J. Austral. Math. Soc. **22** (Series A) (1976), 212–220.

GRAPHS WITH STABILITY INDEX ONE

Dedicated to George Szekeres on the occasion of his 65th birthday
(in anticipation of his $(65 + n)^{\text{th}}$ birthday, where n is a large finite integer)

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(Received 18 December 1974; revised 11 March 1975)

Communicated by Jennifer Seberry Wallis

Abstract

We consider the effect on the stability properties of a graph G, of the presence in the automorphism group of G of automorphisms $(uv)h$, where u and v are vertices of G, and h is a permutation of vertices of G excluding u and v. We find sufficient conditions for an arbitrary graph and a cartesian product to have stability index one, and conjecture in the latter case that they are necessary. Finally we exhibit explicitly a large class of graphs which have stability index one.

1. Introduction

In this paper, all graphs G considered are finite, undirected and are without loops and multiple edges. $V(G)$ denotes the vertex set of the graph G, and $|S|$ the cardinality of the set S. All basic graph-theoretical concepts used in this paper are defined in Harary (1969) and the permutation group terminology is as in Wielandt (1964).

If u and v are adjacent vertices of G, we write $u \sim v$; otherwise $u \not\sim v$. If $u \in V(G)$, the *neighbourhood*, $N_G[u]$, of u in G is the set $\{v \in V(G) : u \sim v\}$. The *degree* of u, denoted $\deg_G u$, is the cardinality of $N_G[u]$. If the set $S = \{v_1, \cdots, v_n\} \subseteq V(G)$, then $G_S = G_{v_1, \cdots, v_n}$ denotes the subgraph of G induced by $V(G)\backslash S$.

$\Gamma(G)$ denotes the automorphism group of G. If $\Gamma(G)$ is a transitive group, we say G is *vertex-transitive*. $\Gamma(G)_S = \Gamma(G)_{v_1,\cdots,v_n}$, where $S = \{v_1, \cdots, v_n\} \subseteq V(G)$, is the subgroup of $\Gamma(G)$ all of whose elements fix each element of S. $\Gamma(G)_S$ is considered to act only on $V(G)\backslash S$. We say that G is *semistable* at $v \in V(G)$ if $\Gamma(G_v) = \Gamma(G)_v$ (see Holton (1973a)). If G is semistable at v for all $v \in V(G)$, then G is said to be *completely semistable*. G is said to have *stability index* n, denoted s.i. $(G) = n$, if there is a sequence $\{v_1, \cdots, v_n\}$ of distinct vertices of $V(G)$ of maximal length such that

$$\Gamma(G_{v_1,v_2,\cdots,v_k}) = \Gamma(G)_{v_1,v_2,\cdots,v_k} \qquad \text{for all} \qquad k = 1,2,\cdots,n$$

(see Grant (1974)). If $n = |V(G)|$ then G is said to be *stable* (see Holton (1973)). If no such sequence exists, G is said to have stability index zero.

If G_1 and G_2 are graphs, then $G_1[G_2]$ denotes the *composition* of G_1 round G_2, $G_1 \times G_2$ denotes the *product* (or cartesian product) of G_1 and G_2, and nG_1 denotes the union of n copies of G_1 (see Harary (1969), pp. 21f).

A graph G is called *prime* if G is nontrivial and $G = G_1 \times G_2$ implies either G_1 or G_2 is trivial. If G is not prime and is nontrivial, then G is *composite*.

In this paper, we investigate stability index one graphs. We show that the presence of $(v_1v_2\cdots v_n)h$ and the absence of $(v_1v_2\cdots v_n)$ in $\Gamma(G)$ where h fixes $v_1, v_2, \cdots, v_n$ indicates that $G_{v_1,v_2,\cdots,v_{n-1}}$ is not semistable at the vertex v_n. Then we find sufficient conditions for a graph to have stability index one. These are (i) Property A: for all u, $v \in V(G)$ there exists $(uv)h \in \Gamma(G)$, where h fixes u and v, and (ii) Property B: for all u, $v \in V(G)$, $u \neq v$, $(uv) \notin \Gamma(G)$.

We next consider cartesian products and show that, in most cases if Property A holds, so does Property B, so that connected graphs with Property A almost always have stability index one. We conjecture that the converse holds. That is, if s.i. $(G) = 1$ for a connected composite graph G, then G and its factors possess Property A. We conclude with a list of some graphs which have this property. Many vertex-transitive graphs do have this property, but we are, as yet, unable to characterise them.

2. Semistability and specific automorphisms

We use the following result, which is Theorem 2 of Holton and Grant (1975))

LEMMA 1. *The graph G is semistable at the vertex v if and only if $N_G[v]$ is fixed by $\Gamma(G_v)$.*

THEOREM 2. *Let G be a graph, $S = \{v_1, v_2, \cdots v_n\} \subseteq V(G)$ and suppose that $(v_1v_2\cdots v_n)h \in \Gamma(G)$, where S is pointwise fixed by h. If there exists $u \in V(G)$, $u \notin S$ such that $u \sim v_n$ and $u \not\sim v_i$ for some $v_i \in S$, then $G_{v_1,v_2,\cdots,v_{n-1}}$ is not semistable at v_n.*

PROOF. Suppose there exists $u \notin S$ such that $u \sim v_n$ and $u \not\sim v_i$ for some $v_i \in S$. Let $g = [(v_1v_2\cdots v_n)h]^{n-i}$, so $g \in \Gamma(G)$. As $(v_1v_2\cdots v_n)$ acts on S, and h acts on $V(G)\backslash S$, $(v_1v_2\cdots v_n)$ and h commute, so $g = (v_1v_2\cdots v_n)^{n-i}h^{n-i}$. As $u \notin S$, $u^g = u^{h^{n-i}}$, and $u^{h^{n-i}} \in V(G)\backslash S$. Also $v_i^g = v_n$.

g preserves adjacency in G, so, as $u \not\sim v_i$, $u^g \not\sim v_i^g$, i.e. $u^{h^{n-i}} \not\sim v_n$. So $u \in N_{G_{v_1,v_2,\cdots,v_{n-1}}}[v_n]$ and $u^{h^{n-i}} \notin N_{G_{v_1,v_2,\cdots,v_{n-1}}}[v_n]$, i.e. h^{n-i} does not fix $N_{G_{v_1,v_2,\cdots,v_{n-1}}}[v_n]$. Now $(v_1v_2\cdots v_n)^{n-i}h^{n-i}$ preserves adjacency in G, and $(v_1v_2\cdots v_n)^{n-i}$ acts only

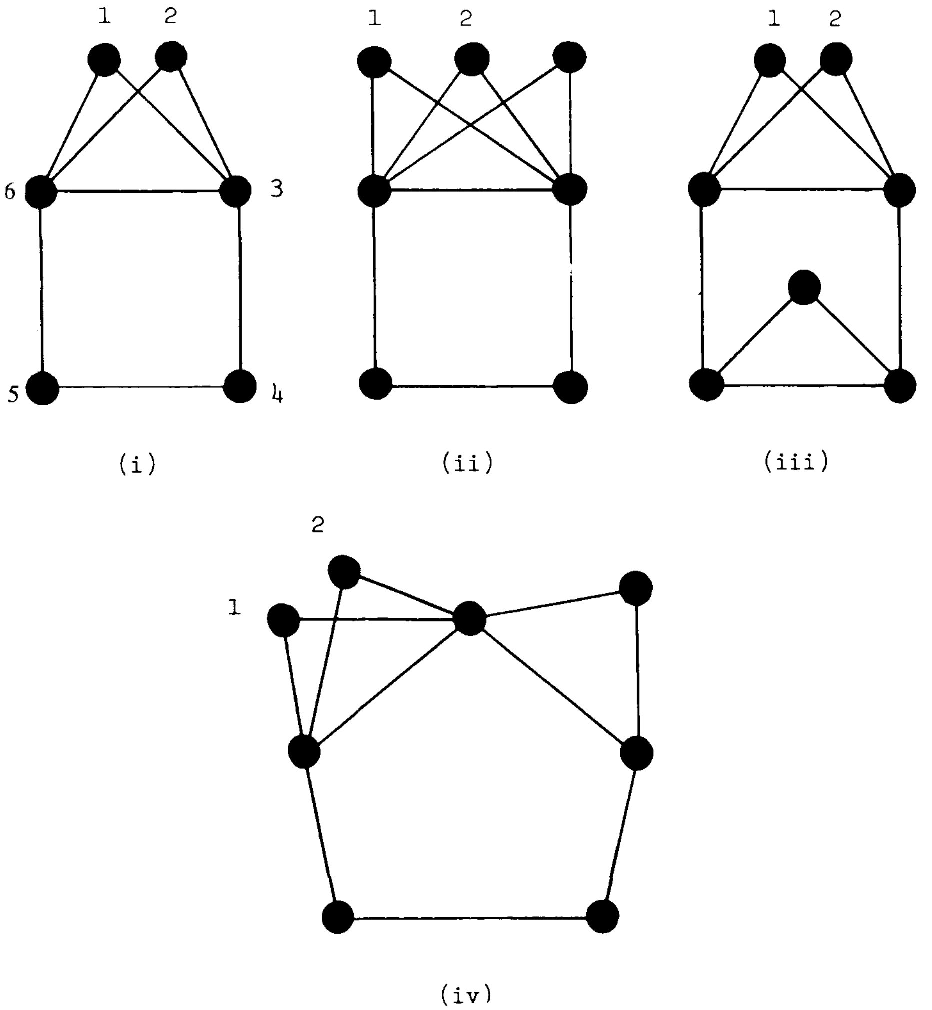

Figure 1

on S, so h^{n-i} preserves adjacency in G_S. Thus $h^{n-i} \in \Gamma(G_S)$, i.e. $h^{n-i} \in \Gamma(G_{v_1,v_2,\cdots v_{n-1},v_n})$. But from above, h^{n-i} does not fix $N_{G_{v_1,v_2,\cdots,v_{n-1}}}[v_n]$. So by Lemma 1, $G_{v_1,v_2,\cdots,v_{n-1}}$ is not semistable at v_n. □

The case $n = 2$ will be useful in the next section.

COROLLARY. *Let G be a graph and $S = \{v_1, v_2, \cdots, v_n\} \subseteq V(G)$. If $(v_1v_2\cdots v_n)h \in \Gamma(G)$, where S is pointwise fixed by h, and $(v_1v_2\cdots v_n) \notin \Gamma(G)$ then $G_{v_1,v_2,\cdots,v_{n-1}}$ is not semistable at v_n.*

PROOF. We show that, if for every $u \notin S$, $u \sim v_n$ implies $u \sim v_i$ for all $v_i \in S$, and if $(v_1 v_2 \cdots v_n)h \in \Gamma(G)$ where S is pointwise fixed by h, then $(v_1 v_2 \cdots v_n) \in \Gamma(G)$.

Let $g = (v_1 v_2 \cdots v_n)h \in \Gamma(G)$. Then if $v, w \in V(G)\backslash S$, $v \sim w$ if and only if $v^g \sim w^g$ which is equivalent to $v^h \sim w^h$. Now if $v_i, v_j \in S$, $v_i \sim v_j$ if and only if $v_i^h \sim v_j^h$, since $v_k^h = v_k$ for all $v_k \in S$. Now if $v_j \in S$ and $v_j \sim v_n$, then $v_j^{g^i} \sim v_n^{g^i}$, i.e. $v_{j+i} \sim v_{n+i} = v_i$ (where the labelling of S is modulo n), and if $v_j \in S$ and $v_j \sim v_i$, then $v_j^{g^{n-i}} \sim v_i^{g^{n-i}}$, i.e. $v_{j+n-i} \sim v_n$. So $|N_G[v_n] \cap S| = |N_G[v_i] \cap S|$. As $v_n^{g^i} = v_i$, and g preserves degree, $|N_G[v_n]| = |N_G[v_i]|$. So

$$|N_G[v_n] \cap (V(G)\backslash S)| = |N_G[v_i] \cap (V(G)\backslash S)|.$$

But if $w \in V(G)\backslash S$, $w \sim v_n$ implies $w \sim v_i$ for all $v_i \in S$, by hypothesis. So $w \sim v_i$ for some $v_i \in S$ implies $w \sim v_n$, which then implies $w \sim v_j$ for all $v_j \in S$.

Thus, if $v_j \in S$, $w \in V(G)\backslash S$, $v_j \sim w$ if and only if $v_j^g \sim w^g$, if and only if $v_{j+1} \sim w^h$, if and only if $v_j \sim w^h$, if and only if $v_j^h \sim w^h$, since $v_j^h = v_j$ for all $v_j \in S$. Then the above results give, for all $x, y \in V(G)$,

$$x \sim y \quad \text{if and only if} \quad x^h \sim y^h \quad \text{i.e.} \quad h \in \Gamma(G).$$

Then $h^{-1} \in \Gamma(G)$. So $(v_1 v_2 \cdots v_n) = (v_1 v_2 \cdots v_n)h \cdot h^{-1} \in \Gamma(G)$, as required.

The statement thus proved is equivalent to: if $(v_1 v_2 \cdots v_n)h \in \Gamma(G)$ where S is pointwise fixed by h, and $(v_1 v_2 \cdots v_n) \notin \Gamma(G)$, then there exists $u \notin S$ such that $u \sim v_n$ and $u \not\sim v_i$ for some $v_i \in S$. Then Theorem 2 gives the result. □

The converse of the theorem does not hold, as shown by the graph of Figure 1 (i), which has $(12)(36)(45) \in \Gamma(G)$. G_1 is not semistable at 2 but $N_{G_1}[2] = N_{G_2}[1]$.

The graphs of Figure 1 show that, if $N_{G_2}[1] = N_{G_1}[2]$, G may or may not be semistable at 1 and G_1 may or may not be semistable at 2: only in graphs (i) and (ii) is G semistable at 1, and only in graphs (ii) and (iii) is G_1 semistable at 2. Similar examples may easily be constructed for $n > 2$, where $n = |S|$ in the above theorem.

3. Some graphs with stability index one

We require the following result, a corollary to Theorem 3 of Holton and Grant (1975)

LEMMA 3. *All regular graphs are completely semistable.*

THEOREM 4. *Let G be a graph such that for all $u, v \in V(G)$, there exists $(uv)h \in \Gamma(G)$, where h (which depends on u and v) fixes u and v. Then s.i.$(G) = 1$ if and only if $N_{G_u}[v] \neq N_{G_v}[u]$ for all $u, v \in V(G)$ such that $u \neq v$.*

PROOF. (i) Assume $N_{G_u}[v] \neq N_{G_v}[u]$ for all $u, v \in V(G)$, $u \neq v$. Take any $u, v \in V(G)$ with $u \neq v$. So there exists $w \in V(G)$ such that $w \neq u, v$ and either

$w \sim u$ and $w \not\sim v$ or $w \not\sim u$ and $w \sim v$. Without loss of generality, assume the latter case. Applying Theorem 2, with $S = \{u, v\}$, we have that G_u is not semistable at v. This holds for arbitrary $u, v \in V(G)$, so s.i. $(G) \leqq 1$.

By hypothesis, G is vertex-transitive, and hence regular, as $\Gamma(G)$ preserves degree. So by Lemma 3, G is completely semistable. Thus s.i. $(G) \geqq 1$.

Thus s.i. $(G) = 1$ as required.

(ii) Assume $N_{G_u}[v] = N_{G_v}[u]$ for some $u, v \in V(G)$, $u \neq v$. As in (i), G is regular and so is semistable at v. Let $\deg_G x = r$ for all $x \in V(G)$, and let $y \in V(G) \backslash \{u, v\}$. Either $y \in N_{G_u}[v] = N_{G_v}[u]$ so $y \sim v$ and $y \sim u$, or $y \notin N_{G_u}[v] = N_{G_v}[u]$ so, as $y \neq u$ and $y \neq v$, $y \not\sim v$ and $y \not\sim u$. So in $G_{v,u}$, $\deg y = r$ or $\deg y = r - 2$. Thus $N_{G_v}[u]$ is the set of vertices of degree $r - 2$ in $G_{v,u}$, and hence is fixed by $\Gamma(G_{v,u})$. Then by Lemma 1, G_v is semistable at u. So s.i. $(G) \geqq 2$, i.e. s.i. $(G) \neq 1$. □

Suppose we have a graph G with Property A. Either $N_{G_u}[v] \neq N_{G_v}[u]$ for all $u, v \in V(G)$, $u \neq v$, so s.i. $(G) = 1$ from the above theorem, or $N_{G_u}[v] = N_{G_v}[u]$ for some $u, v \in V(G)$. Then, in the latter case, it follows directly from Lemma 5 of Holton and Grant (1975) that $(uv) \in \Gamma(G)$, i.e. $\Gamma(G)$ contains a transposition. It was conjectured in Holton and Grant (1975) that if G is vertex-transitive and $\Gamma(G)$ contains a transposition, then G is stable. So in this case we

Conjecture. If for all $u, v \in V(G)$, there exists $(uv)h \in \Gamma(G)$, where h fixes u and v, then either s.i. $(G) = 1$ or G is stable.

It is proved in Holton and Grant (1975), that all graphs G with Property A, for which $\Gamma(G)$ contains a transposition, are of the form $H[K_n]$ or $H[\bar{K}_n]$, for some $n > 1$, where H is a vertex-transitive graph, and K_n and $\bar{K}_n$ denote the complete graph on n vertices and its complement, respectively. The conjecture is supported by the results of Grant (1975) and Holton and Grant (1975) that if H is stable, or $n \geqq 2$ and H is C_m (the cycle on m vertices) for $m \geqq 3$, then $H[K_n]$ and $H[\bar{K}_n]$ are stable.

We now show that, if G has Property A, and is also a cartesian product of nontrivial graphs, then in most cases G is either stable, or s.i. $(G) = 1$.

Lemma 5. *Let G_1 and G_2 be graphs without trivial components. If u, $v \in V(G_1 \times G_2)$, $u \neq v$ and $\deg_{G_1 \times G_2} v > 2$ then*

$$N_{(G_1 \times G_2)_u}[v] \neq N_{(G_1 \times G_2)_v}[u].$$

Proof. Let $v = (v_1, v_2)$, $u = (u_1, u_2)$, where u_i, $v_i \in V(G_i)$, $i = 1, 2$.

As $\deg_{G_1 \times G_2} v = \deg_{G_1} v_1 + \deg_{G_2} v_2$, and $\deg_{G_1 \times G_2} v > 2$ by hypothesis, either $\deg_{G_1} v_1 > 1$ or $\deg_{G_2} v_2 > 1$. Without loss of generality, suppose the latter. Then there exists $v_2' \in V(G_2) : v_2' \neq u_2$ and $v_2' \sim v_2$.

(i) Suppose $u_1 \neq v_1$. Then $(v_1, v_2) \sim (v_1, v_2')$, $(u_1, u_2) \nsim (v_1, v_2')$ and $(v_1, v_2') \neq (u_1, u_2)$,

so
$$N_{(G_1 \times G_2)_u}[v] \neq N_{(G_1 \times G_2)_v}[u].$$

(ii) Suppose $u_1 = v_1$. Then $u_2 \neq v_2$ as $u \neq v$. As the component of G_1 containing v_1 is nontrivial, there exists $v_1' \in V(G_1)$ such that $v_1 \sim v_1'$. Then

$$(v_1, v_2) \sim (v_1', v_2), \qquad (u_1, u_2) = (v_1, u_2) \nsim (v_1', v_2)$$

and $(v_1', v_2) \neq (u_1, u_2)$, so $N_{(G_1 \times G_2)_u}[u] \neq N_{(G_1 \times G_2)_v}[v]$. □

LEMMA 6. *Let G_1 and G_2 be graphs with $(u_i v_i)h_i \in \Gamma(G_i)$, where $u_i, v_i \in V(G_i)$ and h_i fixes u_i $(i = 1, 2)$. Then*

$$((u_1, u_2)(v_1, v_2))h \in \Gamma(G_1 \times G_2),$$

where h fixes (u_1, u_2) and (v_1, v_2).

PROOF. If $g_i \in \Gamma(G_i)$, $i = 1, 2$, define $g: V(G_1 \times G_2) \to V(G_1 \times G_2)$ by $(w_1, w_2)^g = (w_1^{g_1}, w_2^{g_2})$ for all $(w_1, w_2) \in V(G_1 \times G_2)$. Then g is an automorphism of $V(G_1 \times G_2)$, and it is easily verified that g preserves adjacency in $G_1 \times G_2$, so $g \in \Gamma(G_1 \times G_2)$.

In particular, put $g_1 = (u_1 v_1)h_1$, $g_2 = (u_2 v_2)h_2$. Then $g \in \Gamma(G_1 \times G_2)$, where $g = ((u_1, u_2)(v_1, v_2))h$ and h fixes (u_1, u_2) and (v_1, v_2), as required. □

THEOREM 7. *Let G_i, $i = 1, \cdots, r$ be graphs, where r is some positive integer, and let $G = G_1 \times \cdots \times G_r$. If*

(i) *for each i, for all $v_1, v_2 \in V(G_i)$ there exists $(v_1 v_2)h_{12} \in \Gamma(G_i)$ where h_{12} fixes v_1 and v_2, and*

(ii) *$G \neq nH$ where H is K_1, K_2, C_4 or any prime graph, for any $n \geqq 1$, then s.i. $(G) = 1$.*

PROOF. By (i) and Lemma 6, for all $u, v \in V(G)$, there exists $(uv)h_{uv} \in \Gamma(G)$ where h_{uv} fixes u and v. Hence G is vertex-transitive, so all its components are isomorphic. Then by (ii), these components are composite, and are not $C_4 = K_2 \times K_2$. So $G = n(G_1 \times G_2)$ for some $n \geqq 1$, where G_1 and G_2 are regular, nontrivial, connected and are not both K_2. Now

$$G = n(G_1 \times G_2) = \bar{K}_n \times (G_1 \times G_2) = (\bar{K}_n \times G_1) \times G_2 = (nG_1) \times G_2.$$

nG_1 and G_2 have no trivial components, and one has degree $\geqq 2$ and the other degree $\geqq 1$ i.e. G has degree > 2. So by Lemma 5, for all $u, v \in V(G)$, $u \neq v$, $N_{G_u}[v] \neq N_{G_v}[u]$.

Then by Theorem 4, s.i. $(G) = 1$ as required. □

Note that, of the exceptional graphs mentioned in (ii) of the previous theorem, nK_1, nK_2 and nC_4 are stable. The only remaining case is nH where H is prime.

Note also that, in the case $r = 1$, (i) becomes for all $u, v \in V(G)$, there exists $(uv)h_{uv} \in \Gamma(G)$, where h_{uv} fixes u and v. So if G is a connected, composite graph with property A, Theorem 7 gives that s.i. $(G) = 1$ unless $G = C_4$.

4. Some graphs G with property A

1. Any graph G such that $\Gamma(G) \supseteq \Gamma(C_p)$, where $p = |V(G)| \geqq 3$. This includes K_n $(n \geqq 2)$ and C_n $(n \geqq 3)$.

2. The Petersen graph (Figure 2), as

$(12)(35)(1'2')(3'5')$,	$(13)(45)(1'3')(4'5')$,
$(12')(1'4')(43')(55')$,	$(13')(23)(42')(55')$,
$(11')(23')(2'5')(4'5)$	are automorphisms of this graph.

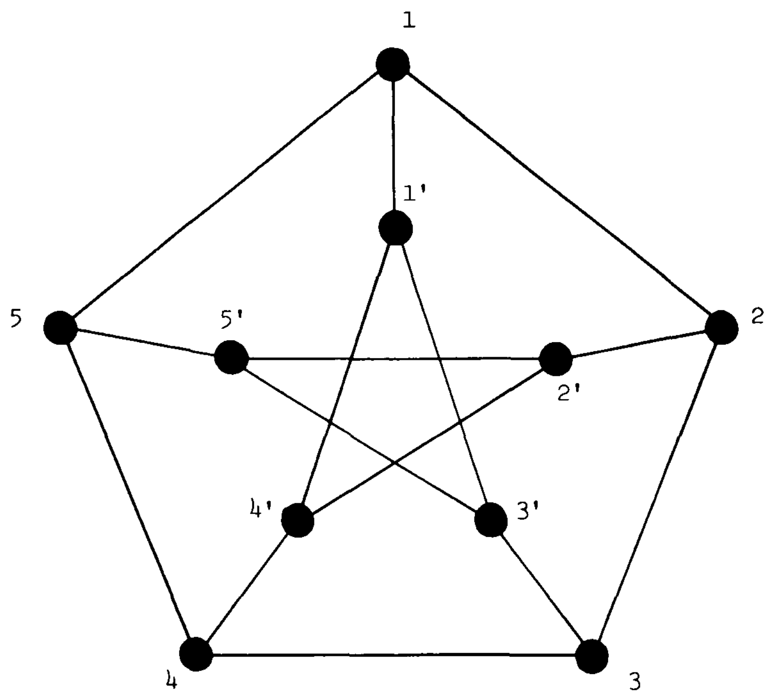

Figure 2

3. Vertex-transitive graphs with a prime number of vertices. From Theorem 1 of Turner (1967):

All connected vertex-transitive graphs having a prime number of vertices p are starred polygons, where a *starred polygon* G is a graph whose vertices may be labelled in such a way that $v_i \sim v_j$ if and only if $v_{i+k} \sim v_{j+k}$, $k = 1, \cdots, p-1$, $p = |V(G)|$.

Let $g_m : v_i \mapsto v_{m-i}$, $i = 1, \cdots, p$ for some positive integer m (subscripts modulo p). g_m is an automorphism of $V(G)$, and $v_i \sim v_j$ if and only if $v_{m-j} \sim v_{m-i}$, $i, j = 1, \cdots, p$ (by putting $k = m - j - i$ in the definition above), so g_m preserves adjacency in G, i.e. $g_m \in \Gamma(G)$. Also $g_m : v_{m-i} \mapsto v_{m-(m-i)} = v_i$, so g_m is a product of transpositions.

Given v_i, $v_j \in V(G)$, put $m = i + j$. Then $g_m : v_i \mapsto v_{m-i} = v_j$, i.e. $g_{i+j} = (v_i v_j)h$, where h fixes v_i and v_j. So G has the Property A. (Note that all vertex-transitive graphs with a prime number of vertices are either connected or $\bar{K}_p$).

Many vertex-transitive graphs have Property A, but not all do. For example, for G as of Figure 3, there is no element of $\Gamma(G)$ of the form $(12)h$. Also, s.i.$(G \times K_2) > 1$ for G of Figure 3, so G, H connected, nontrivial, vertex-transitive graphs, not both K_2 does not imply that s.i. $(G \times H) = 1$.

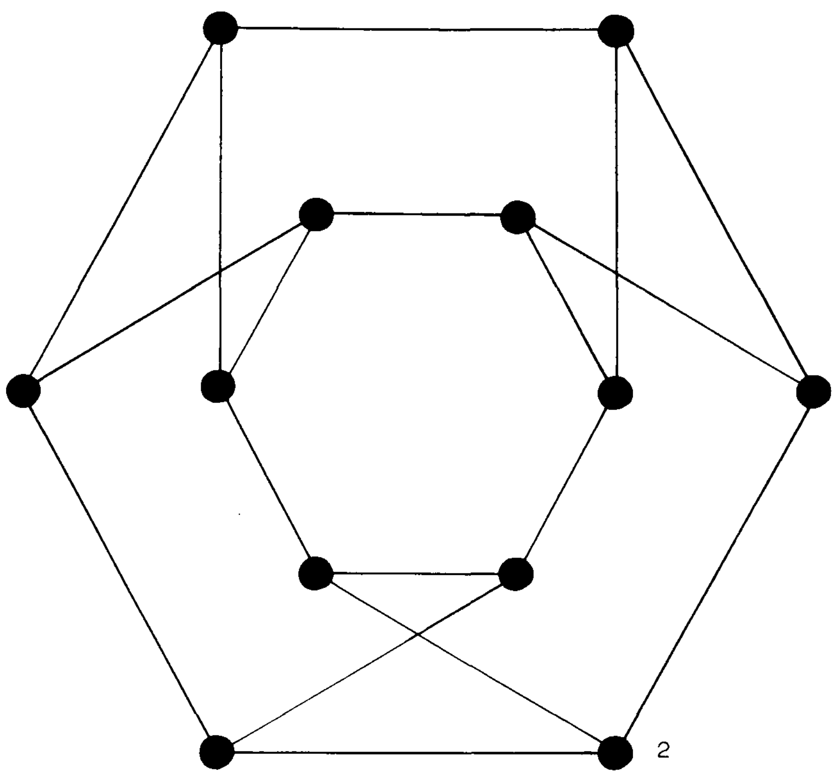

Figure 3

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We conjecture that the converse of Theorem 7 holds.

CONJECTURE. If G_1, $G_2, \cdots, G_r$, $r \geqq 2$, are nontrivial connected graphs for which s.i. $(G_1 \times \cdots \times G_r) = 1$, then for all i, and for all $v_1, v_2 \in V(G_i)$, there exists $(v_1 v_2)h_{12} \in \Gamma(G_i)$, where h_{12} fixes v_1 and v_2, unless $G_1 \times \cdots \times G_r$ is $P_2 \times P_3$ or $P_2 \times P_4$.

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