\partial T \in \mathcal{D}_{k-1}(U)$ given by

$$\partial T(\phi) = T(d\,\phi) \qquad \text{for all } \phi \in \mathcal{D}^k(U),$$

while the *support of* T is defined as

$$\operatorname{spt}(T) = U \setminus \bigcup \{V : \mathrm{V} \subseteq U \text{ open and } T(\phi) = 0 \text{ for all } \phi \in \mathcal{D}^k(V)\}.$$

If $T \in \mathcal{D}_k(U)$ and $\operatorname{spt}(T)$ is a compact subset of U, then T can be uniquely extended to a continuous linear map on $\mathcal{E}^k(U)$. If $\psi \in \mathcal{E}^h(U)$, $T \in \mathcal{D}_k(U)$ and $h \leq k$ we set

$$(T \llcorner \psi)(\phi) = T(\psi \wedge \phi) \qquad \text{for all } \phi \in \mathcal{D}^{k-h}(U).$$

If $T \in \mathcal{D}_k(U)$, V is an open subset of $\mathbf{R}^q$ and $f : U \to V$ is a smooth map such that $f|\operatorname{spt}(T)$ is proper, then noting that $\operatorname{spt} f^\# \phi \subseteq f^{-1}(\operatorname{spt}\phi)$ and $f^{-1}(\operatorname{spt}\phi) \cap \operatorname{spt} T$ is a compact subset of U for each $\phi \in \mathcal{D}^k(V)$, we define $f_\# T \in \mathcal{D}_k(V)$ by the formula

$$f_\# T(\phi) = T\big[\gamma \wedge f^\# \phi\big], \tag{2.6}$$

whenever $\phi \in \mathcal{D}^k(V)$ and $\gamma \in \mathcal{D}^0(U)$ with $f^{-1}(\operatorname{spt}\phi) \cap \operatorname{spt} T \subseteq \operatorname{interior}[\gamma^{-1}(\{1\})]$. If $\operatorname{spt}(T)$ is a compact subset of U then $f_\# T(\phi) = T(f^\# \phi)$ whenever $\phi \in \mathcal{E}^k(V)$. We refer to [10, 4.1.7] for the basic properties of the map $f_\#$.

We say that a k-current $T \in \mathcal{D}_k(U)$ is a *integer multiplicity locally rectifiable k-current of U* provided

$$T(\phi) = \int_M \langle \eta(x), \phi(x) \rangle \, d\mathcal{H}^k(x) \qquad \text{for all } \phi \in \mathcal{D}^k(U), \tag{2.7}$$

where $M \subseteq U$ is $\mathcal{H}^k$-measurable and countably $\mathcal{H}^k$-rectifiable and η is an $\mathcal{H}^k \llcorner M$-measurable k-vectorfield such that:

(1) $\int_{K \cap M} |\eta| \, d\mathcal{H}^k < \infty$ for every compact subset K of U,
(2) $\eta(x)$ is a simple and $|\eta(x)|$ is a positive integer for $\mathcal{H}^k$ a.e. $x \in M$,
(3) $\operatorname{Tan}^k(\mathcal{H}^k \llcorner M, x)$ is associated with $\eta(x)$ for $\mathcal{H}^k$ a.e. $x \in M$.

We refer to M as *carrier of* T.

2.3. Legendrian currents

Here we introduce the central notion of Legendrian cycle and we collect some fundamental facts.

Let $\alpha \in \mathcal{E}^1(\mathbf{R}^{n+1} \times \mathbf{R}^{n+1})$ be the *contact* 1*-form of* $\mathbf{R}^{n+1}$, which is defined by the formula

$$\langle (y, v), \alpha(x, u) \rangle = y \bullet u \quad \text{for } (x, u), (y, v) \in \mathbf{R}^{n+1}.$$

DEFINITION 2.1. *Let $M \subseteq \mathbf{R}^{n+1} \times \mathbf{S}^n$ be a countably $\mathcal{H}^n$ rectifiable set. We say that M is a* Legendrian rectifiable set *if and only if for every $Q \subseteq M$ with $\mathcal{H}^n(Q) < \infty$*

we have that

$$\alpha(x,u)|\operatorname{Tan}^n(\mathcal{H}^n \llcorner Q,(x,u)) = 0$$

for $\mathcal{H}^n$ a.e. $(x,u) \in Q$.

Remark 2.2. In relation with definition 2.1, we recall that $\operatorname{Tan}^n(\mathcal{H}^n \llcorner Q,(x,u))$ is an n-dimensional plane for $\mathcal{H}^n$ a.e. $(x,u) \in Q$.

Definition 2.3. *Let $W \subseteq \mathbf{R}^{n+1}$ be an open set and let T be an integer-multiplicity locally rectifiable n-current of $W \times \mathbf{R}^{n+1}$ with $\operatorname{spt}(T) \subseteq W \times \mathbf{S}^n$.*

We say that T is Legendrian cycle of W *if $T \llcorner \alpha = 0$ and $\partial T = 0$.*

Remark 2.4. If T is a Legendrian cycle and M is a carrier of T, then M is a Legendrian rectifiable set (with $\mathcal{H}^n(K \cap M) < \infty$ for every $K \subseteq \mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$ compact).

Lemma 2.5. *Suppose $W_1, \ldots, W_m \subseteq \mathbf{R}^{n+1}$ are bounded open sets and $T \in \mathcal{D}_n(\mathbf{R}^{n+1} \times \mathbf{R}^{n+1})$ such that $T \llcorner (W_i \times \mathbf{R}^{n+1})$ is a Legendrian cycle of W_i for every $i = 1, \ldots, m$ and $\operatorname{spt}(T)$ is a compact subset of $\bigcup_{i=1}^m W_i \times \mathbf{S}^n$. Then T is a Legendrian cycle of $\mathbf{R}^{n+1}$.*

Proof. For each $i = 1, \ldots, m$ choose an open set V_i with compact closure in W_i and $f_i \in C^\infty(\mathbf{R}^{n+1})$ such that $\operatorname{spt}(f_i)$ is a compact subset of W_i, $\sum_{i=1}^m f_i(x) = 1$ for every $x \in \bigcup_{i=1}^m V_i$ and $\operatorname{spt}(T) \subseteq \bigcup_{i=1}^m V_i \times \mathbf{S}^n$. Then $T = T \llcorner (\sum_{i=1}^m f_i)$, $\sum_{i=1}^m df_i = 0$ on $\bigcup_{i=1}^m V_i$,

$$\langle \phi, T \llcorner \alpha \rangle = \sum_{i=1}^m \langle f_i \phi, T \llcorner \alpha \rangle = 0$$

and

$$\partial T(\phi) = \sum_{i=1}^m \partial T(f_i \phi) + T\Big[\Big(\textstyle\sum_{i=1}^m df_i\Big) \wedge \phi\Big] = 0$$

for every $\phi \in \mathcal{D}^{n-1}(\mathbf{R}^{n+1} \times \mathbf{R}^{n+1})$. □

For the next definition recall (2.3) and (2.5).

Definition 2.6. Lipschitz–Killing forms (cf. [47]) *For $k \in \{0, \ldots, n\}$ the k-th* Lipschitz–Killing form of $\mathbf{R}^{n+1}$, *$\varphi_k \in \mathcal{E}^n(\mathbf{R}^{n+1} \times \mathbf{R}^{n+1})$, is defined by the formula*

$$\langle \xi_1 \wedge \cdots \wedge \xi_n, \varphi_k(x,u) \rangle = \sum_{\sigma \in \Sigma_{n,k}} \langle \pi_{\sigma(1)}(\xi_1) \wedge \cdots \wedge \pi_{\sigma(n)}(\xi_n) \wedge u, E' \rangle,$$

for every $\xi_1, \ldots, \xi_n \in \mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$, where

$$\Sigma_{n,k} = \left\{ \sigma : \{1, \ldots, n\} \to \{0,1\} : \sum_{i=1}^n \sigma(i) = n - k \right\}$$

and π_0 and π_1 are the projections defined in (2.2).

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For $1 \leq k \leq m$, we denote by $\Lambda(m,k)$ the set of all increasing mappings from $\{1,\dots,k\}$ into $\{1,\dots,m\}$.

LEMMA 2.7. (cf. [16, Lemma 3.1]). *The exterior derivatives of the Lipschitz–Killing differential forms satisfy the following equations:*

$$\langle \xi_1 \wedge \cdots \wedge \xi_{n+1}, d\,\varphi_k(x,u)\rangle = \langle \xi_1 \wedge \cdots \wedge \xi_{n+1}, \alpha(x,u) \wedge (n-k+1)\varphi_{k-1}(x,u)\rangle \quad (2.8)$$

for $k = 1,\dots,n$ *and*

$$\langle \xi_1 \wedge \cdots \wedge \xi_{n+1}, d\,\varphi_0(x,u)\rangle = 0, \quad (2.9)$$

whenever $\xi_1,\dots,\xi_{n+1} \in \mathrm{Tan}(\mathbf{R}^{n+1} \times \mathbf{S}^n, (x,u))$ *and* $(x,u) \in \mathbf{R}^{n+1} \times \mathbf{S}^n$.

Proof. We fix $(x,u) \in \mathbf{R}^{n+1} \times \mathbf{S}^n$.

Suppose $k \geq 0$ and notice that $\varphi_k : \mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \to \bigwedge^n(\mathbf{R}^{n+1} \times \mathbf{R}^{n+1})$ is a linear map. Henceforth, we compute (cf. [10, p. 352])

$$\begin{aligned}
&\langle \xi_1 \wedge \cdots \wedge \xi_{n+1}, d\,\varphi_k(x,u)\rangle \qquad (2.10)\\
&= \sum_{j=1}^{n+1}(-1)^{j-1}\big\langle \xi_1 \wedge \cdots \wedge \xi_{j-1} \wedge \xi_{j+1} \wedge \cdots \wedge \xi_{n+1}, \langle \xi_j, D\varphi_k(x,u)\rangle\big\rangle\\
&= \sum_{j=1}^{n+1}(-1)^{j-1}\big\langle \xi_1 \wedge \cdots \wedge \xi_{j-1} \wedge \xi_{j+1} \wedge \cdots \wedge \xi_{n+1}, \varphi_k(\xi_j)\big\rangle\\
&= \sum_{j=1}^{n+1}(-1)^{j-1} \sum_{\sigma \in \Sigma_{n,k}} \langle \pi_{\sigma(1)}(\xi_1) \wedge \cdots \wedge \pi_{\sigma(j-1)}(\xi_{j-1}) \wedge \pi_{\sigma(j)}(\xi_{j+1}) \wedge \cdots\\
&\qquad \cdots \wedge \pi_{\sigma(n)}(\xi_{n+1}) \wedge \pi_1(\xi_j), E'\rangle\\
&= (-1)^n \sum_{j=1}^{n+1} \sum_{\sigma \in \Sigma_{n,k}} \langle \pi_{\sigma(1)}(\xi_1) \wedge \cdots \wedge \pi_{\sigma(j-1)}(\xi_{j-1}) \wedge \pi_1(\xi_j) \wedge \pi_{\sigma(j)}(\xi_{j+1}) \wedge \cdots\\
&\qquad \cdots \wedge \pi_{\sigma(n)}(\xi_{n+1}), E'\rangle
\end{aligned}$$

for $\xi_1,\dots,\xi_{n+1} \in \mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$, whence we readily deduce (2.9). Moreover, if $p_u : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ is the orthogonal projection onto $\mathrm{span}\{u\}$ we use the permutation formula (cf. [10, 1.4.2]) to compute

$$\begin{aligned}
&\langle \xi_1 \wedge \cdots \wedge \xi_{n+1}, \alpha(x,u) \wedge \varphi_{k-1}(x,u)\rangle \qquad (2.11)\\
&= \sum_{j=1}^{n+1}(-1)^{j-1}\langle \xi_j, \alpha(x,u)\rangle\,\langle \xi_1 \wedge \cdots \wedge \xi_{j-1} \wedge \xi_{j+1} \wedge \cdots \wedge \xi_{n+1}, \varphi_{k-1}(x,u)\rangle\\
&= (-1)^n \sum_{j=1}^{n+1} \sum_{\sigma \in \Sigma_{n,k-1}} \langle \pi_{\sigma(1)}(\xi_1) \wedge \cdots \wedge \pi_{\sigma(j-1)}(\xi_{j-1}) \wedge p_u(\pi_0(\xi_j))\\
&\qquad \wedge \pi_{\sigma(j)}(\xi_{j+1}) \wedge \cdots\cdots \wedge \pi_{\sigma(n)}(\xi_{n+1}), E'\rangle
\end{aligned}$$

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whenever $\xi_1, \dots, \xi_{n+1} \in \mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$ and $k \geq 1$. Suppose $\tau_1, \dots, \tau_n \in u^\perp$ is an orthonormal set and we define

$$v_i = (\tau_i, 0) \quad \text{for } i = 1, \dots, n, \quad v_i = (0, \tau_{i-n}) \quad \text{for } i = n+1, \dots, 2n, \quad v_{2n+1} = (u, 0),$$

which form an orthonormal basis of $\mathbf{R}^{n+1} \times u^\perp$. Then we define

$$v_\lambda = v_{\lambda(1)} \wedge \cdots \wedge v_{\lambda(n+1)} \quad \text{whenever } \lambda \in \Lambda(2n+1, n+1)$$

and, recalling that $\{v_\lambda : \lambda \in \Lambda(2n+1, n+1)\}$ is a basis of $\bigwedge_{n+1}(\mathbf{R}^{n+1} \times u^\perp)$ (cf. [10, 1.3.2]), we notice that (2.8) reduces to check

$$(n-k+1)\,\langle v_\lambda, \alpha(x,u) \wedge \varphi_{k-1}(x,u)\rangle = \langle v_\lambda, d\,\varphi_k(x,u)\rangle \quad \text{for } \lambda \in \Lambda(2n+1, n+1). \tag{2.12}$$

Firstly, we notice that if $\lambda \in \Lambda(2n+1, n+1)$ and $2n+1 \notin \mathrm{Im}(\lambda)$ then one can easily check from (2.10) and (2.11) that both sides of (2.12) must be equal to zero.

We fix now $\lambda \in \Lambda(2n+1, n+1)$ such that $\lambda(n+1) = 2n+1$. Then $\pi_1(v_{2n+1}) = 0$ and we employ (2.10) to compute

$$\begin{aligned}
&\langle v_\lambda, d\,\varphi_k(x,u)\rangle \\
&= (-1)^n \sum_{j=1}^{n} \sum_{\sigma \in \Sigma_{n,k}} \langle \pi_{\sigma(1)}(v_{\lambda(1)}) \wedge \cdots \wedge \pi_{\sigma(j-1)}(v_{\lambda(j-1)}) \wedge \pi_1(v_{\lambda(j)}) \\
&\qquad \wedge \pi_{\sigma(j)}(v_{\lambda(j+1)}) \wedge \cdots\cdots \wedge \pi_{\sigma(n-1)}(v_{\lambda(n)}) \wedge \pi_{\sigma(n)}(v_{2n+1}), E'\rangle \\
&= (-1)^n \sum_{j \in \lambda^{-1}\{n+1, \cdots, 2n\}} \sum_{\substack{\sigma \in \Sigma_{n,k} \\ \sigma(n)=0}} \langle \pi_{\sigma(1)}(v_{\lambda(1)}) \wedge \cdots \wedge \pi_{\sigma(j-1)}(v_{\lambda(j-1)}) \wedge \tau_{\lambda(j)-n} \wedge \\
&\qquad \wedge \pi_{\sigma(j)}(v_{\lambda(j+1)}) \wedge \cdots \wedge \pi_{\sigma(n-1)}(v_{\lambda(n)}) \wedge u, E'\rangle,
\end{aligned}$$

while, since $p_u(\pi_0(v_{\lambda(j)})) = 0$ for $j = 1, \dots, n$, we obtain from (2.11)

$$\langle v_\lambda, \alpha(x,u) \wedge \varphi_{k-1}(x,u)\rangle = (-1)^n \sum_{\sigma \in \Sigma_{n,k-1}} \langle \pi_{\sigma(1)}(v_{\lambda(1)}) \wedge \cdots \wedge \pi_{\sigma(n)}(v_{\lambda(n)}) \wedge u, E'\rangle.$$

Therefore, if $\mathcal{H}^0\big(\lambda^{-1}\{1, \dots, n\}\big) \neq k-1$ then

$$\langle v_\lambda, \alpha(x,u) \wedge \varphi_{k-1}(x,u)\rangle = 0 = \langle v_\lambda, d\,\varphi_k(x,u)\rangle.$$

Finally, if $\mathcal{H}^0\big(\lambda^{-1}\{1, \dots, n\}\big) = k-1$ then

$$\begin{aligned}
&(n-k+1)\,\langle v_\lambda, \alpha(x,u) \wedge \varphi_{k-1}(x,u)\rangle \\
&\quad = (n-k+1)\,(-1)^n\,\langle \tau_{\lambda(1)} \wedge \cdots \wedge \tau_{\lambda(k-1)} \wedge \tau_{\lambda(k)-n} \wedge \cdots \wedge \tau_{\lambda(n)-n} \wedge u, E'\rangle \\
&\quad = \langle v_\lambda, d\,\varphi_k(x,u)\rangle.
\end{aligned}$$

□

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We recall that a *local variation* $(F_t)_{t\in I}$ of $\mathbf{R}^{n+1}$ is a smooth map $F : I\times\mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$, where I is an open interval of $\mathbf{R}$ with $0 \in I$, such that $F_0 = F(0,\cdot)$ is the identity of $\mathbf{R}^{n+1}$ and $F_t = F(t,\cdot) : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ is a diffeomorphism for every $t \in I$. For such local variation, we define the initial velocity vector field V by

$$V(x) = \lim_{t\to 0} \frac{F_t(x) - x}{t} \quad \text{for } x \in \mathbf{R}^{n+1}.$$

Moreover, if $F : U \to V$ is a C^2-diffeomorphism between open subsets of $\mathbf{R}^{n+1}$, we define the C^1-diffeomorphism $\Psi_F : U \times \mathbf{S}^n \to V \times \mathbf{S}^n$ by

$$\Psi_F(x,u) = \left(F(x), \frac{(DF(x)^{-1})^*(u)}{|(DF(x)^{-1})^*(u)|}\right) \quad \text{for } (x,u) \in U \times \mathbf{S}^n. \tag{2.13}$$

We define $\mathbf{R}^{n+1}_0 = \mathbf{R}^{n+1} \setminus \{0\}$. For a local variation $(F_t)_{t\in I}$ (where I is an open interval of $\mathbf{R}$ with $0 \in I$) we define (cf. (2.13)) the smooth map $h : \mathbf{R}^{n+1} \times \mathbf{R}^{n+1}_0 \times I \to \mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$ by

$$h(x,u,t) = \Psi_{F_t}(x,u) \qquad \text{for } (x,u,t) \in \mathbf{R}^{n+1} \times \mathbf{R}^{n+1}_0 \times I$$

and we notice that $h(x,u,0) = (x,u)$ for $(x,u) \in \mathbf{R}^{n+1} \times \mathbf{R}^{n+1}_0$. Moreover we define

$$p : \mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \times \mathbf{R} \to \mathbf{R}^{n+1} \times \mathbf{R}^{n+1}, \qquad p(x,u,t) = (x,u),$$

$$q : \mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \times \mathbf{R} \to \mathbf{R}, \qquad q(x,u,t) = t,$$

$$P : \mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \to \mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \times \mathbf{R}, \qquad P(x,u) = (x,u,0).$$

LEMMA 2.8. (cf. [16]). *Suppose T is a Legendrian cycle of $\mathbf{R}^{n+1}$ with* $\operatorname{spt}(T)$ *compact, $(F_t)_{t\in I}$ is a local variation of $\mathbf{R}^{n+1}$ with initial velocity vector field V and $\theta_V : \mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \to \mathbf{R}$ is given by $\theta_V(x,u) = V(x) \bullet u$ for $(x,u) \in \mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$.*
Then (see (2.13))

$$\frac{d}{dt}\big[(\Psi_{F_t})_\# T\big](\varphi_i)\Big|_{t=0} = (n+1-i)T(\theta_V \wedge \varphi_{i-1}) \quad \textit{for } i = 1,\ldots,n$$

and

$$\frac{d}{dt}\big[(\Psi_{F_t})_\# T\big](\varphi_0)\Big|_{t=0} = 0.$$

Proof. Firstly, a simple direct computation leads to

$$h^\# \alpha \circ P = \big(p^\# \alpha + \theta_V\, dq\big) \circ P \tag{2.14}$$

and

$$\big(h^\# \varphi_k \wedge dq\big) \circ P = \big(p^\# \varphi_k \wedge dq\big) \circ P. \tag{2.15}$$

Suppose now $M \subseteq \mathbf{R}^{n+1} \times \mathbf{S}^n$ is a countably $\mathcal{H}^n$-rectifiable set and η is a $\mathcal{H}^n \llcorner M$-measurable simple n-vectorfield such that $|\eta(x,u)|$ is a positive integer for $\mathcal{H}^n$ a.e.

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$(x, u) \in M$,

$$T(\phi) = \int_M \langle \eta(x,u), \phi(x,u) \rangle \, d\mathcal{H}^n(x,u) \quad \text{for each } \phi \in \mathcal{D}^n(\mathbf{R}^{n+1} \times \mathbf{R}^{n+1}),$$

and $\operatorname{Tan}^n(\mathcal{H}^n \llcorner M, (x,u))$ is associated with $\eta(x,u)$ for $\mathcal{H}^n$ a.e. $(x,u) \in M$. For $t > 0$ we define $[\![0,t]\!] \in \mathcal{D}_1(\mathbf{R})$ by the formula

$$[\![0,t]\!](\beta) = \int_0^1 \langle t, \beta(st) \rangle \, ds = \int_0^t \langle 1, \beta(s) \rangle \, ds \quad \text{for } \beta \in \mathcal{E}^1(\mathbf{R}).$$

Denoting by $T \times [\![0,t]\!] \in \mathcal{D}_{n+1}(\mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \times \mathbf{R})$ the Cartesian product of T and $[\![0,t]\!]$, we employ [10, 4.1.8] to compute

$$\big(T \times [\![0,t]\!]\big)(\phi) = \int_M \int_0^t \big\langle \zeta(x,u,s), \phi(x,u,s) \big\rangle \, ds \, d\mathcal{H}^n(x,u)$$

whenever $\phi \in \mathcal{E}^{n+1}(\mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \times \mathbf{R})$, where

$$\zeta(x,u,s) = \big(\textstyle\bigwedge_n P\big)(\eta(x,u)) \wedge w_{2n+3} \quad \text{for } \mathcal{H}^n \text{ a.e. } (x,u) \in M \text{ and for } s \in (0,t).$$

Employing [10, 4.1.8, 4.1.9] and taking into account that $\partial T = 0$ we derive the homotopy formula

$$\big(\Psi_{F_t}\big)_\# T - T = (-1)^n \, \partial\big[h_\#(T \times [\![0,t]\!])\big].$$

Since $\operatorname{spt}\big(h_\#\big(T \times [\![0,t]\!]\big)\big) \subseteq \mathbf{R}^{n+1} \times \mathbf{S}^n$ we use lemma 2.7 to compute

$$\begin{aligned}\big[\big(\Psi_{F_t}\big)_\# T - T\big](\varphi_k) &= (-1)^n \, (n-k+1) \, h_\#(T \times [\![0,t]\!])(\alpha \wedge \varphi_{k-1}) \\ &= (-1)^n \, (n-k+1) \int_M \int_0^t \big\langle \zeta(x,u,s), h^\# \alpha \wedge h^\# \varphi_{k-1}(x,u,s) \big\rangle \, ds \, d\mathcal{H}^n(x,u),\end{aligned}$$

whence we infer that

$$\begin{aligned}&\lim_{t \to 0} \frac{\big[\big(\Psi_{F_t}\big)_\# T - T\big](\varphi_k)}{t} \\ &\qquad = (-1)^n \, (n-k+1) \int_M \big\langle \zeta(x,u,0), h^\# \alpha \wedge h^\# \varphi_{k-1}(x,u,0) \big\rangle \, d\mathcal{H}^n(x,u).\end{aligned}$$

Using (2.14) and (2.15) we deduce that

$$\begin{aligned}&\big\langle \zeta(x,u,0), h^\# \alpha \wedge h^\# \varphi_{k-1}(x,u,0) \big\rangle \\ &\quad = \langle \zeta(x,u,0), p^\# \alpha \wedge h^\# \varphi_{k-1}(x,u,0) \rangle + \langle \zeta(x,u,0), \theta_V(x,u) \, dq \wedge p^\# \varphi_{k-1}(x,u,0) \rangle\end{aligned}$$

for $\mathcal{H}^n$ a.e. $(x,u) \in M$. Moreover, noting that $p(w_{2n+3}) = 0$ and $\langle \tau, \alpha(x,u) \rangle = 0$ whenever $\tau \in \operatorname{Tan}^n(\mathcal{H}^n \llcorner M, (x,u))$ for $\mathcal{H}^n$ a.e. $(x,u) \in M$ by [30, Theorem 9.2],

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we obtain that

$$\langle \zeta(x,u,0), p^{\#}\alpha \wedge h^{\#}\varphi_{k-1}(x,u,0)\rangle = 0 \qquad \text{for } \mathcal{H}^n \text{ a.e. } (x,u) \in M$$

and employing the shuffle formula we compute

$$\langle \zeta(x,u,0), \theta_V(x,u)\, dq \wedge p^{\#}\varphi_{k-1}(x,u,0)\rangle = (-1)^n\, \theta_V(x,u)\, \langle \eta(x,u), \varphi_{k-1}(x,u)\rangle$$

for $\mathcal{H}^n$ a.e. $(x,u) \in M$. Moreover, applying lemma 2.7, we obtain

$$\frac{d}{dt}\big[(\Psi_{F_t})_{\#}T\big](\varphi_0)\Big|_{t=0} = (-1)^n \lim_{t\to 0} \frac{\big[h_{\#}(T \times [\![0,t]\!])\big](d\varphi_0)}{t} = 0\,.$$

□

2.4. The proximal unit normal bundle

The following notion plays a key role in this paper.

Definition 2.9. (cf. [34, p. 212]). *If* $\varnothing \neq C \subseteq \mathbf{R}^{n+1}$ *we define the* proximal unit normal bundle *of* C *as*

$$\operatorname{nor}(C) = \{(x,\nu) \in \overline{C} \times \mathbf{S}^n : \operatorname{dist}(x + s\nu, C) = s \text{ for some } s > 0\}.$$

Remark 2.10. Notice that $\operatorname{nor}(C) = \operatorname{nor}(\overline{C})$. We recall that $\operatorname{nor}(C)$ is a Borel set and it is always countably $\mathcal{H}^n$-rectifiable; see [40, Remark 4.3].[1] However, $\mathcal{H}^n \llcorner \operatorname{nor}(C)$ might not be a Radon measure, even when C is the closure of a smooth submanifold with bounded mean curvature; cf. lemma A.3 and remark A.4.

The following lemma is an extension of well-known results for sets of positive reach (cf. [30, Lemmas 4.23 and 4.24]).

Lemma 2.11. *Suppose* $C \subseteq \mathbf{R}^{n+1}$. *For* $\mathcal{H}^n$ *a.e.* $(x,u) \in \operatorname{nor}(C)$ *there exist numbers*

$$-\infty < \kappa_1(x,u) \leq \ldots \leq \kappa_n(x,u) \leq \infty$$

and vectors $\tau_1(x,u), \ldots, \tau_n(x,u)$ *such that* $\{\tau_1(x,u), \ldots, \tau_n(x,u), u\}$ *is an orthonormal basis of* $\mathbf{R}^{n+1}$ *and the vectors*

$$\zeta_i(x,u) = \left(\frac{1}{\sqrt{1+\kappa_i(x,u)^2}}\tau_i(x,u), \frac{\kappa_i(x,u)}{\sqrt{1+\kappa_i(x,u)^2}}\tau_i(x,u)\right), \quad i = 1, \ldots, n,$$

form an orthonormal basis of $\operatorname{Tan}^n(\mathcal{H}^n \llcorner Q, (x,u))$ *for every* $\mathcal{H}^n$*-measurable set* $Q \subseteq \operatorname{nor}(C)$ *with* $\mathcal{H}^n(Q) < \infty$ *and for* $\mathcal{H}^n$ *a.e.* $(x,u) \in Q$ *(We set* $\frac{1}{\infty} = 0$ *and* $\frac{\infty}{\infty} = 1$*). In particular,*

$$\operatorname{nor}(C) \textit{ is a Legendrian rectifiable set.}$$

Moreover, the maps $\kappa_1, \ldots, \kappa_n$ *can be chosen to be* $\mathcal{H}^n \llcorner \operatorname{nor}(C)$*-measurable and they are* $\mathcal{H}^n \llcorner \operatorname{nor}(C)$*-almost uniquely determined.*

[1] The unit normal bundle of a closed set C in [40] is denoted with $N(C)$.

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Proof. The existence part of the statement and the measurability property are discussed in [14, Section 3] (see in particular [14, Remark 3.7]). Uniqueness can be proved as in [30, Lemma 4.24]. □

Definition 2.12. *If* $C \subseteq \mathbf{R}^{n+1}$*, we denote by* $\kappa_{C,1}, \dots, \kappa_{C,n}$ *the* $\mathcal{H}^n \llcorner \operatorname{nor}(C)$ *measurable maps given by* lemma 2.11*.*

The following Heintze–Karcher-type inequality for arbitrary closed sets is proved in [14].

Theorem 2.13 (cf. [14, Theorem 3.20]). *Let* $C \subseteq \mathbf{R}^{n+1}$ *be a bounded closed set with non empty interior. Let* $K = \mathbf{R}^{n+1} \setminus \operatorname{interior}(C)$ *and assume that*

$$\sum_{i=1}^{n} \kappa_{K,i}(x,u) \leq 0 \quad \text{for } \mathcal{H}^n \text{ a.e. } (x,u) \in \operatorname{nor}(K).$$

Then

$$(n+1)\mathcal{L}^{n+1}(\operatorname{interior}(C)) \leq \int_{\operatorname{nor}(K)} J_n^{\operatorname{nor}(K)} \pi_0(x,u) \, \frac{n}{|\sum_{i=1}^{n} \kappa_{K,i}(x,u)|} \, d\mathcal{H}^n(x,u).$$

Moreover, if the equality holds and there exists $q < \infty$ *such that* $|\sum_{i=1}^{n} \kappa_{K,i}(x,u)| \leq q$ *for* $\mathcal{H}^n$ *a.e.* $(x,u) \in \operatorname{nor}(K)$*, then* $\operatorname{interior}(C)$ *is a finite union of disjointed (possibly mutually tangent) open balls.*

2.5. $W^{2,n}$-functions

Suppose $U \subseteq \mathbf{R}^n$ is open. We denote by $W^{k,p}(U)$ (resp. $W^{k,p}_{\mathrm{loc}}(U)$) the usual Sobolev space of k-times weakly differentiable functions, whose distributional derivatives up to order k belong to the Lebesgue space $L^p(U)$ (resp. $L^p_{\mathrm{loc}}(U)$); cf. [11, Chapter 7]. We denote by $\boldsymbol{\nabla} f$ and $\mathbf{D}^i f$ the distributional gradient and the distributional i-th differential of a Sobolev function f.

We now state two results on the fine properties of $W^{2,n}$-functions, which play an important role in this paper. We start with some definitions.

Definition 2.14. *Given* $U \subseteq \mathbf{R}^n$ *open set and* $f : U \to \mathbf{R}$ *continuous function, we define* $\Gamma^+(f,U)$ *as the set of* $x \in U$ *for which there exists* $p \in \mathbf{R}^n$ *such that*

$$f(y) \leq f(x) + p \bullet (y-x) \quad \forall y \in U.$$

Definition 2.15. *Suppose* $f : U \to \mathbf{R}$ *is a continuous function. We define* $\mathcal{S}^*(f)$*, respectively* $\mathcal{S}_*(f)$*, as the set of* $x \in U$*, where there exists a polynomial function* P *of degree at most* 2 *such that* $P(x) = f(x)$ *and*

$$\limsup_{y \to x} \frac{f(y) - P(y)}{|y-x|^2} < \infty, \qquad \text{respectively} \quad \liminf_{y \to x} \frac{f(y)-P(y)}{|y-x|^2} > -\infty.$$

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Moreover, we set $\mathcal{S}(f)$ to be the set of $x \in U$, where there exists a polynomial function P of degree at most 2 such that $P(x) = f(x)$ and

$$\lim_{y \to x} \frac{f(y) - P(y)}{|y - x|^2} = 0.$$

Lemma 2.16. *If $f \in C(U) \cap W^{2,n}(U)$, then $\mathcal{L}^n\big(U \setminus \mathcal{S}(f)\big) = 0$.*

Proof. This is a special case of [5, Proposition 2.2], which is attributed to Calderon and Zygmund. This result can also be proved by a simple adaptation of the method of [27] (cf. [44]). An interesting generalization is given in [17]. □

The following oscillation estimate plays a key role in lemma 3.5 and was proved by Ulrich Menne, see [24, Appendix B].

Lemma 2.17. *Let $a \in \mathbf{R}^n$, $r > 0$, $f \in C(B_r(a)) \cap W^{2,n}(B_r(a))$ and $g \in C^2(B_r(a))$ such that*

$$g(a) = f(a), \qquad f(x) \geq g(x) \quad \text{for every } x \in B_r(a).$$

Then there exists a constant $c(n)$, depending only on n, such that

$$\|\mathbf{D}f - Dg(a)\|_{L^n(B_r(a))} \leq c(n) r \Big(\|\mathbf{D}^2 f\|_{L^n(B_r(a))} + r \|D^2 g\|_{L^\infty(B_r(a))} \Big), \tag{2.16}$$

$$\|f - L_a\|_{L^\infty(B_r(a))} \leq c(n) r \Big(\|\mathbf{D}^2 f\|_{L^n(B_r(a))} + r \|D^2 g\|_{L^\infty(B_r(a))} \Big), \tag{2.17}$$

where $L_a(x) = f(a) + Dg(a)(x - a)$ for $x \in \mathbf{R}^n$. In particular, f is pointwise differentiable at a with $Df(a) = Dg(a)$.

Proof. See [24, Lemma B.3]. In particular, (2.16) and (2.17) follow from the estimate of [24, Lemma B.3], while the pointwise differentiability of f at a directly follows from (2.17). □

Remark 2.18. It follows from lemma 2.17 that if $a \in \mathcal{S}^*(f) \cup \mathcal{S}_*(f)$, then a is a Lebesgue point of $\mathbf{D}f$, the map f is pointwise differentiable at a, and

$$Df(a) = \mathbf{D}f(a).$$

We conclude with a Lusin-type result for $W^{1,1}$-functions, which can be easily deduced from well-known results of Calderon–Zygmund and Federer. We provide a detailed proof since we were unable to find this precise statement in classical references.

Lemma 2.19. *Suppose U is an open subset of $\mathbf{R}^n$ and $f \in W^{1,1}_{\mathrm{loc}}(U, \mathbf{R}^k)$. Then f is $\mathcal{L}^n \llcorner U$-approximately differentiable at $\mathcal{L}^n$ a.e. $x \in U$ with $\operatorname{ap} Df(x) = \mathbf{D}f(x)$. In particular, there exist countably many $\mathcal{L}^n$-measurable subsets A_i of U such that $\mathcal{L}^n\big(U \setminus \bigcup_{i=1}^\infty A_i\big) = 0$ and $\operatorname{Lip}(f|A_i) < \infty$ for every $i \geq 1$.*

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Proof. By [6, Theorem 12] (or [46, Theorem 3.4.2]), we have that

$$\lim_{r\to 0} r^{-n-1}\int_{B(x,r)} |f(y)-f(x)-\mathbf{D}f(x)(y-x)|\,d\mathcal{L}^n(y)=0 \tag{2.18}$$

for $\mathcal{L}^n$ a.e. $x\in U$. Fix now $x\in U$ such that (2.18) holds, define the affine function $L_x(y)=f(x)+\mathbf{D}f(x)(y-x)$ for $y\in\mathbf{R}^n$ and notice that

$$\frac{\epsilon\mathcal{L}^n(B(x,r)\cap\{y:|f(y)-L_x(y)|\geq\epsilon\, r\})}{r^n}\leq r^{-n-1}\int_{B(x,r)}|f(y)-L_x(y)|\,d\mathcal{L}^n y$$

for every $\epsilon>0$. Henceforth, by (2.18) and [39, Lemma 2.7],

$$\Theta^n\big(\mathcal{L}^n\llcorner\{y:|f(y)-L_x(y)|\geq 2\epsilon|y-x|\},x\big)=0\quad\text{for every }\epsilon>0,$$

whence we conclude that f is $\mathcal{L}^n\llcorner U$-approximately differentiable at x with $\operatorname{ap}Df(x)=\mathbf{D}f(x)$ applying [10, p. 253] with ϕ and m replaced by $\mathcal{L}^n\llcorner U$ and n. Now we can use [10, 3.1.8] to infer the existence of the countable cover A_i. □

3. Legendrian cycles over $W^{2,n}$-graphs

The main result of this section (theorem 3.9) proves that the Legendrian cycle associated with the subgraph of a $W^{2,n}$-function is carried over its proximal unit normal bundle. This is the key result to extend the Alexandrov sphere theorem to $W^{2,n}$-domains.

DEFINITION 3.1. *Suppose $\psi:\mathbf{R}^n\to\mathbf{R}^{n+1}$ is defined as*

$$\psi(y)=\frac{(-y,1)}{\sqrt{1+|y|^2}}.$$

If $f\in W^{2,n}(U)\cap C(U)$, we define

$$\Phi_f(x)=(x,f(x),\psi(\nabla f(x)))\in U\times\mathbf{R}\times\mathbf{S}^n\quad\textit{for every } x\in\operatorname{Diff}(f).$$

REMARK 3.2. Let $\mathbf{S}^n_+=\{(z,t)\in\mathbf{R}^n\times\mathbf{R}:|z|^2+t^2=1,\ t>0\}$. Notice that ψ is a diffeomorphism onto $\mathbf{S}^n_+$ with inverse given by

$$\varphi:\mathbf{S}^n_+\to\mathbf{R}^n,\quad \varphi(z,t)=-\frac{z}{t},$$

and $\|D\psi(y)\|\leq 2$ for $y\in\mathbf{R}^n$.

REMARK 3.3. We recall that $\boldsymbol{\nabla}f$ is $\mathcal{L}^n\llcorner U$-approximately differentiable at $\mathcal{L}^n$ a.e. $a\in U$ by lemma 2.19. Moreover, $\nabla f(a)=\boldsymbol{\nabla}f(a)$ for $\mathcal{L}^n$ a.e. $a\in U$ by remark 2.18. If $a\in\operatorname{Diff}(f)$ and ∇f is $\mathcal{L}^n\llcorner U$-approximately differentiable at a, then Φ_f is

$\mathcal{L}^n \llcorner U$-approximately differentiable at a and

$$\operatorname{ap} D\Phi_f(a)(\tau) = \big(\tau, Df(a)(\tau), \langle \operatorname{ap} D(\nabla f)(a)(\tau), D\psi(\nabla f(a))\rangle\big)$$

for every $\tau \in \mathbf{R}^n$. In particular $\operatorname{ap} D\Phi_f(a)$ is injective for $\mathcal{L}^n$ a.e. $a \in U$ and, recalling (3.2) and noting that $\mathbf{D}(\nabla f) = \operatorname{ap} D(\nabla f)$ by lemma 2.19, we infer that

$$\int_U \|\operatorname{ap} D\Phi_f\|^n \, d\mathcal{L}^n \le c(n)\bigg(\mathcal{L}^n(U) + \int_U \|Df\|^n \, d\mathcal{L}^n + \int_U \|\mathbf{D}^2 f\|^n \, d\mathcal{L}^n\bigg).$$

It follows that Φ_f is a $W^{1,n}$-map over U.

Remark 3.4. The following basic fact of measure theory is used in the proof of lemma 3.5. *Suppose μ is a measure over a set X, C is a positive constant, and $\{E_j : j \in S\}$ is a countable family of μ-measurable sets, such that $\mathcal{H}^0(\{j \in S : E_j \cap E_i \neq \varnothing\}) \le C$ for every $i \in S$. Then*

$$\sum_{i\in S} \mu(E_i) \le C\mu\bigg(\bigcup_{i\in S} E_i\bigg).$$

The following result is proved using a Rado–Reichelderfer-type argument; cf. [21] and references therein.

Lemma 3.5. *If $f \in C(U) \cap W^{2,n}_{\mathrm{loc}}(U)$ then $\mathcal{H}^n(\Phi_f(Z)) = 0$ for every $Z \subseteq \mathcal{S}^*(f)$ such that $\mathcal{L}^n(Z) = 0$.*

Proof. Given $\mu > 0$ and $V \subseteq U$, we define $X(V,\mu)$ as the set of $x \in V$ for which there exists a polynomial function Q of degree at most 2 such that $f(y) \le Q(y)$ for every $y \in V$, $f(x) = Q(x)$, $\|DQ(x)\| \le \mu$ and $\|D^2 Q\| \le \mu$. If $D \subseteq U$ is a countable dense subset of U and $I(c) = \{s \in \mathbf{Q} : B_s(c) \subseteq U\}$ for every $c \in D$, then we notice that

$$\mathcal{S}^*(f) \subseteq \bigcup_{c\in D} \bigcup_{s\in I(c)} \bigcup_{i=1}^{\infty} X(B_s(c), i).$$

Henceforth, it is sufficient to show that $\mathcal{H}^n(\Phi_f(Z)) = 0$ whenever $Z \subseteq X(U,\mu)$ with $\mathcal{L}^n(Z) = 0$, for some $\mu > 0$. Notice that f is pointwise differentiable at each point of the set $X(V,\mu)$, by lemma 2.17.

Now we prove the following estimates: given $c \in U$ and $0 < r < 1$ such that $\overline{B_{3r}(c)} \subseteq U$, then

$$\|Df(a) - Df(b)\| \le c(n)\big(\|\mathbf{D}^2 f\|_{L^n(B_{3r}(c))} + \mu r\big) \tag{3.1}$$

$$|f(a) - f(b)| \le c(n) r\big(\|\mathbf{D}^2 f\|_{L^n(B_{3r}(c))} + \mu\big) \tag{3.2}$$

for every $a, b \in X(U,\mu) \cap B_r(c)$. We fix $a, b \in X(U,\mu) \cap B_r(c)$, $a \neq b$, and we define

$$s = \frac{|a-b|}{2} \quad \text{and} \quad d = \frac{a+b}{2}.$$

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We notice that $s \le r$,

$$B_s(d) \subseteq B_{2s}(a) \cap B_{2s}(b) \quad \text{and} \quad B_{2s}(a) \cup B_{2s}(b) \subseteq B_{3r}(c);$$

consequently it follows from (2.16) of lemma 2.17 that

$$\begin{aligned}\|\mathbf{D}f - Df(e)\|_{L^n(B_s(d))} &\le \|\mathbf{D}f - Df(e)\|_{L^n(B_{2s}(e))} \\ &\le c(n)s\big(\|\mathbf{D}^2 f\|_{L^n(B_{2s}(e))} + \mu s\big) \\ &\le c(n)s\big(|\mathbf{D}^2 f\|_{L^n(B_{3r}(c))} + \mu r\big),\end{aligned}$$

for $e \in \{a, b\}$, whence we infer

$$\begin{aligned}\boldsymbol{\alpha}(n)^{1/n} s\|Df(a) - Df(b)\| &\le \|\mathbf{D}f - Df(a)\|_{L^n(B_s(d))} + \|\mathbf{D}f - Df(b)|_{L^n(B_s(d))} \\ &\le c(n)s\big(\|\mathbf{D}^2 f\|_{L^n(B_{3r}(c))} + \mu r\big)\end{aligned}$$

and (3.1) is proved. Moreover, combining (2.17) of lemma 2.17 with (3.1)

$$\begin{aligned}&|f(a) - f(b)| \\ &\quad \le \|f - L_a\|_{L^\infty(B_s(d))} + \|f - L_b\|_{L^\infty(B_s(d))} + s\|Df(a)\| + s\|Df(b)\| \\ &\quad \le \|f - L_a\|_{L^\infty(B_{2s}(a))} + \|f - L_b\|_{L^\infty(B_{2s}(b))} + 2\mu r \\ &\quad \le c(n)r\big(\|\mathbf{D}^2 f\|_{L^n(B_{3r}(c))} + \mu\big).\end{aligned}$$

We consider the function $\overline{f} \times \nabla f$ mapping $x \in \mathrm{Diff}(f)$ into $(\overline{f}(x), \nabla f(x)) \in \mathbf{R}^{2n+1}$. Then it follows from (3.1) and (3.2) that

$$\operatorname{diam}\big[(\overline{f} \times \nabla f)\big(B_r(c) \cap X(U, \mu)\big)\big] \le c(n, \mu)\big(\|\mathbf{D}^2 f\|_{L^n(B_{3r}(c))} + r\big). \tag{3.3}$$

Let $Z \subseteq X(U, \mu)$ bounded and $\mathcal{L}^n(Z) = 0$. Given $\epsilon > 0$, we choose an open set $V \subseteq U$ such that $Z \subseteq V$, and

$$\mathcal{L}^n(V) \le \epsilon, \quad \|\mathbf{D}^2 f\|^n_{L^n(V)} \le \epsilon. \tag{3.4}$$

We define $R : Z \to \mathbf{R}$ and $\rho : Z \to \mathbf{R}$ as

$$R(x) = \inf\left\{1, \frac{\operatorname{dist}(x, \mathbf{R}^n \setminus V)}{4}\right\}, \quad \text{for } x \in Z,$$

$$\rho(x) = \operatorname{diam}\big((\overline{f} \times \nabla f)(B_{R(x)}(x) \cap X(U, \mu))\big), \quad \text{for } x \in Z.$$

We notice that R is a Lipschitzian function with $\mathrm{Lip}(R) \le \frac{1}{4}$ and, noting that $B_{3R(x)}(x) \subseteq V$ for every $x \in Z$ and combining (3.3) and (3.4), we obtain

$$\rho(x) \le c(n, \mu)\big(\|\mathbf{D}^2 f\|_{L^n(B_{3R(x)}(x))} + R(x)\big) \le c(n, \mu)\epsilon^{1/n}, \tag{3.5}$$

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for $x \in Z$. We prove now the following claim: *there exists $C \subseteq Z$ countable such that*

$$\{B_{R(y)/5}(y) : y \in C\} \text{ is disjointed,}$$

$$Z \subseteq \bigcup_{y \in C} B_{R(y)}(y),$$

and

$$\mathcal{H}^0\big(\{y \in C : B_{3R(y)}(y) \cap B_{3R(x)}(x) \neq \varnothing\}\big) \leq c(n), \quad \text{for every } x \in Z.$$

Applying Besicovitch covering theorem [3, Theorem 2.17] (see also the remark at the beginning of p. 52), there exists a positive constant $\xi(n)$ depending only on n, and there exist $Z_1, \dots, Z_{\xi(n)} \subseteq Z$ such that

$$Z \subseteq \bigcup_{i=1}^{\xi(n)} \bigcup_{x \in Z_i} B_{R(x)/5}(x),$$

and $\{B_{R(x)/5}(x) : x \in Z_i\}$ is disjointed for every $i = 1, \dots, \xi(n)$. We now apply [10, Lemma 3.1.12] with $S = Z_i$, $U = Z$, $h = \frac{R}{5}$, $\lambda = \frac{1}{20}$ and $\alpha = \beta = 15$, to infer that

$$\mathcal{H}^0\big(\{y \in Z_i : B_{3R(y)}(y) \cap B_{3R(x)}(x) \neq \varnothing\}\big) \leq c(n), \quad \text{for every } i = 1, \dots, \xi(n).$$

We define $Z' = \bigcup_{i=1}^{\xi(n)} Z_i$, and we notice that $Z \subseteq \bigcup_{x \in Z'} B_{R(x)/5}(x)$ and

$$\mathcal{H}^0\big(\{y \in Z' : B_{3R(y)}(y) \cap B_{3R(x)}(x) \neq \varnothing\}\big) \leq \xi(n)c(n), \quad \text{for every } x \in Z.$$

Now we apply Vitali covering theorem to find a countable set $C \subseteq Z'$ such that $\{B_{R(x)/5}(x) : x \in C\}$ is disjointed and

$$Z \subseteq \bigcup_{x \in C} B_{R(x)}(x),$$

which proves the claim.

Denoting with ϕ_δ the size δ approximating measure of the n-dimensional Hausdorff measure $\mathcal{H}^n$ of $\mathbf{R}^{2n+1}$ (cf. [10, 2.10.1, 2.10.2]), and combining (3.5) with the claim above and with remark 3.4, we have

$$\begin{aligned}
\phi_{c(n,\mu)\epsilon^{1/n}}((\overline{f} \times \nabla f)(Z)) &\leq c(n) \sum_{y \in C} \rho(y)^n \\
&\leq c(n,\mu) \sum_{y \in C} \Big(\|\mathbf{D}^2 f\|_{L^n(B_{3R(y)}(y))} + R(y) \Big)^n \\
&\leq c(n,\mu) \sum_{y \in C} R(y)^n + c(n,\mu) \sum_{y \in C} \int_{B_{3R(y)}(y)} \|\mathbf{D}^2 f\|^n \, d\mathcal{L}^n \\
&\leq c(n,\mu)\mathcal{L}^n(V) + c(n,\mu) \int_V \|\mathbf{D}^2 f\|^n \, d\mathcal{L}^n \\
&\leq c(n,\mu)\epsilon.
\end{aligned}$$

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Henceforth, letting $\epsilon \to 0$ we deduce that $\mathcal{H}^n((\overline{f}\times\nabla f)(Z)) = 0$. Since $\Phi_f = (\mathbf{1}_{\mathbf{R}^{n+1}} \times \psi) \circ (\overline{\mathbf{f}} \times \nabla \mathbf{f})$ (see remark 3.2), we conclude that $\mathcal{H}^n(\Phi_f(Z)) = 0$. □

LEMMA 3.6. *Suppose $U \subseteq \mathbf{R}^n$ is open, $\gamma > \frac{1}{2}$, $f \in C^{0,\gamma}(U)$, $x \in U$, $\nu \in \mathbf{S}^n \subseteq \mathbf{R}^n \times \mathbf{R}$ such that $B^{n+1}(\overline{f}(x) + s\nu, s) \cap \overline{f}(U) = \varnothing$ for some $\mathrm{s} > 0$.*

Then $\nu \notin \mathbf{R}^n \times \{0\}$. In particular, this is always true for $f \in W^{2,n}_{\mathrm{loc}}(U) \cap C(U)$.

Proof. We prove the assertion by contradiction. Suppose $0 \in U$ and $f(0) = 0$; hence there exists $c \in U \setminus \{0\}$ such that

$$B^{n+1}(c, |c|) \cap G = \varnothing \quad \text{and} \quad K := \overline{B^{n+1}(c, |c|) \cap (\mathbf{R}^n \times \{0\})} \subseteq U.$$

Suppose $L > 0$ such that $|f(x) - f(y)| \leq L|x - y|^\gamma$ for every $x, y \in K$, and define

$$h_\pm(x) = \pm\sqrt{|c|^2 - |x - c|^2} = \pm\sqrt{2c \bullet x - |x|^2}$$

for $x \in K$. Henceforth, either $h_+(x) \leq f(x)$ for every $x \in K$, or $f(x) \leq h_-(x)$ for every $x \in K$. In both cases, replacing x with tc and $0 < t < 1$, one obtains

$$t^{1-2\gamma}(2 - t) \leq L^2|c|^{2\gamma-2}$$

for $0 < t < 1$. This is clearly impossible, since $1 - 2\gamma < 0$. □

DEFINITION 3.7. *If $U \subseteq \mathbf{R}^n$ is an open set and $f : U \to \mathbf{R}$ is a function, we define*

$$E_f = \{(x, u) \in U \times \mathbf{R} : u \leq f(x)\}$$

and

$$N_f = \mathrm{nor}(E_f) \cap (U \times \mathbf{R} \times \mathbf{R}^{n+1}).$$

LEMMA 3.8. *If U is a bounded open set and $f \in W^{2,n}(U) \cap C(U)$ then*

$$N_f \cap (A \times \mathbf{R} \times \mathbf{R}^{n+1}) = \Phi_f\left[A \cap \mathcal{S}^*(f)\right] \tag{3.6}$$

for every $A \subseteq U$, and

$$\int_{N_f} \beta \, d\mathcal{H}^n = \int_U \beta(\Phi_f(x)) \, \mathrm{ap}\, J_n\Phi_f(x) \, d\mathcal{L}^n(x), \tag{3.7}$$

whenever $\beta : U \times \mathbf{R} \times \mathbf{R}^{n+1} \to \mathbf{R}$ is a $\mathcal{H}^n$-measurable non-negative function. In particular, $\mathcal{H}^n(N_f) < \infty$ and

$$\mathrm{ap}\, D\Phi_f(x)[\mathbf{R}^n] = \mathrm{Tan}^n(\mathcal{H}^n \llcorner N_f, \Phi_f(x)) \tag{3.8}$$

for $\mathcal{L}^n$ a.e. $x \in \mathcal{S}^*(f)$.

Proof. Suppose $(z, \nu) \in N_f$, where $z = (x, f(x))$ with $x \in A$, and $s > 0$ such that $B^{n+1}(z + s\nu, s) \cap E_f = \varnothing$. Since $\nu \notin \mathbf{R}^n \times \{0\}$ by lemma 3.6, we can easily see that

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there exists an open set $W \subseteq U \times \mathbf{R}$ with $z \in W$, an open set $V \subseteq U$ with $x \in V$, and a smooth function $g : V \to \mathbf{R}$ such that $f(x) = g(x)$ and

$$W \cap B^{n+1}(z + s\nu, s) = \{(y, u) : y \in V,\ u > g(y)\};$$

in particular $f(y) \leq g(y)$ for every $y \in V$. It follows that $x \in \mathcal{S}^*(f)$, $x \in \mathrm{Diff}(f)$ and $Df(x) = Dg(x)$ by lemma 2.17. Noting that

$$\nu = \frac{(-\nabla g(x), 1)}{\sqrt{1 + |\nabla g(x)|^2}}$$

we conclude $(z, \nu) = \Phi_f(x)$ and $N_f \cap (A \times \mathbf{R} \times \mathbf{R}^{n+1}) \subseteq \Phi_f(\mathcal{S}^*(f) \cap A)$.

The opposite inclusion is clear, since for every $x \in \mathcal{S}^*(f)$ there exists a polynomial function P and an open neighbourhood V of x, such that $P(x) = f(x)$ and $P(y) \geq f(y)$ for every $y \in V$.

We prove now the area formula in (3.7). Since Φ_f is a $W^{1,n}$-map (see remark 3.3), we apply lemma 2.19 with f replaced by Φ_f and we find countably many $\mathcal{L}^n$-measurable sets B_i of U such that $\mathrm{Lip}\big(\Phi_f | B_i\big) < \infty$ for every $i \geq 1$ and

$$\mathcal{L}^n\left(U \setminus \bigcup_{i=1}^{\infty} B_i\right) = 0.$$

Then we set

$$A_1 = B_1 \cap \mathcal{S}^*(f), \qquad A_i = B_i \cap \mathcal{S}^*(f) \setminus \textstyle\bigcup_{\ell=1}^{i-1} B_\ell \quad \text{for } i \geq 2$$

and we notice that

$$\mathcal{L}^n\big(U \setminus \textstyle\bigcup_{i=1}^{\infty} A_i\big) = 0 \quad \text{and} \quad \mathcal{H}^n\big(N_f \setminus \textstyle\bigcup_{i=1}^{\infty} \Phi_f(A_i)\big) = 0 \tag{3.9}$$

by lemma 2.16, lemma 3.5, and (3.6). If $\beta : U \times \mathbf{R} \times \mathbf{R}^{n+1} \to \mathbf{R}$ is a $\mathcal{H}^n$-measurable non-negative function, firstly we notice that (cf. [10, 2.4.8])

$$\int_U \beta(\Phi_f(x))\ \mathrm{ap}\, J_n \Phi_f(x)\, d\mathcal{L}^n(x) = \sum_{i=1}^{\infty} \int_{A_i} \beta(\Phi_f(x))\ \mathrm{ap}\, J_n \Phi_f(x)\, d\mathcal{L}^n(x),$$

then, recalling [10, 2.10.43], we use the area formula for Lipschitzian maps in [10, 3.2.5] and the injectivity of Φ_f to compute

$$\int_{A_i} \beta(\Phi_f(x))\ \mathrm{ap}\, J_n \Phi_f(x)\, d\mathcal{L}^n(x) = \int_{\Phi_f(A_i)} \beta(y)\, d\mathcal{H}^n(y) \qquad \text{for } i \geq 1$$

and we use (3.9) to conclude

$$\begin{aligned}\int_U \beta(\Phi_f(x))\ \mathrm{ap}\, J_n \Phi_f(x)\, d\mathcal{L}^n(x) &= \sum_{i=1}^{\infty} \int_{\Phi_f(A_i)} \beta(y)\, d\mathcal{H}^n(y) \\ &= \int_{N_f} \beta(y)\, d\mathcal{H}^n(y).\end{aligned}$$

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Choosing $\beta = 1$ we conclude that $\mathcal{H}^n(N_f) < \infty$ and we deduce that $\operatorname{Tan}^n(\mathcal{H}^n \llcorner N_f, (z,\nu))$ is an n-dimensional plane for $\mathcal{H}^n$ a.e. $(z,\nu) \in N_f$. Let D_i be the set of $x \in A_i$ such that $\boldsymbol{\Theta}^n(\mathcal{L}^n \llcorner \mathbf{R}^n \setminus A_i, x) = 0$, $\operatorname{ap} D\Phi_f(x)$ is injective and $\operatorname{Tan}^n(\mathcal{H}^n \llcorner N_f, \Phi_f(x))$ is an n-dimensional plane. Noting [10, 2.10.19] and remark 3.3, we deduce that

$$\mathcal{H}^n\big(A_i \setminus D_i\big) = 0.$$

Noting that $\operatorname{Tan}^n(\mathcal{L}^n \llcorner A_i, x) = \mathbf{R}^n$ for $x \in D_i$, and noting that $\Psi|A_i$ is a bi-lipschitz homeomorphism onto $\Phi_f(A_i)$, we employ [40, Lemma B.2] to conclude

$$\begin{aligned}\operatorname{ap} D\Phi_f(x)[\mathbf{R}^n] &= D\Psi_i(x)\big[\operatorname{Tan}^n(\mathcal{L}^n \llcorner A_i, x)\big]\\ &\subseteq \operatorname{Tan}^n(\mathcal{H}^n \llcorner \Psi_i(A_i), \Phi_f(x)) \subseteq \operatorname{Tan}^n(\mathcal{H}^n \llcorner N_f, \Phi_f(x))\end{aligned}$$

for every $x \in D_i$. Since $\operatorname{Tan}^n(\mathcal{H}^n \llcorner N_f, \Phi_f(x))$ is an n-dimensional plane and $\operatorname{ap} D\Phi_f(x)$ is injective for every $x \in D_i$, we conclude that

$$\operatorname{ap} D\Phi_f(x)[\mathbf{R}^n] = \operatorname{Tan}^n(\mathcal{H}^n \llcorner N_f, \Phi_f(x)) \quad \text{for every } x \in D_i.$$

□

THEOREM 3.9. *If $U \subseteq \mathbf{R}^n$ is a bounded open set and $f \in W^{2,n}(U) \cap C(U)$, then there exists a Borel n-vectorfield η on N_f such that*

$$(\mathcal{H}^n \llcorner N_f) \wedge \eta \text{ is a Legendrian cycle of } U \times \mathbf{R}$$

and, for $\mathcal{H}^n$ a.e. $(z,\nu) \in N_f$,

$$|\eta(z,\nu)| = 1, \quad \eta(z,\nu) \text{ is simple},$$

$$\operatorname{Tan}^n(\mathcal{H}^n \llcorner N_f, (z,\nu)) \text{ is associated with } \eta(z,\nu),$$

$$\langle \big[\textstyle\bigwedge_n \pi_0\big](\eta(z,\nu)) \wedge \nu, E' \rangle > 0.$$

Proof. We identify $\mathbf{R}^{n+1} \simeq \mathbf{R}^n \times \mathbf{R}$ and we consider the orthonormal basis $\epsilon_1, \dots, \epsilon_n$ of $\mathbf{R}^n$ such that $(\epsilon_i, 0) = e_i$ for $i = 1, \dots, n$ (cf. (2.3)). We use the notation $D_1, \dots D_n$ and $\operatorname{ap} D_1, \dots, \operatorname{ap} D_n$ for the partial derivatives and the approximate partial derivatives with respect to $\epsilon_1, \dots, \epsilon_n$, respectively. We notice by [10, 3.1.4] that $\operatorname{ap} D_i \Phi_f$ is a $\mathcal{L}^n \llcorner U$-measurable map for $i = 1, \dots, n$. Henceforth, by the classical Lusin theorem (cf. [10, 2.3.5, 2.3.6]) there exists a Borel map $\xi_i : U \to \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^{n+1}$ such that ξ_i is $\mathcal{L}^n \llcorner U$ almost equal to $\operatorname{ap} D_i \Phi_f$ for $i = 1, \dots, n$.

Since $\int_U |\xi_1 \wedge \cdots \wedge \xi_n| \, d\mathcal{L}^n < \infty$ by remark 3.3, we define

$$T(\phi) = \int_U \langle \xi_1(x) \wedge \cdots \wedge \xi_n(x), \phi(\Phi_f(x)) \rangle \, d\mathcal{L}^n(x) \tag{3.10}$$

for $\phi \in \mathcal{D}^n(U \times \mathbf{R} \times \mathbf{R}^{n+1})$ and we notice that $T \in \mathcal{D}_n(U \times \mathbf{R} \times \mathbf{R}^{n+1})$. We choose now a sequence $f_k \in C^\infty(U) \cap W^{2,n}(U)$ such that $f_k \to f$ in $W^{2,n}(U)$,

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$f_k(x) \to f(x)$ and $\mathbf{D} f_k(x) \to \mathbf{D} f(x)$ for $\mathcal{L}^n$ a.e. $x \in U$; see [11, Theorem 7.9]. Since $\Phi_{f_k} : U \to U \times \mathbf{R} \times \mathbf{R}^{n+1}$ is a smooth proper map, we define

$$T_k = (\Phi_{f_k})_\# \big((\mathcal{L}^n \llcorner U) \wedge \epsilon_1 \wedge \cdots \wedge \epsilon_n \big) \in \mathcal{D}_n(U \times \mathbf{R} \times \mathbf{R}^{n+1})$$

and we prove that

$$T_k \to T \text{ in } \mathcal{D}_n(U \times \mathbf{R} \times \mathbf{R}^{n+1}). \tag{3.11}$$

Noting that

$$T_k(\phi) = \int_U \langle D_1 \Phi_{f_k}(x) \wedge \cdots \wedge D_n \Phi_{f_k}(x), \phi(\Phi_{f_k}(x)) \rangle \, d\mathcal{L}^n(x)$$

whenever $\phi \in \mathcal{D}^n(U \times \mathbf{R} \times \mathbf{R}^{n+1})$, we estimate

$$\begin{aligned}
&|T_k(\phi) - T(\phi)| \qquad (3.12)\\
&\quad \le \int_U \big| \langle D_1 \Phi_{f_k}(x) \wedge \cdots \wedge D_n \Phi_{f_k}(x) - \xi_1(x) \wedge \cdots \wedge \xi_n(x), \phi(\Phi_{f_k}(x)) \rangle \big| \, d\mathcal{L}^n(x) \\
&\qquad + \int_U \big| \langle \xi_1(x) \wedge \cdots \wedge \xi_n(x), \phi(\Phi_{f_k}(x)) - \phi(\Phi_f(x)) \rangle \big| \, d\mathcal{L}^n(x) \\
&\quad \le \|\phi\|_{L^\infty(U)} \int_U \big| D_1 \Phi_{f_k}(x) \wedge \cdots \wedge D_n \Phi_{f_k}(x) - \xi_1(x) \wedge \cdots \wedge \xi_n(x) \big| \, d\mathcal{L}^n(x) \\
&\qquad + \int_U \operatorname{ap} J_n \Phi_f(x) \, \|\phi(\Phi_f(x)) - \phi(\Phi_{f_k}(x))\| \, d\mathcal{L}^n(x)
\end{aligned}$$

for $\phi \in \mathcal{D}^n(U \times \mathbf{R} \times \mathbf{R}^{n+1})$. Moreover, noting that $\operatorname{ap} J_n \Phi_f \in L^1(U)$ (see remark 3.3) and

$$\operatorname{ap} J_n \Phi_f(x) \, \|\phi(\Phi_f(x)) - \phi(\Phi_{f_k}(x))\| \le 2 \|\phi\|_{L^\infty(U \times \mathbf{R} \times \mathbf{R}^n)} \operatorname{ap} J_n \Phi_f(x)$$

for $\mathcal{L}^n$ a.e. $x \in U$ and for every $k \ge 1$, it follows from the dominated convergence theorem that

$$\lim_{k \to \infty} \int_U \operatorname{ap} J_n \Phi_f(x) \, \|\phi(\Phi_f(x)) - \phi(\Phi_{f_k}(x))\| \, d\mathcal{L}^n(x) = 0. \tag{3.13}$$

We observe that

$$\begin{aligned}
&D_1 \Phi_{f_k} \wedge \cdots \wedge D_n \Phi_{f_k} - \xi_1 \wedge \cdots \wedge \xi_n \\
&\quad = \sum_{i=1}^n \xi_1 \wedge \cdots \wedge \xi_{i-1} \wedge \big(D_i \Phi_{f_k} - \xi_i \big) \wedge D_{i+1} \Phi_{f_k} \wedge \cdots \wedge D_n \Phi_{f_k}
\end{aligned}$$

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and we use the generalized Holder's inequality to estimate

$$\begin{aligned}
&\int_U \big|D_1\Phi_{f_k}\wedge\cdots\wedge D_n\Phi_{f_k} - \xi_1\wedge\cdots\wedge\xi_n\big|\,d\mathcal{L}^n \qquad (3.14)\\
&\quad\leq \sum_{i=1}^n \int_U \big|\xi_1\wedge\cdots\wedge\xi_{i-1}\big|\cdot\big|\xi_i - D_i\Phi_{f_k}\big|\cdot\big|D_{i+1}\Phi_{f_k}\wedge\cdots\wedge D_n\Phi_{f_k}\big|\,d\mathcal{L}^n\\
&\quad\leq \sum_{i=1}^n \int_U \|\operatorname{ap}D\Phi_f\|^{i-1}\cdot\|\operatorname{ap}D\Phi_f - D\Phi_{f_k}\|\cdot\|D\Phi_{f_k}\|^{n-i}\,d\mathcal{L}^n\\
&\quad\leq \sum_{i=1}^n \Big(\int_U \|\operatorname{ap}D\Phi_f\|^n\,d\mathcal{L}^n\Big)^{\frac{i-1}{n}}\cdot\Big(\int_U \|\operatorname{ap}D\Phi_f - D\Phi_{f_k}\|^n\,d\mathcal{L}^n\Big)^{\frac{1}{n}}\\
&\qquad\cdot\Big(\int_U \|D\Phi_{f_k}\|^n\,d\mathcal{L}^n\Big)^{\frac{n-i}{n}}.
\end{aligned}$$

Moreover, by (3.2),

$$\begin{aligned}
&\int_U \|D\Phi_{f_k} - \operatorname{ap}D\Phi_f\|^n\,d\mathcal{L}^n\\
&\quad\leq c(n)\Big(\int_U \|D\overline{f}_k - \mathbf{D}\overline{f}\|^n\,d\mathcal{L}^n + \int_U \|D(\psi\circ\nabla f_k) - \mathbf{D}(\psi\circ\boldsymbol{\nabla} f)\|^n\,d\mathcal{L}^n\Big)\\
&\quad\leq c(n)\Big(\int_U \|D\overline{f}_k - \mathbf{D}\overline{f}\|^n\,d\mathcal{L}^n + \int_U \|D\psi(\nabla f_k)\|^n\|D^2 f_k - \mathbf{D}^2 f_k\|^n\,d\mathcal{L}^n\\
&\qquad + \int_U \|D\psi(\nabla f_k) - D\psi(\boldsymbol{\nabla} f)\|^n\|\mathbf{D}^2 f\|^n\,d\mathcal{L}^n\Big)\\
&\quad\leq c(n)\Big(\int_U \|D\overline{f}_k - \mathbf{D}\overline{f}\|^n\,d\mathcal{L}^n + \int_U \|D^2 f_k - \mathbf{D}^2 f_k\|^n\,d\mathcal{L}^n\\
&\qquad + \int_U \|D\psi(\nabla f_k) - D\psi(\boldsymbol{\nabla} f)\|^n\|\mathbf{D}^2 f\|^n\,d\mathcal{L}^n\Big)
\end{aligned}$$

and

$$\lim_{k\to\infty}\int_U \|D\psi(\nabla f_k) - D\psi(\boldsymbol{\nabla} f)\|^n\|\mathbf{D}^2 f\|^n\,d\mathcal{L}^n = 0$$

by dominated convergence theorem. Consequently, $\|D\Phi_{f_k} - \mathbf{D}\Phi_f\|_{L^n(U)} \to 0$, and combining (3.12), (3.13), and (3.14) we obtain (3.11).

Since $\partial T_k = 0$ for every $k \geq 1$, we readily infer from (3.11) that $\partial T = 0$. Define $G = [\Phi_f|\mathcal{S}^*(f)]^{-1} : N_f \to U$ and notice that G is simply the restriction on N_f of the linear function that maps a point of $\mathbf{R}^{n+1}\times\mathbf{R}^{n+1}$ onto its first n coordinates; in particular G is a Borel map (recall that N_f is a Borel set). We employ lemma 3.8 to see that

$$T(\phi) = \int_{N_f} \langle \xi\big[G(z,\nu)\big], \phi(z,\nu)\rangle\,d\mathcal{H}^n(z,\nu) \qquad (3.15)$$

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for every $\phi \in \mathcal{D}^n(U \times \mathbf{R} \times \mathbf{R}^{n+1})$, where $\xi : U \to \bigwedge_n(\mathbf{R}^{n+1} \times \mathbf{R}^{n+1})$ is the Borel map defined as

$$\xi(x) = \frac{\xi_1(x) \wedge \cdots \wedge \xi_n(x)}{|\xi_1(x) \wedge \cdots \wedge \xi_n(x)|}.$$

Then we define the Borel map

$$\eta : N_f \to \bigwedge_n(\mathbf{R}^{n+1} \times \mathbf{R}^{n+1}), \quad \eta = \xi \circ G,$$

and we readily infer that $\mathrm{Tan}^n(\mathcal{H}^n \llcorner N_f, (z, \nu))$ is associated with $\eta(z, \nu)$ for $\mathcal{H}^n$ a.e. $(z, \nu) \in N_f$ by lemma 3.8, and $(\mathcal{H}^n \llcorner N_f) \wedge \eta$ is a Legendrian cycle by lemma 2.11. Finally, we use remark 3.3 and shuffle formula (see [10, p. 19]) to compute

$$\begin{aligned}
&\left\langle \left[\textstyle\bigwedge_n \pi_0\right](\xi(x)) \wedge \psi(\nabla f(x)), E' \right\rangle \\
&\qquad = \frac{1}{\mathrm{ap}\, J_n \Phi_f(x)} \left\langle D_1\overline{f}(x) \wedge \cdots \wedge D_n\overline{f}(x) \wedge \psi(\nabla f(x)), E' \right\rangle \\
&\qquad = \frac{e'_{n+1}(\psi(\nabla f(x)))}{\mathrm{ap}\, J_n \Phi_f(x)} > 0
\end{aligned}$$

for $\mathcal{L}^n$ a.e. $x \in U$. □

4. The support of Legendrian cycles

Suppose $C \subseteq \mathbf{R}^{n+1}$. We define $\mathrm{Unp}(C)$ as the set of $x \in \mathbf{R}^{n+1} \setminus \overline{C}$ such that there exists a *unique* $y \in \overline{C}$ with $\mathrm{dist}(x, C) = |y - x|$. It is well known that $\mathbf{R}^{n+1} \setminus (\overline{C} \cup \mathrm{Unp}(C))$ is the set of points in $\mathbf{R}^{n+1} \setminus \overline{C}$, where $\mathrm{dist}(\cdot, C)$ is not differentiable, see [20, Lemma 2.41(c)] and references therein. In particular, Rademacher theorem ensures that

$$\mathcal{L}^{n+1}(\mathbf{R}^{n+1} \setminus (\overline{C} \cup \mathrm{Unp}(C))) = 0. \tag{4.1}$$

The nearest point projection ξ_C is multivalued function mapping $x \in \mathbf{R}^{n+1}$ onto

$$\xi_C(x) = \{a \in C : |a - x| = \mathrm{dist}(x, C)\}.$$

Notice that $\xi_C | \mathrm{Unp}(C)$ is single-valued and we define

$$\nu_C(x) = \frac{x - \xi_C(x)}{\mathrm{dist}(x, C)} \quad \text{and} \quad \psi_C(x) = (\xi_C(x), \nu_C(x))$$

for $x \in \mathrm{Unp}(C)$. It is well known (see [9, Theorem 4.8(4)]) that ξ_C, ν_C and ψ_C are continuous functions over $\mathrm{Unp}(C)$ and it is easy to see that

$$\mathrm{nor}(C) = \psi_C(\mathrm{Unp}(C)). \tag{4.2}$$

We also define

$$\rho_C(x) = \sup\{s > 0 : \mathrm{dist}(a + s(x - a), C) = s\mathrm{dist}(x, C)\}$$

for $x \in \mathbf{R}^{n+1} \setminus \overline{C}$ and $a \in \xi_C(x)$. This definition does not depend on the choice of $a \in \xi_C(x)$, the function $\rho_C : \mathbf{R}^{n+1} \setminus \overline{C} \to [1, \infty]$ is upper-semicontinuous and we

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set

$$\mathrm{Cut}(C) = \{x \in \mathbf{R}^{n+1} \setminus \overline{C} : \rho_C(x) = 1\};$$

see [20, Remark 2.32 and Lemma 2.33]. Finally, we define

$$S_t(C) = \{x \in \mathbf{R}^{n+1} : \mathrm{dist}(x, C) = t\} \qquad \text{for } t > 0$$

and we recall from [14, Lemma 4.2(53)] that

$$\mathcal{H}^n(S_t(C) \cap \mathrm{Unp}(C) \cap \mathrm{Cut}(C)) = 0 \qquad \text{for every } t > 0. \tag{4.3}$$

We are ready to prove the following lemma.

Lemma 4.1. *If $C \subseteq \mathbf{R}^{n+1}$ and $W \subseteq \mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$ is an open set such that $W \cap \mathrm{nor}(C) \neq \varnothing$, then $\mathcal{H}^n(W \cap \mathrm{nor}(C)) > 0$.*

Proof. Notice that $\mathbf{R}^{n+1} \setminus \overline{C} \neq \varnothing$ since $\mathrm{nor}(C) \neq \varnothing$. It follows from the continuity of ψ_C that $\psi_C^{-1}(W \cap \mathrm{nor}(C))$ is relatively open in $\mathrm{Unp}(C)$. Henceforth, we conclude from (4.1) that

$$\mathcal{L}^{n+1}\big(\psi_C^{-1}(W \cap \mathrm{nor}(C))\big) > 0.$$

We define $T = \psi_C^{-1}(W \cap \mathrm{nor}(C))$. Since $J_1\mathrm{dist}(\cdot, C)$ is $\mathcal{L}^{n+1}$ almost equal to the constant function 1 on $\mathbf{R}^{n+1} \setminus \overline{C}$, we use coarea formula to compute

$$0 < \mathcal{L}^{n+1}(T) = \int \mathcal{H}^n(T \cap S_t(C))\, dt$$

we infer there exists $\tau > 0$ so that $\mathcal{H}^n(T \cap S_\tau(C)) > 0$, and we use (4.3) to conclude

$$\mathcal{H}^n\big((T \cap S_\tau(C)) \setminus \mathrm{Cut}(C)\big) > 0.$$

Consequently there exists $s > \tau$ so that

$$\mathcal{H}^n\big(T \cap S_\tau(C) \cap \{\rho_C \geq s/\tau\}\big) > 0.$$

Since $\psi_C | S_\tau(C) \cap \{\rho_C \geq s/\tau\}$ is a bi-lipschitz homeomorphism by [14, Theorem 3.16], we conclude that

$$\mathcal{H}^n\big(\psi_C\big[T \cap S_\tau(C) \cap \{\rho_C \geq s/\tau\}\big]\big) > 0,$$

whence we infer that $\mathcal{H}^n(W \cap \mathrm{nor}(C)) > 0$. □

Remark 4.2. In relation with lemma 4.1, the following example is particularly appropriate.

Suppose $0 < \alpha < 1$ and $f : \mathbf{R}^n \to \mathbf{R}$ is a $C^{1,\alpha}$-function such that

$$\mathcal{H}^n(\{(x, f(x)) : x \in \mathbf{R}^n\} \cap B) = 0$$

whenever B is an n-dimensional C^2-submanifold of $\mathbf{R}^{n+1}$. The existence of this type of functions is proved in [33]. We define $M = \{(x, f(x)) : x \in \mathbf{R}^n\}$, we choose

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$\nu : M \to \mathbf{S}^n$ a unit-normal vector field of M of class $C^{0,\alpha}$ and we define

$$N^+ = \{(x, \nu(x)) : x \in M\} \quad \text{and} \quad N^- = \{(x, -\nu(x)) : x \in M\}.$$

Clearly, N^+ and N^- are disjointed closed n-dimensional $C^{0,\alpha}$-submanifolds without boundary of $\mathbf{R}^{n+1} \times \mathbf{S}^n$ and $\operatorname{nor}(M) \subseteq N^+ \cup N^-$. Moreover, we notice that $\mathcal{H}^n(\pi_0(\operatorname{nor}(M))) = 0$ since $\pi_0(\operatorname{nor}(M))$ is $\mathcal{H}^n$-rectifiable of class 2 by [25]. Noting that $\pi_0|N^+$ and $\pi_0|N^-$ are homeomorphisms and recalling remark 2.10, we conclude that $\operatorname{nor}(M)$ *is a countably $\mathcal{H}^n$-rectifiable subset of $N^+ \cup N^-$ with empty relative interior.*

For the next proof we recall that, for a subset $C \subseteq \mathbf{R}^{n+1}$, the normal cone $\operatorname{Nor}(C, z)$ (see [10, 3.1.21]) coincides with the cone of *regular normals* of C at z introduced in [34, Definition 6.3] (and denoted there by $\hat{N}_C(z)$), while $\operatorname{nor}(C, z)$ is the cone of proximal normals of unit length of C at z defined in [34, Example 6.16].

Lemma 4.3. *Suppose $U \subseteq \mathbf{R}^n$ is open and $f \in C(U)$ such that $\overline{\nabla f}[\operatorname{Diff}(f)]$ is a dense subset of $U \times \mathbf{R}^n$. Then $\operatorname{nor}(E_f)$ is dense in $G \times \mathbf{S}^n_+$, where $G = \{(x, f(x)) : x \in U\}$.*

Proof. First, we observe that

$$\psi(\nabla f(x)) \in \operatorname{Nor}(E_f, \overline{f}(x)) \qquad \text{for every } x \in \operatorname{Diff}(f).$$

Since ψ is a diffeomorphism of $\mathbf{R}^n$ onto $\mathbf{S}^n_+$ and f is continuous, the set $\{(\overline{f}(x), \psi(\nabla f(x))) : x \in U\}$ is dense in $G \times \mathbf{S}^n_+$; consequently we infer that $\operatorname{nor}(E_f)$ is dense in $G \times \mathbf{S}^n_+$ by standard approximation of regular normals, see [34, Exercise 6.18(a)]. □

Fu in [17, p. 2260] observed that there exist continuous functions as in lemma 4.3 that belong to $W^{2,n}(U)$. Consequently, combining lemma 4.1, lemma 4.3, and theorem 3.9, we conclude that there exists n-dimensional Legendrian cycles (of open subsets on $\mathbf{R}^{n+1}$) whose support has positive $\mathcal{H}^{2n}$-measure. This answers a question implicit in [29, Remark 2.3].

5. Reilly-type variational formulae for $W^{2,n}$-domains

In this section, we study the structure of the unit normal bundle of a $W^{2,n}$-domain (see theorem 5.7), and we prove the variational formulae for their mean curvature functions (see theorem 5.15). The latter extends the well known variational formulae obtained by Reilly in [31] for smooth domains. As a corollary Minkowski–Hsiung formulae are also proved; see theorem 5.17.

Definition 5.1. (Viscosity boundary). *Suppose $\Omega \subseteq \mathbf{R}^{n+1}$ be an open set. We define $\partial^v_+ \Omega$ to be the set of all $p \in \partial\Omega$ such that there exists $\nu \in \mathbf{S}^n$ and $r > 0$ such that*

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$$B^{n+1}(p+r\nu,r)\cap\Omega=\varnothing \quad \textit{and} \quad B^{n+1}(p-r\nu,r)\subseteq\Omega.$$

[Notice $\{p\}=\partial B(p+r\nu,r)\cap\partial B(p-r\nu,r)$*.] Clearly for each* $p\in\partial^v_+\Omega$ *the unit vector* ν *is unique. This defines an exterior unit-normal vector field on* $\partial^v_+\Omega$*,*

$$\nu_\Omega:\partial^v_+\Omega\to\mathbf{S}^n.$$

We introduce the notion of second-order rectifiability. Suppose $X\subseteq\mathbf{R}^m$ and μ is a positive integer such that $\mathcal{H}^\mu(X)<\infty$. We say that X is *$\mathcal{H}^\mu$-rectifiable of class* 2 if and only if there exists countably many μ-dimensional submanifolds $\Sigma_i\subseteq\mathbf{R}^m$ of class 2 such that

$$\mathcal{H}^\mu\big(X\setminus\textstyle\bigcup_{i=1}^\infty\Sigma_i\big)=0.$$

LEMMA 5.2. *Suppose* $X\subseteq\mathbf{R}^{n+1}$ *is* $\mathcal{H}^n$*-measurable and* $\mathcal{H}^n$*-rectifiable of class* 2*, and* $\nu:X\to\mathbf{S}^n$ *is a* $\mathcal{H}^n\llcorner X$*-measurable map such that*

$$\nu(a)\in\operatorname{Nor}^n(\mathcal{H}^n\llcorner X,a)\quad\textit{for } \mathcal{H}^n \textit{ a.e. } a\in X.$$

Then there exist countably many $\mathcal{H}^n$*-measurable sets* $X_i\subseteq X$ *such that* $\mathcal{H}^n\big(X\setminus\bigcup_{i=1}^\infty X_i\big)=0$ *and* $\operatorname{Lip}(\nu|X_i)<\infty$*; moreover,* ν *is* $\mathcal{H}^n\llcorner X$*-approximately differentiable at* $\mathcal{H}^n$ *a.e.* $a\in X$ *and* $\operatorname{ap}D\nu(a)$ *is a symmetric endomorphism of* $\operatorname{Tan}^n(\mathcal{H}^n\llcorner X,a)$*.*

Proof. Suppose $\{\Sigma_i\}_{i\geq1}$ is a countable family of C^2-hypersurfaces such that

$$\mathcal{H}^n\big(X\setminus\textstyle\bigcup_{i=1}^\infty\Sigma_i\big)=0$$

and $\eta_i:\Sigma_i\to\mathbf{S}^n$ is a continuously differentiable unit-normal vector field with $\operatorname{Lip}(\eta_i)<\infty$ for $i\geq1$. By [10, 2.10.19(4)]

$$\Theta^n(\mathcal{H}^n\llcorner X\setminus\Sigma_i,a)=\Theta^n(\mathcal{H}^n\llcorner\Sigma_i\setminus X,a)=0\quad\text{for } \mathcal{H}^n \text{ a.e. } a\in\Sigma_i\cap X,$$

whence we infer that $\operatorname{Tan}^n(\mathcal{H}^n\llcorner X,a)=\operatorname{Tan}(\Sigma_i,a)$ for $\mathcal{H}^n$ a.e. $a\in\Sigma_i\cap X$. In particular, if we define

$$\Sigma_i^+=\{a\in\Sigma_i\cap X:\eta_i(a)=\nu(a)\}\quad\text{and}\quad\Sigma_i^-=\{a\in\Sigma_i\cap X:\eta_i(a)=-\nu(a)\}$$

we infer that $\mathcal{H}^n\big(\Sigma_i\cap X\setminus(\Sigma_i^+\cup\Sigma_i^-)\big)=0$ for each $i\geq1$. Moreover, employing again [10, 2.10.19(4)] we infer that

$$\Theta^n(\mathcal{H}^n\llcorner X\setminus\Sigma_i^\pm,a)=0\quad\text{for } \mathcal{H}^n \text{ a.e. } a\in\Sigma_i^\pm,$$

whence we deduce that ν is $\mathcal{H}^n\llcorner X$ approximately differentiable at a with

$$\operatorname{ap}D\nu(a)=\pm D\eta_i(a)$$

for $\mathcal{H}^n$ a.e. $a\in\Sigma_i^\pm$. Finally, since $D\eta_i(a)$ is symmetric, we conclude the proof. □

We introduce now the class of $W^{2,n}$-domains in a slightly more general fashion than the notion given in the Introduction.

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DEFINITION 5.3. *An open set $\Omega \subseteq \mathbf{R}^{n+1}$ is a $W^{2,n}$-domain if and only there exists a couple (Ω', F), where*

(1) $\Omega' \subseteq \mathbf{R}^{n+1}$ *is an open set such that for each $p \in \partial\Omega'$ there exist $\epsilon > 0$, $\nu \in \mathbf{S}^n$, a bounded open set $U \subseteq \nu^\perp$ with $0 \in U$ and a continuous function $f \in W^{2,n}(U)$ with $f(0) = 0$ such that*

$$\{p + b + \tau\nu : b \in U, -\epsilon < \tau \le f(b)\} = \overline{\Omega'} \cap \{p + b + \tau\nu : b \in U, -\epsilon < \tau < \epsilon\},$$

(2) *F is a C^2-diffeomorphism defined over an open set $V \subseteq \mathbf{R}^{n+1}$ such that $\overline{\Omega'} \subseteq V$, and*

(3) $F(\Omega') = \Omega$.

REMARK 5.4. This class of domains is invariant under images of C^2-diffeomorphisms, which is clearly a necessary condition in order to provide a natural framework to generalize Reilly's variational formulae. We do not know if we really need to introduce the diffeomorphism F in the definition above; in other words, if Ω' belongs to the class $\mathcal{S}$ of domains satisfying only condition (1) of definition 5.3, is it true that $F(\Omega')$ belongs to $\mathcal{S}$ too?

We collect some basic properties of $W^{2,n}$-domains.

LEMMA 5.5. *If $\Omega \subseteq \mathbf{R}^{n+1}$ is a $W^{2,n}$-domain, then the following statements hold.*

(1) *$\mathcal{H}^n(\partial\Omega \setminus \partial^v_+\Omega) = 0$ and $K \cap \partial\Omega$ is $\mathcal{H}^n$-rectifiable of class 2 for every compact set $K \subseteq \mathbf{R}^{n+1}$.*

(2) *For $\mathcal{H}^n$ a.e. $p \in \partial\Omega$,*

$$\operatorname{Tan}^n(\mathcal{H}^n \llcorner \partial\Omega, p) = \operatorname{Tan}(\partial\Omega, p) = \nu_\Omega(p)^\perp.$$

(3) *For every $p \in \partial^v_+\Omega$,*

$$\operatorname{Tan}^{n+1}(\mathcal{L}^{n+1} \llcorner \Omega, p) = \operatorname{Tan}(\Omega, p) = \{v \in \mathbf{R}^{n+1} : v \bullet \nu_\Omega(p) \le 0\}.$$

Proof. Suppose $\Omega = F(\Omega')$, where Ω' and F are as in definition 5.3. Clearly, $F(\partial\Omega') = \partial\Omega$ and $F(\partial^v_+\Omega') = \partial^v_+\Omega$. Therefore, assertion (1) follows from theorem 2.16, theorem 7.6, and remark 7.7.

If $p \in \partial^v_+\Omega$, we have $\operatorname{Tan}(\partial\Omega, p) \subseteq \nu_\Omega(p)^\perp$; since $\operatorname{Tan}^n(\mathcal{H}^n \llcorner \partial\Omega, p) \subseteq \operatorname{Tan}(\partial\Omega, p)$ for every $p \in \partial\Omega$ and $\operatorname{Tan}^n(\mathcal{H}^n \llcorner \partial\Omega, p)$ is an n-dimensional plane for $\mathcal{H}^n$ a.e. $p \in \partial\Omega$, we obtain (2). Finally it follows from definitions that $\operatorname{Tan}(\Omega, p) = \{v \in \mathbf{R}^{n+1} : v \bullet \nu_\Omega(p) \le 0\}$ and $\operatorname{Tan}^{n+1}(\mathcal{L}^{n+1} \llcorner \Omega, p) = \{v \in \mathbf{R}^{n+1} : v \bullet \nu_\Omega(p) \le 0\}$ for every $p \in \partial^v_+\Omega$. □

By lemma 5.2 the map ν_Ω is $\mathcal{H}^n \llcorner \partial\Omega$-approximately differentiable with a symmetric approximate differential $\operatorname{ap} D\nu_\Omega(x)$ at $\mathcal{H}^n$ a.e. $x \in \partial\Omega$. Consequently we introduce the following definition.

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DEFINITION 5.6. (Approximate principal curvatures). *Suppose $\Omega \subseteq \mathbf{R}^{n+1}$ is a $W^{2,n}$-domain. The* approximate principal curvatures *of Ω are the* **R***-valued $(\mathcal{H}^n \llcorner \partial\Omega)$-measurable maps*

$$\chi_{\Omega,1}, \ldots, \chi_{\Omega,n},$$

defined so that $\chi_{\Omega,1}(p) \leq \ldots \leq \chi_{\Omega,n}(p)$ are the eigenvalues of $\operatorname{ap} D\nu_\Omega(p)$ for $\mathcal{H}^n$ a.e. $p \in \partial\Omega$.

We prove now the main structure theorem for the unit normal bundle $\operatorname{nor}(\Omega)$ of a $W^{2,n}$-domain.

THEOREM 5.7 *If $\Omega \subseteq \mathbf{R}^{n+1}$ is a $W^{2,n}$-domain then the following statements hold.*

(1) *$\mathcal{H}^n(\overline{\nu_\Omega}(Z)) = 0$, whenever $Z \subseteq \partial^v_+\Omega$ with $\mathcal{H}^n(Z) = 0$.*
(2) *$\mathcal{H}^n\big(\operatorname{nor}(\Omega) \setminus \overline{\nu_\Omega}(\partial^v_+\Omega)\big) = 0$.*
(3) *$\kappa_{\Omega,i}(x,u) = \chi_{\Omega,i}(x)$ for every $i = 1,\ldots,n$ and for $\mathcal{H}^n$ a.e. $(x,u) \in \operatorname{nor}(\Omega)$. In particular, $\kappa_{\Omega,i}(x,u) < \infty$ for $\mathcal{H}^n$ a.e. $(x,u) \in \operatorname{nor}(\Omega)$.*
(4) *If $\partial\Omega$ is compact, then $\mathcal{H}^n(\operatorname{nor}(\Omega)) < \infty$ and there exists a unique Legendrian cycle T of $\mathbf{R}^{n+1}$ such that*

$$T = (\mathcal{H}^n \llcorner \operatorname{nor}(\Omega)) \wedge \eta,$$

where η is a $\mathcal{H}^n \llcorner \operatorname{nor}(\Omega)$ measurable n-vectorfield such that

$$|\eta(x,u)| = 1, \quad \eta(x,u) \text{ is simple,}$$

$$\operatorname{Tan}^n(\mathcal{H}^n \llcorner \operatorname{nor}(\Omega), (x,u)) \text{ is associated with } \eta(x,u)$$

and

$$\langle \big[\textstyle\bigwedge_n \pi_0\big](\eta(x,u)) \wedge u, E'\rangle > 0$$

for $\mathcal{H}^n$ a.e. $(x,u) \in \operatorname{nor}(\Omega)$. In this case, $\eta = \zeta_1 \wedge \ldots \wedge \zeta_n$, where

$$\zeta_i = \left(\frac{1}{\sqrt{1+\kappa^2_{\Omega,i}}}\tau_i, \frac{\kappa_{\Omega,i}}{\sqrt{1+\kappa^2_{\Omega,i}}}\tau_i \right) \quad \text{for } i = 1,\ldots,n$$

and $\tau_1(x,u),\ldots,\tau_n(x,u)$ are an orthonormal basis of $u^\perp$ such that $\tau_1(x,u) \wedge \ldots \wedge \tau_n(x,u) \wedge u = E$ for $\mathcal{H}^n$ a.e. $(x,u) \in \operatorname{nor}(\Omega)$.

Proof. Suppose $\Omega = F(\Omega')$, where Ω' and F are as in definition 5.3. We recall the definition of Ψ_F from (2.13) and notice that

$$\Psi_F(\operatorname{nor}(\Omega')) = \operatorname{nor}(\Omega) \tag{5.1}$$

by [41, Lemma 2.1]. Since $F(\partial^v_+\Omega') = \partial^v_+\Omega$, we readily infer from (5.1) that

$$\Psi_F(x, \nu_{\Omega'}(x)) = (F(x), \nu_\Omega(F(x))) \quad \text{for every } x \in \partial^v_+\Omega'$$

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and

$$\Psi_F\big(\overline{\nu_{\Omega'}}(F^{-1}(S))\big) = \overline{\nu_\Omega}(S) \quad \text{for every } S \subseteq \partial^v_+\Omega. \tag{5.2}$$

To prove the assertions in (1) and (2) we notice, firstly, that they are true for Ω' as a consequence of lemma 3.5 and (3.6) of lemma 3.8; then we apply (5.1) and (5.2).

To prove (3) we first employ lemma 5.2 to find a countable family $X_i \subseteq \partial^v_+\Omega$ such that $\mathcal{H}^n\big(\partial\Omega \setminus \bigcup_{i=1}^\infty X_i\big) = 0$ and $\mathrm{Lip}(\nu_\Omega|X_i) < \infty$ for every $i \geq 1$; then we define Y_i to be the set of $x \in X_i$ such that ν_Ω is $\mathcal{H}^n \llcorner \partial\Omega$ approximately differentiable at x, $\mathrm{Tan}^n(\mathcal{H}^n \llcorner \partial\Omega, x)$ and $\mathrm{Tan}^n(\mathcal{H}^n \llcorner \mathrm{nor}(\Omega), \overline{\nu_\Omega}(x))$ are n-dimensional planes, and $\Theta^n(\mathcal{H}^n \llcorner \partial\Omega \setminus X_i, x) = 0$. We notice that $\overline{\nu_\Omega}|X_i$ is bi-lipschitz and, since $\mathrm{Tan}^n(\mathcal{H}^n \llcorner \mathrm{nor}(\Omega), (x,u))$ is an n-dimensional plane for $\mathcal{H}^n$ a.e. $(x,u) \in \mathrm{nor}(\Omega)$, we conclude that

$$\mathcal{H}^n(X_i \setminus Y_i) = 0 \quad \text{for every } i \geq 1.$$

It follows from (1) and (2) that

$$\mathcal{H}^n\Big(\mathrm{nor}(\Omega) \setminus \bigcup_{i=1}^\infty \overline{\nu_\Omega}(Y_i)\Big) = 0. \tag{5.3}$$

We fix now $x \in Y_i$. Then there exists a map $g : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$ pointwise differentiable at x such that $\Theta^n(\mathcal{H}^n \llcorner \partial\Omega \setminus \{g = \overline{\nu_\Omega}\}, x) = 0$ and $\mathrm{ap}\, D\overline{\nu_\Omega}(x) = Dg(x)|\,\mathrm{Tan}^n(\mathcal{H}^n \llcorner \partial\Omega, x)$; noting that $\mathrm{ap}\, D\overline{\nu_\Omega}(x)$ is injective, $g|X_i \cap \{g = \overline{\nu_\Omega}\}$ is bi-lipschitz and

$$\mathrm{Tan}^n(\mathcal{H}^n \llcorner \partial\Omega, x) = \mathrm{Tan}^n(\mathcal{H}^n \llcorner X_i \cap \{g = \overline{\nu_\Omega}\}, x),$$

we readily infer by [40, Lemma B.2] that

$$\mathrm{ap}\, D\overline{\nu_\Omega}(x)[\mathrm{Tan}^n(\mathcal{H}^n \llcorner \partial\Omega, x)] = \mathrm{Tan}^n(\mathcal{H}^n \llcorner \mathrm{nor}(\Omega), \overline{\nu_\Omega}(x)).$$

Henceforth, if $\tau_1, \dots, \tau_n$ is an orthonormal basis of $\mathrm{Tan}^n(\mathcal{H}^n \llcorner \partial\Omega, x)$ with $\mathrm{ap}\, D\nu_\Omega(x)(\tau_i) = \chi_{\Omega,i}(x)\tau_i$ for $i = 1, \dots, n$, we conclude that

$$\left\{\left(\frac{1}{\sqrt{1+\chi_{\Omega,i}(x)^2}}\tau_i, \frac{\chi_{\Omega,i}(x)}{\sqrt{1+\chi_{\Omega,i}(x)^2}}\tau_i\right) : i = 1, \dots, n\right\}$$

is an orthonormal basis of $\mathrm{Tan}^n(\mathcal{H}^n \llcorner \mathrm{nor}(\Omega), \overline{\nu_\Omega}(x))$. Since x is arbitrarily chosen in Y_i, thanks to (5.3), we deduce from the uniqueness stated in lemma 2.11 that

$$\kappa_{\Omega,i}(x,u) = \chi_{\Omega,i}(x) \quad \text{for } \mathcal{H}^n \text{ a.e. } (x,u) \in \mathrm{nor}(\Omega).$$

Finally, we prove (4). By lemma 2.11 we can choose maps $\tau_1, \dots, \tau_n$ defined $\mathcal{H}^n$ a.e. on $\mathrm{nor}(\Omega')$ such that $\tau_1(x,u), \dots, \tau_n(x,u), u$ is an orthonormal basis of $\mathbf{R}^{n+1}$,

$$\tau_1(x,u) \wedge \cdots \tau_n(x,u) \wedge u = e_1 \wedge \cdots \wedge e_{n+1} \quad \text{for } \mathcal{H}^n \text{a.e. } (x,u) \in \mathrm{nor}(\Omega') \tag{5.4}$$

and the vectors

$$\zeta_i'(x,u) = \left(\frac{1}{\sqrt{1+\kappa_{\Omega',i}(x,u)^2}}\tau_i(x,u), \frac{\kappa_{\Omega',i}(x,u)}{\sqrt{1+\kappa_{\Omega',i}(x,u)^2}}\tau_i(x,u)\right), \quad i = 1, \dots, n,$$

form an orthonormal basis of $\operatorname{Tan}^n(\mathcal{H}^n \llcorner \operatorname{nor}(\Omega'), (x,u))$ for $\mathcal{H}^n$ a.e. $(x,u) \in \operatorname{nor}(\Omega')$. Then we define

$$\eta' = \zeta_1' \wedge \cdots \wedge \zeta_n'$$

and notice that

$$|\eta'(x,u)| = 1, \quad \eta'(x,u) \text{ is simple},$$

$$\operatorname{Tan}^n(\mathcal{H}^n \llcorner \operatorname{nor}(\Omega'), (x,u)) \text{ is associated with } \eta'(x,u)$$

and (see (2.2) and (2.5))

$$\langle [\textstyle\bigwedge_n \pi_0](\eta'(x,u)) \wedge u, E' \rangle > 0 \quad \text{(by 5.4)} \tag{5.5}$$

for $\mathcal{H}^n$ a.e. $(x,u) \in \operatorname{nor}(\Omega')$. If $p \in \partial\Omega'$, $\epsilon > 0$, $\nu \in \mathbf{S}^n$, $U \subseteq \nu^\perp$ is a bounded open set with $0 \in U$ and $f \in W^{2,n}(U)$ is a continuous function with $f(0) = 0$ such that

$$\{p + b + \tau\nu : b \in U, \, -\epsilon < \tau \leq f(b)\} = \overline{\Omega'} \cap C_{U,\epsilon},$$

where $C_{U,t} = \{p + b + \tau\nu : b \in U, \, -t < \tau < t\}$ for each $0 < t \leq \infty$, then we observe that

$$N_f = \operatorname{nor}(\Omega') \cap (C_{U,\epsilon} \times \mathbf{S}^n),$$

where $N_f = \operatorname{nor}(E_f) \cap (C_{U,\infty} \times \mathbf{S}^n)$ and $E_f = \{p + b + \tau\nu : b \in U, \, -\infty < \tau \leq f(b)\}$. It follows from (5.5) and theorem 3.9 that $\eta' | [\operatorname{nor}(\Omega') \cap (C_{U,\epsilon} \times \mathbf{S}^n)]$ is $\mathcal{H}^n$ almost equal to a Borel n-vectorfield defined over $\operatorname{nor}(\Omega') \cap (C_{U,\epsilon} \times \mathbf{S}^n)$ and $(\mathcal{H}^n \llcorner [\operatorname{nor}(\Omega') \cap (C_{U,\epsilon} \times \mathbf{S}^n)]) \wedge \eta'$ is an n-dimensional Legendrian cycle of $C_{U,\epsilon}$. Henceforth, we define the integer multiplicity locally rectifiable n-current

$$T' = \big(\mathcal{H}^n \llcorner \operatorname{nor}(\Omega')\big) \wedge \eta'$$

and we conclude by lemma 2.5 that $T^{'}$ is a Legendrian cycle of $\mathbf{R}^{n+1}$.

We define now $\psi = \Psi_F | \operatorname{nor}(\Omega')$ and recalling (5.1) and noting that

$$\operatorname{ap} D\psi(\psi^{-1}(y,v)) = D\Psi_F(\psi^{-1}(y,v)) \tag{5.6}$$

for $\mathcal{H}^n$ a.e. $(y,v) \in \operatorname{nor}(\Omega)$, we define

$$\eta(y,v) = \frac{[\bigwedge_n \operatorname{ap} D\psi(\psi^{-1}(y,v))]\eta'(\psi^{-1}(y,v))}{J_n^{\operatorname{nor}(\Omega')}\psi(\psi^{-1}(y,v))}$$

for $\mathcal{H}^n$ a.e. $(y,v) \in \operatorname{nor}(\Omega)$. Since Ψ_F is a diffeomorphism we have that $\eta(y,v) \neq 0$ for $\mathcal{H}^n$ a.e. $(y,v) \in \operatorname{nor}(\Omega)$. We now apply [10, 4.1.30] with U, K, W, ξ, G and g replaced by $\mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$, $\partial\Omega' \times \mathbf{S}^n$, $\operatorname{nor}(\Omega')$, Ψ_F and ψ respectively. We infer that

$$(\Psi_F)_\# [(\mathcal{H}^n \llcorner \operatorname{nor}(\Omega')) \wedge \eta'] = \big(\mathcal{H}^n \llcorner \operatorname{nor}(\Omega)\big) \wedge \eta$$

and that $|\eta(y,v)| = 1$ and $\operatorname{Tan}^n(\mathcal{H}^n \llcorner \operatorname{nor}(\Omega), (y,v))$ is associated with $\eta(y,v)$ for $\mathcal{H}^n$ a.e. $(y,v) \in \operatorname{nor}(\Omega)$. Clearly, $\big(\mathcal{H}^n \llcorner \operatorname{nor}(\Omega)\big) \wedge \eta$ is a cycle, and $\big[\big(\mathcal{H}^n \llcorner \operatorname{nor}(\Omega)\big) \wedge$

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$\eta] \llcorner \alpha = 0$ by lemma 2.11. Finally, if $*$ is the Hodge-star operator with respect to E (cf. remark 5.8), since $\tau_1(x,u) \wedge \cdots \wedge \tau_n(x,u) = (-1)^n\,(*u)$ and

$$\begin{aligned}
&\big[\textstyle\bigwedge_n \pi_0\big](\eta(\Psi_F(x,u))) \\
&\quad = \frac{1}{J_n^{\mathrm{nor}(\Omega')}\psi(x,u)} \left(\prod_{i=1}^n \frac{1}{\sqrt{1+\kappa_{\Omega',i}(x,u)^2}}\right) \\
&\qquad \big[DF(x)(\tau_1(x,u)) \wedge \cdots \wedge DF(x)(\tau_n(x,u))\big] \\
&\quad = \frac{(-1)^n}{J_n^{\mathrm{nor}(\Omega')}\psi(x,u)} \left(\prod_{i=1}^n \frac{1}{\sqrt{1+\kappa_{\Omega',i}(x,u)^2}}\right) \big[\textstyle\bigwedge_n DF(x)\big](*u)
\end{aligned}$$

for $\mathcal{H}^n$ a.e. $(x,u) \in \mathrm{nor}(\Omega')$, it follows by remark 5.8 below and (5.1) that either

$$\langle \big[\textstyle\bigwedge_n \pi_0\big](\eta(y,v)) \wedge v, E' \rangle > 0 \quad \text{for } \mathcal{H}^n \text{ a.e. } (y,v) \in \mathrm{nor}(\Omega)$$

or

$$\langle \big[\textstyle\bigwedge_n \pi_0\big](\eta(y,v)) \wedge v, E' \rangle < 0 \quad \text{for } \mathcal{H}^n \text{a.e. } (y,v) \in \mathrm{nor}(\Omega).$$

This settles the existence part in statement (4). Uniqueness easily follows from the defining conditions of T and the representation of η follows from lemma 2.11. □

Remark 5.8. Let $* : \mathbf{R}^{n+1} \to \bigwedge_n \mathbf{R}^{n+1}$ be the Hodge-star operator, taken with respect to E; cf. [10, 1.7.8]. We notice that if $u \in \mathbf{S}^n$ and $\tau_1, \ldots, \tau_n$ is an orthonormal basis of $u^\perp$ such that $u \wedge \tau_1 \wedge \cdots \wedge \tau_n = E$, then it follows from the shuffle formula [10, p. 18] that

$$*u = \tau_1 \wedge \cdots \wedge \tau_n.$$

Using this remark, we prove that *if* $F : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ *is a diffeomorphism, then either*

$$\langle \big[\textstyle\bigwedge_n DF(x)\big](*u) \wedge (DF(x)^{-1})^*(u), E' \rangle > 0 \quad \text{for every } (x,u) \in \mathbf{R}^{n+1} \times \mathbf{S}^n$$

or

$$\langle \big[\textstyle\bigwedge_n DF(x)\big](*u) \wedge (DF(x)^{-1})^*(u), E' \rangle < 0 \quad \text{for every } (x,u) \in \mathbf{R}^{n+1} \times \mathbf{S}^n.$$

By contradiction, assume that there exists $(x,u) \in \mathbf{R}^{n+1} \times \mathbf{S}^n$ such that

$$\langle \big[\textstyle\bigwedge_n DF(x)\big](*u) \wedge (DF(x)^{-1})^*(u), E' \rangle = 0$$

and choose an orthonormal basis $\tau_1, \ldots, \tau_n$ of $u^\perp$ such that $u \wedge \tau_1 \wedge \cdots \wedge \tau_n = e_1 \wedge \cdots \wedge e_{n+1}$. Henceforth, $DF(x)(\tau_1) \wedge \cdots \wedge DF(x)(\tau_n) \wedge (DF(x)^{-1})^*(u) = 0$ and, since $\{DF(x)(\tau_i) : i = 1, \ldots n\}$ are linearly independent, we conclude that there

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exists $c_1, \dots, c_n \in \mathbf{R}$ such that

$$(DF(x)^{-1})^*(u) = \sum_{i=1}^{n} c_i\, DF(x)(\tau_i).$$

Applying $DF(x)^{-1}$ to both sides and taking the scalar product with u, we get

$$\big[DF(x)^{-1} \circ (DF(x)^{-1})^*\big](u) \bullet u = 0,$$

whence we infer that $(DF(x)^{-1})^*(u) = 0$, a contradiction.

Definition 5.9. *Suppose $\Omega \subseteq \mathbf{R}^{n+1}$ is a $W^{2,n}$-domain. We denote by N_Ω the Legendrian cycle given by* theorem 5.7(4)*.*

Remark 5.10. The proof of theorem 5.7(4) proves that *if $F : U \to V$ is a C^2-diffeomorphism between open subsets of $\mathbf{R}^{n+1}$ and Ω is a bounded $W^{2,n}$-domain such that $\overline{\Omega} \subseteq U$, then*

$$(\Psi_F)_\#(N_\Omega) = N_{F(\Omega)}.$$

Definition 5.11. (r-th elementary symmetric function). *Suppose $r \in \{1, \dots, n\}$. The r-th symmetric function $\sigma_r : \mathbf{R}^n \to \mathbf{R}$ is defined as*

$$\sigma_r(t_1, \dots, t_n) = \frac{1}{\binom{n}{r}} \sum_{\lambda \in \Lambda_{n,r}} t_{\lambda(1)} \cdots t_{\lambda(r)},$$

where $\Lambda_{n,r}$ is the set of all increasing functions from $\{1, \dots, r\}$ to $\{1, \dots, n\}$. We set

$$\sigma_0(t_1, \dots, t_n) = 1 \quad \textit{for } (t_1, \dots, t_n) \in \mathbf{R}^n.$$

Definition 5.12. (r-th mean curvature function). *Suppose $\Omega \subseteq \mathbf{R}^{n+1}$ is a $W^{2,n}$-domain and $r \in \{0, \dots, n\}$. Then we define the r-th* mean curvature function of Ω *as*

$$H_{\Omega,r}(z) = \sigma_r(\chi_{\Omega,1}(z), \dots, \chi_{\Omega,n}(z))$$

for $\mathcal{H}^n$ a.e. $z \in \partial\Omega$.

Lemma 5.13. *If $\Omega \subseteq \mathbf{R}^{n+1}$ is a bounded $W^{2,n}$-domain and ϕ is a smooth $\mathbf{R}$-valued function over $\mathbf{R}^{n+1} \times \mathbf{S}^n$ then*

$$[N_\Omega \llcorner \varphi_{n-k}](\phi) = \binom{n}{k} \int_{\partial\Omega} H_{\Omega,k}(x)\, \phi(x, \nu_\Omega(x))\, d\mathcal{H}^n(x) \quad \textit{for } k = 0, \dots, n.$$

Proof. We know by theorem 5.7(4) that $N_\Omega = (\mathcal{H}^n \llcorner \operatorname{nor}(\Omega)) \wedge (\zeta_1 \wedge \dots \wedge \zeta_n)$. Noting that

$$J_n^{\operatorname{nor}(\Omega)} \pi_0(x, u) = \prod_{i=1}^{n} \frac{1}{\sqrt{1 + \kappa_{\Omega,i}(x, u)^2}} \quad \text{for } \mathcal{H}^n \text{ a.e. } (x, u) \in \operatorname{nor}(\Omega),$$

we employ theorem 5.7(3) to compute

$$[N_\Omega \llcorner \varphi_{n-k}](\phi) = \binom{n}{k} \int_{\mathrm{nor}(\Omega)} J_n^{\mathrm{nor}(\Omega)} \pi_0(x,u)\, \phi(x,u)\, H_{\Omega,k}(x)\, d\mathcal{H}^n(x,u),$$

whence we conclude using area formula in combination with theorem 5.7(2) and lemma 5.5(1). □

DEFINITION 5.14. (r-th total curvature measure). *If $\Omega \subseteq \mathbf{R}^{n+1}$ is a bounded $W^{2,n}$-domain and $r = 0, \dots, n$, we define*

$$\mathcal{A}_r(\Omega) = \int_{\partial\Omega} H_{\Omega,r}\, d\mathcal{H}^n.$$

Now we can quickly derive the following extension of Reilly's variational formulae (cf. [31]) to $W^{2,n}$-domain.

THEOREM 5.15. *Suppose $\Omega \subseteq \mathbf{R}^{n+1}$ is a bounded $W^{2,n}$-domain and $(F_t)_{t\in I}$ is a local variation of $\mathbf{R}^{n+1}$ with initial velocity vector field V. Then*

$$\frac{d}{dt}\mathcal{A}_{k-1}(F_t(\Omega))\Big|_{t=0} = (n-k+1)\int_{\partial\Omega} H_{\Omega,k}\,(\nu_\Omega \bullet V)\, d\mathcal{H}^n \quad \textit{for } k = 1, \dots, n$$

and

$$\frac{d}{dt}\mathcal{A}_n(F_t(\Omega))\Big|_{t=0} = 0. \tag{5.7}$$

Proof. Combining remark 5.10 and lemma 5.13 we obtain

$$\big[(\Psi_{F_t})_\# N_\Omega\big](\varphi_{n-k+1}) = N_{F_t(\Omega)}(\varphi_{n-k+1}) = \binom{n}{k-1}\mathcal{A}_{k-1}(F_t(\Omega))$$

for $k = 1, \dots, n+1$. Hence we use lemma 2.8 and again lemma 5.13 to compute

$$\frac{d}{dt}\big[(\Psi_{F_t})_\# N_\Omega\big](\varphi_{n-k+1})\Big|_{t=0} = k\binom{n}{k}\int_{\partial\Omega}(V(x)\bullet\nu_\Omega(x))\, H_{\Omega,k}(x)\, d\mathcal{H}^n(x)$$

for $k = 1, \dots, n$ and

$$\frac{d}{dt}\big[(\Psi_{F_t})_\# N_\Omega\big](\varphi_0)\Big|_{t=0} = 0.$$

□

REMARK 5.16. If Ω is a C^2-domain, then (5.7) follows from the Gauss–Bonnet theorem. The validity of the Gauss–Bonnet theorem for bounded $W^{2,n}$-domains is an interesting open question, and (5.7) seems to point to a possible positive answer.

The following integral formulae can be easily deduced from theorem 5.15 by a standard procedure. For the C^2-regular domains these formulae are classic, see [13].

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Corollary 5.17. *If $\Omega \subseteq \mathbf{R}^{n+1}$ is a bounded $W^{2,n}$-domain and $r \in \{1, \dots, n\}$ then*

$$\int_{\partial\Omega} H_{\Omega,r-1}(x)\, d\mathcal{H}^n(x) = \int_{\partial\Omega} (x \bullet \nu_\Omega(x))\, H_{\Omega,r}(x)\, d\mathcal{H}^n(x).$$

Proof. We consider the local variation $F_t(x) = e^t x$ for $(x,t) \in \mathbf{R}^n \times \mathbf{R}$ and we notice that

$$\mathrm{Tan}^n(\mathcal{H}^n \llcorner \partial\Omega, x) = \mathrm{Tan}^n(\mathcal{H}^n \llcorner F_t(\partial\Omega), F_t(x))$$

$$\nu_{F_t(\Omega)}(F_t(x)) = \nu_\Omega(x) \quad \text{and} \quad \chi_{F_t(\Omega),i}(F_t(x)) = e^{-t}\chi_{\Omega,i}(x)$$

for $\mathcal{H}^n$ a.e. $x \in \partial\Omega$ and $i = 1, \dots, n$. Henceforth, we compute by area formula

$$\begin{aligned}\mathcal{A}_{r-1}(F_t(\Omega)) &= \int_{\partial F_t(\Omega)} H_{F_t(\Omega),r-1}\, d\mathcal{H}^n \\ &= e^{-(r-1)t} \int_{F_t(\partial\Omega)} H_{\Omega,r-1}(F_t^{-1}(y))\, d\mathcal{H}^n(y) \\ &= e^{(n-r+1)t} \int_{\partial\Omega} H_{\Omega,r-1}(x)\, d\mathcal{H}^n(x)\end{aligned}$$

and we apply theorem 5.15. □

Corollary 5.18. *Suppose $\Omega \subseteq \mathbf{R}^{n+1}$ is a bounded $W^{2,n}$-domain, $k \in \{1, \dots, n\}$ and*

$$H_{\Omega,i}(z) \geq 0 \quad \text{for } i = 1, \dots, k-1 \text{ and for } \mathcal{H}^n \text{ a.e. } z \in \partial\Omega. \tag{5.8}$$

Then there exists $P \subseteq \partial\Omega$ such that $\mathcal{H}^n(P) > 0$ and $H_{\Omega,k}(z) \neq 0$ for $z \in P$.

Proof. Suppose $H_{\Omega,k}(z) = 0$ for $\mathcal{H}^n$ a.e. $z \in \partial\Omega$. Then we can employ corollary 5.17 (with $r = k$) and use (5.8) (for $i = k-1$) to infer that $H_{\Omega,k-1}(z) = 0$ for $\mathcal{H}^n$ a.e. $z \in \partial\Omega$. Now we repeat this argument with $r = k-1$ and $i = k-2$ to infer that $H_{\Omega,k-2}(z) = 0$ for $\mathcal{H}^n$ a.e. $z \in \partial\Omega$, and we continue until we obtain that $H_{\Omega,0}(z) = 0$ for $\mathcal{H}^n$ a.e. $z \in \partial\Omega$, which means $\mathcal{H}^n(\partial\Omega) = 0$. Since the latter is clearly impossible, we have proved the assertion. □

6. Sphere theorems for $W^{2,n}$-domains

The results in the previous section in combination with the Heintze–Karcher inequality proved below can be used to generalize classical sphere theorems to $W^{2,n}$-domains.

Theorem 6.1 (Heintze–Karcher inequality). *Suppose $\Omega \subseteq \mathbf{R}^{n+1}$ is a bounded and connected $W^{2,n}$-domain such that $H_{\Omega,1}(z) \geq 0$ for $\mathcal{H}^n$ a.e. $z \in \partial\Omega$. Then*

$$(n+1)\mathcal{L}^{n+1}(\Omega) \leq \int_{\partial\Omega} \frac{1}{H_{\Omega,1}(x)}\, d\mathcal{L}^n(x).$$

Moreover, if $H_{\Omega,1}(z) \geq \frac{\mathcal{H}^n(\partial\Omega)}{(n+1)\mathcal{L}^{n+1}(\Omega)}$ for $\mathcal{H}^n$ a.e. $z \in \partial\Omega$ then Ω is a round ball.

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Proof. We define $\Omega' = \mathbf{R}^{n+1} \setminus \overline{\Omega}$ and notice that Ω' is a $W^{2,n}$-domain. Since $\partial^v_+ \Omega' = \partial^v_+ \Omega$ and $\nu_{\Omega'} = -\nu_\Omega$, it follows from theorem 5.7 that

$$\mathcal{H}^n\big(\operatorname{nor}(\Omega') \setminus \{(z, -\nu_\Omega(z)) : z \in \partial^v_+\Omega\}\big) = 0,$$

and

$$-\chi_{\Omega,i}(z) = \chi_{\Omega',i}(z) = \kappa_{\Omega',i}(z, -\nu_\Omega(z)) \quad \text{for } \mathcal{H}^n \text{ a.e. } z \in \partial^v_+\Omega.$$

Henceforth,

$$\sum_{i=1}^{n} \kappa_{\Omega',i}(z,u) = -n\, H_{\Omega,1}(z) \le 0 \quad \text{for } \mathcal{H}^n \text{ a.e. } (z,u) \in \operatorname{nor}(\Omega') \tag{6.1}$$

and we infer from theorem 2.13 and area formula [10, 3.2.20] that

$$\begin{aligned}(n+1)\mathcal{L}^{n+1}(\Omega) &\le \int_{\operatorname{nor}(\Omega')} J_n^{\operatorname{nor}(\Omega')}\pi_0(z,u)\, \frac{n}{|\sum_{i=1}^n \kappa_{\Omega',i}(z,u)|}\, d\mathcal{H}^n(z,u) \\ &= \int_{\partial\Omega} \frac{1}{H_{\Omega,1}(z)}\, d\mathcal{H}^n(z). \end{aligned} \tag{6.2}$$

We assume now that $H_{\Omega,1}(z) \ge \frac{\mathcal{H}^n(\partial\Omega)}{(n+1)\mathcal{L}^{n+1}(\Omega)}$ for $\mathcal{H}^n$ a.e. $z \in \partial\Omega$. Then, we observe that

$$\mathcal{H}^n\left(\left\{ z \in \partial\Omega : H_{\Omega,1}(z) \ge (1+\epsilon)\frac{\mathcal{H}^n(\partial\Omega)}{(n+1)\mathcal{L}^{n+1}(\Omega)}\right\}\right) = 0 \quad \text{for every } \epsilon > 0,$$

otherwise we would obtain a contradiction with the inequality (6.2) (cf. proof of [14, Corollary 5.16]). This implies that

$$H_{\Omega,1}(z) = \frac{\mathcal{H}^n(\partial\Omega)}{(n+1)\mathcal{L}^{n+1}(\Omega)} \quad \text{for } \mathcal{H}^n \text{ a.e. } z \in \partial\Omega,$$

whence we infer that (6.2) holds with equality. Recalling (6.1) we deduce from theorem 2.13 that Ω must be a round ball. □

THEOREM 6.2. *If $k \in \{1, \dots, n\}$, $\lambda \in \mathbf{R}$ and $\Omega \subseteq \mathbf{R}^{n+1}$ is a bounded and connected $W^{2,n}$-domain such that*

$$H_{\Omega,i}(z) \ge 0 \quad \text{for } i = 1, \dots, k-1 \tag{6.3}$$

and

$$H_{\Omega,k}(z) = \lambda \tag{6.4}$$

for $\mathcal{H}^n$ a.e. $z \in \partial\Omega$, then Ω is a round ball.

Proof. Combining theorem 5.17 and divergence theorem for sets of finite perimeter, (it is clear by lemma 5.5 that Ω is a set of finite perimeter whose reduced boundary

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is $\mathcal{H}^n$ almost equal to the topological boundary) we obtain

$$\int_{\partial\Omega} H_{\Omega,k-1}\, d\mathcal{H}^n = \lambda \int_{\partial\Omega} x \bullet \nu_\Omega(x)\, d\mathcal{H}^n(x) = \lambda(n+1)\mathcal{L}^{n+1}(\Omega) \tag{6.5}$$

and we infer that $\lambda \geq 0$. Hence we deduce from [14, Lemma 2.2] and corollary 5.18 that

$$H_{\Omega,1}(z) \geq \ldots \geq H_{\Omega,k-1}(z)^{\frac{1}{k-1}} \geq H_{\Omega,k}(z)^{\frac{1}{k}} = \lambda^{\frac{1}{k}} > 0 \tag{6.6}$$

for $\mathcal{H}^n$ a.e. $z \in \partial\Omega$. By (6.6),

$$\int_{\partial\Omega} H_{\Omega,k-1}(z)\, d\mathcal{H}^n(z) \geq \lambda^{\frac{k-1}{k}} \mathcal{H}^n(\partial\Omega)$$

and combining with (6.5) we obtain

$$\lambda(n+1)\mathcal{L}^{n+1}(\Omega) \geq \lambda^{\frac{k-1}{k}} \mathcal{H}^n(\partial\Omega).$$

Since $\lambda > 0$, we obtain from (6.6) that

$$H_{\Omega,1}(z) \geq \frac{\mathcal{H}^n(\partial\Omega)}{(n+1)\mathcal{L}^{n+1}(\Omega)} \quad \text{for } \mathcal{H}^n \text{ a.e. } z \in \partial\Omega \tag{6.7}$$

and we conclude applying lemma 6.1. □

Remark 6.3. Hypothesis (6.3) in theorem 6.2 can be equivalently replaced by the following assumption:

$$\frac{\partial \sigma_k}{\partial t_i}(\chi_{\Omega,1}(z), \ldots, \chi_{\Omega,n}(z)) \geq 0 \quad \text{for } i = 1, \ldots, n \text{ and for } \mathcal{H}^n \text{ a.e. } z \in \partial\Omega. \tag{6.8}$$

Assume (6.8) in place of (6.5) in theorem 6.2. Then, first we use [38, eq. (1.15)] to infer that

$$H_{\Omega,k-1}(z) = \frac{1}{k}\sum_{i=1}^{n} \frac{\partial \sigma_k}{\partial t_i}(\chi_{\Omega,1}(z), \ldots, \chi_{\Omega,n}(z)) \geq 0$$

for $\mathcal{H}^n$ a.e. $z \in \partial\Omega$. Then, as in (6.5), we deduce that $\lambda \geq 0$. Finally, we employ [38, Proposition 1.3.2] to infer (6.3).

7. Nabelpunksatz for Sobolev graphs

In this final section, we extend the Nabelpunktsatz to graphs of twice weakly differentiable functions in terms of the approximate curvatures of their graphs. In particular, theorem 7.6 provides a general version of the Nabelpunktsatz for $W^{2,1}$-graphs. In view of well-known examples of convex functions, this result is sharp; see remark 7.8. In this section we use the symbols $\mathbf{D}_i$ and $\mathbf{D}^2_{ij}$ (respectively D_i and D^2_{ij}) for the distributional partial derivatives of a Sobolev function (respectively the classical partial derivatives of a function) with respect to the standard base $e_1, \ldots, e_n$ of $\mathbf{R}^n$.

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REMARK 7.1. Let $U \subseteq \mathbf{R}^n$ be a connected open and $f \in C^2(U)$. We define $G = \{(x, f(x)) : x \in U\}$, and $\nu : G \to \mathbf{S}^n \subseteq \mathbf{R}^{n+1}$ so that

$$\nu(\overline{f}(x)) = \frac{(-\nabla f(x), 1)}{\sqrt{1 + |\nabla f(x)|^2}} \tag{7.1}$$

for every $x \in U$. Differentiating (7.1) we get

$$D\nu(\overline{f}(x))(v, Df(x)(v)) = \frac{(-D(\nabla f)(x)(v), 0)}{\sqrt{1 + |\nabla f(x)|^2}} - \frac{\nabla f(x) \bullet D(\nabla f)(x)(v)}{1 + |\nabla f(x)|^2}\nu(\overline{f}(x))$$

for every $v \in \mathbf{R}^n$. We recall that G is umbilical if and only if there exists a function $\lambda : G \to \mathbf{R}$ such that

$$D\nu(z) = \lambda(z)\mathrm{Id}_{\,\mathrm{Tan}(G,z)} \quad \forall z \in G.$$

Therefore, noting that $\mathrm{Tan}(G, \overline{f}(x)) = \{(v, Df(x)(v)) : v \in \mathbf{R}^n\}$, we conclude that *$G$ is umbilical if and only if*

$$\lambda(\overline{f}(x))\big[e_i \bullet e_j + D_i f(x) D_j f(x)\big] = -\frac{D^2_{ij} f(x)}{\sqrt{1 + |\nabla f(x)|^2}} \tag{7.2}$$

for every $x \in U$ and for every $i, j = 1, \ldots, n$. It follows from [42] that *if $U \subseteq \mathbf{R}^n$ is a connected open set, $f \in C^2(U)$ and $\lambda : G \to \mathbf{R}$ is a function such that* (7.2) *holds for every $x \in U$, then either $\overline{f}(U)$ is contained in an n-dimensional plane or $\overline{f}(U)$ is contained in an n-dimensional sphere.*

The first result of this section generalizes remark 7.1 to $W^{2,1}$-functions. Suppose $U \subseteq \mathbf{R}^n$ is an open set, $\nu \in \mathbf{S}^{n-1}$ and π_ν is the orthogonal projection onto $\nu^\perp$. Then we define

$$U_\nu = \pi_\nu[U]$$

and

$$U^\nu_y = \{t \in \mathbf{R} : y + t\nu \in U\} \subseteq \mathbf{R} \quad \text{for } y \in U_\nu.$$

Notice that U_ν is an open subset of $\nu^\perp$ and U^ν_y is an open subset of $\mathbf{R}$ for every $y \in U_\nu$.

LEMMA 7.2. *Suppose $U \subseteq \mathbf{R}^n$ be an open set, $g \in W^{1,1}_{\mathrm{loc}}(U)$ and $k \in \{1, \ldots, n\}$ such that*

$$\mathbf{D}_k g(x) = 0 \quad \textit{for } \mathcal{L}^n \textit{ a.e. } x \in U.$$

Then for $\mathcal{L}^{n-1}$ a.e. $y \in U_{e_k}$ the function mapping $t \in U^{e_k}_y$ into $g(y + te_k)$ is $\mathcal{L}^1$ almost equal to a constant function.

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Proof. It follows from [46, Theorem 2.1.4] that there exists a representative $\tilde{g}$ of g such that the restriction of $\tilde{g}$ on $U_y^{e_k}$ is absolutely continuous and

$$\mathbf{D}_k g(y + te_k) = \frac{d}{dt}\tilde{g}(y + te_k) \quad \text{for } \mathcal{L}^1 a.e.\ t \in U_y^{e_k}$$

for $\mathcal{L}^{n-1}$ a.e. $y \in U_{e_k}$. It follows from the hypothesis that

$$\frac{d}{dt}\tilde{g}(y + te_k) = 0$$

for $\mathcal{L}^1$ a.e. $t \in U_y^{e_k}$ and for $\mathcal{L}^{n-1}$ a.e. $y \in U_{e_k}$, and we readily obtain the conclusion from the absolute continuity hypothesis of $\tilde{g}$. ☐

We prove now the first result of this section.

THEOREM 7.3. *Suppose $U \subseteq \mathbf{R}^n$ is a connected open set, $f \in W^{2,1}_{\mathrm{loc}}(U)$ and $\mu : U \to \mathbf{R}$ is a function such that*

$$\mu(x)\big[e_i \bullet e_j + \mathbf{D}_i f(x)\mathbf{D}_j f(x)\big] = -\frac{\mathbf{D}^2_{ij} f(x)}{\sqrt{1 + |\nabla f(x)|^2}} \tag{7.3}$$

for $\mathcal{L}^n$ a.e. $x \in U$ and for every $i, j = 1, \ldots, n$.

Then, either f is $\mathcal{L}^n$ almost equal to a linear function on U, or there exists an n-dimensional sphere S in $\mathbf{R}^{n+1}$ such that $\overline{f}(x) \in S$ for $\mathcal{L}^n$ a.e. $x \in U$.

Proof. Recall the diffeomorphism ψ from remark 3.2 and define $\eta = \psi \circ \boldsymbol{\nabla} f$. By the classical chain rule for Sobolev maps (cf. [11]), $\eta \in W^{1,1}_{\mathrm{loc}}(U, \mathbf{R}^{n+1})$ and

$$\begin{aligned}\mathbf{D}\eta(x)(v) &= \big[D\psi(\boldsymbol{\nabla} f(x)) \circ \mathbf{D}(\boldsymbol{\nabla} f)(x)\big](v) \\ &= \frac{(-\mathbf{D}(\boldsymbol{\nabla} f)(x)(v), 0)}{\sqrt{1 + |\boldsymbol{\nabla} f(x)|^2}} - \frac{\boldsymbol{\nabla} f(x) \bullet \mathbf{D}(\boldsymbol{\nabla} f)(x)(v)}{1 + |\boldsymbol{\nabla} f(x)|^2}\eta(x)\end{aligned}$$

for $\mathcal{L}^n$ a.e. $x \in U$. In particular, noting that $\eta(x) \bullet (e_j, \mathbf{D}_j f(x)) = 0$ for every $j = 1, \ldots, n$ and for $\mathcal{L}^n$ a.e. $x \in U$, we employ the umbilicality condition to obtain

$$\begin{aligned}\mathbf{D}_i\eta(x) \bullet (e_j, \mathbf{D}_j f(x)) &= -\frac{\mathbf{D}^2_{ij} f(x)}{\sqrt{1 + |\boldsymbol{\nabla} f(x)|^2}} \\ &= \mu(x)(e_i, \mathbf{D}_i f(x)) \bullet (e_j, \mathbf{D}_j f(x))\end{aligned}$$

for $\mathcal{L}^n$ a.e. $x \in U$ and for every $i, j = 1, \ldots, n$. Consequently, for every $i = 1, \ldots, n$ and for $\mathcal{L}^n$ a.e. $x \in U$ there exists $\lambda_i(x) \in \mathbf{R}$ such that

$$\mathbf{D}_i\eta(x) - \mu(x)(e_i, \mathbf{D}_i f(x)) = \lambda_i(x)\eta(x). \tag{7.4}$$

On the other hand, since η is a unit-length vector, we see (again from the chain rule for Sobolev maps) that $\eta(x) \bullet \mathbf{D}_i\eta(x) = 0$ for $\mathcal{L}^n$ a.e. $x \in U$ and for $i = 1, \ldots, n$.

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Therefore, we infer from (7.4) that $\lambda_i(x) = 0$ and

$$\mathbf{D}_i\eta(x) = \mu(x)(e_i, \mathbf{D}_i f(x)) = \mu(x)\mathbf{D}_i\overline{f}(x) \tag{7.5}$$

for $\mathcal{L}^n$ a.e. $x \in U$. For $k = 1, \dots, n$ let $g_k \in W^{1,1}_{\mathrm{loc}}(U)$ be given by

$$g_k = -\frac{\mathbf{D}_k f}{\sqrt{1 + |\boldsymbol{\nabla} f|^2}},$$

and we notice from (7.5) that

$$\mathbf{D}_i g_j = 0, \qquad \text{whenever } i, j \in \{1, \dots, n\} \text{ and } i \neq j, \tag{7.6}$$

$$\mathbf{D}_i g_i = \mu, \qquad \text{whenever } i \in \{1, \dots, n\}. \tag{7.7}$$

We fix now an open cube $Q \subseteq U$ with sides parallel to the coordinate axes, $\phi \in C^\infty_c(Q)$ and $k = 1, \dots, n$, and we prove that

$$\int_Q \mu \, D_k\phi \, d\mathcal{L}^n = 0. \tag{7.8}$$

Choose $j \in \{1, \dots, n\}$ with $k \neq j$. Since by (7.6) we have that $\mathbf{D}_k g_j = 0$, it follows from lemma 7.2 that for $\mathcal{L}^{n-1}$ a.e. $y \in U_{e_k}$ there exists $v_j(y) \in \mathbf{R}$ such that

$$g_j(y + te_k) = v_j(y) \quad \text{for } \mathcal{L}^1 \text{ a.e. } t \in U^{e_k}_y.$$

Now we use (7.7) to obtain

$$\begin{aligned}\int_Q \mu \, D_k\phi \, d\mathcal{L}^n &= \int_Q \mathbf{D}_j g_j \, D_k\phi \, d\mathcal{L}^n \\ &= -\int_Q g_j \, D_j\big(D_k\phi\big) \, d\mathcal{L}^n \\ &= -\int_{Q_{e_k}} v_j(y) \int_{Q^y_{e_k}} D_k\big(D_j\phi\big)(y + te_k) \, d\mathcal{L}^1(t) \, d\mathcal{L}^{n-1}(y) = 0,\end{aligned}$$

where the last equality follows from the fact that the function mapping $t \in Q^{e_k}_y$ into $D_j\phi(y + te_k)$ has compact support in $Q^{e_k}_y$.

Since (7.8) holds for every open cube Q with sides parallel to the coordinate axes and for every $\phi \in C^\infty_c(Q)$, and since U is connected, we infer from [10, 4.1.4] that

$$\mu \text{ is } \mathcal{L}^n \text{ almost equal to a constant function on } U. \tag{7.9}$$

Since U is connected, we combine (7.9) and (7.5) to infer that there exists $c \in \mathbf{R}$ and $w \in \mathbf{R}^{n+1}$ such that

$$\eta(x) - c\overline{f}(x) = w \quad \text{for } \mathcal{L}^n \text{ a.e. } x \in U.$$

If $c \neq 0$ the last equation evidently implies that $\overline{f}(x) \in \partial B^{n+1}(-w/c, 1/|c|)$ for $\mathcal{L}^n$ a.e. $x \in U$. If $c = 0$, we have that $w \bullet e_{n+1} = (1 + |\nabla f|^2)^{-1/2}$ and

$$\mathbf{D}_i f(x) = -\frac{w \bullet e_i}{w \bullet e_{n+1}} \quad \text{for } \mathcal{L}^n \text{ a.e. } x \in U \text{ and } i = 1, \ldots, n.$$

This implies that f is $\mathcal{L}^n$ almost equal to a linear function on U, since U is connected. □

Definition 7.4. *Suppose $X \subseteq \mathbf{R}^{n+1}$ is $\mathcal{H}^n$-measurable and $\mathcal{H}^n$-rectifiable of class 2. We say that* X *is* approximate totally umbilical *if there exists a $\mathcal{H}^n \llcorner X$-measurable map ν such that $\nu(x) \in \operatorname{Nor}^n(\mathcal{H}^n \llcorner X, x) \cap \mathbf{S}^n$ and there exists a function $\mu : X \to \mathbf{R}$ such that*

$$\operatorname{ap} D\nu(x)(\tau) = \mu(x)\tau \quad \textit{for every } \tau \in \operatorname{Tan}^n(\mathcal{H}^n \llcorner X, x), \tag{7.10}$$

for $\mathcal{H}^n$ a.e. $x \in X$ (keep in mind lemma 5.2).

Definition 7.5. (Lusin (N) condition). *Suppose $U \subseteq \mathbf{R}^n$ is open and $g : U \to \mathbf{R}^k$ $(k \geq n)$. We say that g satisfies the* Lusin's condition (N) *if and only if $\mathcal{H}^n(g(Z)) = 0$ for every $Z \subseteq U$ with $\mathcal{L}^n(Z) = 0$.*

We are now ready to prove the second result of this section.

Theorem 7.6. *Suppose $U \subseteq \mathbf{R}^n$ is a bounded open set, $f \in W^{2,1}(U)$, $\overline{f}$ satisfies the Lusin's condition (N) and $G = \overline{f}(U)$.*

Then G is $\mathcal{H}^n$-rectifiable of class 2. Moreover, if G is approximate totally umbilical then, up to a $\mathcal{H}^n$-negligible set, either G is a subset of an n-dimensional plane or a subset of an n-dimensional sphere.

Proof. By [6, Theorem 13] and [10, 2.10.19(4), 2.10.43] we can find countably many functions $g_1, g_2, \ldots \in C^2(\mathbf{R}^n)$ such that $\mathcal{L}^n(U \setminus \bigcup_{i=1}^{\infty}\{g_i = f\}) = 0$ and $\operatorname{Lip}(g_i) < \infty$ for every $i \geq 1$. Henceforth, thanks to the Lusin's condition (N), we readily infer that G is $\mathcal{H}^n$-rectifiable of class 2. We define D_i as the set of $x \in \{g_i = f\}$ such that

$$\Theta^n(\mathcal{L}^n \llcorner U \setminus \{g_i = f\}, x) = 0, \quad Dg_i(x) = \mathbf{D}f(x)$$

and $\operatorname{Tan}^n(\mathcal{H}^n \llcorner G, \overline{f}(x))$ is an n-dimensional plane. Since $Dg_i(x) = \operatorname{ap} Df(x)$ for every $x \in D_i$, it follows from [10, 2.10.19(4), 3.2.19] and lemma 2.19 that

$$\mathcal{L}^n(\{g_i = f\} \setminus D_i) = 0 \quad \text{for every } i \geq 1.$$

Since $\operatorname{Tan}^n(\mathcal{L}^n \llcorner \{g_i = f\}, x) = \mathbf{R}^n$ for every $x \in D_i$, and noting that $\overline{g_i} : \mathbf{R}^n \to \overline{g_i}(\mathbf{R}^n)$ is a bi-lipschitz homeomorphism, we use [40, Lemma B.2] to conclude

$$\mathbf{D}\overline{f}(x)[\mathbf{R}^n] = D\overline{g_i}(x)[\operatorname{Tan}^n(\mathcal{L}^n \llcorner \{g_i = f\}, x)] \subseteq \operatorname{Tan}^n(\mathcal{H}^n \llcorner G, \overline{f}(x))$$

for every $x \in D_i$. Since $\mathbf{D}\overline{f}(x)$ is injective whenever it exists, we conclude that

$$\mathbf{D}\overline{f}(x)[\mathbf{R}^n] = \operatorname{Tan}^n(\mathcal{H}^n \llcorner G, \overline{f}(x))$$

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and

$$\psi(\boldsymbol{\nabla} f(x)) \in \operatorname{Nor}^n(\mathcal{H}^n \llcorner G, \overline{f}(x))$$

for every $x \in D_i$ and $i \geq 1$. Let $D = \bigcup_{i=1}^{\infty} D_i$ and notice that $\mathcal{H}^n(G \setminus \overline{f}(D)) = 0$ (again by Lusin's condition (N)). Let ν be the $\mathcal{H}^n \llcorner G$-measurable map defined by

$$\nu = \psi \circ \boldsymbol{\nabla} f \circ (\pi | G),$$

where $\pi : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$ is the orthogonal projection onto $\mathbf{R}^n$. We observe that if $z \in \overline{f}(D_i)$ and $\Theta^n(\mathcal{H}^n \llcorner G \setminus \overline{f}(D_i), z) = 0$, then ν is $\mathcal{H}^n \llcorner G$-approximately differentiable at z (since $\nu | \overline{f}(D_i) = (\psi \circ \nabla g_i \circ \pi) | \overline{f}(D_i))$ and

$$\begin{aligned}
\operatorname{ap} D\nu(z) &= D(\psi \circ \nabla g_i \circ \pi)(z) \\
&= D(\psi \circ \nabla g_i)(\pi(z)) \circ \big(\pi | \operatorname{Tan}^n(\mathcal{H}^n \llcorner G, z)\big) \\
&= \operatorname{ap} D(\psi \circ \boldsymbol{\nabla} f)(\pi(z)) \circ \big(\pi | \operatorname{Tan}^n(\mathcal{H}^n \llcorner G, z)\big) \\
&= \mathbf{D}(\psi \circ \boldsymbol{\nabla} f)(\pi(z)) \circ \big(\pi | \operatorname{Tan}^n(\mathcal{H}^n \llcorner G, z)\big),
\end{aligned}$$

whence we infer

$$\operatorname{ap} D\nu(z) \circ \mathbf{D}\overline{f}(\pi(z)) = \mathbf{D}(\psi \circ \boldsymbol{\nabla} f)(\pi(z)). \tag{7.11}$$

By [10, 2.10.19(4)] we conclude that (7.11) is true for $\mathcal{H}^n$ a.e. $z \in G$.

If G is approximate totally umbilical then it is easy to see that the unit normal vector field ν defined above fulfils the umbilicalilty condition in (7.10) with some function μ. Henceforth,

$$\begin{aligned}
\mu(\overline{f}(x))(e_i \bullet e_j + \mathbf{D}_i f(x) \mathbf{D}_j f(x)) &= \big(\operatorname{ap} D\nu(\overline{f}(x)) \circ D\overline{f}(x)\big)(e_i) \bullet (e_j, \mathbf{D}_j f(x)) \\
&= \mathbf{D}_i(\psi \circ \nabla f)(x) \bullet (e_j, \mathbf{D}_j f(x)) \\
&= -\frac{\mathbf{D}^2_{ij} f(x)}{\sqrt{1 + |\nabla f(x)|^2}}
\end{aligned}$$

for every $i, j = 1, \ldots, n$ and for $\mathcal{L}^n$ a.e. $x \in U$. By theorem 7.3 and the Lusin's condition (N), we deduce that, up to a $\mathcal{H}^n$-negligible set, G is either a subset of an n-dimensional plane or a subset of an n-dimensional sphere of $\mathbf{R}^{n+1}$. □

Remark 7.7. If $f \in W^{2,p}(U)$ with $p > \frac{n}{2}$, then the Sobolev embedding theorem [11, Theorem 7.26] ensures that $f \in W^{1,p^*}(U)$ with $p^* > n$. Henceforth, $\overline{f}$ satisfies the Lusin's condition (N) by [23, Theorem 1.1].

Remark 7.8. It is easy to find convex functions $f \in C^{1,\alpha}(\mathbf{R}^n)$ such that the approximate principal curvatures of the graph are zero $\mathcal{H}^n$ almost everywhere and the conclusion of theorem 7.6 fails. (Notice that the graph is $\mathcal{H}^n$-rectifiable of class 2 and $\overline{f}$ satisfies the Lusin's condition (N).) Indeed the gradient of these functions are continuous maps of bounded variation whose distributional derivative is not a function. An example of such functions is given by the primitive of the ternary Cantor function.

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Appendix A. Area of the proximal unit normal bundle

LEMMA A.1. *If $C \subseteq \mathbf{R}^{n+1}$ is a closed set and $M \subseteq \mathbf{R}^{n+1}$ is a k-dimensional submanifold of class 2 then there exists $R \subseteq M \cap C$ such that:*

(1) $\mathrm{nor}(C) \cap (R \times \mathbf{S}^n) \subseteq \mathrm{nor}(M)$;
(2) $\mathcal{H}^k\big((M \cap C) \setminus R\big) = 0$.

Proof. See the proof of [40, Lemma 6.1]. □

Suppose $C \subseteq \mathbf{R}^{n+1}$ is closed with $\mathcal{H}^n(C) < \infty$ and $\Sigma = \pi_0(\mathrm{nor}(C))$. It follows from [25] that Σ is $\mathcal{H}^n$-rectifiable of class 2. We fix a $\mathcal{H}^n \llcorner \Sigma$-measurable map $\nu : \Sigma \to \mathbf{S}^n$ such that $\nu(a) \in \mathrm{Nor}^n(\mathcal{H}^n \llcorner \Sigma, a)$ for $\mathcal{H}^n$ a.e. $a \in \Sigma$ and we notice that it is $\mathcal{H}^n \llcorner \Sigma$-approximately differentiable at $\mathcal{H}^n$ a.e. $a \in \Sigma$ with a symmetric approximate differential $\mathrm{ap}\, D\nu(a)$ by lemma 5.2. We denote by

$$\chi_{\Sigma,1}(a) \leq \ldots \leq \chi_{\Sigma,n}(a)$$

the eigenvalues of $\mathrm{ap}\, D\nu(a)$ and we define (cf. definition 2.12)

$$E = \{(x,u) \in \mathrm{nor}(C) : \kappa_{C,n}(x,u) < \infty\}.$$

LEMMA A.2. *If $A \subseteq \mathbf{R}^{n+1}$ is a Borel set, then*

$$\mathcal{H}^n(E \cap (A \times \mathbf{S}^n)) \geq \int_{\Sigma \cap A} \prod_{\ell=1}^{n} \sqrt{1 + \chi_{\Sigma,\ell}(x)^2}\, d\mathcal{H}^n(x).$$

Proof. Let $\{\Sigma_i\}_{i \geq 1}$ be a sequence of C^2-hypersurfaces such that $\mathcal{H}^n(\Sigma \setminus \bigcup_{i=1}^{\infty} \Sigma_i) = 0$. Employing lemma A.1 we can find a *disjointed* sequence of Borel subsets $\{R_i\}_{i \geq 1}$ of $\Sigma_i \cap \Sigma$ such that

$$\mathcal{H}^n\big(\Sigma \setminus \textstyle\bigcup_{i=1}^{\infty} R_i\big) = 0 \quad \text{and} \quad \mathrm{nor}(C) \cap (R_i \times \mathbf{S}^n) \subseteq \mathrm{nor}(\Sigma_i)$$

for every $i \geq 1$. It follows from [10, 2.10.19(4)] that

$$\mathrm{Tan}^n\left(\mathcal{H}^n \llcorner (\mathrm{nor}(C) \cap (R_i \times \mathbf{S}^n)), (x,u)\right) = \mathrm{Tan}(\mathrm{nor}(\Sigma_i), (x,u)) \tag{A.1}$$

for $\mathcal{H}^n$ a.e. $(x,u) \in \mathrm{nor}(C) \cap (R_i \times \mathbf{S}^n)$ (recall that $\mathrm{nor}(\Sigma_i) \cap (\Sigma_i \times \mathbf{S}^n)$ is an n-dimensional submanifold of $\mathbf{R}^{n+1} \times \mathbf{S}^n$ of class 1). Since $\pi_0\big(\mathrm{Tan}(\mathrm{nor}(\Sigma_i), (x,u))\big) = \mathrm{Tan}(\Sigma_i, x)$ is an n-dimensional linear space for every $(x,u) \in \mathrm{nor}(\Sigma_i) \cap (\Sigma_i \times \mathbf{S}^n)$, we deduce from (A.1) and [14, Lemma 3.9] that

$$\kappa_{C,n}(a,u) < \infty \quad \text{for } \mathcal{H}^n \text{ a.e. } (a,u) \in \bigcup_{i=1}^{n} \big[\mathrm{nor}(C) \cap (R_i \times \mathbf{S}^n)\big].$$

For each $i \geq 1$ let $\chi_{\Sigma_i,1} \leq \ldots \leq \chi_{\Sigma_i,n}$ be the principal curvatures of Σ_i, and we notice that

$$J_n^{\mathrm{nor}(\Sigma_i)} \pi_0(x,u) = \prod_{\ell=1}^{n} \frac{1}{\sqrt{1 + \chi_{\Sigma_i,\ell}(x)^2}}$$

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for $(x,u) \in \mathrm{nor}(\Sigma_i) \cap (\Sigma_i \times \mathbf{S}^n)$. Applying area formula we can estimate

$$\begin{aligned}
&\mathcal{H}^n(E \cap (A \times \mathbf{S}^n)) \\
&\qquad \geq \sum_{i=1}^{\infty} \mathcal{H}^n(\mathrm{nor}(C) \cap ((A \cap R_i) \times \mathbf{S}^n)) \\
&\qquad = \sum_{i=1}^{\infty} \int_{\mathrm{nor}(C) \cap ((A \cap R_i) \times \mathbf{S}^n)} J_n^{\mathrm{nor}(\Sigma_i)} \pi_0(x,u) \\
&\qquad\qquad \prod_{\ell=1}^{n} \sqrt{1 + \chi_{\Sigma_i,\ell}(x)^2}\, d\mathcal{H}^n(x,u) \\
&\qquad \geq \sum_{i=1}^{\infty} \int_{A \cap R_i} \prod_{\ell=1}^{n} \sqrt{1 + \chi_{\Sigma_i,\ell}(x)^2}\, d\mathcal{H}^n(x).
\end{aligned}$$

From the proof of lemma 5.2 we obtain that if $i \geq 1$ then

$$\prod_{\ell=1}^{n} \sqrt{1 + \chi_{\Sigma_i,\ell}(x)^2} = \prod_{\ell=1}^{n} \sqrt{1 + \chi_{\Sigma,\ell}(x)^2} \quad \text{for } \mathcal{H}^n \text{ a.e. } x \in \Sigma \cap \Sigma_i.$$

Henceforth, we conclude

$$\mathcal{H}^n(E \cap (A \times \mathbf{S}^n)) \geq \int_{A \cap \Sigma} \prod_{\ell=1}^{n} \sqrt{1 + \chi_{\Sigma,\ell}(x)^2}\, d\mathcal{H}^n(x).$$

□

LEMMA A.3. *There exist a smooth two-dimensional submanifold $M \subseteq \mathbf{R}^3$ with bounded mean curvature such that $\mathcal{H}^2 \llcorner \overline{M}$ is a Radon measure over $\mathbf{R}^3$, and a set $P \subseteq \mathbf{R}^3 \times \mathbf{S}^2$ such that $\mathcal{H}^2(P) > 0$ and*

$$\mathcal{H}^2\big(\mathrm{nor}(\overline{M}) \cap \{(x,u) \in \mathbf{R}^{n+1} \times \mathbf{S}^n : |x-b| < r,\ |u-v| < r\}\big) = \infty$$

for every $(b,v) \in P$ and for every $r > 0$.

Proof. By [19, Example 10.8 and Remark 10.10] (see also [18, 6.1]) there exists a smooth two-dimensional submanifold $M \subseteq \mathbf{R}^3$ such that $\mathcal{H}^2 \llcorner \overline{M}$ is a Radon measure, and if $\chi_1 \leq \chi_2$ are the principal curvatures of M with respect to a unit-normal vector field $\eta : M \to \mathbf{S}^n$ then

$$|\chi_1(a) + \chi_2(a)| \leq 1 \quad \text{for every } a \in M$$

and there exists a Borel set $B \subseteq \overline{M} \setminus M$ such that

$$\int_{M \cap B(b,r)} \big(\chi_1^2 + \chi_2^2\big)^{q/2}\, d\mathcal{H}^2 = \infty \quad \text{for every } 1 < q < \infty, b \in B \text{ and } r > 0.$$

Henceforth, for every $b \in B$ and $r > 0$,

$$+\infty = \int_{M \cap B(b,r)} \big(\chi_1^2 + \chi_2^2\big)\, d\mathcal{H}^2 \leq \mathcal{H}^2(M \cap B(b,r)) - 2\int_{M \cap B(b,r)} \chi_1\, \chi_2\, d\mathcal{H}^2$$

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and

$$\int_{M \cap B(b,r)} \chi_1 \, \chi_2 \, d\mathcal{H}^2 = -\infty.$$

By lemma A.2 we infer that

$$\mathcal{H}^2\big(\operatorname{nor}(\overline{M}) \cap (B(b,r) \times \mathbf{S}^n)\big) \geq \int_{M \cap B(b,r)} \prod_{\ell=1}^{2} (1 + \chi_\ell(x)^2)^{1/2} \, d\mathcal{H}^n(x) = \infty$$

for every $r > 0$ and $b \in B$.

Now the compactness of $\mathbf{S}^n$ guarantees that for each $b \in B$ there exists $\upsilon(b) \in \mathbf{S}^n$ such that

$$\mathcal{H}^2\big(\operatorname{nor}(\overline{M}) \cap \{(x,u) \in \mathbf{R}^{n+1} \times \mathbf{S}^n : |x - b| < r, \ |u - \upsilon(b)| < r\}\big) = \infty$$

for each $r > 0$. Henceforth, setting $P = \{(b, \upsilon(b)) : b \in B\}$ we have that $\mathcal{H}^2(P) > 0$ and the proof is complete. □

Remark A.4. In particular, $\operatorname{nor}(\overline{M})$ is a Legendrian rectifiable set by lemma 2.11, but it cannot be the carrier an integer-multiplicity rectifiable n-current of $\mathbf{R}^3 \times \mathbf{S}^2$.

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