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COINCIDENCES AND FIXED POINTS IN LOCALLY G-CONVEX SPACES

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A new coincidence point theorem is proved for a pair of multivalued mappings operating between G-convex spaces. From this theorem, a generalisation of the classical Fan-Glicksberg fixed point theorem is established.

1. INTRODUCTION

In recent years many researchers have been interested in various notions of convexity on topological spaces which do not rely on a linear structure of the underlying space. The first work in this direction may be Aronszajn and Panitchpakdi [**1**] where the authors introduced a convexity structure on metric spaces; *hyperconvex* metric spaces. Subsequently this property has been found to be important in the study of nonexpansive mappings, see [**6, 14, 15**].

Some time later Horvath [**9, 10, 11**] defined a convexity structure in topological spaces and proved several important results in the theory of nonlinear analysis. The structure determining convexity in this space is a multivalued monotone operator mapping the finite subsets to contractible subsets of the topological space. Note that a contractible set in a topological space is one in which the identity map, restricted to the set in question, is homotopic to a constant map. This structure replaces the convex hull in vector spaces. Such a space has since been called an H-convex space (or simply H-space) by Bardaro and Ceppitelli [**2**] where amongst other results, a KKM type theorem is established.

The so-called G-convex spaces were introduced in [**12**] to allow for a convexity structure that need not have contractible values. These spaces generalise the notion of H-convexity (see Definition 1 below) as well as hyperconvexity. We refer to [**18, 6**] for further discussion on the relations between these concepts of convexity.

This study examines the existence of coincidence points for multivalued operators acting between different G-convex spaces. The first result, Lemma 1, is a fixed point result for the composition of a single valued continuous function and a multivalued operator with G-convex values. A selection theorem proved in [**16**], Theorem 2.1 below, is fundamental

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in the proof, and its use replaces linear approximation arguments used when the ambient space is linear (see for example [**8**, Lemma 2]). From this, a coincidence point theorem is proved and then a fixed point theorem which generalises the classical Fan-Glicksberg fixed point theorem. This study concludes with a fixed point theorem in which the compactness condition on the space is relaxed.

It should be noted that Yuan [**18**] has generalised the Fan-Glicksberg fixed point theorem for multifunctions with acyclic values, and in Ding and Tarafdar [**4**], a coincidence point theorem has been proved (in H-spaces) for a pair of multifunctions, one of which has acyclic values. The emphasis of this work is to study multifunctions with G-convex values instead of acyclic values. Therefore the results established here are proved by different means and they do not compare with the results in [**18, 4**].

2. G-convex Spaces

First we elucidate the notations and definitions used in this paper. Let X be a set. 2^X denotes the family of all nonempty subsets of the space X and $\mathcal{F}(X)$ denotes the family of nonempty finite subsets of X. Δ_n is the standard n-dimensional simplex with verticies $e_0, \dots, e_n$ where $e_0 = 0$ and e_i, for $i = 1, \dots, n$, is the i-th unit vector in $\mathbb{R}^n$; that is, $\Delta_n = \operatorname{co}\{e_0, \dots, e_n\}$. If $a_0, \dots, a_n$ are points in some vector space X, then $a_0 \dots a_n$ will denote the simplex with verticies $a_0, \dots, a_n$. Let X and Y be topological spaces. A multifunction $T : X \to 2^Y$ is said to be upper semicontinuous if $T^{-1}(C) = \{x \in X : T(x) \cap C \neq \emptyset\}$ is closed in X whenever C is closed in Y.

The following definition originally appeared in [**12**].

Definition 1. *A generalised convex, or G-convex space $(X, D; \Gamma)$ is a topological space X, a nonempty subset D of X and a function $\Gamma : \mathcal{F}(X) \to 2^X$ with the following properties:*

1. *for any $A, B \in \mathcal{F}(X)$ with $A \subset B$, we have $\Gamma(A) \subset \Gamma(B)$;*
2. *for any $A \in \mathcal{F}(X)$ with elements $a_0, \dots, a_n$ there exists a continuous function $\psi : \Delta_n \to \Gamma(A)$ such that for each $0 \leqslant i_0 < \dots < i_k \leqslant n$ it follows that*

$$\psi\big(\operatorname{co}\{e_{i_0}, \cdots, e_{i_k}\}\big) \subset \Gamma\big(\{a_{i_0}, \cdots, a_{i_k}\}\big).$$

(X, Γ) is an H-space when $D = X$, condition 1 is satisfied and the operator Γ has contractible values. It has been shown in [**10**, Theorem 1] that such an operator satisfies condition 2.

A subset K of a G-convex space $(X, D; \Gamma)$ is said to be G-convex if, for any $A \in \mathcal{F}(K \cap D)$, $\Gamma(A) \subset K$. Note that the intersection of G-convex sets remains G-convex. The G-convex hull of a subset Y of a G-convex space, denoted $G\text{-}\operatorname{co}(Y)$, is defined to be the intersection of all G-convex sets containing the set Y. So the G-convex hull of Y is the smallest G-convex set containing Y, which is evidently G-convex. Further properties of G-convex spaces and sets can be found in [**12**] and [**16**]. In this study, the set D in the definition of G-convex will be all of X and $(X, X; \Gamma)$ will be denoted $(X; \Gamma)$.

The following definition relates the G-convex sets with the topology of X, and it generalises the concept of a locally convex topological vector space.

Definition 2. *A G-convex space $(X; \Gamma)$ is said to be a locally G-convex uniform space if X is a uniform space with uniformity $\mathcal{U}$ having base β of open symmetric entourages such that each $W \in \beta$ satisfies the property that*

$$W(x) = \{y \in X : (x, y) \in W\}$$

is G-convex.

An arbitrary entourage satisfying this property will be said to be G-convex.

An alternative definition of local G-convexity is to assume that for any $W \in \beta$, $W(K) = \{x \in X : (y, x) \in W \text{ for some } y \in K\}$ is G-convex whenever K is G-convex. A locally G-convex space satisfying this property has fewer G-convex sets than one satisfying Definition 2. This follows as Definition 2 implies each singleton is G-convex (simply note $\{x\} = \bigcap_{V \in \beta} V(x)$ and the intersection of G-convex sets is G-convex), whereas the second notion does not necessarily imply this. Although the alternative definition gives rise to a more general space, it may be the case that there are fewer multifunctions with G-convex values (for example, single valued functions may not have G-convex values). Thus we restrict our analysis to locally G-convex spaces as in Definition 2. Note that both concepts coincide if the G-convex space $(X; \Gamma)$ is such that $\Gamma(x) = \{x\}$ for all $x \in X$.

It is well know that in uniform spaces, the closure of a set $K \subset X$ is given by

$$\overline{K} = \bigcap \{V(K) : V \in \beta\}$$

where β is any base for $\mathcal{U}$. It follows that in locally G-convex uniform spaces, the closure of a G-convex set, being the intersection of G-convex sets, is G-convex.

The following selection theorem is a weaker formulation of [**16**, Theorem 2.4], though sufficient for our purpose.

Theorem 2.1. *Let X be a compact topological space and $(Y; \Gamma)$ a G-convex space. Suppose $T : X \to 2^Y$ satisfies*

1. *$T(x)$ is G-convex for all $x \in X$;*
2. *for each $x \in X$ there exists $y \in Y$ such that $x \in \operatorname{int}(T^{-1}(y))$.*

Then there exists $A \in \mathcal{F}(Y)$ and continuous functions $g : \Delta_n \to Y$ and $\phi : X \to \Delta_n$, where $n + 1 = |A|$, such that the composition $f = g \circ \phi$ is a continuous selection of T; that is, $f(x) \in T(x)$ for all $x \in X$.

3. A Coincidence Theorem

The first result is the G-convex version of [**8**, Lemma 2] and is similar to the fixed point theorems of Eilenberg and Montgomery [**5**], Gòrniewicz [**7**] and Shioji [**13**], although

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the setting is a locally G-convex space and the multifunction has G-convex values rather than contractible or acyclic values.

Lemma 1. *Let $(X;\Gamma)$ be a compact locally G-convex uniform space. Suppose $p : X \to \Delta_n$ is continuous and $q : \Delta_n \to 2^X$ is upper semicontinuous with compact G-convex values. Then $p \circ q : \Delta_n \to 2^{\Delta_n}$ has a fixed point.*

Proof: For $k = 1, 2, \dots$, denote by S^k the k-th barycentric subdivision of the simplex Δ_n. For each k define a multivalued mapping $T_k : \Delta_n \to 2^X$ by

$$T_k(v) = G\text{-co}\left\{\bigcup_{i=0}^{r} q\left(a_k^i\right)\right\}$$

where a_k^i, for $i = 0, \dots, r$, $0 \leqslant r \leqslant n$, are the verticies of the simplex in S^k of least dimension containing the point v. The values of T_k are clearly G-convex.

We prove condition 2 of Theorem 2.1 is satisified for T_k. So we show each $v \in \Delta_n$ belongs to the interior of $T_k^{-1}(y)$ for some $y \in X$. To this end, let $v \in \Delta_n$ be arbitrary. For $a_k^0 \dots a_k^r$ the simplex in S^k of least dimension containing v, choose $\varepsilon > 0$ such that $\varepsilon < \text{dist}(v, \Lambda_k)$ for all simplexes $\Lambda_k \in S^k$ with $v \notin \Lambda_k$. We claim the open ball $B_\varepsilon(v)$ in Δ_n is a subset of

$$\Phi = \bigcup\left\{\Lambda_k^n \in S^k : a_k^0 \dots a_k^r \text{ is a face of } \Lambda_k^n \text{ and } \dim\Lambda_k^n = n\right\}.$$

To see this, suppose z is not an element of Φ. Then $z \in \Delta_n \backslash \Lambda_k^n$ for all n-dimensional $\Lambda_k^n \in S^k$ having $a_k^0 \dots a_k^r$ as a face. Hence z belongs to an n-dimensional simplex $\widehat{\Lambda}_k^n$ and $a_k^0 \dots a_k^r$ is not a face of $\widehat{\Lambda}_k^n$. Either $a_k^0 \dots a_k^r \cap \widehat{\Lambda}_k^n = \emptyset$ or not. In the first case it immediately follows that $v \notin \widehat{\Lambda}_k^n$ so $z \notin B_\varepsilon(v)$ from the definition of ε. If $a_k^0 \dots a_k^r \cap \widehat{\Lambda}_k^n \neq \emptyset$ then the intersection is a face common to both. As $a_k^0 \dots a_k^r$ is not a face of $\widehat{\Lambda}_k^n$, the intersection must be a simplex of dimension strictly less than r. As r is the smallest integer such that $v \in a_k^0 \dots a_k^r$ then $v \notin \widehat{\Lambda}_k^n$ so again $z \notin B_\varepsilon(v)$.

Thus the inclusion $B_\varepsilon(v) \subset \Phi$ has been established. This implies that for each $w \in B_\varepsilon(v)$, $T_k(v) \subset T_k(w)$ by the definition of T_k and Φ. By choosing $y \in T_k(w)$, it follows that $B_\varepsilon(v) \subset T_k^{-1}(y)$ and condition 2 of Theorem 2.1 is satisfied.

By Theorem 2.1 there exists a continuous selection f_k of T_k. The composition $p \circ f_k : \Delta_n \to \Delta_n$ is continuous and so by Brouwer's fixed point theorem, there exists $v_k \in \Delta_n$ such that $v_k = p\big(f_k(v_k)\big)$. Let $x_k = f_k(v_k)$. As X is compact we may assume the net x_k converges to $x_0 \in X$. As p is continuous, $v_k = p(x_k) \to p(x_0) = v_0$. We claim $x_0 \in q(v_0)$ so that v_0 is a fixed point of the multivalued composition $p \circ q$.

As $q(v_0)$ is closed it is enough to show $x_0 \in V\big(q(v_0)\big)$ for any V in any base for the uniformity $\mathcal{U}$. So let V be a fixed element of some base for the uniformity. As all the closed symmetric entourages form a base for $\mathcal{U}$, there exists a closed symmetric entourage $W \subset V$. Similarly as all the open symmetric G-convex entourages form a base for the uniformity, there exists an open symmetric G-convex $W_1 \subset W$. Therefore $W_1\big(q(v_0)\big)$ is an open G-convex neighbourhood of $q(v_0)$. By upper semicontinuity of q, there exists a neighbourhood $N(v_0)$ such that $q(v) \subset W_1\big(q(v_0)\big)$ for all $v \in N(v_0)$.

For each barycentric subdivision S^k of Δ_n there exists an n-simplex $a_k^0 \dots a_k^n$ containing the point v_k and moreover $a_k^i \to v_0$ for each $i = 0, 1, \dots, n$ as $k \to \infty$. For k sufficiently large, $a_k^i \in N(v_0)$ for each $i = 0, 1, \dots, n$ and

$$x_k = f_k(v_k) \in G\text{-co}\left\{\bigcup_{i=0}^{n} q\left(a_k^i\right)\right\}.$$

As $W_1\big(q(v_0)\big)$ is G-convex and $a_k^i \in N(v_0)$ it follows that

$$x_k \in G\text{-co}\left\{\bigcup_{i=0}^{n} q\left(a_k^i\right)\right\} \subset W_1\big(q(v_0)\big) \subset W\big(q(v_0)\big).$$

This implies $x_0 \in W\big(q(v_0)\big) \subset V\big(q(v_0)\big)$ as W is closed and $q(v_0)$ is compact. As V is arbitrary, $x_0 \in q(v_0)$. ∎

Using this, the following coincidence point theorem is established.

Theorem 3.1. *Let $(X;\Gamma)$ be a compact locally G-convex space and $(Y;\Sigma)$ an arbitrary G-convex space. Suppose $F : X \to 2^Y$ is such that*

1. *$F(x)$ is G-convex for all $x \in X$;*
2. *$F^{-1}(y)$ contains an open set O_y (which may be empty for some y);*
3. *$\bigcup_{y \in Y} O_y = X$.*

Then for each upper semicontinuous $g : Y \to 2^X$ with compact G-convex values there exists a coincidence point; that is, a point $x_0 \in X$ such that

$$F(x_0) \cap g^{-1}(x_0) \neq \emptyset.$$

Proof: By Theorem 2.1 there exists $n \in \mathbb{N}$ and continuous maps $h : \Delta_n \to Y$ and $\phi : X \to \Delta_n$ such that $f = h \circ \phi$ is a continuous selection of F. The composition $g \circ h : \Delta_n \to 2^X$ is upper semicontiuous with compact G-convex values. From Lemma 1 there exists $v_0 \in \Delta_n$ with $v_0 \in \phi\big(g\big(h(v_0)\big)\big)$. Letting $y_0 = h(v_0)$, we have $y_0 \in h\big(\phi\big(g(y_0)\big)\big)$; that is, $y_0 = h\big(\phi(z)\big) = f(z)$ for some $z \in g(y_0)$. Hence $y_0 \in F(z) \cap g^{-1}(z)$ as required. ∎

4. Fixed Points

As an application of Theorem 3.1, the Fan-Glicksberg fixed point theorem is generalised to locally G-convex spaces as follows.

Theorem 4.1. *Let $(X;\Gamma)$ be a compact locally G-convex uniform space. Then any upper semicontinuous $g : X \to 2^X$ with closed G-convex values has a fixed point.*

PROOF: For $W \in \beta$ arbitrary, so W is an open symmetric G-convex entourage, define a multifunction $F_W : X \to 2^X$ by $F_W(x) = W(x)$. It is clear that F_W has G-convex values. Also $F_W^{-1}(y) = W^{-1}(y) = W(y)$ as W is symmetric. By Theorem 3.1 there exists $x_W \in X$ such that $g(x_W) \cap F_W^{-1}(x_W) \neq \emptyset$. Let z_W be an element of this intersection. Thus $x_W \in F_W(z_W) \subset F_W\big(g(x_W)\big) = W\big(g(x_W)\big)$.

For each $W \in \beta$, let $H_W = \big\{x \in X : x \in \overline{W}\big(g(x)\big)\big\}$ which is nonempty by the above arguments. Moreover H_W is closed. Indeed, let $\{x_\delta\}$ be a net in H_W converging to x_0. Then there exists a net $\{u_\delta\}$ such that $x_\delta \in \overline{W}(u_\delta)$ and $u_\delta \in g(x_\delta)$. As X is compact, without loss of generality we may assume $u_\delta \to u_0 \in X$. As g is upper semicontinuous, it has a closed graph so $u_0 \in g(x_0)$. Also $(x_\delta, u_\delta) \in \overline{W}$ so $(x_0, u_0) \in \overline{W}$, that is, $x_0 \in \overline{W}(u_0) \subset \overline{W}\big(g(x_0)\big)$ and H_W is closed.

As any finite intersection of elements in β is again an element of β, the compactness of X implies $\bigcap\{H_W : W \in \beta\} \neq \emptyset$. For x_0 a member of this intersection, $x_0 \in \overline{W}\big(g(x_0)\big)$ for all $W \in \beta$. We claim x_0 is a fixed point of g. As in the proof of Lemma 1, it is enough to show $x_0 \in V\big(g(x_0)\big)$ for any V in an arbitrary basis for the uniformity $\mathcal{U}$. So let V be arbitrary but fixed. We may choose a closed symmetric entourage W_1 and a $W_2 \in \beta$ such that $W_2 \subset W_1 \subset V$. Then $x_0 \in \overline{W_2}\big(g(x_0)\big) \subset W_1\big(g(x_0)\big) \subset V\big(g(x_0)\big)$, which completes the proof. ∎

This result extends [**17**, Theorem 2.1] to G-convex spaces as well as considering upper semicontinuous rather that continuous multifunctions.

When the domain X is not compact, under stronger conditions for the mapping $g : X \to 2^X$ we have:

THEOREM 4.2. *Let $(X;\Gamma)$ be a locally G-convex space, $D \subset X$ closed and G-convex, and $g : D \to 2^D$ upper semicontinuous with compact G-convex values. If for some $e \in D$ the following implication holds:*

$$\big(V = G\text{-co}\, g(V) \text{ or } V \subset g(V) \cup \{e\}\big) \Rightarrow V \text{ is relatively compact}$$

for any subset V of D,

then g has a fixed point.

PROOF: In the proof we employ some ideas from the paper of Daneš [**3**]. Define a net $\{y_n\}$ as follows: $y_0 = e$ and $y_{n+1} \in g(y_n)$. Let $Y = \{y_n : n \geqslant 0\}$. Then $Y \subset g(Y) \cup \{e\}$ so by assumption, Y is relatively compact. The set Z of limit points of Y is therefore nonempty and moreover $Z \subset g(Z)$. Indeed, for arbitrary $z_0 \in Z$, there exists a subnet $y_{n_i} \to z_0$, $y_{n_i} \in Y$. By construction of the net Y, we have $(y_{n_i}, y_{n_i - 1}) \in \text{Graph}(g|_{\overline{Y}})$ which is compact by the compactness of $\overline{Y}$ and upper semicontinuity of g. Therefore $(y_{n_i}, y_{n_i-1}) \to (z_0, z_1)$ for some $z_1 \in Z$. This means $z_0 \in g(z_1)$ and so $Z \subset g(Z)$.

Let Ω be the family of all subsets $K \subset D$ such that $Z \subset K$ and $G\text{-co}\, g(K) \subset K$. Then $\Omega \neq \emptyset$ as $D \in \Omega$. Let $V = \bigcap\{K : K \in \Omega\}$, which is nonempty as $Z \subset V$.

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Also $G\text{-co}\,g(V) \subset G\text{-co}\,g(K) \subset K$ for all $K \in \Omega$. Therefore $G\text{-co}\,g(V) \subset V$ and since $G\text{-co}\,g(V) \in \Omega$ is clear, then $V \subset G\text{-co}\,g(V)$. Thus $V = G\text{-co}\,g(V)$ so by assumption, V is relatively compact. Applying now Theorem 4.1 with $X = \overline{V}$, we conclude the mapping g has a fixed point. $\square$

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